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Categorification of invariants in gauge theory and symplectic geometry*

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Abstract. This is a mixture of survey article and research announcement. We discuss instanton Floer homology for 3 manifolds with boundary. We also discuss a categorification of the Lagrangian Floer theory using the unobstructed immersed Lagrangian correspondence as a morphism in the category of symplectic manifolds.

During the year 1998–2012, those problems have been studied emphasizing the ideas from analysis such as degeneration and adiabatic limit (instanton Floer homology) and strip shrinking (Lagrangian correspondence). Recently we found that replacing those analytic approach by a combination of cobordism type argument and homological algebra, we can resolve various difficulties in the analytic approach. It thus solves various problems and also simplify many of the proofs.

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1. Introduction and review

The research defining invariants by using moduli spaces in differential geometry and topology started around 1980's. One of its first examples is Donaldson's polynomial invariant of smooth 4 manifolds [D3]. Various 'quantum' invariants of knots which appeared around the same time have a similar flavor and actually they turn out to be closely related to each other. (Instanton) Floer homology of 3 manifolds (homology 3 spheres) appeared late 1980's [Fl2] and it was soon realized that instanton Floer homology provides the basic framework to define a relative version of the Donaldson invariant. The notion of topological field theory was introduced by Witten [Wi1] inspired by this relative Donaldson invariant. Soon after that Witten [Wi2] found an invariant of 3 manifolds (possibly equipped with knot and link) and its relative version. This is a generalization of quantum invariant of knot. The relative version of Witten's invariant uses conformal block as its 2 dimensional counterpart. Segal [Se1], [Se2] introduced categorical formulation of conformal field theory and of several related theories. Since then various categorifications have been introduced and studied by many mathematicians. In this article the author surveys some of them where A_∞ category appears.

Table 1.

	invariants	Case $n = 4$	Case $n = 3$
n	number	Donaldson invariant	Witten's invariant
$n-1$	group	Floer homology	Conformal block
$n-2$	category	$\mathfrak{Fuk}(R(\Sigma))$	representation of loop group

The gauge theory invariant we discuss in this article is one in the column $n = 4$, of the Table 1.

We begin with a quick review of 4 and 3 dimensional invariants.

1.1. Donaldson invariant

Let X be an oriented 4 manifold and $\mathcal{P}_X \to X$ either a principal $SO(3)$ or $SU(2)$ bundle. (We denote $G = SO(3)$ or $SU(2)$.) We take a Riemannian metric on X, which induces the Hodge $*$ operator on differential k forms.

$$* : \Omega^k(X) \longrightarrow \Omega^{4-k}(X).$$

On 2 forms we have $** = 1$. Therefore $\Omega^2(X)$ is decomposed into a direct sum

$$\Omega^2(X) = \Omega^2_+(X) \oplus \Omega^2_-(X), \quad \Omega^2_\pm(X) = \{u \in \Omega^2(X) \mid *u = \pm u\}.$$

Let $\mathrm{ad}(\mathcal{P}_X) = \mathcal{P}_X \times_G \mathfrak{g}$ be the Lie algebra ($\mathfrak{g} = \mathfrak{so}(3)$ or $\mathfrak{su}(2)$) bundle associated to $\mathcal{P}_X$ by the adjoint representation $G \to \mathrm{Aut}(\mathfrak{g})$. For a connection A of $\mathcal{P}_X$ its curvature F_A is a section of $\Omega^2(X) \otimes \mathrm{ad}(\mathcal{P}_X)$. We decompose it into

$$F_A = F_A^+ + F_A^-,$$

where $F_A^\pm$ is a section of $\Omega^2_\pm(X) \otimes \mathrm{ad}(\mathcal{P}_X)$.

A connection A is called an Anti-Self-Dual (or ASD) connection if $F_A^+ = 0$.

We denote by $\mathcal{A}(\mathcal{P}_X)$ the set of all (smooth) connections on $\mathcal{P}_X$ and $\mathcal{G}(\mathcal{P}_X)$ the set of all (smooth) $SU(2)$-gauge transformations of $\mathcal{P}_X$. (The latter is the set of all smooth sections of the bundle $\mathrm{Ad}(\mathcal{P}_X) = \mathcal{P}_X \times_G SU(2)$ which is associated to $\mathcal{P}_X$ by the adjoint action of G on $SU(2)$.) The group $\mathcal{G}(\mathcal{P}_X)$ acts on $\mathcal{A}(\mathcal{P}_X)$ and we denote by $\mathcal{B}(\mathcal{P}_X)$ the quotient space.

Remark 1.1. Note the adjoint action of $SU(2)$ on $SU(2)$ is induced by an action of $SO(3) = SU(2)/\{\pm 1\}$ on $SU(2)$. Therefore the bundle $\mathcal{P}_X \times_G SU(2)$ is defined in the case $G = SO(3)$, too.

A section g of $\mathcal{P}_X \times_G SO(3)$ lifts to a section of $\mathcal{P}_X \times_G SU(2)$ if and only if the restriction of g to one skeleton is homotopic to the identity. So our choice of gauge transformation group coincides with the one in [BD, page 198].

Hereafter, we mean by gauge transformation $SU(2)$ gauge transformation in both cases $G = SU(2), SO(3)$, unless otherwise specified.

We denote by $\mathcal{M}(\mathcal{P}_X) \subset \mathcal{B}(\mathcal{P}_X)$ the set of all $\mathcal{G}(\mathcal{P}_X)$ equivalence classes of ASD connections.

In the simplest case, the Donaldson invariant of X is the order of the set $\mathcal{M}(\mathcal{P}_X)$ (counted with appropriate sign), and is an integer. More generally it is regarded as a polynomial map $H^*(\mathcal{B}(\mathcal{P}_X)) \to \mathbb{Z}$ obtained by

$$(1.1) \qquad c \longmapsto \int_{\mathcal{M}(\mathcal{P}_X)} c.$$

Actually since $\mathcal{M}(\mathcal{P}_X)$ is non-compact, we need to study the behavior of the cohomology class c at infinity of $\mathcal{M}(\mathcal{P}_X)$, carefully. Another problem is that $\mathcal{M}(\mathcal{P}_X)$ has in general a singularity. We do not discuss these points in this section. Donaldson used a map $\mu : H_2(X) \to H^2(\mathcal{B}(\mathcal{P}_X))$. This map is defined by the slant product $c \mapsto p_1/c$, where p_1 is the 1st Pontryagin or the second Chern class of the universal bundle on $\mathcal{B}(\mathcal{P}_X) \times X$.

On the subring generated by the image of this map μ, the integration (1.1) behaves nicely and defines an invariant. (We need to assume that the number b_2^+ which is the sum of the multiplicities of positive eigenvalues of the intersection form on $H_2(X; \mathbb{Q})$, is not smaller than 2, for this invariant to be well defined.) In that case, we have a multi-linear map on $H_2(X)$ which is called Donaldson's polynomial invariant. We denote it by

$$(1.2) \qquad Z(c_1, \ldots, c_k; \mathcal{P}_X) = \int_{\mathcal{M}(\mathcal{P}_X)} \mu(c_1) \wedge \cdots \wedge \mu(c_k) \in \mathbb{Z}$$

for $c_i \in H_2(X)$. Note the integration makes sense only when

$$\dim \mathcal{M}(\mathcal{P}_X) = 2k,$$

($\deg c_i = 2$). The dimension of $\mathcal{M}(\mathcal{P}_X)$ is determined by the second Chern (or the first Pontryagin) number of $\mathcal{P}_X$. So if we fix the second Stiefel–Whitney class of $\mathcal{P}_X$ the isomorphism class of bundle $\mathcal{P}_X$ for which (1.2) can be non-zero is determined by k. So we omit $\mathcal{P}_X$ and write $Z(c_1, \ldots, c_k)$ sometimes.

1.2. Floer homology (instanton homology)

Let M be a 3 manifold and $\mathcal{P}_M$ a principal $G = SO(3)$ or $SU(2)$ bundle on it.

Assumption 1.2. We assume that one of the following two conditions is satisfied.

(1) $G = SO(3)$ and $w^2(\mathcal{P}_M) \neq 0 \in H^2(M; \mathbb{Z}_2)$. (Here $w^2(\mathcal{P}_M)$ is the second Stiefel–Whitney class.)
(2) $G = SU(2)$ and $H(M; \mathbb{Z}) \cong H(S^3; \mathbb{Z})$.

The notation $\mathcal{A}(\mathcal{P}_M)$, $\mathcal{G}(\mathcal{P}_M)$, $\mathcal{B}(\mathcal{P}_M)$ are defined in the same way as the 4 dimensional case. We denote by $R(M;\mathcal{P}_M) \subset \mathcal{B}(\mathcal{P}_M)$ the set of all $\mathcal{G}(\mathcal{P}_M)$ equivalence classes of flat connections.

We assume the following for the simplicity of description.

Assumption 1.3. (1) The set $R(M;\mathcal{P}_M)$ is a finite set.
(2) For any $[a] \in R(M;\mathcal{P}_M)$ the cohomology group $H^1(M;\mathrm{ad}_a(\mathcal{P}_M))$ vanishes. Here the first cohomology group $H^1(M;\mathrm{ad}_a(\mathcal{P}_M))$ is defined by the complex

$$\mathrm{ad}\,\mathcal{P}_M \otimes \Omega^0 \xrightarrow{\;d_a\;} \mathrm{ad}\,\mathcal{P}_M \otimes \Omega^1 \xrightarrow{\;d_a\;} \mathrm{ad}\,\mathcal{P}_M \otimes \Omega^2.$$

Here d_a is defined by $d_a b = db + b \wedge a$. (Note $d_a \circ d_a = 0$ since $F_a = 0$.)

Remark 1.4. We can remove this assumption by appropriately perturbing the defining equation $F_a = 0$ of $R(M;\mathcal{P}_M)$ in a way similar to [D2], [Fl2], [He].

In case Assumption 1.2 (2) is satisfied we put $R_0(M;\mathcal{P}_M) = R(M;\mathcal{P}_M) \setminus \{[0]\}$, where $[0]$ is the gauge equivalence class of the trivial connection. In case of Assumption 1.2 (1) we put $\mathcal{B}_0(\mathcal{P}_M) = R(M;\mathcal{P}_M)$.

We define $\mathbb{Z}_2$ vector space $CF(M;\mathcal{P}_M)$ whose basis is identified with $R(M;\mathcal{P}_M)$. We define a boundary operator

$$\partial : CF(M;\mathcal{P}_M) \longrightarrow CF(M;\mathcal{P}_M)$$

as follows. Let $[a],[b] \in R_0(M;\mathcal{P}_M)$. We fix their representatives a,b. We consider the set of connections A of the bundle $\mathcal{P}_M \times \mathbb{R}$ on $M \times \mathbb{R}$ with the following properties. We use τ as the coordinate of $\mathbb{R}$.

(IF.1) $F_A^+ = 0$.
(IF.2) The L^2-norm of the curvature

$$\int_{M \times \mathbb{R}} \|F_A\|^2 \mathrm{vol}_M \, d\tau$$

is finite.
(IF.3) We require

$$\lim_{\tau \to -\infty} A = a, \qquad \lim_{\tau \to +\infty} A = b.$$

Remark 1.5. We can use Assumption 1.3 (2) to show that if (IF.3) is satisfied then the convergence is automatically of exponential order.

We denote by $\widetilde{\mathcal{M}}(M \times \mathbb{R};a,b)$ the set of all gauge equivalence classes of the connections A satisfying the above conditions (IF.1)–(IF.3).

The $\mathbb{R}$ action induced by the translation of $\mathbb{R}$ factor in $M \times \mathbb{R}$ induces an $\mathbb{R}$ action on $\widetilde{\mathcal{M}}(M \times \mathbb{R};a,b)$. We denote the quotient space by this action by $\mathcal{M}(M \times \mathbb{R};a,b)$.

Theorem 1.6 (Floer). *We can define a map $\mu : R_0(M; \mathcal{P}_M) \to \mathbb{Z}_8$ such that:*

(1) $\mathcal{M}(M \times \mathbb{R}; a, b)$ is decomposed into pieces $\mathcal{M}(M \times \mathbb{R}; a, b; k)$ where $k + 1$ is a natural number congruent to $\mu(b) - \mu(a)$.

(2) By 'generic' perturbation we may assume that $\mathcal{M}(M \times \mathbb{R}; a, b; k)$ is compactified to a manifold with corners of dimension k, outside the singularity set of codimension ≥ 2.

(3) Moreover the boundary of $\mathcal{M}(M \times \mathbb{R}; a, b; k)$ is identified with the disjoint union of the direct product

$$(1.3) \qquad \mathcal{M}(M \times \mathbb{R}; a, c; k_1) \times \mathcal{M}(M \times \mathbb{R}; c, b; k_2),$$

where $c \in R_0(M; \mathcal{P}_M)$ and $k_1 + k_2 + 1 = k$.

Remark 1.7. In the $SO(3)$ case there is an involution of degree 4 and so Floer homology is $\mathbb{Z}_4$ graded. Involution is induced by an $SO(3)$ gauge transformation which does not lift to an $SU(2)$ gauge transformation. See [BD, page 198].

Now we define

$$\langle \partial[a], [b] \rangle \equiv \sharp \mathcal{M}(M \times \mathbb{R}; a, b; 0) \quad \mod 2,$$

and

$$(1.4) \qquad \partial[a] = \sum_{[b]; \mu(b) = \mu(a) - 1} \langle \partial([a]), [b] \rangle [b].$$

Theorem 1.6 (3) implies that the union of the spaces (1.3) over c and k_1, k_2 with $k_1 + k_2$ given is a boundary of some space. Especially in case $k_1 = k_2 = 0$ the union of (1.3) over c is a boundary of 1 dimensional manifold and so its order is even. It implies:

$$\sum_c \sharp \langle \partial[a], [c] \rangle \langle \partial[c], [b] \rangle = 0.$$

Namely $\partial\partial[a] = 0$.

Definition 1.8.

$$HF(M; \mathcal{P}_M) \cong \frac{\mathrm{Ker} \left(\partial : CF(M; \mathcal{P}_M) \longrightarrow CF(M; \mathcal{P}_M) \right)}{\mathrm{Im} \left(\partial : CF(M; \mathcal{P}_M) \longrightarrow CF(M; \mathcal{P}_M) \right)}.$$

We call this group ($\mathbb{Z}_2$ vector space) the instanton Floer homology of $(M, \mathcal{P}_M)$. Actually we can orient the moduli spaces we use and then define Floer homology group over $\mathbb{Z}$.

Floer ([Fl2], [Fl3]) proved that the group $HF(M; \mathcal{P}_M)$ is independent of various choices made in the definitions.

The idea behind this definition is to study the following Chern–Simons functional. We consider the case when $\mathcal{P}_M$ is an $SU(2)$ bundle which is necessary trivial as a smooth $SU(2)$ bundle. We fix a trivialization and then an element of $\mathcal{A}(\mathcal{P}_M)$ is identified with an $su(2)$ valued one form a. We may regard it as a 2×2 matrix valued one form. We then put

$$(1.5) \qquad \mathrm{cs}(a) = \frac{1}{4\pi^2} \int_M Tr\left(\frac{1}{2}a \wedge da + \frac{1}{3}a \wedge a \wedge a\right).$$

This functional descents to a map $\mathcal{B}(\mathcal{P}_M) \to \mathbb{R}/\mathbb{Z}$. In fact if we regard a gauge transformation as a map $M \to SU(2)$ we have

$$\mathrm{cs}(g^*a) = \mathrm{cs}(a) + \deg g.$$

On the other hand any connection A of $\mathcal{P}_M \times \mathbb{R}$ on $M \times \mathbb{R}$ can be transformed to a connection without $d\tau$ component by a gauge transformation. (Note τ is the coordinate of $\mathbb{R}$.) We call it the *temporal gauge*. If we take the temporal gauge and $A|_{M \times \{\tau\}} = a(\tau)$ then the equation $F_A^+ = 0$ is equivalent to

$$(1.6) \qquad \frac{d}{d\tau}a(\tau) = \mathrm{grad}_{a(\tau)}\mathrm{cs}.$$

Here the right hand side is defined by

$$\langle \mathrm{grad}_a\mathrm{cs}, a' \rangle = \frac{d}{ds}\mathrm{cs}(a + sa')\Big|_{s=0}.$$

($\langle \cdot, \cdot \rangle$ is the L^2 inner product.) So $HF(M; \mathcal{P}_M)$ is regarded as a Morse homology of cs. There is a similar functional in the $SO(3)$ case.

1.3. Relative Donaldson invariant

The relation between Donaldson invariant and Floer homology is described as follows.

Let X_1 and X_2 be oriented 4 manifolds with boundary M and $-M$, respectively. We glue X_1 and X_2 at M to obtain a closed oriented 4 manifold X. We consider the case $H_1(M; \mathbb{Q}) = 0$. Then

$$(1.7) \qquad H_2(X; \mathbb{Q}) = H_2(X_1; \mathbb{Q}) \oplus H_2(X_2; \mathbb{Q}).$$

We also assume $b_2^+(X_i)$, the sum of multiplicities of positive eigenvalues of the intersection form on $H_2(X_i; \mathbb{Q})$, is at least 2.

We remark that

$$R_0(M; \mathcal{P}_M) \cong R_0(-M; \mathcal{P}_M).$$

We define a pairing

$$\langle \cdot, \cdot \rangle : CF(M; \mathcal{P}_M) \otimes CF(-M; \mathcal{P}_M) \longrightarrow \mathbb{Z}$$

by

$$\langle [b], [b'] \rangle = \begin{cases} 1 & \text{if } [b] = [b'], \\ 0 & \text{if } [b] \neq [b']. \end{cases}$$

Here we use Floer's chain complex over $\mathbb{Z}$ coefficient. The map $\mathbb{R} \times M \to \mathbb{R} \times -M$ which sends (τ, x) to $(-\tau, x)$ is an orientation preserving diffeomorphism. So

$$\langle \partial_M a, b \rangle = \langle a, \partial_{-M} b \rangle.$$

Therefore the boundary operator ∂_{-M} is the dual to ∂_M. We thus obtain a pairing:

$$\langle \cdot, \cdot \rangle : HF(M; \mathcal{P}_M) \times HF(-M; \mathcal{P}_M) \longrightarrow \mathbb{Z}.$$

Theorem 1.9 (Floer–Donaldson, see [Fu3], [D5]).

(1) If $\partial X_1 = M$ and $\mathcal{P}_{X_1}|_M = \mathcal{P}_M$, then there exists a multilinear map

$$Z(\cdot; X_1, \mathcal{P}_{X_1}) : H_2(X_1; \mathbb{Z})^{\otimes k} \longrightarrow HF(M; \mathcal{P}_M).$$

(2) In the situation we mentioned at the beginning of this subsection we have

$$(1.8) \qquad \begin{aligned} &\langle Z(c_{1,1}, \ldots, c_{1,k_1}; \mathcal{P}_{X_1}), Z(c_{2,1}, \ldots, c_{2,k_2}; \mathcal{P}_{X_2}) \rangle \\ &= Z(c_{1,1}, \ldots, c_{1,k_1}, c_{2,1}, \ldots, c_{2,k_2}; \mathcal{P}_X). \end{aligned}$$

The construction of relative invariant in Theorem 1.9 roughly goes as follows. We take a Riemannian metric on $X_1 \setminus \partial X_1$ such that it is isometric to the direct product $M \times [0, \infty)$ outside a compact set. Let a be a flat connection with $[a] \in R_0(M; \mathcal{P}_M)$.

We consider the set of connections A of $\mathcal{P}_X$ such that

(1) $F_A^+ = 0$.
(2) The L^2-norm of the curvature $\int_{X_1} \|F_A\|^2 \text{vol}_{X_1}$ is finite.
(3) We require $\lim_{\tau \to +\infty} A|_{M \times \{\tau\}} = a$.

We denote the set of gauge equivalence classes of such A by $\mathcal{M}(X_1; a; \mathcal{P}_{X_1})$ and define

$$Z(c_{1,1}, \ldots, c_{1,k_1}; X_1, \mathcal{P}_{X_1})$$

$$= \sum_{a \in R_0(M; \mathcal{P}_M)} \left(\int_{\mathcal{M}(X_1; a; \mathcal{P}_{X_1})} \mu(c_{1,1}) \wedge \cdots \wedge \mu(c_{1,k_1}) \right) [a].$$

We can show that this is a cycle in $CF(M; \mathcal{P}_M)$ by studying the boundary of the moduli space $\mathcal{M}(X_1; a; \mathcal{P}_{X_1})$, that is,

$$\partial \mathcal{M}(X_1; a; \mathcal{P}_{X_1}) = \bigcup_b \mathcal{M}(X_1; b; \mathcal{P}_{X_1}) \times \mathcal{M}(M \times \mathbb{R}; b, a).$$

To show (1.8) we consider the following sequence of metrics on X. We take compact subsets K_i of X_i such that

$$\text{Int } X_1 \setminus K_1 \cong M \times (0, \infty), \quad \text{Int } X_2 \setminus K_2 \cong M \times (-\infty, 0).$$

We put

$$X(T) = (K_1 \cup M \times (0, T/2]) \cup (M \times [-T/2, 0] \cup K_2)$$

where we identify $M \times \{T/2\} \cong M \times \{-T/2\}$. $X(T)$ is diffeomorphic to X and has an obvious Riemannian metric. So we obtain the moduli space $\mathcal{M}(X(T); \mathcal{P}_X)$. We have

$$Z(c_{1,1}, \dots, c_{1,k_1}, c_{2,1}, \dots, c_{2,k_2}; \mathcal{P}_X) = \int_{\mathcal{M}(X(T); \mathcal{P}_X)} \mu(c_{1,1}) \wedge \cdots \wedge \mu(c_{2,k_2})$$

for any T.[1]

Then (1.8) will be a consequence of the next equality.

$$(1.9) \quad \lim_{T \to \infty} \mathcal{M}(X(T); \mathcal{P}_X) = \bigcup_{a \in R_0(M; \mathcal{P}_M)} \mathcal{M}(X_1; a; \mathcal{P}_{X_1}) \times \mathcal{M}(X_2; a; \mathcal{P}_{X_2}).$$

2. Invariant in dimension 4-3-2

The idea to extend the story of Subsects. 1.1, 1.2, 1.3 so that it includes dimension 2 was studied by various mathematicians in 1990's. (See for example [Fu2], [Fu5].) It can be summarized as follows.

Problem 2.1. (1) For each pair of an oriented 2 manifold Σ and a principal G-bundle $\mathcal{P}_\Sigma$ on it, associate a category $\mathcal{C}(\Sigma; \mathcal{P}_\Sigma)$, such that for each two objects of $\mathcal{C}(\Sigma; \mathcal{P}_\Sigma)$ the set of morphisms between them is an abelian group.
(2) For any pair $(M, \mathcal{P}_M)$ of an oriented 3 manifold M with boundary and a principal G-bundle $\mathcal{P}_M$ on it, associate an object $HF_{(M, \mathcal{P}_M)}$ of $\mathcal{C}(\Sigma; \mathcal{P}_M|_\Sigma)$, where $\Sigma = \partial M$.

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[1] More precisely we cannot expect that $\mathcal{M}(X(T); \mathcal{P}_X)$ is a smooth manifold for arbitrary T. However we can expect that it is a smooth manifold for T outside a finite set. But the union of $\mathcal{M}(X(T); \mathcal{P}_X)$ for $T \in [T_1, T_2]$ is again a manifold. So the integral for $T = T_1$ and $T = T_2$ coincides by Stokes' theorem.

(3) Let $(M_1, \mathcal{P}_{M_1})$, $(M_2, \mathcal{P}_{M_2})$ be pairs as in (2) such that $\partial M_1 = -\partial M_2 = \Sigma$, $\mathcal{P}_\Sigma = \mathcal{P}_{M_1}|_\Sigma = \mathcal{P}_{M_2}|_\Sigma$. We glue them to obtain $(M, \mathcal{P}_M)$. Then show:

(2.1) $$HF(M; \mathcal{P}_M) = \mathcal{C}(HF_{(M_1, \mathcal{P}_{M_1})}, HF_{(M_2, \mathcal{P}_{M_2})}).$$

Here the left hand side is the instanton Floer homology as in Definition 1.8 and the right hand side is the set of morphisms in the category $\mathcal{C}(\Sigma; \mathcal{P}_M|_\Sigma)$, which is an abelian group.

There is a formulation which includes the case

$$\partial(M, \mathcal{P}_M) = -(\Sigma_1, \mathcal{P}_{\Sigma_1}) \sqcup (\Sigma_2, \mathcal{P}_{\Sigma_2}).$$

See Sect. 8. We may join it with 4+3 dimensional picture so that we include the case of 4 manifold with corner.

An idea to find such category $\mathcal{C}(\Sigma; \mathcal{P}_\Sigma)$ is based on the fact that the space of all flat connections $R(\Sigma; \mathcal{P}_\Sigma)$ has a symplectic structure, which we define below.

Let $[\alpha] \in R(\Sigma; \mathcal{P}_\Sigma)$. The tangent space $T_\alpha R(\Sigma; \mathcal{P}_\Sigma)$ is identified with the first cohomology

$$H^1(\Sigma, \mathrm{ad}_\alpha(\mathcal{P}_\Sigma)) = \frac{\mathrm{Ker}(d_\alpha : \mathrm{ad}\, \mathcal{P}_M \otimes \Omega^1 \xrightarrow{d_\alpha} \mathrm{ad}\, \mathcal{P}_M \otimes \Omega^2)}{\mathrm{Im}(d_\alpha : \mathrm{ad}\, \mathcal{P}_M \xrightarrow{d_\alpha} \mathrm{ad}\, \mathcal{P}_M \otimes \Omega^1)}.$$

The symplectic form ω at $T_\alpha R(\Sigma; \mathcal{P}_\Sigma) \cong H^1(\Sigma, \mathrm{ad}_\alpha(\mathcal{P}_\Sigma))$ is given by

$$\omega([u], [v]) = \int_\Sigma Tr(u \wedge v).$$

(See [Go].) We can prove that this 2 form ω is a closed two form based on the fact that $R(\Sigma; \mathcal{P}_\Sigma)$ is regarded as a symplectic quotient

$$\mathcal{A}(\Sigma, \mathcal{P}_\Sigma) /\!\!/ \mathcal{G}(\Sigma, \mathcal{P}_\Sigma).$$

In fact we may regard the curvature

$$\alpha \longmapsto F_\alpha \in C^\infty(\Sigma; \mathrm{ad}_\alpha \mathcal{P} \otimes \Omega^2)$$

as the moment map of the action of gauge transformation group $\mathcal{G}(\Sigma, \mathcal{P}_\Sigma)$ on $\mathcal{A}(\Sigma, \mathcal{P}_\Sigma)$. (See [AB].)

We next consider $(M, \mathcal{P}_M)$ as in Problem 2.1 (2). By the same reason as Assumption 1.3 we assume:

Assumption 2.2. (1) The set $R(M; \mathcal{P}_M)$ has a structure of a finite dimensional manifold.

(2) For any $[a] \in R(M; \mathcal{P}_M)$ the second cohomology $H^2(M; \mathrm{ad}_a(\mathcal{P}_M))$ vanishes. Here the second cohomology group $H^2(M; \mathrm{ad}_a(\mathcal{P}_M))$ is the cokernel of

$$d_a : \mathrm{ad}\,\mathcal{P}_M \otimes \Omega^1 \longrightarrow \mathrm{ad}\,\mathcal{P}_M \otimes \Omega^2.$$

Remark 2.3. In Assumption 1.3 we assumed all the cohomology groups vanish. In fact in case $\partial M = \emptyset$ (and M is 3 dimensional) we have

$$H^2(M; \mathrm{ad}_a(\mathcal{P}_M)) \cong (H^1(M; \mathrm{ad}_a(\mathcal{P}_M)))^*$$

by Poincaré duality. So vanishing of the 2nd cohomology implies the vanishing of the 1st cohomology. The zero-th cohomology vanishes if the connection is irreducible. (Namely the set of all gauge transformations which preserve the connection a is zero dimensional.)

We also remark that actually (2) implies (1).

We then have the next lemma.

Lemma 2.4. *We assume $\partial(M, \mathcal{P}_M) = (\Sigma, \mathcal{P}_\Sigma)$ and Assumption 2.2. Let $i : R(M; \mathcal{P}_M) \to R(\Sigma; \mathcal{P}_\Sigma)$ be the map induced by the restriction of the connection. Then*

$$i^*\omega = 0.$$

Here ω is the symplectic form of $R(\Sigma; \mathcal{P}_\Sigma)$.

Proof. This is an immediate consequence of Stokes' theorem. $\square$

Let $i(a) = \alpha$. We consider the exact sequence

$$(2.2) \quad \begin{aligned} 0 \longrightarrow H^1(M; \mathrm{ad}_a(\mathcal{P}_M)) &\longrightarrow H^1(\Sigma; \mathrm{ad}_\alpha(\mathcal{P}_\Sigma)) \\ &\longrightarrow H^2(M, \Sigma; \mathrm{ad}_a(\mathcal{P}_M)) \longrightarrow 0. \end{aligned}$$

Note $H^2(M; \mathrm{ad}_a(\mathcal{P}_M)) \cong H^1(M, \Sigma; \mathrm{ad}_a(\mathcal{P}_M)) \cong 0$ by Assumption 2.2 (2) and Poincaré duality. Moreover $H^1(M; \mathrm{ad}_a(\mathcal{P}_M)) \cong H^2(M, \Sigma; \mathrm{ad}_a(\mathcal{P}_M))^*$ by Poincaré duality. Thus (2.2) implies that

$$(2.3) \qquad \dim R(M; \mathcal{P}_M) = \frac{1}{2} \dim R(\Sigma; \mathcal{P}_\Sigma)$$

if $\partial M = \Sigma$.

Corollary 2.5. *In the situation of Lemma 2.4, $R(M; \mathcal{P}_M)$ is an immersed Lagrangian submanifold of $R(\Sigma; \mathcal{P}_\Sigma)$ if $i : R(M; \mathcal{P}_M) \to R(\Sigma; \mathcal{P}_\Sigma)$ is an immersion.*

We can again perturb the defining equation $F_a = 0$ of $R(M; \mathcal{P}_M)$ so that the assumption of Corollary 2.5 is satisfied in the modified form. Namely we obtain a Lagrangian immersion to $R(\Sigma; \mathcal{P}_\Sigma)$ from a moduli space that is a perturbation of $R(M; \mathcal{P}_M)$. This is proved by Herald [He]. He also proved that the Lagrangian cobordism class of the perturbed immersed Lagrangian submanifold is independent of the choice of the perturbation.

The above observations let the Donaldson make the next:

Proposal 2.6 (Donaldson [D4]). The category $\mathcal{C}(\Sigma; \mathcal{P}_\Sigma)$ is defined such that:

(1) Its object is a Lagrangian submanifold of $R(\Sigma; \mathcal{P}_\Sigma)$.
(2) The set of morphisms from L_1 to L_2 is the Lagrangian Floer homology $HF(L_1, L_2)$.
(3) The composition of the morphism is defined by counting the pseudo holomorphic triangle as in Fig. 1 below.

The first approximation of the object which we assign to $(M, \mathcal{P}_M)$ is the immersed Lagrangian submanifold $R(\Sigma; \mathcal{P}_\Sigma)$.

This proposal is made in 1992 at University of Warwick. There are various problems to realize this proposal which was known already at that stage to experts.
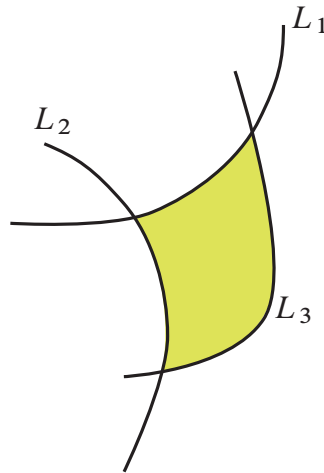

Fig. 1. Holomorphic triangle

Difficulty 2.7. (1) The space $R(\Sigma; \mathcal{P}_\Sigma)$ is in general singular. Symplectic Floer theory on a singular symplectic manifold is difficult to study.
(2) Even in the case of smooth compact symplectic manifold, the Floer homology of two Lagrangian submanifolds is not defined in general and there are various conceptional and technical difficulties in doing so.
(3) It is known that (instanton) Floer homology of $M = M_1 \cup_\Sigma M_2$ is *not* determined by the pair of Lagrangian submanifolds $R(M_1; \mathcal{P}_{M_1})$ and $R(M_2; \mathcal{P}_{M_2})$. So we need some additional information than the immersed

Lagrangian submanifold $R(M; \mathcal{P}_M)$ to obtain actual relative invariant. (This is the reason why Donaldson mentioned $R(M; \mathcal{P}_M)$ as a *first approximation* and is not a relative invariant itself.)

I will explain how much in 25 years, after the proposal made in 1992, our understanding on these problems have been improved.

3. Relation to Atiyah–Floer conjecture

In this section we explain the relation of the discussion in the previous section to a famous conjecture called Atiyah–Floer conjecture [At]. In its original form Atiyah–Floer conjecture can be stated as follows. Let M be a closed oriented 3 manifold such that $H_1(M; \mathbb{Z}) = 0$. We represent M as

$$M = H_g^1 \cup_{\Sigma_g} H_g^2,$$

where H_g^1 and H_g^2 are handle-bodies with genus g and $\Sigma_g = \partial H_g^1 = \partial H_g^2$. We consider the trivial $SU(2)$ bundle $\mathcal{P}_M$ on M and on H_g^i, Σ_g. Let $R(\Sigma_g; \mathcal{P}_{\Sigma_g})$, $R(H_g^i; \mathcal{P}_{H_g^i})$, $R(M; \mathcal{P}_M)$ be the spaces of gauge equivalence classes of flat connections of the trivial $SU(2)$ bundle on H_g^i, Σ_g, M, respectively. (2.3) holds in this case without perturbation. Namely

$$(3.1) \qquad \dim R(\Sigma_g; \mathcal{P}_{\Sigma_g}) = 3(g-1) = \frac{1}{2} \dim R(H_g^i; \mathcal{P}_{H_g^i})$$

and $R(H_g^i; \mathcal{P}_{H_g^i})$ ($i = 1, 2$) are Lagrangian 'submanifolds' of $R(\Sigma_g; \mathcal{P}_{\Sigma_g})$.

Conjecture 3.1 (Atiyah–Floer). The instanton Floer homology of M is isomorphic to the Lagrangian Floer homology between $R(H_g^1; \mathcal{P}_{H_g^1})$ and $R(H_g^2; \mathcal{P}_{H_g^2})$. Namely

$$(3.2) \qquad HF(M; \mathcal{P}_M) \cong HF(R(H_g^1; \mathcal{P}_{H_g^1}), R(H_g^2; \mathcal{P}_{H_g^2})).$$

Remark 3.2. In this remark we mention various problems around Conjecture 3.1.

(1) As we explain in later section (Subsect. 4.1), Lagrangian Floer homology $HF(L_1, L_2)$ is defined as a cohomology of the chain complex whose basis is identified with the intersection points $L_1 \cap L_2$. (This is the case when L_1 is transversal to L_2.) It is easy to see that

$$R(M; \mathcal{P}_M) \cong R(H_g^1; \mathcal{P}_{H_g^1}) \times_{R(\Sigma_g; \mathcal{P}_{\Sigma_g})} R(H_g^2; \mathcal{P}_{H_g^2}).$$

Note instanton Floer homology $HF(M; \mathcal{P}_M)$ is the homology group of a chain complex whose basis is identified with $R_0(M; \mathcal{P}_M)$. So *roughly*

speaking two Floer homology groups $HF(M; \mathcal{P}_M)$ and $HF(R(H_g^1; \mathcal{P}_{H_g^1}),$ $R(H_g^2; \mathcal{P}_{H_g^2}))$ are homology groups of the chain complexes whose underlying groups are isomorphic. So the important point of the proof of Conjecture 3.1 is comparing boundary operators.

The boundary operator defining $HF(M; \mathcal{P}_M)$ is obtained by counting the order of the moduli space $\mathcal{M}(M \times \mathbb{R}; a, b)$, as we explained in Subsect. 1.2. The boundary operator defining $HF(R(H_g^1; \mathcal{P}_{H_g^1}), R(H_g^2; \mathcal{P}_{H_g^2}))$ is obtained by counting the order of the moduli space of pseudo holomorphic strips in $R(\Sigma_g; \mathcal{P}_{\Sigma_g})$ with boundary condition defined by $R(H_g^1; \mathcal{P}_{H_g^1})$ and $R(H_g^2; \mathcal{P}_{H_g^2})$. Various attempts to relate these two moduli spaces directly have never been successful for more than twenty years.

(2) Another problem, which is actually more serious, is related to Difficulty 2.7 (1). In fact the space $R(\Sigma_g; \mathcal{P}_{\Sigma_g})$ is singular. The singularity corresponds to the reducible connections. (Here a connection a is called reducible if the set of gauge transformations g such that $g^*a = a$ has positive dimension.) Moreover the intersection $R(H_g^1; \mathcal{P}_{H_g^1}) \cap R(H_g^2; \mathcal{P}_{H_g^2})$ contains a reducible connection, which is nothing but the trivial connection. Note we assumed Assumption 1.2 (2). In this situation the only reducible connection in $R(M; \mathcal{P}_M)$ is the trivial connection. The singularity of $R(\Sigma_g; \mathcal{P}_{\Sigma_g})$ makes the study of pseudo holomorphic strip in $R(\Sigma_g; \mathcal{P}_{\Sigma_g})$ with boundary condition defined by $R(H_g^1; \mathcal{P}_{H_g^1})$ and $R(H_g^2; \mathcal{P}_{H_g^2})$ very hard.

In other words, the right hand side of the isomorphism (3.2) has *never* been defined. In that sense Conjecture 3.1 is not even a rigorous mathematical conjecture yet.

We like to mention that there is an interesting work [MW] by Manolescu and Woodward on this point. They used extended moduli space studied previously by [Hu], [HL], [Je]. A proposal to resolve the problem of singularity of $R(\Sigma_g; \mathcal{P}_{\Sigma_g})$ using the idea of [MW] is written in [DF].

There are various variants of Conjecture 3.1 which are solved and/or which can be stated rigorously and/or which are more accessible.

Among those variants the most important result is one by Dostoglou–Salamon [DS]. It studies Problem 2.1 in the following case. $M_1 = M_2 = \Sigma \times [0, 1]$ where Σ is a Riemann surface. $\mathcal{P}_\Sigma$ is an $SO(3)$ bundle with $w_2(\mathcal{P}_\Sigma) = [\Sigma]$. Note $\partial M_1 = \partial M_2 \cong \Sigma \sqcup -\Sigma$. When we glue M_1 and M_2 along their boundaries we obtain a closed 3 manifold M of the form

$$\Sigma \longrightarrow M \longrightarrow S^1.$$

Namely M is a fiber bundle over S^1 with fiber Σ. The diffeomorphism class of M is determined by $\varphi : \Sigma \to \Sigma$. Namely

$$M = M_\varphi = (\Sigma \times [0, 1])/ \sim$$

and the equivalence relation $\sim$ is defined by $(1, x) \sim (0, \varphi(x))$.

The diffeomorphism φ induces a diffeomorphism

$$\varphi^* : R(\Sigma; \mathcal{P}_\Sigma) \longrightarrow R(\Sigma; \mathcal{P}_\Sigma)$$

which is a symplectic diffeomorphism. Its graph

$$\text{graph}(\varphi^*) = \{(x, \varphi^* x) \in R(\Sigma; \mathcal{P}_\Sigma) \times R(\Sigma; \mathcal{P}_\Sigma) \mid x \in R(\Sigma; \mathcal{P}_\Sigma)\}$$

is a Lagrangian submanifold of $R(\Sigma; \mathcal{P}_\Sigma) \times R(\Sigma; \mathcal{P}_\Sigma)$ equipped with symplectic form $-\omega \oplus \omega$.

Theorem 3.3 (Dostoglou–Salamon [DS]). *The instanton Floer homology $HF(M_\varphi; \mathcal{P}_{M_\varphi})$ is isomorphic to the Lagrangian Floer homology $HF(\Delta, \text{graph}(\varphi^*))$, where $\Delta \subset R(\Sigma; \mathcal{P}_\Sigma) \times R(\Sigma; \mathcal{P}_\Sigma)$ is the diagonal.*

There is another case which is actually simpler. We consider $\Sigma = T^2$ (2 torus) with non-trivial $SO(3)$ bundle $\mathcal{P}_\Sigma$. Then it is easy to see that the space of flat connections $R(T^2; \mathcal{P}_{T^2})$ is one point. The following is known in this case.

Theorem 3.4 (Braam–Donaldson [BD]).

(1) Let M be an oriented 3 manifolds with boundary such that each of the connected component of ∂M is T^2. Let $\mathcal{P}_M$ be a principal $SO(3)$ bundle such that $w^2(\mathcal{P}_M)|_{\partial M} = [\partial M]$.
Then we can define a Floer homology $HF(M; \mathcal{P}_M)$ which is a $\mathbb{Z}_2$ vector space.

(2) Suppose M_1 and M_2 are both as in (1). We assume $\partial M_1 \cong \partial M_2$. We glue them to obtain $(M, \mathcal{P}_M)$. Then

$$(3.3) \qquad HF(M, \mathcal{P}_M) \cong HF(M_1, \mathcal{P}_{M_1}) \otimes HF(M_2, \mathcal{P}_{M_2}).$$

Note in the situation of Theorem 3.4 (1) the set of flat connections $R(M; \mathcal{P}_M)$ is a finite set if Assumption 2.2 is satisfied. In that case $HF(M; \mathcal{P}_M)$ is the cohomology of a chain complex $CF(M; \mathcal{P}_M)$ whose underlying vector space has a basis identified with $R(M; \mathcal{P}_M)$.

Note (3.3) is the case of $\mathbb{Z}_2$ coefficient. In the case of $\mathbb{Z}$ coefficient there is a Künneth type split exact sequence.

In the situation of Theorem 3.4 (2) we assume both $R(M_i; \mathcal{P}_{M_i})$ $(i = 1, 2)$ satisfy Assumption 2.2. Then we can identify

$$R(M; \mathcal{P}_M) = R(M_1; \mathcal{P}_{M_1}) \times R(M_2; \mathcal{P}_{M_2})$$

and hence

$$CF(M; \mathcal{P}_M) = CF(M_1; \mathcal{P}_{M_1}) \otimes CF(M_2; \mathcal{P}_{M_2})$$

as vector spaces. It is proved in [BD] that the boundary operators coincide.

There are two similar cases which were studied around the same time. One is the case $\Sigma = S^2$. In this case the bundle $\mathcal{P}_{S^2}$ on S^2 is necessarily trivial if it carries a flat connection.[2] In this case (3.2) and (3.3) correspond to the study of the Floer homology of connected sum.

Theorem 3.5 (Fukaya, Li [Fu4], [Li]). *Let $(M_1; \mathcal{P}_{M_1})$ and $(M_2; \mathcal{P}_{M_2})$ both satisfy Assumption 1.2 (2). We put $M = M_1 \sharp M_2$ (the connected sum). $\mathcal{P}_{M_1}$ and $\mathcal{P}_{M_2}$ induce a principal bundle $\mathcal{P}_M$ on M in an obvious way so that Assumption 1.2 (2) is satisfied. Then for each field F there exists a spectral sequence with the following properties:*

(1) Its E^2 page is

$$HF(M_1; \mathcal{P}_{M_1}; F) \oplus HF(M_2; \mathcal{P}_{M_2}; F)$$
$$\oplus \left(HF(M_1, \mathcal{P}_{M_1}; F) \otimes HF(M_2, \mathcal{P}_{M_2}; F) \otimes H(SO(3); F) \right).$$

(2) It converges to $HF(M; \mathcal{P}_M; F)$.

We can prove a similar statement in the case of Assumption 1.2 (1). (It was not explored 25 years ago.)

Note in the situation of Theorem 3.5 we have the following isomorphism if Assumption 2.2 is satisfied.

$$(3.4) \qquad R_0(M; \mathcal{P}_M) \cong R_0(M_1; \mathcal{P}_{M_1}) \sqcup R_0(M_2; \mathcal{P}_{M_2})$$
$$\sqcup \left(R_0(M_1; \mathcal{P}_{M_1}) \times R_0(M_2; \mathcal{P}_{M_2}) \times SO(3) \right).$$

Note $R_0(M; \mathcal{P}_M) = R(M; \mathcal{P}_M) \setminus \{\text{trivial connection}\}$ in our situation. The first and the second term of the right hand side of (3.4) correspond to the flat connection on M which is trivial either on M_1 or on M_2. The third term of (3.4) corresponds to the flat connection on M which is non-trivial both on M_1 and M_2. In this case there is extra freedom to twist the connections on S^2 where we glue M_1 and M_2. It is parametrized by $SO(3)$. (3.4) explains Theorem 3.5 (1).

The spectral sequence in general does not degenerate in E^2 level. In fact, there is a non-trivial differential which is related to the fundamental homology class of $H(SO(3))$. One such example is the case when M_1 is Poincaré homology sphere and M_2 is Poincaré homology sphere with reverse orientation.

The next simplest case is one when $\Sigma = T^2$ with the trivial $SU(2)$ bundle. The following Floer's exact sequence is closely related to this case. Suppose $(M; \mathcal{P}_M)$ is as in Assumption 1.2 (1) and we take $S^1 = K \subset M$, a knot. We remove a tubular neighborhood $S^1 \times D^2$ of K from M and reglue $S^1 \times D^2$ along the boundaries to obtain M'. This process is called Dehn surgery. There are

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[2] If for a pair of closed 3 manifolds M and $SO(3)$ bundle $\mathcal{P}_M$, there exists $S^2 \subset M$ with $w^2(\mathcal{P}_M) \cap S^2 \neq 0$, then $HF(M; \mathcal{P}_M) = 0$.

several different ways to reglue, which is parametrized by the diffeomorphism $T^2 \to T^2$. Composing with the diffeomorphism which extends to $S^1 \times D^2$ does not change the diffeomorphism type of M' and so M' is parametrized by a pair of integers (p, q) (which are coprime) or $p/q \in \mathbb{Q} \cup \{\infty\}$. We consider the case when this rational number is $0, 1, \infty$ and write them M_0, M_{+1} and M_∞. The manifold M_∞ is actually M itself. M_{+1} is another 3 manifold which is a homology 3 sphere. M_0 is homology $S^1 \times S^2$. (It satisfies Assumption 1.2 (1).) We can extend $\mathcal{P}_M|_{M \setminus K}$ to it and obtain $\mathcal{P}_{M_0}$ such that the flat connection of $\mathcal{P}_{M_0}$ corresponds to the group homomorphism $\pi_1(M \setminus K) \to SU(2)$ which sends the meridian to -1. Here meridian is a small circle which has linking number 1 with the knot K.

Theorem 3.6 (Floer [Fl3], Braam–Donaldson [BD]). *There exists a long exact sequence:*

$$(3.5) \quad \cdots \longrightarrow HF(M_{+1}; \mathcal{P}_{M_{+1}}) \longrightarrow HF(M_\infty; \mathcal{P}_{M_\infty}) \longrightarrow HF(M_0; \mathcal{P}_{M_0}) \longrightarrow \cdots .$$

The relation of Theorem 3.6 to the gluing problem such as Theorems 3.3–3.5 is as follows. We put

$$\overset{\circ}{M} = M \setminus (S^1 \times \operatorname{Int} D^2).$$

(Here $S^1 \times D^2$ is the tubular neighborhood of the knot K.) The boundary $\partial \overset{\circ}{M}$ is T^2 on which $\mathcal{P}_M$ is trivial. Therefore $R(\partial \overset{\circ}{M}, \mathcal{P}_{\partial \overset{\circ}{M}})$ is the set of gauge equivalence classes of flat $SU(2)$ connections on T^2, which is identified with $T^2/\pm 1$. So, according to Proposal 2.6, the relative invariant $HF_{\overset{\circ}{M}}$ 'is' a Lagrangian submanifold of $T^2/\pm 1$, which is a sum of immersed circles in it.

On the other hand, the manifolds M_0, M_{+1}, M_∞ are obtained by gluing $S^1 \times D^2$ in various ways to $\overset{\circ}{M}$. The relative invariant $HF_{S^1 \times D^2}$ is the set of flat connections on M which can be identified with various circles. (See Fig. 2.) Let C_0, C_{+1} and C_∞ be the circles in $T^2/\pm 1$ corresponding to $0, +1$ and ∞ surgeries, respectively.

The Floer homologies appearing in (3.5) is obtained as the set of 'morphisms' from the object $HF_{\overset{\circ}{M}}$ to those circles C_0, C_{+1} and C_∞. The proof then goes by using the identity

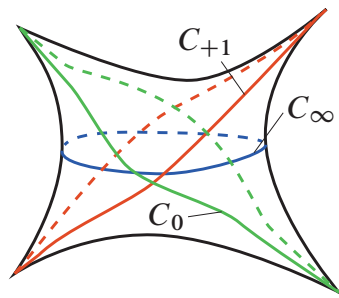$$[C_0] + [C_{+1}] = [C_\infty]$$

as cycles.

Fig. 2. C_0, C_{+1}, C_∞

4. Biased review of Lagrangian Floer theory I

In this and the next sections we review Lagrangian Floer theory with emphasis on its application to gauge theory. Another review of Lagrangian Floer theory which put more emphasis on its application to Mirror symmetry is [Fu8].

4.1. General idea of Lagrangian Floer homology

Let (X, ω) be a $2n$ dimensional compact symplectic manifold (namely X is a $2n$ dimensional manifold and ω is a closed 2 form on it such that ω^n never vanishes.) Let L_1, L_2 be Lagrangian submanifolds of X (namely they are n dimensional submanifolds such that $\omega|_{L_i} = 0$.) We first consider the case when L_1 and L_2 are both embedded. We assume for simplicity that L_1 is transversal to L_2. It implies that the set $L_1 \cap L_2$ is a finite set. Let $CF(L_1, L_2; \mathbb{Z}_2)$ be the $\mathbb{Z}_2$ vector space whose basis is identified with $L_1 \cap L_2$. We take and fix an almost complex structure J which is compatible with ω. (Namely we assume $\omega(JX, JY) = \omega(X, Y)$ and $g(X, Y) = \omega(X, JY)$ is a Riemannian metric.[3]) For $a, b \in L_1 \cap L_2$ we consider the set of maps

$$u : \mathbb{R} \times [0, 1] \longrightarrow X$$

with the following properties. (Here τ and t are coordinate of $\mathbb{R}$ and $[0, 1]$ respectively.)

(LF.1) u is J holomorphic. Namely

$$(4.1) \qquad \frac{\partial u}{\partial \tau} = J \frac{\partial u}{\partial t}.$$

(LF.2) $u(\tau, 0) \in L_0$ and $u(\tau, 1) \in L_1$.

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[3] Actually the first condition is a consequence of the second condition.

(LF.3)
$$\int u^*\omega < \infty.$$

Moreover
$$\lim_{\tau\to-\infty} u(\tau,t) = a, \qquad \lim_{\tau\to+\infty} u(\tau,t) = b.$$

We remark that these conditions are similar to (IF.1)–(IF.3) we put to define the moduli space $\mathcal{M}(M \times \mathbb{R}; a, b)$ in Subsect. 1.2. We denote the set of the maps u satisfying these conditions by $\widetilde{\mathcal{M}}(L_1, L_2; a, b)$. The translation on $\mathbb{R}$ direction of the source $\mathbb{R} \times [0, 1]$ induces an $\mathbb{R}$ action on $\widetilde{\mathcal{M}}(L_1, L_2; a, b)$. We denote by $\mathcal{M}(L_1, L_2; a, b)$ the quotient space of this action.

We decompose $\mathcal{M}(L_1, L_2; a, b)$ as

(4.2)
$$\mathcal{M}(L_1, L_2; a, b) = \bigcup_{k,E} \mathcal{M}(L_1, L_2; a, b; k, E)$$

where $k \in \mathbb{Z}$ and $E \in \mathbb{R}_{\geq 0}$. Here $\mathcal{M}(L_1, L_2; a, b; k, E)$ consists of the maps u such that:

(1) $\int u^*\omega = E$.
(2) The index of the linearized equation (4.1) at u is k.

Roughly speaking the Floer's boundary operator is defined by

(4.3)
$$\langle \partial a, b \rangle = \sum_E \sharp \mathcal{M}(L_1, L_2; a, b; 1, E)[b].$$

The proof of $\partial\partial = 0$ would be based on the equality

(4.4)
$$\partial\mathcal{M}(L_1, L_2; a, c; 2, E)$$
$$= \bigcup_{E_1+E_2=E} \mathcal{M}(L_1, L_2; a, c; 1, E_1) \times \mathcal{M}(L_1, L_2; c, b; 1, E_2),$$

which is similar to (1.3). Actually (4.4) does not hold in general. There is so called disk bubble which corresponds to another type of the boundary component of $\mathcal{M}(L_1, L_2; a, c; 2, E)$. Floer [Fl1] put Condition 4.1 below to avoid it.

Condition 4.1. For any $u : (D^2, \partial D^2) \to (X, L_i)$ $(i = 1, 2)$ we have

$$\int_{D^2} u^*\omega = 0.$$

Theorem 4.2 (Floer [Fl1]). *Under Condition 4.1, the following holds for generic compatible almost complex structure J.*

(1) The moduli space $\mathcal{M}(L_1, L_2; a, b; 1, E)$ satisfies an appropriate transversality condition.

(2) We can define the boundary operator by (4.3).

(3) (4.4) holds and we can use it to prove $\partial\partial = 0$. So we can define Floer homology

$$HF(L_1, L_2) \cong \frac{\text{Ker}\,(\partial : CF(L_1, L_2) \longrightarrow CF(L_1, L_2))}{\text{Im}\,(\partial : CF(L_1, L_2) \longrightarrow CF(L_1, L_2))}.$$

(4) If $\varphi : X \to X$ is a Hamiltonian diffeomorphism then

$$HF(\varphi(L_1), L_2) \cong HF(L_1, L_2).$$

(5) If $L_1 = L_2 = L$ and $\varphi : X \to X$ is a Hamiltonian diffeomorphism then $HF(\varphi(L), L)$ is isomorphic to the ordinary homology $H(L; \mathbb{Z}_2)$ of L.

Remark 4.3. We do not explain the notion of Hamiltonian diffeomorphism here since our description of Lagrangian Floer homology is biased to the direction which is related to gauge theory.

   If L_i is spin we can define Floer homology over $\mathbb{Z}$ coefficient. (We can relax this condition to relative spinness.) See [FOOO2, Chapter 8].

4.2. Monotone Lagrangian submanifold

Condition 4.1 is too much restrictive. Especially we can not work under this condition in our application to gauge theory (for example to realize Proposal 2.6). The next step to relax this condition is due to Y.-G. Oh.

Theorem 4.4 (Oh [Oh]). *Instead of Condition 4.1 we assume that L_i is monotone and has minimal Maslov number > 2, for $i = 1, 2$. Then Theorem 4.2 (1)–(4) holds.*

   We will explain the notion of monotonicity and minimal Maslov number later in this subsection.

Remark 4.5. (1) Under the assumption of Theorem 4.4, Theorem 4.2 (5) may not hold. There exists however a spectral sequence whose E^2 term is $H(L; \mathbb{Z}_2)$ and which converges to $HF(\varphi(L), L)$.

(2) If we assume L_i to be spin (or more generally (X, L_1, L_2) are relatively spin) Theorem 4.4 (and item (1) of this remark) holds over $\mathbb{Z}$ coefficient. This is proved in [FOOO1, Chapters 2 and 6] and [FOOO2, Chapter 8].

   To apply Theorem 4.4 to gauge theory, we can use the next fact.

Proposition 4.6. *Let $(M, \mathcal{P}_M)$ be a pair of an oriented 3 manifold with boundary and a principal $SO(3)$ bundle on it such that $w^2(\mathcal{P}_M)|_{\partial M}$ is the fundamental class $[\partial M]$. We also assume Assumption 2.2. Moreover we assume $R(M; \mathcal{P}_M) \to R(\partial M; \mathcal{P}_{\partial M})$ is an embedding (that is, injective).*
   Then $R(M; \mathcal{P}_M)$ is a monotone Lagrangian submanifold. Its minimal Maslov number is in $4\mathbb{Z}_+$.

This fact was known to many researchers at least in late 1990's.

Sketch of the proof. We put $R(\partial M; \mathcal{P}_{\partial M}) = R(\Sigma; \mathcal{P}_\Sigma)$. Let $u : (D^2, \partial D^2) \to (R(\Sigma; \mathcal{P}_\Sigma), R(M; \mathcal{P}_M))$ be a holomorphic map. Since D^2 is contractible, u is induced by a D^2 parametrized family of flat connections $z \mapsto A(z)$ on Σ. We use $u|_{\partial D^2} \to R(M; \mathcal{P}_M)$ to obtain an $S^1 = \partial D^2$ parametrized family of flat connections $z \mapsto \mathcal{A}(z)$ on M.

We regard $A(z)$ as a connection of $\mathcal{P}_\Sigma \times D^2$ and denote it by A. We also regard $\mathcal{A}(z)$ as a connection of $\mathcal{P}_M \times S^1$ and denote it by $\mathcal{A}$. We remark that

$$\mathcal{A}|_{\partial M \times S^1} \sim A|_{\Sigma \times \partial D^2}.$$

Here $\sim$ means $SO(3)$ gauge equivalent. We glue $\mathcal{P}_\Sigma \times D^2$ and $\mathcal{P}_M \times S^1$ on $\partial D^2 \times \Sigma$ by the above gauge equivalence to obtain a bundle $\mathcal{P}_X$ on a 4 manifold $X = (\Sigma \times D^2) \cup_{\Sigma \times S^1} (M \times S^1)$. The connections $\mathcal{A}$ and A can be glued to give a connection $\mathfrak{A}$ of $\mathcal{P}_X$. (We can modify u by an arbitrary small amount so that $\mathfrak{A}$ becomes smooth.)

By using [Fu6, Formula (2.7)] we can show that the symplectic area of u is proportional to the Yang–Mills functional of $\mathfrak{A}$. (Note Yang–Mills functional of $\mathfrak{A}$ is equal to one of $\mathcal{A}$.)

Note the Maslov index of $[u]$ is defined by comparing two trivializations of $TR(\Sigma; \mathcal{P}_\Sigma)$ on $u(\partial D^2)$, one is obtained by contractibility D^2 and the map u, the other is obtained by

$$TR(\Sigma; \mathcal{P}_\Sigma)|_{R(M; \mathcal{P}_M)} = TR(M; \mathcal{P}_M) \otimes \mathbb{C}.$$

The first trivialization corresponds to the triviality of $\mathcal{P}_X|_{\Sigma \times D^2}$ and the second trivialization corresponds to the triviality of $\mathcal{P}_X|_{M \times S^1}$.

The difference between two trivializations of bundles on $\Sigma \times S^1$ is a gauge transformation on $\Sigma \times S^1$ whose degree is nothing but the first Pontryagin number of $\mathcal{P}_X$. Thus the Maslov index of u is proportional to the first Pontryagin number of $\mathcal{P}_X$.

Using the holomorphicity of u we can show that Yang–Mills functional and first Pontryagin number is proportional. We thus proved the monotonicity.

The fact that Maslov index is divisible by 4 can be proved by showing that Maslov index is 4 times the first Pontryagin number. We can prove it for example from the formula [DS, page 592 the 6th line from the bottom]. It describes the way how $\langle c^1(R(\Sigma; \mathcal{P}_\Sigma)), [u] \rangle$ changes when we twist the bundle by a gauge transformation g on S^3. (Here twisting bundle on 4 manifolds corresponding to changing the map u from 2 manifold to $R(\Sigma; \mathcal{P}_\Sigma)$.) It claims that the change of $\langle c^1(R(\Sigma; \mathcal{P}_\Sigma)), [u] \rangle$ is twice of the degree of g. Note the degree of g is the amount of the change of the first Pontryagin number and Maslov index is twice of c^1.

Theorem 4.7. *Suppose $(M_i, \mathcal{P}_{M_i})$ satisfies the assumptions of Proposition 4.6 for $i = 1, 2$. Theorem 4.4 implies that Floer homology $HF(R(M_1; \mathcal{P}_{M_1}),$ $R(M_2; \mathcal{P}_{M_2}))$ is well-defined. We assume that $\partial M_1 = -\partial M_2$ and $\mathcal{P}_{M_1}|_{\partial M_1} \cong \mathcal{P}_{M_2}|_{\partial M_2}$.*

Then instanton Floer homology is isomorphic to Lagrangian Floer homology:

$$(4.5) \qquad HF(M; \mathcal{P}_M) \cong HF(R(M_1; \mathcal{P}_{M_1}), R(M_2; \mathcal{P}_{M_2})).$$

Here $(M, \mathcal{P}_M)$ is obtained by gluing $(M_1, \mathcal{P}_{M_1})$ and $(M_2, \mathcal{P}_{M_2})$ along their boundaries.

Remark 4.8. Theorem 4.7 is claimed as [Fu9, Corollary 1.2]. The outline of its proof is given in [Fu9, Sect. 5]. The detail of the proof of Theorem 4.7 will be written in a subsequent joint paper [DFL] with Aliakbar Daemi and Max Lipyanskiy.

The cobordism argument used in [Fu9, Sect. 5] appeared also in [Fu5, Sect. 8] as the proof of [Fu5, Theorem 8.7], which claims that there exists a homomorphism from the left hand side of (4.5) to the right hand side of (4.5). It was conjectured but not proved in [Fu5, Conjecture 8.9] that this homomorphism is an isomorphism. We use the same map to prove Theorem 4.7. The idea which was missing in 1997 when [Fu5] was written is the following.

(1) When $\partial(M, \mathcal{P}_M) = (\Sigma, \mathcal{P}_\Sigma)$, we use $R(M; \mathcal{P}_M)$ itself as a 'test object' of the functor: $\mathfrak{Fuk}(R(\Sigma; \mathcal{P}_\Sigma)) \to \mathcal{CH}$. (Here $\mathcal{CH}$ is the A_∞ category whose object is a chain complex.) This is the functor which associates to L (a Lagrangian submanifold of $R(\Sigma; \mathcal{P}_\Sigma)$) the chain complex $CF((M; \mathcal{P}_M); L)$ the 'Floer's chain complex of $(M; \mathcal{P}_M)$ with boundary condition given by L'. (See Theorem 6.12.) In other words we take $R(M; \mathcal{P}_M)$ as the Lagrangian submanifold L in (6.6).

(2) If we take $L = R(M_i; \mathcal{P}_{M_i})$ then the chain complex $CF((M; \mathcal{P}_M); L)$ in (6.6) can be identified with the de Rham complex of L.

Something equivalent to these two points appeared in the paper [LL] by Lekili–Lipyanskiy. After that it was used more explicitly in [Fu9]. The same argument applied in a similar situation as Theorem 4.7 was explicitly mentioned in a talk by Lipyanskiy [Ly2] done in 2012.

It seems to the author that [LL] is the paper which revives the idea using the cobordism argument in this and related problems and became the turning point of the direction of the research. During the years 1998–2010 the cobordism argument proposed in [Fu5] was not studied and instead a more analytic approach using adiabatic limit had been pursued.

In the rest of this subsection we explain the notion of monotone Lagrangian submanifold and minimal Maslov number and a part of the idea of the proof of Theorem 4.4.

We first review the definition of Maslov index of a Lagrangian submanifold. (See [FOOO1, Subsect. 2.1.1] for detail.) Let (X, L) be a pair of a symplectic manifold X and its (embedded) Lagrangian submanifold L. We take a compatible almost complex structure J on X. For a map $u : (D^2, \partial D^2) \to (X, L)$ we consider the equation $\bar{\partial} u = 0$. Its linearization defines an operator

$$D_u \bar{\partial} : C^\infty((D^2, \partial D^2); (u^*TX, u^*TL)) \longrightarrow C^\infty(D^2; u^*TX \otimes \Omega^{0,1}).$$

We can show that there exists $\mu : H_2(X, L) \to \mathbb{Z}$ such that

(4.6) $$\text{Index } D_u \bar{\partial} = \mu([u]) + \dim L.$$

The number $\mu([u])$ is called the Maslov index of u.

Definition 4.9. *We call L a monotone Lagrangian submanifold, if there exists a positive number c independent of u such that*

(4.7) $$\mu([u]) = c \int_{D^2} u^* \omega$$

for any $u : (D^2, \partial D^2) \to (X, L)$.
    The minimum Maslov number L is by definition:

(4.8) $$\inf\{\mu([u]) \mid u : (D^2, \partial D^2) \longrightarrow (X, L), \ \mu([u]) \neq 0\}.$$

Now we briefly explain the reason why the equality (4.4) holds in the situation of Theorem 4.4. Let $u_i : \mathbb{R} \times [0, 1] \to X$ be a sequence of elements of $\mathcal{M}(L_1, L_2; a, b; 2, E)$. By taking a subsequence if necessary we may assume that the limit looks like either Fig. 3 or Fig. 4. The case of Fig. 3 corresponds to the right hand side of (4.4). So it suffices to see that Fig. 4 does not occur. The limit drawn in Fig. 4 can be regarded as a pair of $[u_\infty] \in \mathcal{M}(L_1, L_2; a, b; k, E')$ for some k and E' and $u'' : (D^2, \partial D^2) \to (X, L_i)$. Since u'' is pseudo holomorphic we have $\int_{D^2} (u'')^* \omega > 0$. Therefore (4.7) implies $\mu([u'']) > 0$. Since the minimal Maslov number is greater than 2 we have $\mu([u'']) > 2$. By (4.6) and index sum formula we can show

$$k = 2 - \mu([u'']) < 0.$$

Using this fact we can show $\mathcal{M}(L_1, L_2; a, b; k, E')$ is an empty set when appropriate transversality is satisfied. This implies that Fig. 4 does not happen.

Remark 4.10. Note in the case when the minimal Maslov number is 2 we may have $k = 2 - 2 = 0$. Since the (virtual) dimension of $\mathcal{M}(L_1, L_2; a, b; k, E')$ is $k - 1 = -1$, one might imagine that this is enough to show that $\mathcal{M}(L_1, L_2; a, b; k, E')$ is an empty set. However there is a case $a = b$. In that case $\mathcal{M}(L_1, L_2; a, a; 0, 0)$ consists of one point (the constant map u to a). This is an element of $\mathcal{M}(L_1, L_2; a, a; 0, 0)$. So it is non-empty. Note this element is a fixed point of
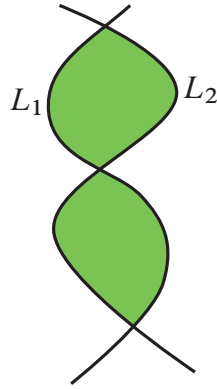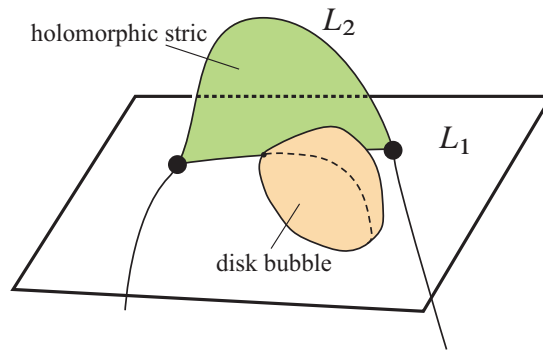
Fig. 3. End 1

Fig. 4. End 2

the $\mathbb{R}$ action. So even though the virtual dimension of $\mathcal{M}(L_1, L_2; a, a; 0, 0)$ is -1 it can still be non-empty. This is the reason why we need to assume that minimal Maslov index is *strictly* larger than 2 in Theorem 4.4. This point appeared in [Oh] and was studied in great detail in [FOOO1, Sect. 3.6.3, etc.]. The notion of potential function introduced in [FOOO1, Definition 3.6.33] is related to this point.

As we have seen in this section, in the situation when the space $R(M; \mathcal{P}_M)$ is embedded in $R(\partial M; \mathcal{P}_{\partial M})$ we can use its monotonicity to define Lagrangian Floer homology and prove Theorem 4.7.

In general $R(M; \mathcal{P}_M)$ is an immersed Lagrangian submanifold in $R(\partial M; \mathcal{P}_{\partial M})$ even after making appropriate perturbation. However we still have a kind of monotonicity.

Definition 4.11. *Let* (X, ω) *be a symplectic manifold and* $i_L : \tilde{L} \to X$ *a Lagrangian immersion. Namely* i_L *is an immersion,* $\dim \tilde{L} = \frac{1}{2} \dim X$ *and* $i_L^* \omega = 0$.

We say $L = (\tilde{L}, i_L)$ *is monotone in the weak sense if for each pair* (u, γ) *such that* $u : D^2 \to X$ *and* $\gamma : \partial S^1 \to \tilde{L}$ *with* $u|_{\partial D^2} = i_L \circ \gamma$ *the equality*

$$(4.9) \qquad \qquad \mu([u]) = c \int_{D^2} u^* \omega$$

holds. (Note the Maslov index $\mu([u])$ can be defined in a similar way as embedded case.)

The minimal Maslov number is defined in the same way.

Proposition 4.12. *Let $(M, \mathcal{P}_M)$ be a pair of an oriented 3 manifolds with boundary and a principal $SO(3)$ bundle on it such that $w^2(\mathcal{P}_M)|_{\partial M}$ is the fundamental class $[\partial M]$. We also assume Assumption 2.2. Moreover we assume $R(M; \mathcal{P}_M)$ $\to R(\partial M; \mathcal{P}_{\partial M})$ is an immersion.*

Then $R(M; \mathcal{P}_M)$ is an immersed monotone Lagrangian submanifold in the weak sense. Its minimal Maslov number is in $4\mathbb{Z}_+$.

The proof is the same as the proof of Proposition 4.6.

However we cannot generalize Theorem 4.4 to the case of immersed monotone Lagrangian submanifold in the weak sense with minimal Maslov number > 2. In fact other than those drawn in Figs. 3 and 4 there exists another type of boundary of the moduli space $\mathcal{M}(L_1, L_2; a, b; 2, E)$, which is drawn in Fig. 5 below. So (4.4) does not hold in this generality.

Nevertheless actually we can still define the right hand side of (4.5) and can prove the isomorphism (4.5). For this purpose we need various new ideas which was developed in Lagrangian Floer theory during 1996–2009. We describe some of them in the next section.
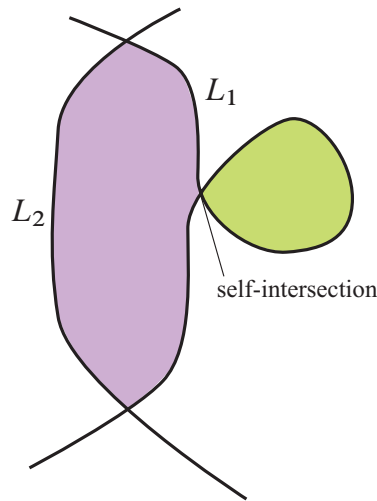

Fig. 5. End 3

5. Biased review of Lagrangian Floer theory II

5.1. Filtered A_∞ category

This subsection is a brief introduction to filtered A_∞ category and algebra. For more detail, see [AFOOO], [Fu10], [Fu7], [FOOO1], [FOOO3], [Ke], [Le], [Ly], [Sei] and, etc.

For the application to gauge theory such as Problem 2.1 we can work on $\mathbb{Z}$ or $\mathbb{Z}_2$ coefficient as we will explain in Subsect. 6.1. However for the general story it is more natural to use universal Novikov ring which was introduced in [FOOO1]. We first define it below. Let R be a commutative ring with unit, which we call ground ring. The reader may consider $R = \mathbb{Z}_2, \mathbb{Z}, \mathbb{R}$ or $\mathbb{C}$. We take a formal variable T and consider the formal sum

$$(5.1) \qquad \sum_{k=0}^{\infty} a_k T^{\lambda_k}$$

such that:

(NR.1) $a_k \in R$,
(NR.2) $\lambda_k \in \mathbb{R}_{\geq 0}$,
(NR.3) $\lambda_k < \lambda_{k+1}$,
(NR.4) $\lim_{k \to \infty} \lambda_k = \infty$.

We call the totality of such formal sum (5.1) the *universal Novikov ring* and denote it by Λ_0^R. We replace (NR.2) by $\lambda_k \in \mathbb{R}$ (resp. $\lambda_k > 0$) to define Λ^R (resp. Λ_+^R.) Λ_0^R and Λ^R become rings with unit in an obvious way. Λ_+^R is an ideal of Λ_0^R. In case R is a field, Λ^R becomes a field, which we call (universal) Novikov field. In case R is a field, Λ_+^R is a maximal ideal of Λ_0^R. In fact $\Lambda_0^R / \Lambda_+^R = R$.

The ring Λ^R has a filtration $F^{\lambda}\Lambda^R$ which consists of (5.1) with $\lambda_0 \geq \lambda$. It induces a filtration $F^{\lambda}\Lambda_0^R$ of Λ_0^R. This filtration defines a metric on these rings, by which they become complete.

Hereafter we omit R and write Λ_0, Λ, Λ_+ in place of Λ_0^R, Λ^R, Λ_+^R, sometimes.

Definition 5.1. *A filtered A_∞ category $\mathscr{C}$ consists of the following objects.*

(1) A set $\mathfrak{Ob}(\mathscr{C})$, which is the set of objects.

(2) A graded Λ_0 module $\mathscr{C}(c_1, c_2)$ for each $c_1, c_2 \in \mathfrak{Ob}(\mathscr{C})$. We call $\mathscr{C}(c_1, c_2)$ the module of morphisms. It is a T-adic completion of a free Λ_0 module. (We may consider $\mathbb{Z}$ grading or $\mathbb{Z}_{2N}$ grading for some $N \in \mathbb{Z}_{>0}$. In our application in Sect. 6, we have $\mathbb{Z}_4$ grading. The number $2N$ in our geometric applications is a minimal Maslov number explained in the last section.)

(3) For each $c_0, \ldots, c_k \in \mathfrak{Ob}(\mathscr{C})$, we are given operations (Λ_0 linear homomorphisms)

$$(5.2) \qquad \mathfrak{m}_k : \mathscr{C}[1](c_0, c_1) \widehat{\otimes} \cdots \widehat{\otimes} \mathscr{C}[1](c_{k-1}, c_k) \longrightarrow \mathscr{C}[1](c_0, c_k),$$

of degree $+1$ for $k = 0, 1, 2, \ldots$ and $c_i \in \mathfrak{Ob}(\mathscr{C})$, which preserves the filtration. We call it structure operations. Here $\mathscr{C}[1](c, c')$ is the degree shift of $\mathscr{C}(c, c')$. Namely the degree k part of $\mathscr{C}[1](c, c')$ is degree $k + 1$ part of $\mathscr{C}(c, c')$. The symbol $\widehat{\otimes}$ denotes the T-adic completion of the algebraic tensor product.

(4) The following A_∞ relation is satisfied.

(5.3)
$$0 = \sum_{k_1+k_2=k+1} \sum_{i=0}^{k_1-1} (-1)^* \mathfrak{m}_{k_1}(x_1, \ldots, x_i, \mathfrak{m}_{k_2}(x_{i+1}, \ldots, x_{k_2}), \ldots, x_k),$$

where $* = i + \sum_{j=1}^{i} \deg x_j$, $x_i \in \mathscr{C}[1](c_{i-1}, c_i)$, $c_0, \ldots, c_k \in \mathfrak{Ob}(\mathscr{C})$.
(5) We require
$$\mathfrak{m}_0(1) \equiv 0 \mod T^\epsilon,$$

for some $\epsilon > 0$.
(6) An element $\mathbf{e}_c \in \mathscr{C}(c, c)$ of degree 0 is given for each $c \in \mathfrak{Ob}(\mathscr{C})$ such that:
   (a) If $x_1 \in \mathscr{C}(c, c')$, $x_2 \in \mathscr{C}(c', c)$ then

$$\mathfrak{m}_2(\mathbf{e}_c, x_1) = x_1, \quad \mathfrak{m}_2(x_2, \mathbf{e}_c) = (-1)^{\deg x_2} x_2.$$

   (b) If $k + \ell \neq 1$, $x_1 \otimes \cdots \otimes x_\ell \in B_\ell \mathscr{C}[1](a, c)$, $y_1 \otimes \cdots \otimes y_k \in B_k \mathscr{C}[1](c, b)$ then

   (5.4) $$\mathfrak{m}_{k+\ell+1}(x_1, \ldots, x_\ell, \mathbf{e}_c, y_1, \ldots, y_k) = 0.$$

   We call $\mathbf{e}_c$ the unit.[4]

A filtered A_∞ category with one object is called a filtered A_∞ algebra.
A filtered A_∞ category or algebra is called strict if $\mathfrak{m}_0 = 0$. It is called curved otherwise.

Note $\mathfrak{m}_1 : \mathscr{C}(c, c') \to \mathscr{C}(c, c')$ is degree one and is regarded as a 'boundary operator'. However in general $\mathfrak{m}_1 \circ \mathfrak{m}_1 = 0$ does *not hold*. In fact (5.3) implies

(5.5) $$\mathfrak{m}_1(\mathfrak{m}_1(x)) + \mathfrak{m}_2(\mathfrak{m}_0(1), x) + (-1)^{\deg x+1} \mathfrak{m}_2(x, \mathfrak{m}_0(1)) = 0.$$

On the other hand (5.5) implies that $\mathfrak{m}_1 \circ \mathfrak{m}_1 = 0$ if filtered A_∞ category $\mathscr{C}$ is strict. The algebraic point about the well-definedness of $\mathfrak{m}_1$ homology, which we mentioned above, is closely related to the geometric problem to define Lagrangian Floer theory, which we mentioned at the end of the last section. We will go back to this point in the next subsection.

An idea introduced in [FOOO1] is to deform Floer's boundary operator $\mathfrak{m}_1$ to $\mathfrak{m}_1^b$ so that $\mathfrak{m}_1^b \circ \mathfrak{m}_1^b = 0$ holds.

Definition 5.2. *Let $c \in \mathfrak{Ob}(\mathscr{C})$. A bounding cochain (or Maurer–Cartan element) of c is an element $b \in \mathscr{C}(c, c)$ such that:*

(1) The degree of b is 1.
(2) $b \equiv 0 \mod \Lambda_+$.

---

[4] In some reference we do not assume unit to exist for filtered A_∞ category. In this article we assume it to simplify the notation.

(3)

(5.6)
$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b, \ldots, b) = 0.$$

Note (2) implies that the left hand side of (5.6) converges in T-adic topology.
We denote by $\widetilde{\mathcal{M}}(c)$ the set of all bounding cochains. We say an object c is unobstructed if $\widetilde{\mathcal{M}}(c)$ is non-empty.

Remark 5.3. (1) We can define appropriate notion of gauge equivalence among elements of $\widetilde{\mathcal{M}}(c)$. (See [FOOO1, Sect. 4.3].) The set of all gauge equivalence classes is called *Maurer–Cartan moduli space* and is written as $\mathcal{M}(c)$.[5]
(2) In certain situation we may relax the condition Definition 5.2 (2) and can use a class $b \in \mathscr{C}(c, c)$ of degree 1 which satisfies (5.6). (In such a case the left hand side of (5.6) should be defined carefully.) We write $\widetilde{\mathcal{M}}(c; \Lambda_+)$ in place of $\widetilde{\mathcal{M}}(c)$ when we want to clarify that we consider only the elements satisfying Definition 5.2 (2).

Definition-Lemma 5.4. *Let $c, c' \in \mathfrak{Ob}(\mathscr{C})$ and $b \in \widetilde{\mathcal{M}}(c)$, $b' \in \widetilde{\mathcal{M}}(c')$. We define*

(5.7)
$$\mathfrak{m}_1^{b,b'} : \mathscr{C}(c, c') \longrightarrow \mathscr{C}(c, c')$$

by the formula:

(5.8)
$$\mathfrak{m}_1^{b,b'}(x) = \sum_{k,\ell=0}^{\infty} \mathfrak{m}_{k+\ell+1}(\underbrace{b, \ldots, b}_{k}, x, \underbrace{b', \ldots, b'}_{\ell}).$$

(5.3) and (5.6) imply
$$\mathfrak{m}_1^{b,b'} \circ \mathfrak{m}_1^{b,b'} = 0.$$

We define

(5.9)
$$HF((c, b), (c', b')) = \frac{\mathrm{Ker}(\mathfrak{m}_1^{b,b'} : \mathscr{C}(c, c') \longrightarrow \mathscr{C}(c, c'))}{\mathrm{Im}(\mathfrak{m}_1^{b,b'} : \mathscr{C}(c, c') \longrightarrow \mathscr{C}(c, c'))}$$

and call it the Floer cohomology of (c, b) and (c', b').

We can deform $\mathfrak{m}_k$ in the same way as follows. Hereafter we write $\mathfrak{m}_{k+\ell+1}(b^k, x, (b')^\ell)$, etc. in place of $\mathfrak{m}_{k+\ell+1}(\underbrace{b, \ldots, b}_{k}, x, \underbrace{b', \ldots, b'}_{\ell})$, etc.

Definition-Lemma 5.5. *Let $\mathscr{C}$ be a curved filtered A_∞ category. We define a strict filtered A_∞ category $\mathscr{C}'$ as follows.*

---

[5] It can be regarded as a set of rigid points of certain rigid analytic stack.

(1) An object of $\mathscr{C}'$ is a pair (c, b) where $c \in \mathfrak{DB}(\mathscr{C})$ and $b \in \widetilde{\mathfrak{M}}(c)$.

(2) If $(c, b), (c', b')$ are objects of $\mathscr{C}'$ then $\mathscr{C}'((c, b), (c', b')) = \mathscr{C}(c, c')$ by definition.

(3) If $(c_i, b_i) \in \mathfrak{DB}(\mathscr{C}')$ for $i = 0, \ldots, k$ and $x_i \in \mathscr{C}'((c_{i-1}, b_{i-1}), (c_i, b_i)) = \mathscr{C}(c_{i-1}, c_i)$ for $i = 1, \ldots, k$. Then we define the structure operations $\mathfrak{m}_k^{(b_0, \ldots, b_k)}$ of $\mathscr{C}'$ as follows.

$$
\begin{aligned}
&\mathfrak{m}_k^{(b_0, \ldots, b_k)}(x_1, \ldots, x_k) \\
(5.10) \quad &= \sum_{\ell_0, \cdots, \ell_k} \mathfrak{m}_{k+\ell_0+\cdots+\ell_k}(b_0^{\ell_0}, x_1, b_1^{\ell_1}, \ldots, b_{k-1}^{\ell_{k-1}}, x_k, b_k^{\ell_k}).
\end{aligned}
$$

We call $\mathscr{C}'$ the associated strict category to $\mathscr{C}$.

Note

$$
\mathfrak{m}_0^{(b)}(1) = \sum_{k=0}^{\infty} \mathfrak{m}_k(b, \ldots, b) = 0
$$

by (5.6). We omit the proof that the structure operations (5.10) satisfies the relation (5.3), which is an easy calculation.

5.2. Immersed Lagrangian submanifold and its Floer homology

Let $i_L : \tilde{L} \to X$ be an n dimensional immersed submanifold of a symplectic manifold X of dimension $2n$. We say $L = (\tilde{L}, i_L)$ is an *immersed Lagrangian submanifold* if $i_L^* \omega = 0$.

Definition 5.6. *We say that L has clean self-intersection if the following holds.*

(1) The fiber product $\tilde{L} \times_X \tilde{L}$ is a smooth submanifold of $\tilde{L} \times \tilde{L}$.
(2) For each $(p, q) \in \tilde{L} \times_X \tilde{L}$ we have

$$
\{(V, W) \in T_p\tilde{L} \oplus T_q\tilde{L} \mid di_L(V) = di_L(W)\} = T_{(p,q)}(\tilde{L} \times_X \tilde{L}).
$$

We say L has transversal self-intersection if L has clean self-intersection and

$$
(\tilde{L} \times_X \tilde{L}) \setminus \tilde{L}
$$

is a finite set. (Note the fiber product $\tilde{L} \times_X \tilde{L}$ contains the diagonal $\cong \tilde{L}$.)

A finite set of immersed Lagrangian submanifolds $\{L_i \mid i = 1, \ldots, N\}$ is said a clean collection (resp. transversal collection) if the disjoint union $\bigcup_{i=1}^{N} L_i$ has clean self-intersection (resp. has transversal self-intersection).

Lagrangian Floer theory in [FOOO1], [FOOO2] associates a filtered A_∞ algebra to an embedded Lagrangian submanifold L. This A_∞ algebra as a Λ_0 module is taken to be the cohomology group of L or any of its chain model. Namely it defines a homomorphism $\mathfrak{m}_k : H(L; \Lambda_0)^{\otimes k} \to H(L; \Lambda_0)$ satisfying (5.3). It also associates a filtered A_∞ category $\mathfrak{Fuk}(\mathfrak{L})$ to a transversal collection of embedded Lagrangian submanifolds $\mathfrak{L} = \{L_i \mid i = 1, \ldots, N\}$ as follows.

(L.C1) The set of object is $\mathfrak{L}$.
(L.C2) For $L_i, L_j \in \mathfrak{L}$, the module of morphisms, which we write $CF(L_i, L_j)$, is defined as follows:
    (a) If $i \neq j$ then it is a free Λ_0 module whose basis is identified with $L_i \cap L_j$.
    (b) If $i = j$ then $CF(L_i, L_i) = H(L_i; \Lambda_0)$. [6]
(L.C3) The structure operations (5.2) is defined by using the moduli space of pseudo holomorphic $k + 1$ gons.

Fig. 6 below is the moduli space of $k + 1$ gons, which calculate the coefficient of $[x_0]$ in $\mathfrak{m}_7(x_1, \ldots, x_7)$.
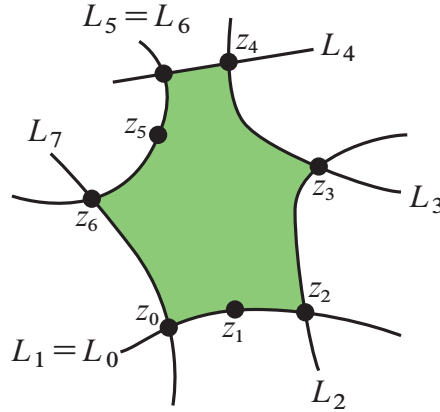

Fig. 6. $\langle \mathfrak{m}_7(x_1, \ldots, x_7), x_0 \rangle$

Akaho–Joyce [AJ] generalized this story to the immersed case as follows. Let $\mathfrak{L}$ be a transversal collection of immersed Lagrangian submanifolds $\mathfrak{L} = \{L_i \mid i = 1, \ldots, N\}$. Then we have a filtered A_∞ category satisfying (L.C1)–(L.C3), except (L.C2) (b) is replaced by

    (b') If $i = j$ then

$$(5.11) \qquad CF(L_i, L_i) = H(L_i; \Lambda_0) \oplus \bigoplus \Lambda_0(p, q)$$

---

[6] We may also take $CF(L_i, L_i)$ (in principle any) chain model of the cohomology of L_i. For example in [FOOO1], [FOOO2], [FOOO3], [Fu6] the singular chain complex is used. In [AFOOO], [Fu10], [FOOO5], [FOOO6], etc. de Rham complex is used.

where the direct sum is taken over all (p, q) such that $i_{L_i}(p) = i_{L_i}(q)$ and $p \neq q$.

We say $\bigoplus \Lambda_0(p, q)$ the *switching part* and $H(L_i; \Lambda_0)$ the *diagonal part* of $CF(L_i, L_i)$.

We remark that (p, q) is an *ordered* pair. Namely $(p, q) \neq (q, p)$. So we associate extra two generators to each self-intersection.

The definition of structure operation including the generator (p, q) is similar and by using the moduli space drawn below. Note the 3rd marked point in
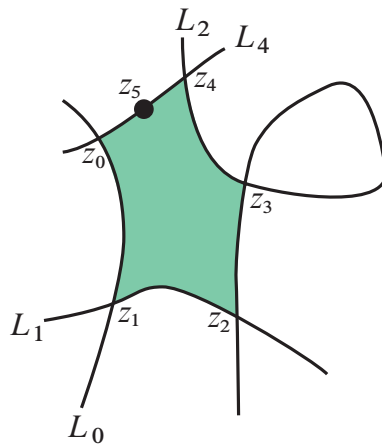

Fig. 7. $\langle \mathfrak{m}_7(x_1, \ldots, x_5), x_0 \rangle$

the figure corresponds (p, q) which is the extra generator appearing in item (b') above. Those generators correspond to the switchings at the boundary curve. We remark if $x = i_{L_i}(p) = i_{L_i}(q)$ is the self-intersection point of L_i, the intersection $L_i \cap U$ of L_i with a small neighborhood U of x consists of two smooth n dimensional submanifolds. They are image of neighborhoods of p and of q respectively. Switching at the 3rd marked point means that the boundary value is on one of the components of $L_i \cap U$ for $z \in \partial D^2$, $z < z_3$ and on the other component of $L_i \cap U$ for $z \in \partial D^2$, $z > z_3$. There are two different ways how the switching occurs. One is the switching from the component containing p to the component containing q, and the other is the switching from the component containing q to the component containing p. Those two different ways of switching correspond to the two generators appearing in (5.11).

5.3. The case of immersed Lagrangian submanifold which is monotone in the weak sense

Note (5.5) shows that the term $\mathfrak{m}_0(1)^7$ causes the problem to define Floer homology. Namely if $\mathfrak{m}_0(1) = 0$ then $\mathfrak{m}_1 \mathfrak{m}_1 = 0$. In other words we are looking

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[7] which is called curvature sometimes.

for an appropriate element b of $CF(L_i, L_i)$ such that $\mathfrak{m}_0^b(1) = 0$. We called such b a *bounding cochain*. The progress we made recently in our study of 3+2 dimensional Donaldson–Floer theory, is that we can now prove the existence of such b in the situation appearing in it. We explain it in Sect. 6. Before doing so we go back to the situation of Subsect. 4.2.

Let $\mathfrak{L}$ be a transversal collection of immersed Lagrangian submanifolds. We put $\mathfrak{L} = \{L_i \mid i = 1, \ldots, N\}$.

Proposition 5.7. *Suppose each of L_i are immersed monotone Lagrangian submanifolds in the weak sense and its minimal Maslov index is greater than 2. Then the element*

$$\mathfrak{m}_0(1) \in CF(L_i, L_i)$$

lies in the switching part.

Proof. The diagonal part of $\mathfrak{m}_0(1)$ is defined as the (virtual) fundamental class of the moduli space of the pair $(((D^2, \partial D^2), z_0), u)$ where $u : (D^2, \partial D^2) \to (X, L_i)$ is a pseudo holomorphic curve such that $\partial D^2 \to L_i$ lifts to $\tilde{L}_i$. Therefore by assumption the (virtual) dimension of such moduli space is $\mu(\beta) + n - 3 + 1$. Since $\mu(\beta) > 2$ by assumption the dimension is greater than n. Hence it lies in $H_*(L_i)$ with $* > n = \dim L_i$ and is 0. $\square$

By Proposition (5.7) and (5.5) the right hand side of

$$\mathfrak{m}_1\mathfrak{m}_1(x) = (-1)^{\deg x}\mathfrak{m}_2(x, \mathfrak{m}_0(1)) - \mathfrak{m}_2(\mathfrak{m}_0(1), x)$$

is defined by the moduli space as in Fig. 5. In other words this formula is an algebraic way to explain the difficulty to define the Floer homology of a pair of immersed Lagrangian submanifolds which is monotone but immersed. (We explained it in Subsect. 4.2 in a geometric way.)

We also have the following:

Proposition 5.8. *In the situation of Proposition 5.7 the sum appearing in the definition of structure operations are all finite sum and the construction works over the ground ring $\mathbb{Z}_2$ (or $\mathbb{Z}$ in the case when we can find orientations of the moduli space which is compatible at the boundaries).*

The proof of the first half is similar to the proof of Proposition 5.7. The proof of the second half is similar to [FOOO4].

6. Instanton Floer homology for 3 manifolds with boundary

6.1. Main theorems

The next two theorems are the main results explained in this article. We assume the following for the simplicity of the statement. Let M be a 3 manifold with boundary $\Sigma = \partial M$ and let $\mathcal{P}_M$ be a principal $SO(3)$ bundle. We denote $\mathcal{P}_\Sigma$ the restriction of $\mathcal{P}_M$ to Σ. We assume $w^2(\mathcal{P}_\Sigma) = [\Sigma]$. (*cf.* Assumption 1.2 (1).)

Assumption 6.1. We assume Assumption 2.2. Moreover we assume that the restriction map defines an immersion $R(M; \mathcal{P}_M) \to R(\Sigma; \mathcal{P}_\Sigma)$.

We remark that we can reduce the general case to the case when this assumption is satisfied by perturbing the equation $F_a = 0$ in the same way as [D2], [Fl2], [He].

Theorem 6.2. *We assume Assumption 6.1. Then the object $R(M; \mathcal{P}_M)$ is unobstructed. Namely there exists a bounding cochain b_M in $CF(R(M; \mathcal{P}_M))$.*

Moreover we can find b_M in a canonical way. In other words, its gauge equivalence class is an invariant of $(M; \mathcal{P}_M)$.

Remark 6.3. (1) We omit the definition of gauge equivalence between bounding cochains. See [FOOO1, Definition 4.3.1].
(2) We can remove Assumption 6.1 in Theorem 6.2 by perturbing the equation $F_a = 0$ (which defines $R(M; \mathcal{P}_M)$) on a compact subset of M. The same remark applies to the next theorem.
(3) We can also prove that b_M is supported in the switching components.

Theorem 6.4. *Suppose $(M_1, \mathcal{P}_{M_1})$ and $(M_2, \mathcal{P}_{M_2})$ satisfy Assumption 2.2. We also assume*

$$\partial(M_1, \mathcal{P}_{M_1}) = (\Sigma, \mathcal{P}_\Sigma), \quad \partial(M_2, \mathcal{P}_{M_2}) = (-\Sigma, \mathcal{P}_\Sigma).$$

We glue $(M_1, \mathcal{P}_{M_1})$ and $(M_2, \mathcal{P}_{M_2})$ along their boundaries to obtain $(M, \mathcal{P}_M)$. Then we have an isomorphism:

$$(6.1) \quad HF((R(M_1; \mathcal{P}_{M_1}), b_{M_1}), (R(M_2; \mathcal{P}_{M_2}), b_{M_2})) \cong HF(M, \mathcal{P}_M) \otimes \Lambda_0^{\mathbb{Z}_2}.$$

Note the Lagrangian Floer homology in the left hand side has $\Lambda_0^{\mathbb{Z}_2}$ coefficient. However actually it can be defined over $\mathbb{Z}_2$ coefficient. In fact we have:

Proposition 6.5. *In the situation of Theorem 6.2 we may choose the bounding cochain b_M so that it is entirely in the switching part.*

Then using Proposition 5.8 we can define Floer homology

$$HF((R(M_1; \mathcal{P}_{M_1}), b_{M_1}), (R(M_2; \mathcal{P}_{M_2}), b_{M_2}))$$

over $\mathbb{Z}_2$ coefficient and (6.1) holds over $\mathbb{Z}_2$ coefficient.

Theorems 6.2 and 6.4 will be proved in a forthcoming joint paper with Aliakbar Daemi. (See [DF] for possible generalizations of them.)

6.2. A possible way to obtain bounding cochain b_M directly from moduli spaces of ASD-connections

In this subsection we explain a conjecture which provides a way to obtain the bounding cochain b_M in Theorem 6.2, directly by counting the order of certain moduli space of ASD-connections. The author is unable to prove this conjecture, which looks rather hard. The proof of Theorem 6.2 is performed in a different way as we will sketch in Subsects. 6.3 and 6.4.

Let $(M, \mathcal{P}_M)$ be as in Theorem 6.2. We consider the interior of M, that is $M \setminus \partial M$. We write it M for simplicity of notation in this subsection. We choose its Riemannian metric such that M minus a compact set is isometric to the direct product $\Sigma \times (0, \infty)_t$. We use t as the coordinate of $(0, \infty)_t$. We put the suffix t to clarify this point. We take direct product $M \times \mathbb{R}_\tau$ and use τ as the coordinate of $\mathbb{R}$ factor and consider a principal $SO(3)$ bundle $\mathcal{P}_M \times \mathbb{R}_\tau$.

Definition 6.6. *We consider the set of all connections A of $\mathcal{P}_M \times \mathbb{R}_\tau$ with the following properties.*

(1) $F_A^+ = 0$. Namely A is an Anti-Self-Dual connection.
(2) We have

$$(6.2) \qquad \int_{M \times \mathbb{R}_\tau} \|F_A\|^2 \, \mathrm{Vol}_M \, d\tau = E < \infty.$$

We denote by $\widetilde{\mathcal{M}}(M \times \mathbb{R}_\tau; \mathcal{P}_M \times \mathbb{R}_\tau; E)$ the set of all gauge equivalence classes of such A. (Here E is as in (6.2).)

We can define an $\mathbb{R}_\tau$ action on $\widetilde{\mathcal{M}}(M \times \mathbb{R}_\tau; \mathcal{P}_M \times \mathbb{R}_\tau; E)$ by the translation of $\mathbb{R}_\tau$ direction. We denote by $\mathcal{M}(M \times \mathbb{R}_\tau; \mathcal{P}_M \times \mathbb{R}_\tau; E)$ the quotient space of this $\mathbb{R}_\tau$ action.

Conjecture 6.7. We assume Assumption 6.1. Moreover we assume that $R(M; \mathcal{P}_M)$ has transversal self-intersection. Then we can find an $\mathbb{R}_\tau$ invariant perturbation supported on a compact subset of M so that the following holds.

(1) $\mathcal{M}(M \times \mathbb{R}_\tau; \mathcal{P}_M \times \mathbb{R}_\tau; E)$ becomes a finite dimensional manifold.
(2) We can compactify it so that the singularity of the compactification has codimension 4.
(3) Any element of $\mathcal{M}(M \times \mathbb{R}_\tau; \mathcal{P}_M \times \mathbb{R}_\tau; E)$ is gauge equivalent to a connection A with the following properties.
  (a) There exist $[a], [b] \in R(M; \mathcal{P}_M)$ such that

$$\lim_{\tau \to -\infty} A|_{M \times \{\tau\}} = a, \qquad \lim_{\tau \to +\infty} A|_{M \times \{\tau\}} = b.$$

  (b) There exists $[\alpha] \in R(\Sigma; \mathcal{P}_\Sigma)$ such that the following holds for any τ.

$$\lim_{t \to \infty} A|_{\Sigma \times \{(t, \tau)\}} = \alpha.$$

   Note we require α to be independent of τ.

(c) In particular the restriction of a and b to Σ are both gauge equivalent to α.

(4) If the dimension of $\mathcal{M}(M \times \mathbb{R}_\tau; \mathcal{P}_M \times \mathbb{R}_\tau; E)$ is zero then $[a] \neq [b]$ in item (3).

(5) We put Switch $= \{([a], [b]) \mid [a] \neq [b], \ [a|_\Sigma] = [b|_\Sigma]\}$. For each $([a], [b]) \in$ Switch, let $c([a], [b])$ be the number of elements $[A]$ as in item (3) such that it is in the zero dimensional component. Then the sum

$$\sum_{([a],[b]) \in \text{Switch}} c([a], [b])([a], [b]) \in CF(R(M; \mathcal{P}_M), R(M; \mathcal{P}_M))$$

is the bounding cochain b_M in Theorem 6.2.

This conjecture is difficult to prove. It seems that item (3) is a kind of gauge theory analogue of a result by Bottman [Bo].

6.3. Right A_∞ module and cyclic element

In this and the next subsections we explain a way to go around the difficult analysis to study the moduli space $\mathcal{M}(M \times \mathbb{R}_\tau; \mathcal{P}_M \times \mathbb{R}_\tau; E)$ but use an algebraic lemma to obtain the bounding cochain b_M. To state it we need notations.

Definition 6.8. *Let $(C, \{\mathfrak{m}_k\})$ be a filtered A_∞ algebra, which may be curved. A right filtered A_∞ module over $(C, \{\mathfrak{m}_k\})$ is $(D, \{\mathfrak{n}_k\})$ such that:*

(1) D is a graded Λ_0 module which is a completion of free Λ_0 module,
(2) For $k = 0, 1, 2, \ldots,$

$$\mathfrak{n}_k : D \otimes \underbrace{C[1] \otimes \cdots \otimes C[1]}_{k} \longrightarrow D$$

is a Λ_0 module homomorphism which preserves filtration and has degree 1. Here $C[1]^d = C^{d+1}$ by definition.
(3) The next relation holds for $y \in D$ and $x_1, \ldots, x_k \in C$.
    (6.3)

$$0 = \sum_{\ell=0}^{k} \mathfrak{n}_{k-\ell}(\mathfrak{n}_\ell(y; x_1, \ldots, x_\ell), x_{\ell+1}, \ldots, x_k)$$

$$+ \sum_{i=-1}^{k} \sum_{j=i}^{k} (-1)^* \mathfrak{n}_{k-j+i+1}(y; x_1, \ldots, \mathfrak{m}(x_{i+1}, \ldots, x_j), x_{j+1}, \ldots, x_k),$$

where $* = \deg y + \deg x_1 + \cdots + \deg x_i + i + 1$.

Let $(D, \{\mathfrak{n}_k\})$ be a right filtered A_∞ module over $(C, \{\mathfrak{m}_k\})$ and b a bounding cochain of $(C, \{\mathfrak{m}_k\})$. We define $d^b : D \to D$ by

$$(6.4) \qquad\qquad d^b(y) = \sum_{k=0}^{\infty} \mathfrak{n}_k(y; b, \ldots, b).$$

It is easy to see from (6.3) that $d^b \circ d^b = 0$.

Definition 6.9. *Let $(D, \{\mathfrak{n}_k\})$ be a right filtered A_∞ module over $(C, \{\mathfrak{m}_k\})$. An element $\mathbf{1} \in D$ is said to be a cyclic element if the following holds.*

(1) The map $x \longmapsto \mathfrak{n}_1(\mathbf{1}, x)$ is a Λ_0 module isomorphism: $C \longrightarrow D$,
(2) $\mathfrak{n}_0(\mathbf{1}) \equiv 0 \mod \Lambda_+$.

Definition 6.10. *Let G be a submonoid of $\mathbb{R}$ which is discrete.*
Let C be a completion of a free Λ_0 module. An element x of C is said to be G-gapped if it is of the form

$$x = \sum_{i,j} a_{i,j} T^{\lambda_i} e_j,$$

where $a_{i,j} \in R$ (the ground ring), $\lambda_i \in G \subset \mathbb{R}_{\geq 0}$, and e_j is a basis of C.
Suppose C_i $(i = 1, 2)$ are completions of free Λ_0 modules. A filtered Λ_0 module homomorphism from C_1 to C_2 is said to be G-gapped if it sends G-gapped elements to G-gapped elements.
A filtered A_∞ algebra (resp. category, module) are said to be G-gapped if all of its structure operations are G-gapped.

The filtered A_∞ categories we obtain in Lagrangian Floer theory are always G-gapped for some G.

Proposition 6.11. *Let $(D, \{\mathfrak{n}_k\})$ be a right filtered A_∞ module over $(C, \{\mathfrak{m}_k\})$. We assume that they are G-gapped. Let $\mathbf{1} \in D$ be a cyclic element, which is also G-gapped.*
Then there exists a unique G-gapped element b of C such that:

(1) b is a bounding cochain of C,
(2)

$$(6.5) \qquad\qquad\qquad d^b(\mathbf{1}) = 0.$$

Here d^b is as in (6.4).

The proof is actually easy. We regard (6.5) as an equation for b. We can solve it by induction on energy filtration. (We use G-gappedness here so that the filtration is parametrized by a discrete set.) Then using $d^b(d^b(\mathbf{1})) = 0$, we can show that b is a bounding cochain. See [Fu9, Proposition 3.5] for the proof.

We remark that we do not assume that C is unobstructed. Namely a priori there may not exist bounding cochain. In other words, we can use Proposition 6.11 to show the existence of bounding cochain. This turn out to be a useful tool to prove such existence. We remark that we are unable to define Lagrangian Floer homology unless we have some bounding cochain. So proving the existence of bounding cochain is a crucial step for various applications of Lagrangian Floer theory.

6.4. Existence of bounding cochain

In this subsection we show an outline of the way how we use Proposition 6.11 to prove Theorem 6.2.

To put the discussion in an appropriate perspective, we consider the following situation. Let $(M; \mathcal{P}_M)$ be as in Theorem 6.2. Let L be an immersed Lagrangian submanifold of $R(\Sigma; \mathcal{P}_\Sigma)$. We assume that $\{R(M; \mathcal{P}_M), L\}$ is a clean collection in the sense of Definition 5.6.

We put

$$(6.6) \qquad CF((M; \mathcal{P}_M); L) = C_*(R(M; \mathcal{P}_M) \times_{R(\Sigma; \mathcal{P}_\Sigma)} L) \, \hat{\otimes} \, \Lambda_0.$$

Here $R(M; \mathcal{P}_M) \times_{R(\Sigma; \mathcal{P}_\Sigma)} L$ is the fiber product of two immersed Lagrangian submanifolds and is a smooth manifold by our assumption. $C_*(R(M; \mathcal{P}_M) \times_{R(\Sigma; \mathcal{P}_\Sigma)} L)$ is certain chain model of the homology group of this manifold. $\hat{\otimes}$ is the T-adic completion of algebraic tensor product.

Note $CF((M; \mathcal{P}_M); L)$ as a Λ_0 module is the same as the underlying Λ_0 module $CF(R(M; \mathcal{P}_M); L)$ of the chain complex which we use to define Floer homology $HF(R(M; \mathcal{P}_M); L)$. We recall that we associated to L a filtered A_∞ algebra (as in Subsect. 5.2). We write it $CF(L)$.

Theorem 6.12. *On $CF((M; \mathcal{P}_M); L)$, there exists a structure of right filtered A_∞ module over $CF(L)$.*

We use the next proposition together with Theorem 6.12 to prove Theorem 6.2.

We take $L = R(M; \mathcal{P}_M)$. Then by definition

$$CF((M; \mathcal{P}_M); L) = C_*(R(M; \mathcal{P}_M) \times_{R(\Sigma; \mathcal{P}_\Sigma)} R(M; \mathcal{P}_M)) \, \hat{\otimes} \, \Lambda_0.$$

We remark $R(M; \mathcal{P}_M)$ is an open and closed submanifold of $R(M; \mathcal{P}_M) \times_{R(\Sigma; \mathcal{P}_\Sigma)} R(M; \mathcal{P}_M)$. We denote by $\mathbf{1} \in CF((M; \mathcal{P}_M); R(M; \mathcal{P}_M))$ the differential 0 form which is 1 on $R(M; \mathcal{P}_M)$ and is zero on other part.

Proposition 6.13. $\mathbf{1} \in CF((M; \mathcal{P}_M); R(M; \mathcal{P}_M))$ *is a cyclic element of the right filtered A_∞ module $CF((M; \mathcal{P}_M); R(M; \mathcal{P}_M))$ over $CF(R(M; \mathcal{P}_M))$.*

Theorem 6.2 is an immediate consequence of Proposition 6.11, Theorem 6.12 and Proposition 6.13.

In the rest of this subsection we briefly explain the idea of the proof of Theorem 6.12. The idea is to use Lagrangian submanifold L as a boundary condition for an ASD-equation on $M \times \mathbb{R}$. (It appeared in [Fu2] in the year 1992. It was elaborated in [Fu5]. Actually this was the motivation of the author when he introduced the notion of A_∞ category in the study of gauge theory and of symplectic geometry.)

We put a metric on M such that it is of product type $\Sigma \times (-1, 0]$ near the neighborhood of the boundary $\partial M = \Sigma$. (Note this metric is different from one we used in Subsect. 6.2. In Subsect. 6.2 we take a Riemannian metric on $M \setminus \Sigma$ which is isometric to $\Sigma \times (0, \infty)$ outside a compact set.)

We then consider the product and $M \times \mathbb{R}_\tau$ and a connection A on it such that:

(1) A is an ASD-connection. Namely $F_A^+ = 0$.
(2)

$$\int_{M \times \mathbb{R}_\tau} \|F_A\|^2 \mathrm{Vol}_M \, d\tau < \infty.$$

(3) There exists $(a, a'), (b, b') \in R(M; \mathcal{P}_M) \times_{R(\Sigma; \mathcal{P}_\Sigma)} L$ such that

$$\lim_{\tau \to -\infty} A|_{M \times \{\tau\}} = a, \quad \lim_{\tau \to +\infty} A|_{M \times \{\tau\}} = b.$$

(4) We put $L = (\tilde{L}, i_L)$ where $i_L : \tilde{L} \to R(\Sigma; \mathcal{P}_\Sigma)$ is an immersion. There exist $\vec{z} = (\tau_1, \ldots, \tau_k) \in \mathbb{R}_\tau$ with $\tau_1 < \cdots < \tau_k$ and a continuous map $\gamma : \mathbb{R}_\tau \setminus \{\tau_1, \ldots, \tau_k\} \to \tilde{L}$ such that

$$\lim_{\tau \to -\infty} \gamma(\tau) = a', \quad \lim_{\tau \to +\infty} \gamma(\tau) = b'$$

and for each τ

$$[A|_{\partial M \times \{\tau\}}] = i_L(\gamma(\tau)) \in R(\Sigma; \mathcal{P}_\Sigma).$$

Moreover $i_L \circ \gamma : \mathbb{R}_\tau \setminus \{\tau_1, \ldots, \tau_k\} \to R(\Sigma; \mathcal{P}_\Sigma)$ extends continuously to $\mathbb{R}_\tau$.

Remark 6.14. The way how we write Condition (4) above is not precise. There are three different ways known to set this boundary conditions at the stage of 2016. (One which the author proposed in [Fu2] in the year 1992 does not seem to give a correct moduli space.)

  (i) The method to use a Riemannian metric which is degenerate near the boundary. This was introduced by the author in [Fu6] in 1997.

(ii) The method to require that A is flat on each $\Sigma \times \{\tau\}$ and its gauge equivalence class is in L. This was introduced and used by Salamon–Wehrheim [SaWe], [We1], [We2] at the beginning of the 21st century.

(iii) Using pseudo holomorphic curve equation near the boundary and a matching condition. This is introduced by Lipyanskiy [Ly1] around 2010.

The methods (i), (iii) both can be used for our purpose. The compactness and removable singularity results which are needed for our purpose are proved in [Fu6] and in [Ly1]. The detail of the Fredholm theory is not yet written.

The method (ii) works for our purpose if $R(M; \mathcal{P}_M)$ and L are both embedded and monotone. In such a case [SaWe] gives a proof of Theorem 6.12. On the other hand it seems difficult to generalize this method beyond the case when $R(M; \mathcal{P}_M)$ is embedded, by the reason explained in [Fu9, Sect. 6].

We consider the pair $(A, \vec{z})$ where A is a connection on $M \times \mathbb{R}_\tau$ satisfying the above conditions (1)–(4) and $\vec{z} = (\tau_1, \ldots, \tau_k)$ as in item (4). We denote the totality of (the gauge equivalence class of) such pair by $\mathcal{M}((M; \mathcal{P}_M), L)$. We use the boundary value of A at $\Sigma \times \{\tau_i\}$ to define evaluation maps

$$\mathrm{ev} = (\mathrm{ev}_1, \ldots, \mathrm{ev}_k) : \mathcal{M}((M; \mathcal{P}_M), L) \longrightarrow (\tilde{L} \times_{R(\Sigma; \mathcal{P}_\Sigma)} \tilde{L})^k.$$

Namely

$$\mathrm{ev}_i([A]) = (\lim_{\tau \downarrow \tau_i} \gamma(\tau), \lim_{\tau \uparrow \tau_i} \gamma(\tau)),$$

where γ is as in item (4) above.

Using asymptotic limit as $\tau \to \pm\infty$ we obtain

$$\mathrm{ev}_{\pm\infty} : \mathcal{M}((M; \mathcal{P}_M), L) \longrightarrow R(M; \mathcal{P}_M) \times_{R(\Sigma; \mathcal{P}_\Sigma)} \tilde{L},$$

namely

$$\mathrm{ev}_{\pm\infty}([A]) = (\lim_{\tau \to \pm\infty} [A(\tau)], \lim_{\tau \to \pm\infty} \gamma(\tau)).$$

Now let $h_{\pm\infty}$ is a differential form on $R(M; \mathcal{P}_M) \times_{R(\Sigma; \mathcal{P}_\Sigma)} \tilde{L}$ and $h_1, \ldots, h_k$ be differential forms on $\tilde{L} \times_{R(\Sigma; \mathcal{P}_\Sigma)} \tilde{L}$. The right filtered A_∞ module structure is defined roughly speaking by

$$(6.7) \quad \begin{aligned} &\langle \mathfrak{n}_k([h_{-\infty}]; [h_1], \ldots, [h_k]), [h_{+\infty}] \rangle \\ &= \int_{\mathcal{M}((M; \mathcal{P}_M), L)} \mathrm{ev}^*(h_1 \times \cdots \times h_k) \wedge \mathrm{ev}_{-\infty}^* h_{-\infty} \wedge \mathrm{ev}_{+\infty}^* h_{+\infty}. \end{aligned}$$

(Note we use integration in (6.7). When we work on $\mathbb{Z}_2$ coefficient we actually need to use, say, singular homology rather than de Rham homology.)

We can prove the relation (6.3) by studying the compactification of the moduli space $\mathcal{M}((M; \mathcal{P}_M), L)$ and its codimension one boundary.

6.5. *Proof of Gluing theorem*

In this subsection we sketch the proof of Theorem 6.4. This proof is a 'gauge theory analogue' of the proof by Lekili–Lipyanskiy [LL] of a similar result in Lagrangian correspondence. (See Sect. 7.) We consider a domain W of $\mathbb{C}$ as in
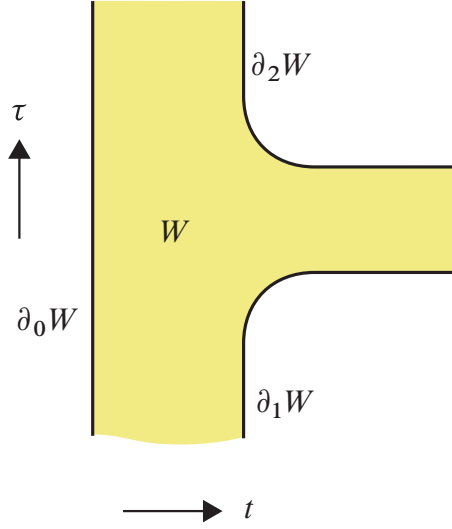

Fig. 8. The domain W

Fig. 8. It has three boundary components $\partial_0 W$, $\partial_1 W$, $\partial_2 W$, which lie in the part $t = 0$, $\tau < 0$, $\tau > 0$, respectively.

We consider the direct product $\Sigma \times W$ with the direct product metric. We glue $M_1 \times \mathbb{R}_\tau$ with $W \times \Sigma$ by the diffeomorphism $\partial M_1 \times \mathbb{R}_\tau \cong \Sigma \times \partial_1 W$. We also glue $M_2 \times \mathbb{R}_\tau$ with $W \times \Sigma$ by the diffeomorphism $\partial M_2 \times \mathbb{R}_\tau \cong \Sigma \times \partial_2 W$. We then obtain a 4 manifold X with boundary and ends. X has a boundary

$$\partial X = \Sigma \times \partial_0 W \cong \Sigma \times \mathbb{R}_\tau.$$

X has three ends.

(End.1) $M_1 \times (-\infty, 0]_\tau$. This lies in the part where $\tau \to -\infty$.

(End.2) $M \times [0, \infty)_t$. Here M is obtained by gluing M_1 and M_2 along Σ. This lies in the part where $t \to \infty$.

(End.3) $M_2 \times [0, +\infty)_\tau$. This lies in the part where $\tau \to +\infty$.

See Fig. 9.

Note the $SO(3)$-bundles $\mathcal{P}_{M_1}$ and $\mathcal{P}_{M_2}$ induce an $SO(3)$-bundle on X, which we denote by $\mathcal{P}_X$.

We take

$$\alpha, \beta \in R(M_1; \mathcal{P}_{M_1}) \times_{R(\Sigma; \mathcal{P}_\Sigma)} R(M_2; \mathcal{P}_{M_2}).$$

Note

$$R(M_1; \mathcal{P}_{M_1}) \times_{R(\Sigma; \mathcal{P}_\Sigma)} R(M_2; \mathcal{P}_{M_2}) \cong R(M; \mathcal{P}_M).$$
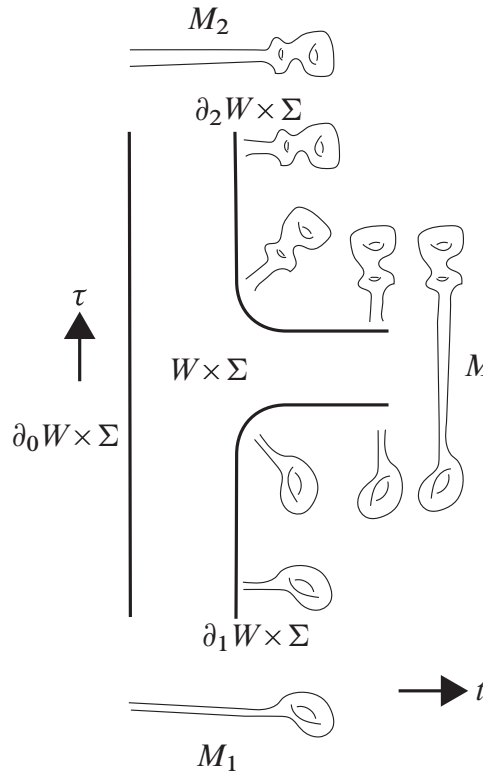
Fig. 9. The 4 manifold X

We regard β as an element of $R(M;\mathcal{P}_M)$ by this isomorphism. We consider the connection A of $\mathcal{P}_X$ with the following properties.

(1) A is an ASD connection. Namely $F_A^+ = 0$.
(2) On boundary $(-\infty, 0] \times \Sigma$ we use the Lagrangian submanifold $R(M_1;\mathcal{P}_{M_1})$ to set the boundary condition for A.
(3) On boundary $[0, +\infty) \times \Sigma$ we use the Lagrangian submanifold $R(M_2;\mathcal{P}_{M_2})$ to set the boundary condition for A.
(4) At $\{(0,0)\} \times \Sigma$ we require that the restriction of A is α.
(5) At the end (End.1) we require that A is asymptotic to an element of

$$R(M_1;\mathcal{P}_{M_1}) \subset R(M_1;\mathcal{P}_{M_1}) \times_{R(\Sigma;\mathcal{P}_\Sigma)} R(M_1;\mathcal{P}_{M_1}).$$

(6) At the end (End.2) we require that A is asymptotic to the flat connection β of $(M, \mathcal{P}_M)$.
(7) At the end (End.3) we require that A is asymptotic to an element of the diagonal

$$R(M_2;\mathcal{P}_{M_2}) \subset R(M_2;\mathcal{P}_{M_2}) \times_{R(\Sigma;\mathcal{P}_\Sigma)} R(M_2;\mathcal{P}_{M_2}).$$

(8)
$$\int_X \|F_A\|^2 \text{Vol}_M \, d\tau < \infty.$$

Note items (5) and (7) mean that A is asymptotic to the cyclic element **1** there. See Fig. 10.
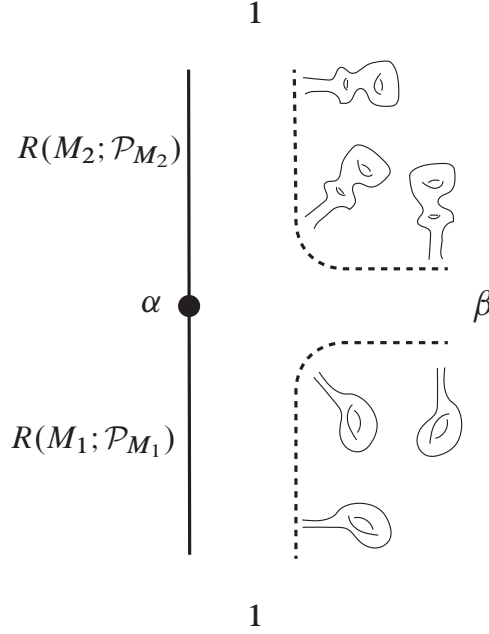

Fig. 10. Condition for connection A

Definition 6.15. *We denote by $\mathcal{M}((X, \mathcal{P}_X); \alpha, \beta)$ the moduli space of gauge equivalence classes of the connections A satisfying the conditions (1)–(8) above.*

We use this moduli space to define a map

$$CF((R(M_1; \mathcal{P}_{M_1}), b_{M_1}), (R(M_2; \mathcal{P}_{M_2}), b_{M_2})) \longrightarrow CF(M; \mathcal{P}_M)$$

by

(6.8) $$[\alpha] \longmapsto \sum_{\beta} \sharp \mathcal{M}((X, \mathcal{P}_X); \alpha, \beta)[\beta].$$

Here we use the component of the moduli space $\mathcal{M}((X, \mathcal{P}_X); \alpha, \beta)$ with (virtual) dimension 0.

To show that (6.8) becomes a chain map, we study the compactification of $\mathcal{M}((X, \mathcal{P}_X); \alpha, \beta)$ and show that its codimension one boundary is classified as follows.

(bdry.1) The disk bubble at the boundary point $(\tau, 0)$ with $\tau < 0$.

(bdry.2) Disk bubble at the boundary point $(\tau, 0)$ with $\tau > 0$.

(bdry.3) Disk bubble at $(0, 0)$.

(bdry.4) Sliding ends as $\tau \to -\infty$.

(bdry.5) Sliding ends as $\tau \to +\infty$.
(bdry.6) Sliding ends as $t \to +\infty$.

See Fig. 11. Note there are two kinds of disk bubbles in (bdry.1), (bdry.2). One is similar to the bubble in Fig. 4 and the other is similar to one in Fig. 5. They are drawn as the left (resp. right) among two figures in Fig. 11 (bdry.1). Actually in our situation where $R(M_i; \mathcal{P}_{M_i})$ are monotone in weak sense, only the bubble in the right figure of Fig. 11 (bdry.1) occurs in codimension 1.
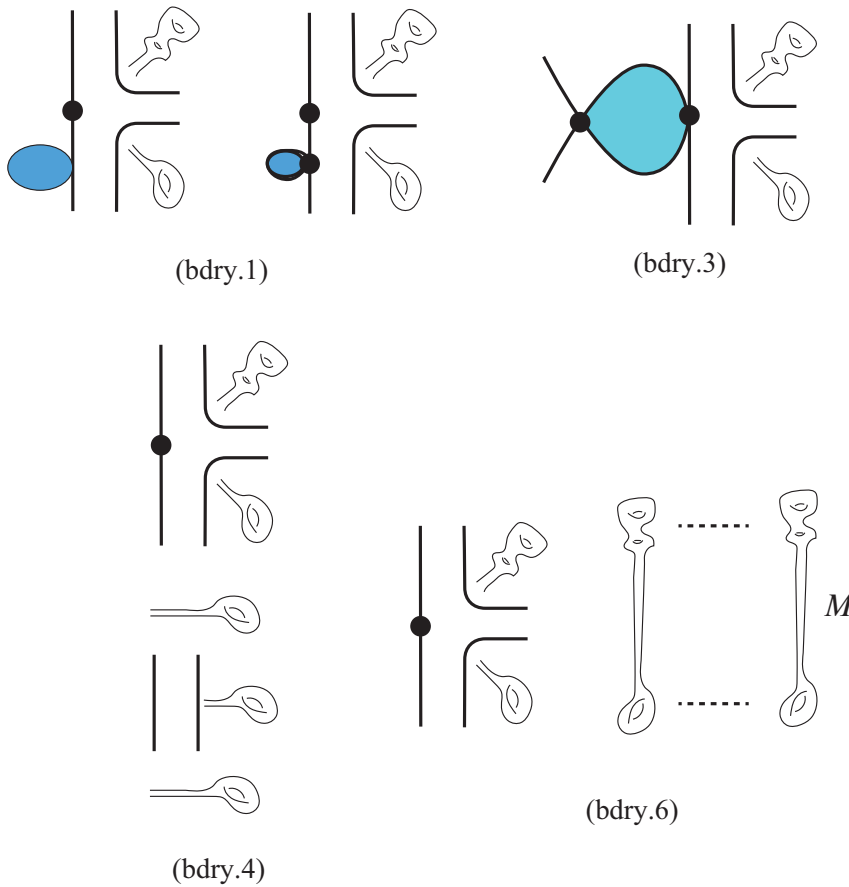

(bdry.1) (bdry.3)

(bdry.4) (bdry.6)

Fig. 11. The boundary of $\mathcal{M}((X, \mathcal{P}_X); \alpha, \beta)$

Remark 6.16. We use the method (i) or (iii) in Remark 6.14 to set boundary condition in Definition 6.15. In those methods we switch from ASD-equation to the pseudo holomorphic curve equation on a line which is close to but is different from the boundary $\partial_0 W$. (See [Fu6], [Ly1] [DF], [DFL], etc.) So we have usual pseudo holomorphic curve equation with Lagrangian boundary condition in a neighborhood of $\partial_0 W$. We use this fact to classify the bubble on $\partial M \times \mathbb{R}_\tau$ as above.

We can cancel the effect of the boundaries as in (bdry.1) and (bdry.2) using bounding cochains b_{M_1} and b_{M_2}. (This part of the argument is the same as the

Lagrangian Floer theory [FOOO1], [AJ].) The effect of boundaries as in (bdry.4) and (bdry.5) is zero because of the equality

$$d^{b_{M_1}}\mathbf{1} = d^{b_{M_2}}\mathbf{1} = 0.$$

(This is the equality (6.5), which we required when we defined b_{M_1} and b_{M_2}.)

Therefore the remaining boundary components are ones of (bdry.3) and (bdry.6).

(bdry.3) is described by the sum of the product

$$(6.9) \qquad \mathcal{M}(R(M_1; \mathcal{P}_{M_1}), R(M_2; \mathcal{P}_{M_2}); \alpha, \alpha') \times \mathcal{M}((X, \mathcal{P}_X); \alpha', \beta)$$

for various α'. Here the first factor is a special case of the moduli space $\mathcal{M}(L_1, L_2; a, b)$, which we introduced in Subsect. 4.1, using (LF.1)–(LF.3). (In Subsect. 4.1 it was assumed that Lagrangian submanifolds are embedded. We can however define this moduli space in the same way in our embedded case, requiring the boundary value to lift to $R(M_1; \mathcal{P}_{M_1})$, $R(M_2; \mathcal{P}_{M_2})$.)

(bdry.6) is described by the sum of the product

$$(6.10) \qquad \mathcal{M}((X, \mathcal{P}_X); \alpha, \beta') \times \mathcal{M}(M \times \mathbb{R}; \beta', \beta)$$

for various β'. Here the second factor is the moduli space introduced in Subsect. 1.2 using condition (IF.1)–(IF.3).

In the simplest case where $b_{M_1} = b_{M_2} = 0$ the above argument implies that the sum of (6.9) and (6.10) is 0 (in $\mathbb{Z}_2$ coefficient in our situation). It implies that the map (6.8) is a chain map.

In the general case we need additional term which is related to the correction by b_{M_i}. It is described by the moduli space drawn in Fig. 12. (We omit the detail.) We again obtain a chain map
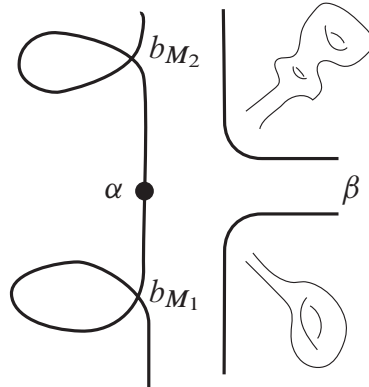

Fig. 12. The correction to the map (6.8)

$$\mathscr{F} : CF((R(M_1; \mathcal{P}_{M_1}), b_{M_1}), (R(M_2; \mathcal{P}_{M_2}), b_{M_2})) \longrightarrow CF(M; \mathcal{P}_M).$$

To show that $\mathscr{F}$ induces isomorphism we use energy filtration. The leading order term of the map $\mathscr{F}$ with respect to the energy filtration is the case of energy 0. It consists of connections A which are flat. In that case α is necessary equal to β. Thus the leading order term of the map $\mathscr{F}$ is the identity map, by using the identification

$$R(M_1; \mathcal{P}_{M_1}) \times_{R(\Sigma; \mathcal{P}_\Sigma)} R(M_2; \mathcal{P}_{M_2}) \cong R(M; \mathcal{P}_M).$$

Therefore $\mathscr{F}$ induces the required isomorphism

$$HF((R(M_1; \mathcal{P}_{M_1}), b_{M_1}), (R(M_2; \mathcal{P}_{M_2}), b_{M_2})) \cong HF(M; \mathcal{P}_M).$$

This is the outline of the proof of Theorem 6.4.

7. Lagrangian correspondence and A_∞ functors.

The story we described in Sect. 6 has an analogue in the study of Lagrangian correspondence, which we will outline in this section. See [Fu10] for detail. In this section we work over the ground ring $\mathbb{R}$. We need to take a spin or relative spin structure of a Lagrangian submanifold to use such ground ring. Spin or relative spin structures are necessary to orient the moduli spaces of pseudo holomorphic disks. (See [FOOO2, Chapter 8].) When (X, ω) is a symplectic manifold and V is an oriented real vector bundle on it the V-relative spin structure of an oriented submanifold $L \subset X$ is by definition the spin structure of the bundle $TL \oplus V|_L$. Under this assumption the construction of Subsect. 5.2 works over the ground ring $\mathbb{R}$ (or $\mathbb{Q}$).

7.1. The main results

Let (X_1, ω_1) and (X_2, ω_2) be compact symplectic manifolds.

Definition 7.1. *An immersed Lagrangian correspondence from X_1 to X_2 is an immersed Lagrangian submanifold of $(X_1 \times X_2, -\omega_1 \oplus \omega_2)$.*

Let $L_{12} = (\tilde{L}_{12}, i_{L_{12}})$ be an immersed Lagrangian correspondence from X_1 to X_2 and $L_1 = (\tilde{L}_1, i_{L_1})$ be an immersed Lagrangian submanifold of X_1. If the fiber product $\tilde{L}_2 = \tilde{L}_1 \times_{X_1} \tilde{L}_{12}$ is transversal then, together with the composition $\tilde{L}_1 \times_{X_1} \tilde{L}_{12} \to \tilde{L}_{12} \to X_1 \times X_2 \to X_2$, the manifold $\tilde{L}_{12}$ defines an immersed Lagrangian submanifold of X_2. We call it the geometric transformation of L_1 by L_{12} and write it as $L_1 \times_{X_2} L_{12}$.

Let (X_i, ω_i) $(i = 1, 2, 3)$ be symplectic manifolds. Let L_{12} be an immersed Lagrangian submanifold of $(X_1 \times X_2, -\omega_1 \oplus \omega_2)$ and L_{23} a Lagrangian submanifold of $(X_2 \times X_3, -\omega_2 \oplus \omega_3)$. We assume that the fiber product

$$\tilde{L}_{12} \times_{X_2} \tilde{L}_{23}$$

is transversal and write the fiber product as $\tilde{L}_{13}$. Together with the obvious map $i_{L_{13}} : \tilde{L}_{13} \to X_1 \times X_3$, the manifold $\tilde{L}_{13}$ defines an immersed Lagrangian submanifold L_{13} of $(X_1 \times X_3, -\omega_1 \oplus \omega_3)$. We call L_{13} the geometric composition of L_{12} and L_{23}. We write it as $L_{12} \times_{X_2} L_{23}$.

See [We].

Definition 7.2. *The (immersed) Weinstein category is defined as follows. Its objects are symplectic manifolds (X, ω). A morphism from X_1 to X_2 is an immersed Lagrangian correspondence from X_1 to X_2.*

The composition of morphisms is defined as their geometric composition.

A slight issue is that, actually, we can define geometric composition only for a transversal pair. However, for the purpose of most of the applications, we can go around this problem by considering only composable pair of morphisms. In other words, Weinstein category is rather a 'topological category' where morphisms can be composed only on certain dense open subset. We can thus go around the problem by carefully stating various theorems in this subsection in such a way using only transversal pair for compositions. Another possible way to proceed is to introduce certain equivalence relation between Lagrangian submanifolds such as Hamiltonian isotopy or Lagrangian cobordism so that we can compose Lagrangian correspondences after perturbing them in the equivalence classes.

In Subsect. 5.2 we started with a finite set of immersed Lagrangian submanifolds $\mathfrak{L}$ of (X, ω) (a clean collection) and obtained a filtered A_∞ category, the set of whose objects is $\mathfrak{L}$. We denote it by $\mathfrak{Fuk}(\mathfrak{L})$. Roughly speaking we can take 'all' immersed Lagrangian submanifolds and define $\mathfrak{Fuk}(X, \omega)$. An issue in doing so is perturbing all the Lagrangian submanifolds simultaneously to obtain some clean collection. We do not discuss this point. Using $\mathfrak{Fuk}(\mathfrak{L})$ instead of $\mathfrak{Fuk}(X, \omega)$ is enough for the purpose of most of the applications. To simplify the notation we pretend as if we defined the filtered A_∞ category $\mathfrak{Fuk}(X, \omega)$. The actual result we prove is one which is restated by using $\mathfrak{Fuk}(\mathfrak{L})$ instead.

Note the filtered A_∞ category $\mathfrak{Fuk}(X, \omega)$ is in general curved. We denote by $\mathfrak{Fuks}(X, \omega)$ the strict category associated to $\mathfrak{Fuk}(X, \omega)$.

In fact to take care of the problem of orientation and sign we need to use relative spin structure. We fix V and consider a set of triples (L, σ, b) where L is a Lagrangian submanifold of X, σ is a V-relative spin structure and b is a bounding cochain of $CF(L)$, that is, the (curved) A_∞ algebra obtained by using (L, σ). The strict filtered A_∞ category whose object is such triple (L, σ, b) is abbreviated by $\mathfrak{Fuks}(X, \omega, V)$.

Definition 7.3. *An unobstructed immersed Weinstein category is defined as follows.*

(1) Its object is a triple (X, ω, V) where (X, ω) is a compact symplectic manifold and V is a real oriented vector bundle on X.

(2) A morphism from (X_1, ω_1, V_1) to (X_2, ω_2, V_2) is a triple $(L_{12}, \sigma_{12}, b_{12})$ where

  (a) L_{12} is an immersed Lagrangian submanifold of $(X_1 \times X_2, -\omega_1 \oplus \omega_2)$.
  (b) σ_{12} is a $\pi_1^* V_1 \oplus \pi_1^* TX_1 \oplus \pi_2^* V_2$-relative spin structure of L_{12}.
  (c) b_{12} is a bounding cochain of $CF(L_{12})$. (Note the filtered A_∞ algebra $CF(L_{12})$ is defined by using the relative spin structure in (b).)

(3) See Theorem 7.6 for the composition of the morphisms.

The main result of [Fu10] is a construction of the (2-)functor from the unobstructed immersed Weinstein category to the (2-)category of all filtered A_∞ categories. We will state it as Theorems 7.4, 7.6, 7.7 below.

Theorem 7.4 ([Fu10]). *Let $(L_{12}, \sigma_{12}, b_{12})$ be as in Definition 7.3 (2).*

(1) Let (L_1, σ_1, b_1) be an object of $\mathfrak{Fuks}(X_1, \omega_1, V_1)$. Then the geometric transformation $L_1 \times_{X_1} L_{12}$ has a canonical choice of V_2 relative spin structure σ_2 and a bounding cochain b_2.

(2) There exists a strict filtered A_∞ functor

$$\mathfrak{W}_{L_{12}} : \mathfrak{Fuks}(X_1, \omega_1, V_1) \longrightarrow \mathfrak{Fuks}(X_2, \omega_2, V_2)$$

of which the map $(L_1, \sigma_1, b_1) \mapsto (L_1 \times_{X_1} L_{12}, V_2, b_2)$ in item (1) is the object part.

(3) There is a strict filtered A_∞ bi-functor

$$\mathfrak{Fuks}(X_1 \times X_2, -\omega_1 \oplus \omega_2, \pi_1^* V_1 \oplus \pi_1^* TX_1 \oplus \pi_2^* V_2) \times \mathfrak{Fuks}(X_1, \omega_1, V_1)$$
$$\longrightarrow \mathfrak{Fuks}(X_2, \omega_2, V_2)$$

which induces $\mathfrak{W}_{L_{12}}$ when we fix an object of the first factor $\mathfrak{Fuks}(X_1 \times X_2, -\omega_1 \oplus \omega_2, \pi_1^* V_1 \oplus \pi_1^* TX_1 \oplus \pi_2^* V_2)$.

The notions of filtered A_∞ functor and bi-functor (and its strictness) are explained in the next subsection.

Remark 7.5. If we assume all the Lagrangian submanifolds involved (including those appearing as fiber products among the Lagrangian submanifolds) are embedded, monotone and have minimal Maslov number > 2, Theorem 7.4 follows from the earlier results by Wehrheim–Woodward [WW1], [WW2], [WW4], [WW3] and Ma'u–Wehrheim–Woodward [MWW]. The same remark applies to Theorems 7.6 and 7.7.

Note Theorem 7.4 (3) implies that there exists a strict filtered A_∞ functor

$$(7.1) \qquad \begin{aligned} &\mathfrak{Fuks}(X_1 \times X_2, -\omega_1 \oplus \omega_2, \pi_1^* V_1 \oplus \pi_1^* TX_1 \oplus \pi_2^* V_2) \\ &\longrightarrow \mathfrak{Funcs}(\mathfrak{Fuks}(X_1, \omega_1, V_1), \mathfrak{Fuks}(X_2, \omega_2, V_2)). \end{aligned}$$

Here $\mathfrak{Funcs}(\mathscr{C}_1, \mathscr{C}_2)$ is the strict filtered A_∞ category whose object is a strict filtered A_∞ functor $: \mathscr{C}_1 \to \mathscr{C}_2$.

Theorem 7.6 ([Fu10]). *Let $(L_{12}, \sigma_{12}, b_{12})$ (resp. $(L_{23}, \sigma_{23}, b_{23})$) be morphisms from (X_1, ω_1, V_1) to (X_2, ω_2, V_2) (resp. from (X_2, ω_2, V_2) to (X_3, ω_3, V_3)).*

(1) Let $L_{13} = L_{12} \times_{X_2} L_{23}$ be the geometric composition. Then we can define a $\pi_2^* V_2 \oplus \pi_2^* T X_2 \oplus \pi_3^* V_3$-relative spin structure σ_{13} on it and a bounding cochain b_{13} of $CF(L_{13})$. (In particular L_{13} is unobstructed.)

(2) Let $\mathfrak{W}_{L_{12}}$, $\mathfrak{W}_{L_{23}}$ and $\mathfrak{W}_{L_{13}}$ be strict filtered A_∞ functors associated to $(L_{12}, \sigma_{12}, b_{12})$, $(L_{23}, \sigma_{23}, b_{23})$ and $(L_{13}, \sigma_{13}, b_{13})$, respectively, by Theorem 7.4 (2). Then

(7.2) $$\mathfrak{W}_{L_{23}} \circ \mathfrak{W}_{L_{12}} \sim \mathfrak{W}_{L_{13}}.$$

Here the left hand side is the composition of strict filtered A_∞ functors and $\sim$ is the homotopy equivalence of two strict filtered A_∞ functors.

(3) The next diagram commutes up to homotopy equivalence of strict filtered A_∞ bi-functors.

$$
\begin{array}{ccc}
\mathfrak{Fuks}(X_1 \times X_2) \times \mathfrak{Fuks}(X_2 \times X_3) & \longrightarrow & \mathfrak{Fuks}(X_1 \times X_3) \\
\downarrow & & \downarrow \\
\begin{array}{c} \mathfrak{Funcs}(\mathfrak{Fuks}(X_1), \mathfrak{Fuks}(X_2)) \\ \times \mathfrak{Funcs}(\mathfrak{Fuks}(X_2), \mathfrak{Fuks}(X_3)) \end{array} & \longrightarrow & \mathfrak{Funcs}(\mathfrak{Fuks}(X_1), \mathfrak{Fuks}(X_3)).
\end{array}
$$

Here the vertical arrows are (7.1). (We omit the bundle V_i in the notation.) The first horizontal line is a strict filtered A_∞ bi-functor whose object part sends $((L_{12}, \sigma_{12}, b_{12}), (L_{23}, \sigma_{23}, b_{23}))$ to $(L_{13}, \sigma_{13}, b_{13})$ in item (1).

The second horizontal line is a strict filtered A_∞ bi-functor whose object part is a composition of filtered A_∞ functors.

(Note the commutativity of this diagram in the object level is (7.2).)

Theorem 7.7 ([Fu10]). *The next diagram commutes up to homotopy equivalence of strict filtered A_∞ tri-functors.*

$$
\begin{array}{ccc}
\begin{array}{c} \mathfrak{Fuks}(X_1 \times X_2) \times \mathfrak{Fuks}(X_2 \times X_3) \\ \times \mathfrak{Fuks}(X_3 \times X_4) \end{array} & \longrightarrow & \mathfrak{Fuks}(X_1 \times X_3) \times \mathfrak{Fuks}(X_3 \times X_4) \\
\downarrow & & \downarrow \\
\mathfrak{Fuks}(X_1 \times X_2) \times \mathfrak{Fuks}(X_2 \times X_4) & \longrightarrow & \mathfrak{Fuks}(X_1 \times X_4),
\end{array}
$$

where all the arrows are defined by composition functor in Theorem 7.6.

Theorems 7.4, 7.6, 7.7 provide the functorial picture of the construction of A_∞ categories out of symplectic manifolds.

Remark 7.8. Theorems 7.4, 7.6, 7.7 provide the functorial picture up to the level of 2-category. Since the diagram in Theorem 7.7 commutes only up to homotopy, we may continue and may have a compatibility as ∞-categories.

7.2. A_∞ functor, A_∞ bi-functor, and Yoneda's lemma

The proof of Theorem 6.2 which we explained in Subsect. 6.4 could be regarded as a proof using the idea of representable functors. For the proof of Theorems 7.4, 7.6, 7.7 we use a similar idea in a more systematic way. In this subsection we explain certain definitions and results in the story of A_∞ category needed for this purpose. See [Fu7], [Ke] ,[Le], [Ly], [Sei], etc. for homological algebra of A_∞ category. The notion of A_∞ bi-functor is discussed in more detail in [Fu10].

Let $\mathscr{C}$ be a filtered A_∞ category and $c, c' \in \mathfrak{OB}(\mathscr{C})$. We define

$$(7.3) \qquad B_k\mathscr{C}[1](c, c') = \bigoplus \bigotimes_{i=1}^{k} \mathscr{C}[1](c_{i-1}, c_i),$$

where direct sum is taken over all $c_0, \ldots, c_k$ such that $c_0 = c$ and $c_k = c'$. Note the tensor product here is a T-adic completion of the algebraic tensor product. We also put:

$$(7.4) \qquad B\mathscr{C}[1](c, c') = \bigoplus_{k=0}^{\infty} B_k\mathscr{C}[1](c, c').$$

Here $B_0\mathscr{C}[1](c, c') = 0$ if $c \neq c'$ and $B_0\mathscr{C}[1](c, c') = \Lambda_0$ if $c = c'$.

$B\mathscr{C}[1](c, c')$ has a formal[8] coalgebra structure

$$\Delta : B\mathscr{C}[1](c, c') \longrightarrow \bigoplus_{c''} B\mathscr{C}[1](c, c'') \hat{\otimes} B\,\mathscr{C}[1](c'', c')$$

defined by

$$\Delta(x_1 \otimes \cdots \otimes x_k) = \sum_{i=1}^{k-1} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_k).$$

The operation $\mathfrak{m}_k$ induces a unique coderivation $B\mathscr{C}[1](c, c') \to B\mathscr{C}[1](c, c')$ so that its $Hom(B_k\mathscr{C}[1](c, c'), B_1\mathscr{C}[1](c, c'))$ component is $\mathfrak{m}_k$. We denote it by $\hat{d}_k$ and put

$$\hat{d} = \sum \hat{d}_k : B\mathscr{C}[1](c, c') \longrightarrow B\mathscr{C}[1](c, c').$$

The A_∞ relation (5.3) is equivalent to the equality $\hat{d} \circ \hat{d} = 0$.

Definition 7.9. *Let $\mathscr{C}_1, \mathscr{C}_2$ be filtered A_∞ categories. A filtered A_∞ functor $\mathscr{F}$: $\mathscr{C}_1 \to \mathscr{C}_2$ consists of objects:*

---

[8] Here our structure is a formal coalgebra structure rather than a coalgebra structure since we take completion of the tensor product.

(1) A map $\mathscr{F}_{ob} : \mathfrak{DB}(\mathscr{C}_1) \to \mathfrak{DB}(\mathscr{C}_2)$,
(2) A series of maps

$$\mathscr{F}_{c,c'} : B\mathscr{C}_1[1](c,c') \longrightarrow \mathscr{C}_2[1](\mathscr{F}_{ob}(c), \mathscr{F}_{ob}(c')),$$

which preserves filtrations.
(3) We require that the coalgebra homomorphism

$$\widehat{\mathscr{F}}_{c,c'} : B\mathscr{C}_1[1](c,c') \longrightarrow B\mathscr{C}_2[1](\mathscr{F}_{ob}(c), \mathscr{F}_{ob}(c'))$$

induced by $\mathscr{F}_{c,c'}$ is a chain map with respect to the boundary operator $\hat{d}$.
(4) We require

$$\widehat{\mathscr{F}}_{c,c'}(x_1, \ldots, \mathbf{e}, \ldots, x_k) = 0$$

except

$$\widehat{\mathscr{F}}_{c,c}(\mathbf{e}_c) = \mathbf{e}_{\mathscr{F}_{ob}(c)}.$$

A filtered A_∞ functor is said to be strict if its restriction to $B_0\mathscr{C}_1[1](c,c)$ is 0.

For a pair of filtered A_∞ categories $\mathscr{C}_1, \mathscr{C}_2$ we can define a filtered A_∞ category $\mathfrak{Func}(\mathscr{C}_1, \mathscr{C}_2)$ whose object is a filtered A_∞ functor : $\mathscr{C}_1 \to \mathscr{C}_2$. We can define its strict version $\mathfrak{Funcs}(\mathscr{C}_1, \mathscr{C}_2)$ in the same way.

Definition 7.10. *Let $\mathscr{C}_1, \ldots, \mathscr{C}_n$ and $\mathscr{C}$ be filtered A_∞ categories. A filtered A_∞ multi-functor*

$$\mathscr{F} : \mathscr{C}_1 \times \cdots \times \mathscr{C}_n \longrightarrow \mathscr{C}$$

consists of the following objects.

(1) A map $\mathscr{F}_{ob} : \prod_{i=1}^{n} \mathfrak{DB}(\mathscr{C}_i) \to \mathfrak{DB}(\mathscr{C})$.
(2) A series of Λ_0 module homomorphisms

$$\mathscr{F}_{(c_1,\ldots,c_n),(c'_1,\ldots,c'_n)} : \bigotimes_{i=1}^{n} B\mathscr{C}_i[1](c_i, c'_i)$$
$$\longrightarrow \mathscr{C}(\mathscr{F}_{ob}(c_1, \ldots, c_n), \mathscr{F}_{ob}(c'_1, \ldots, c'_n)),$$

which preserve filtrations.
(3) We require that the coalgebra homomorphism

$$\hat{\mathscr{F}}_{(c_1,\ldots,c_n),(c'_1,\ldots,c'_n)} : \bigotimes_{i=1}^{n} B\mathscr{C}_i[1](c_i, c'_i)$$
$$\longrightarrow B\mathscr{C}[1](\mathscr{F}_{ob}(c_1, \ldots, c_n), \mathscr{F}_{ob}(c'_1, \ldots, c'_n))$$

induced by $\mathscr{F}_{(c_1,\ldots,c_n),(c'_1,\ldots,c'_n)}$ is a chain map with respect to the boundary operator $\hat{d}$.

(4) $\hat{\mathscr{F}}_{(c_1,\ldots,c_n),(c_1',\ldots,c_n')}(\mathbf{x})$ *is zero if* $\mathbf{x}$ *contains the unity except*

$$\hat{\mathscr{F}}_{(c_1,\ldots,c_n),(c_1,\ldots,c_n)}(\mathbf{e}_{c_1},\ldots,\mathbf{e}_{c_n}) = \mathbf{e}_{\mathscr{F}_{ob}(c_1,\ldots,c_n)}.$$

We use coalgebra structures of $B\mathscr{C}_i[1](c_i,c_i')$ to define a coalgebra structure on $\bigotimes_{i=1}^{n} B\mathscr{C}_i[1](c_i,c_i')$ in an obvious way.

We can define strictness of multi-functor in the same way.

Lemma 7.11. *The following two objects can be identified.*

(1) A filtered A_∞ bi-functor $\mathscr{F} : \mathscr{C}_1 \times \mathscr{C}_2 \to \mathscr{C}$,
(2) A filtered A_∞ functor $\mathscr{F} : \mathscr{C}_1 \to \mathfrak{Funk}(\mathscr{C}_2, \mathscr{C})$.

Definition 7.12. *(1) Let $\mathscr{C}$ be a strict filtered A_∞ category and $c, c' \in \mathfrak{OB}(\mathscr{C})$. We say c and c' are homotopy equivalent if there exists $x \in \mathscr{C}(c, c')$ and $x' \in \mathscr{C}(c', c)$ such that $\mathfrak{m}_1(x) = \mathfrak{m}_1(x') = 0$, $\mathfrak{m}_2(x, x') - \mathbf{e}_c \in \mathrm{Im}(\mathfrak{m}_1)$, $\mathfrak{m}_2(x', x) - \mathbf{e}_{c'} \in \mathrm{Im}(\mathfrak{m}_1)$.*

(2) Two strict filtered A_∞ functors $\mathscr{F}_1, \mathscr{F}_2 : \mathscr{C}_1 \to \mathscr{C}_2$ are said to be homotopy equivalent each other if they are homotopy equivalent in the functor category $\mathfrak{Funcs}(\mathscr{C}_1, \mathscr{C}_2)$ in the sense of (1).

(3) A strict filtered A_∞ functor $\mathscr{F} : \mathscr{C}_1 \to \mathscr{C}_2$ is said to be a homotopy equivalence if there exists a strict filtered A_∞ functor $\mathscr{G} : \mathscr{C}_2 \to \mathscr{C}_1$ such that the compositions $\mathscr{F} \circ \mathscr{G}$ and $\mathscr{G} \circ \mathscr{F}$ are homotopy equivalent to the identity functor.

Basic results in the story of A_∞ category is the following two theorems.

Theorem 7.13 (Whitehead theorem for A_∞ functor). *Let $\mathscr{F} : \mathscr{C}_1 \to \mathscr{C}_2$ be a strict filtered A_∞ functor between strict filtered A_∞ categories. It is a homotopy equivalence if and only if the following two conditions are satisfied.*

(1) For any $c' \in \mathfrak{OB}(\mathscr{C}_2)$ there exists $c \in \mathfrak{OB}(\mathscr{C}_1)$ such that $\mathscr{F}_{ob}(c)$ is homotopy equivalent to c'.

(2) For any $c, c' \in \mathfrak{OB}(\mathscr{C}_1)$ the chain map

$$\mathscr{F}_{c,c'} : \mathscr{C}_1(c, c') \longrightarrow \mathscr{C}_2(\mathscr{F}_{ob}(c), \mathscr{F}_{ob}(c'))$$

is a chain homotopy equivalence.

We omit the proof. See [Fu7].

Let $\mathfrak{D} \subset \mathfrak{OB}(\mathscr{C})$. We define a filtered A_∞ category whose object set is $\mathfrak{D}$ and the module of morphisms and structure operations are obvious restrictions of those of $\mathscr{C}$. We call such a filtered A_∞ category a *full subcategory* of $\mathscr{C}$.

We denote by $\mathscr{CH}$ an A_∞ category whose object is a chain complex and whose module of morphisms between two chain complexes is the set of linear maps among them, which is a chain complex. The boundary operator of this chain complex is the operator induced from $\mathfrak{m}_1$ in an obvious way. The operation $\mathfrak{m}_2$ is the composition of linear maps (up to sign). $\mathfrak{m}_3$ and all higher $\mathfrak{m}_k$ are all zero.

Definition 7.14. *Let $\mathscr{C}$ be a filtered A_∞ category. We define its opposite category $\mathscr{C}^{\mathrm{op}}$ as follows.*

(1) $\mathfrak{DB}(\mathscr{C}^{\mathrm{op}}) = \mathfrak{DB}(\mathscr{C})$,
(2) $\mathscr{C}^{\mathrm{op}}(c, c') = \mathscr{C}(c', c)$,
(3)
$$\mathfrak{m}_k^{\mathrm{op}}(x_1, \ldots, x_k) = (-1)^* \mathfrak{m}_k(x_k, \ldots, x_1),$$

where $* = 1 + \sum_{1 \leq i < j \leq k} (\deg x_i + 1)(\deg x_j + 1)$. *Here* $\mathfrak{m}^{\mathrm{op}}$ *is the structure operation of* $\mathscr{C}^{\mathrm{op}}$.

Theorem 7.15 (Yoneda's lemma for A_∞ categories). *Let $\mathscr{C}$ be a strict filtered A_∞ category.*

(1) There exists a filtered A_∞ functor $\mathfrak{YON} : \mathscr{C} \to \mathfrak{Funcs}(\mathscr{C}^{\mathrm{op}}, \mathscr{CH})$.
(2) Let $c \in \mathfrak{DB}(\mathscr{C}) = \mathfrak{DB}(\mathscr{C}^{\mathrm{op}})$. *Then* $\mathfrak{YON}_{ob}(c) : \mathfrak{DB}(\mathscr{C}) \to \mathfrak{DB}(\mathscr{CH})$ *is a strict filtered A_∞ functor which is defined in the object level by*

$$c' \longmapsto \mathscr{C}(c, c').$$

(3) Let $\mathfrak{Rep}(\mathscr{C}^{\mathrm{op}}, \mathscr{CH})$ *be a full subcategory of* $\mathfrak{Funcs}(\mathscr{C}^{\mathrm{op}}, \mathscr{CH})$, *whose object is an element of* $\mathfrak{DB}(\mathfrak{Funcs}(\mathscr{C}^{\mathrm{op}}, \mathscr{CH}))$ *which is homotopy equivalent to the image of* $\mathfrak{YON}_{ob} : \mathfrak{DB}(\mathscr{C}) \to \mathfrak{DB}(\mathscr{C}^{\mathrm{op}}, \mathscr{CH})$. *Then*

$$\mathfrak{YON} : \mathscr{C} \longrightarrow \mathfrak{Rep}(\mathscr{C}^{\mathrm{op}}, \mathscr{CH}).$$

is a homotopy equivalence.

We omit the proof. See [Fu7]. We call $\mathfrak{YON}$ the *Yoneda functor*. We say an object of $\mathfrak{Rep}(\mathscr{C}^{\mathrm{op}}, \mathscr{CH})$ a *representable* functor.

7.3. Künneth tri-functor and representability

In this subsection we sketch an argument to prove Theorem 7.4 using the algebraic framework of Subsect. 7.2. Let (L_i, σ_i) be a pair of Lagrangian submanifold of X_i and its V_i-relative spin structure for $i = 1, 2$ and (L_{12}, σ_{12}) a pair of Lagrangian submanifold of $(X_1 \times X_2, -\omega_1 \oplus \omega_2)$ and its $\pi_1^* V_1 \oplus \pi_1^* TX_1 \oplus \pi_2^* V_2$-relative spin structure. We consider curved filtered A_∞ algebras $CF(L_i)$ and $CF(L_{12})$. We assume appropriate transversality or cleanness of intersection (or fiber product) among them.

Proposition 7.16. *There exists a left $CF(L_1)$, $CF(L_{12})$ and right $CF(L_2)$ filtered A_∞ tri-module D such that as a Λ_0 module D is*

$$H(\tilde{L}_1 \times_{X_1} \tilde{L}_{12} \times_{X_2} \tilde{L}_2; \Lambda_0)$$

or its chain model.

The notion of tri-module is defined in a similar way as tri-functor (Definition 7.10). More explicitly it gives a series of operators

$$(7.5) \quad \mathfrak{n}_{k_1,k_{12},k_2} : CF(L_1)^{\otimes k_1} \otimes CF(L_{12})^{\otimes k_{12}} \otimes D \otimes CF(L_2)^{\otimes k_2} \longrightarrow D$$

which satisfies a similar relation as right module. We sketch the proof of Proposition 7.16 later in this subsection.

Corollary 7.17. *If b_1 and b_{12} are bounding cochains of $CF(L_1)$ and $CF(L_{12})$ respectively then D in Proposition 7.16 has a structure of right filtered A_∞ module over $CF(L_2)$.*

Proof. Using (7.5) we obtain

$$\mathfrak{n}_k : D \otimes CF(L_2)^{\otimes k} \longrightarrow D$$

by

$$\mathfrak{n}_k(y; x_1, \ldots, x_k)$$
$$= \sum_{k_1,k_{12}=0}^{\infty} \mathfrak{n}_{k_1,k_{12},k}(b_1, \ldots, b_1; b_{12}, \ldots, b_{12}; y; x_1, \ldots, x_k).$$

It is easy to check (6.3). $\qquad\square$

Now we consider the case when L_2 is the geometric transformation $L_1 \times_{X_1} L_{12}$. Then

$$D = H(\tilde{L}_1 \times_{X_1} \tilde{L}_{12} \times_{X_2} \tilde{L}_2; \Lambda_0) = H(\tilde{L}_2 \times_{X_2} \tilde{L}_2; \Lambda_0).$$

The fundamental class of $\tilde{L}_2$ is an element of D, which we write **1**.

Lemma 7.18. *In the situation of Corollary 7.17 we assume that L_2 is the geometric transformation $L_1 \times_{X_1} L_{12}$.*
Then $\mathbf{1} \in D$ is a cyclic element of right filtered A_∞ module D.

Theorem 7.4 (1) is a consequence of Corollary 7.17, Lemma 7.18 and Proposition 6.11.
To prove Theorem 7.4 (2), (3) we enhance Proposition 7.16 as follows.

Proposition 7.19. *There exists a filtered A_∞ tri-functor*

$$\mathfrak{Fuks}(X_1 \times X_2, -\omega_1 \oplus \omega_2, \pi_1^* V_1 \oplus \pi_1^* TX_1 \oplus \pi_2^* V_2)$$
$$\times \mathfrak{Fuks}(X_1, \omega_1, V_1) \times \mathfrak{Fuks}(X_2, \omega_2, V_2)^{\mathrm{op}} \longrightarrow \mathscr{CH}$$

such that the chain complex associated to L_{12}, L_1, L_2 by this tri-functor is D in Proposition 7.16.

By Proposition 7.19, Lemma 7.11 induces a bi-functor

$$(7.6) \quad \begin{aligned} &\mathfrak{Fuks}(X_1 \times X_2, -\omega_1 \oplus \omega_2, \pi_1^* V_1 \oplus \pi_1^* TX_1 \oplus \pi_2^* V_2) \\ &\times \mathfrak{Fuks}(X_1, \omega_1, V_1) \longrightarrow \mathfrak{Funcs}(\mathfrak{Fuks}(X_2, \omega_2, V_2)^{\mathrm{op}}, \mathscr{CH}). \end{aligned}$$

Proposition 7.20. *Let* (L_1, σ_1, b_1) *and* $(L_{12}, \sigma_{12}, b_{12})$ *be objects of* $\mathfrak{Fuks}(X_1, \omega_1, V_1)$ *and* $\mathfrak{Fuks}(X_1 \times X_2, -\omega_1 \oplus \omega_2, \pi_1^* V_1 \oplus \pi_1^* TX_1 \oplus \pi_2^* V_2)$, *respectively.*

Then the strict filtered A_∞ *functor* $: \mathfrak{Fuks}(X_2, \omega_2, V_2)^{\mathrm{op}} \to \mathscr{CH}$ *obtained by applying (7.6) to them is represented by the object* (L_2, σ_2, b_2).

Here L_2 *is the geometric composition and* b_2 *is obtained by Theorem 7.4 (1).*

The proof is similar to the discussion in the next section using a diagram similar to the Y diagram. See [Fu10].

By Proposition 7.20 we obtain a filtered A_∞ bi-functor

$$\begin{aligned} &\mathfrak{Fuks}(X_1 \times X_2, -\omega_1 \oplus \omega_2, \pi_1^* V_1 \oplus \pi_1^* TX_1 \oplus \pi_2^* V_2) \\ &\times \mathfrak{Fuks}(X_1, \omega_1, V_1) \longrightarrow \mathfrak{Rep}(\mathfrak{Fuks}(X_2, \omega_2, V_2)^{\mathrm{op}}, \mathscr{CH}). \end{aligned}$$

Therefore we use Theorem 7.15 (3) and compose homotopy inverse to the Yoneda functor to obtain desired filtered A_∞ functor in Theorem 7.4.

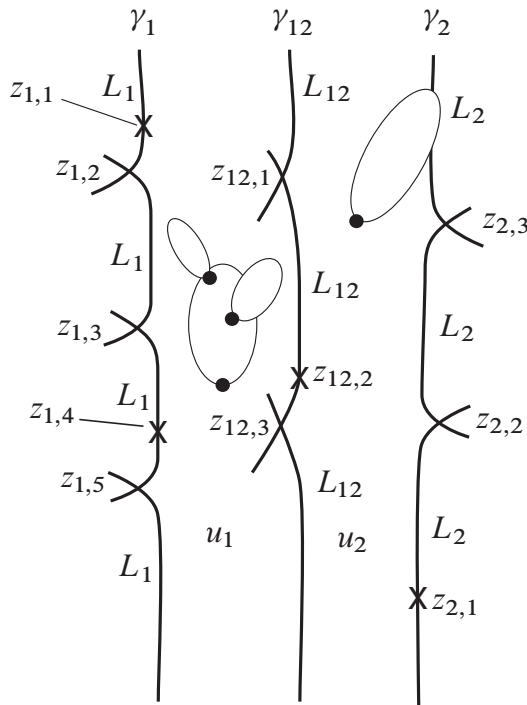We finally sketch a proof of Proposition 7.16. We use the moduli space of objects drawn in the next Fig. 13.

Fig. 13. Moduli space of simple quilt

Here the source curve Σ is the domain $\mathbb{R}_\tau \times [-1,1]_t$ plus possibly some sphere bubbles. We divide Σ into two parts. The first one Σ_1 is the union of $\mathbb{R}_\tau \times [-1,0]_t$ and sphere bubbles rooted on it and the second one Σ_2 is the union of $\mathbb{R}_\tau \times [0,1]_t$ and sphere bubbles rooted on it. (We require the sphere bubbles are not rooted on the part $t = -1, 0, 1$.) The map is a combination of $u_1 : \Sigma_1 \to X_1$ and $u_2 : \Sigma_2 \to X_2$. We include three kinds of marked points $z_{1,1}, \ldots, z_{1,k_1} \in \mathbb{R}_\tau \times \{-1\}$, $z_{12,1}, \ldots, z_{12,k_{12}} \in \mathbb{R}_\tau \times \{0\}$, $z_{2,1}, \ldots, z_{2,k_2} \in \mathbb{R}_\tau \times \{1\}$.

We require the following boundary conditions:

(1) $u_1(\tau, -1) \in L_1$.
(2) $u_2(\tau, +1) \in L_2$.
(3) $(u_1(\tau, 0), u_2(\tau, 0)) \in L_{12}$.
(4)
$$\lim_{\tau \to -\infty} (u_1(\tau, t_1), u_2(\tau, t_2)) = a \in \tilde{L}_1 \times_{X_1} \tilde{L}_{12} \times_{X_2} \tilde{L}_2.$$

(5)
$$\lim_{\tau \to +\infty} (u_1(\tau, t_1), u_2(\tau, t_2)) = b \in \tilde{L}_1 \times_{X_1} \tilde{L}_{12} \times_{X_2} \tilde{L}_2.$$

Moreover we assume

(6)
$$\int_{\Sigma_1} u_1^* \omega_1 + \int_{\Sigma_2} u_2^* \omega_2 = E < \infty.$$

We denote the moduli space of such objects by $\mathcal{M}_{k_1, k_{12}, k_2}(L_1, L_{12}, L_2; a, b; E)$. It comes with evaluation maps:

$$\mathrm{ev} = (\mathrm{ev}_1, \mathrm{ev}_{12}, \mathrm{ev}_2) : \mathcal{M}_{k_1, k_{12}, k_2}(L_1, L_{12}, L_2; a, b; E) \longrightarrow \hat{L}_1^{k_1} \times \hat{L}_{12}^{k_{12}} \times \hat{L}_2^{k_2},$$

where

$$\hat{L}_1 = \tilde{L}_1 \times_{X_1} \tilde{L}_1.$$

$\hat{L}_2$ and $\hat{L}_{12}$ are defined in the same way.

Let $h_{i,j}$ ($j = 1, \ldots, k_i$) be differential forms on $\hat{L}_i$ and $h_{12,j}$ ($j = 1, \ldots, k_{12}$) differential forms on $\hat{L}_{12}$. Then we define the structure operations (7.5) of the tri-module D by

$$\mathfrak{n}_{k_1, k_{12}, k_2}(\vec{h}_1; \vec{h}_{12}; [a]; \vec{h}_2)$$
$$= \sum_{E, b} T^E [b] \int_{\mathcal{M}_{k_1, k_{12}, k_2}(L_1, L_{12}, L_2; a, b; E)} \mathrm{ev}^*(\vec{h}_1 \wedge \vec{h}_{12} \wedge \vec{h}_2).$$

Here $\vec{h}_1 = (h_{1,1}, \ldots, h_{1,k_1}) \in CF(L_1)^{\otimes k_1}$. The notations $\vec{h}_{12}$ and $\vec{h}_2$ are defined in a similar way.

We can show that it satisfies the required relation by studying the boundary of our moduli space $\mathcal{M}_{k_1,k_{12},k_2}(L_1, L_{12}, L_2; a, b; E)$ and using Stokes' theorem.

We remark that our moduli space $\mathcal{M}_{k_1,k_{12},k_2}(L_1, L_{12}, L_2; a, b; E)$ is similar to one we use to define Floer homology group $HF(L_1 \times L_2, L_{12})$ in the product space $-X_1 \times X_2$. For example in case L_1, L_2, L_{12} are embedded and monotone with minimal Maslov number > 2 we may use the case $k_1 = k_{12} = k_2 = 0$ to obtain

$$\mathfrak{n}_{0,0,0} : D \longrightarrow D$$

and D is a free Λ_0 module with basis

$$L_1 \times_{X_1} L_{12} \times_{X_2} L_2 = (L_1 \times L_2) \cap L_{12}.$$

So D is also the underlying vector space of the chain complex calculating Floer homology $HF(L_1 \times L_2, L_{12})$. Moreover the operation $\mathfrak{n}_{0,0,0}$ coincides with Floer's boundary operator.

Remark 7.21. To obtain appropriate Kuranishi structure we need to slightly change the way to compactify the bubble on the line $t = 0$. (See [Fu10, Sect. 12].)

In case k_i or k_{12} are non-zero there is some technical difference between the moduli space $\mathcal{M}_{k_1,k_{12},k_2}(L_1, L_{12}, L_2; a, b; E)$ and the moduli space we use to define $CF(L_1 \times L_2)$-$CF(L_{12})$ bimodule structure on D.

Wehrheim–Woodward [WW1] studied the case of monotone Lagrangian submanifolds using the moduli space $\mathcal{M}_{0,0,0}(L_1, L_{12}, L_2; a, b; E)$. They also generalize this moduli space to the case where the domain is divided into several (not necessary two) domains which are sent to various symplectic manifolds by a pseudo holomorphic curve. They call such objects pseudo holomorphic quilt. Here we use only the simplest case of pseudo holomorphic quilt.

7.4. Y diagram and compatibility of compositions

In this subsection we give a brief explanation of the proof of Theorem 7.6.

The proof of Theorem 7.6 (1) is similar to one of Theorem 7.4 (1). (In fact Theorem 7.4 (1) can be regarded as a special case of Theorem 7.6 (1) where L_3 is a point.) We construct a tri-module D over $CF(L_{12})$, $CF(L_{23})$, $CF(L_{31})$ and use it in the same way as the last subsection. The moduli space we use for this purpose is obtained by replacing Fig. 13 by the next Fig. 14.

Here the source is a circular cylinder divided into three parts. The map is a combination of three maps u_1, u_2, u_3 which sends those three parts to X_1, X_2 and X_3 respectively. We use L_{12}, L_{23}, L_{31} to set the boundary conditions at
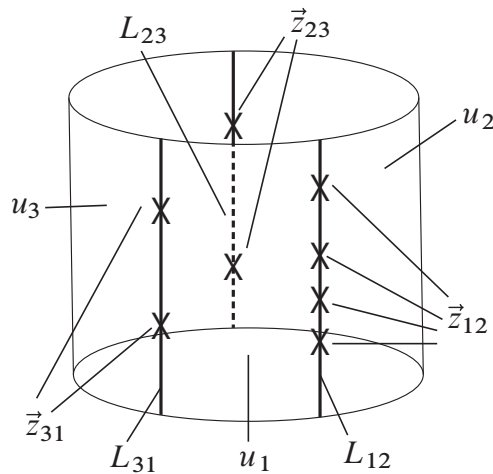
Fig. 14. Composition of unobstructed correspondences

the lines where two of those three subdomains intersect. The underlying vector space of D is the cohomology group of the triple fiber product

$$\left\{ (x, y, z) \in \tilde{L}_{12} \times \tilde{L}_{23} \times \tilde{L}_{31} \;\middle|\; \begin{array}{l} \pi_1(i_{L_{12}}(x)) = \pi_1(i_{L_{31}}(z)), \\ \pi_2(i_{L_{12}}(x)) = \pi_2(i_{L_{23}}(z)), \\ \pi_3(i_{L_{13}}(x)) = \pi_3(i_{L_{23}}(z)) \end{array} \right\}.$$

Here $\pi_i : X_i \times X_j \to X_i$ and $\pi_j : X_i \times X_j \to X_j$ are projections.

We put elements of this triple fiber product at the part $\tau \to \pm\infty$ and use it as the asymptotic boundary condition.

The proof of Theorem 7.6 (2), (3) is based on the moduli space introduced by Lekili–Lipyanskiy [LL] which is drawn in the next Fig. 15.
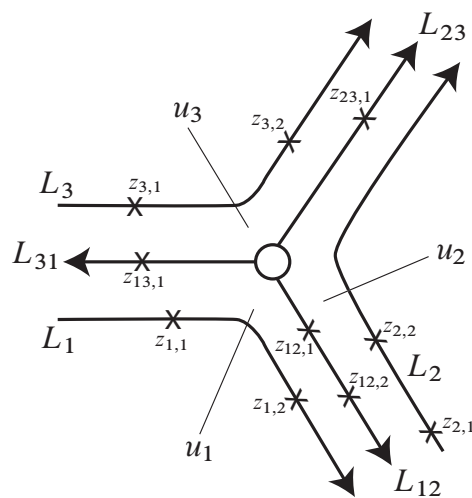

Fig. 15. Lekili–Lipyanskiy's Y-diagram

Here the source is a domain in $\mathbb{C}$ which is divided into three parts. The maps u_1, u_2, u_3 send each of those three parts to X_1, X_2, X_3, respectively and are pseudo holomorphic. Note in our situation of Theorem 7.6 (2), (3) we are given 6 Lagrangian submanifolds $L_{12}, L_{23}, L_{31}, L_1, L_2, L_3$, where $L_i \subset X_i$ and $L_{ij} \subset X_i \times X_j$. We use L_{ij} to set the boundary condition at the curves where two subdomains intersect each other. We use L_i to set the boundary condition at the boundary of our domain. We have six curves and so we put six kinds of marked points. The evaluation maps go to the products of $\hat{L}_i$'s and $\hat{L}_{ij}$'s.

Note our domain has 4 ends. Three of them (left, upper right and lower right) are similar to the ends appearing in Fig. 13 and the fourth one which is a neighborhood of the white circle in the middle of the domain is similar to the end appearing in Fig. 14.

Thus our moduli space defines a map which interpolates tensor products of $CF(L_i)$, $CF(L_{ij})$ and tri-modules which we used to prove Theorem 7.6 (1) and Theorem 7.4 (1).

The Λ_0 linear maps we thus obtain looks rather cumbersome. However when we see them carefully we find that it is exactly the maps we need to show the homotopy commutativity of the diagrams appearing in Theorem 7.6 (2), (3). See [Fu10] for detail.

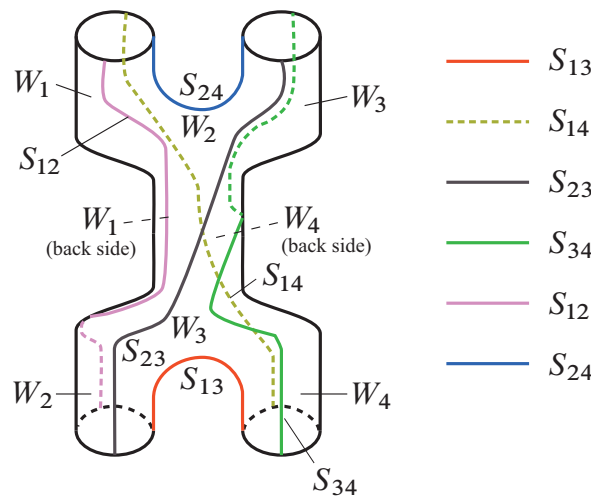We finally mention that Theorem 7.7 is proved by using the objects drawn in the next Fig. 16.

Fig. 16. Associativity of compositions

8. Categorification of Donaldson–Floer theory

We can enhance the construction of Sect. 6 to the topological field theory style results and clarify its relation to Lagrangian correspondence.

For oriented two manifold Σ we always consider an $SO(3)$ bundle $\mathcal{P}_\Sigma$ on it such that $w_2(\mathcal{P}_\Sigma)$ is the fundamental class. Let $R(\Sigma, \mathcal{P}_\Sigma)$ be the moduli space

of the gauge equivalence classes of the flat connections of $\mathcal{P}_\Sigma$. It is a symplectic manifold and is monotone with minimal Chern number 2.

In this subsection we work over $\mathbb{Z}_2$ coefficient.

Definition 8.1. *We consider the strict filtered A_∞ category $\mathfrak{Fuk}(\Sigma)$ as follows.*

(1) The object of $\mathfrak{Fuk}(\Sigma)$ is a pair (L, b). Here L is an immersed Lagrangian submanifold of $R(\Sigma, \mathcal{P}_\Sigma)$, which is monotone in the weak sense (Definition 4.11) and has minimal Maslov number in $4\mathbb{Z}_+$. b is its bounding cochain which is supported in the switching components. (Here we consider the $\Lambda_0^{\mathbb{Z}_2}$ filtered A_∞ algebra associated to L.)

(2) The module of morphisms is $CF((L_1, b_1), (L_2, b_2))$ which is ($\mathbb{Z}_2$ version) of the chain complex introduced in Subsect. 4.1.

(3) The structure operations $\mathfrak{m}_k$ is defined as in Subsect. 5.2.

We remark that we can easily prove the following which is expected by topological field theory.

$$(8.1) \qquad \mathfrak{Fuk}(\Sigma \sqcup \Sigma') = \mathfrak{Fuk}(\Sigma) \otimes \mathfrak{Fuk}(\Sigma'),$$

$$(8.2) \qquad \mathfrak{Fuk}(-\Sigma) = \mathfrak{Fuk}(\Sigma)^{\mathrm{op}}.$$

Let M be a 3 manifold with boundary $\Sigma = \partial M$. We consider an $SO(3)$ bundle $\mathcal{P}_M$ on M such that the restriction of $\mathcal{P}_M$ to $\Sigma = \partial M$ is $\mathcal{P}_\Sigma$.[9] We assume that ∂M is divided into $\partial_- M \sqcup \partial_+ M$ such that a neighborhood of $\partial_- M$ (resp. a neighborhood of $\partial_+ M$) in M is identified with $\partial_- M \times [-\infty, 0)$ (resp. $\partial_+ M \times [0, +\infty))$ as oriented manifolds.

By Theorem 6.2, the (appropriately perturbed) moduli space of flat connections $R(M; E_M)$ is unobstructed. Namely we have a bounding cochain b_M of the $\Lambda_0^{\mathbb{Z}_2}$ linear filtered A_∞ algebra associated to $R(M; \mathcal{P}_M) \subset -R(\Sigma_-; \mathcal{P}_{\Sigma_-}) \times R(\Sigma_+; \mathcal{P}_{\Sigma_+})$.

Definition 8.2. *Suppose $\Sigma_-, \Sigma_+ \neq \emptyset$. We define the strict filtered A_∞ functor*

$$(8.3) \qquad \mathcal{HF}_{(M, \mathcal{P}_M)} : \mathfrak{Fuk}(\Sigma_-) \longrightarrow \mathfrak{Fuk}(\Sigma_+)$$

as the strict filtered A_∞ functor associated to the unobstructed immersed Lagrangian correspondence $(R(M; \mathcal{P}_M), b_M)$ by Theorem 7.4.

We can prove the following compatibility result. We consider $(M_i, \mathcal{P}_{M_i})$ $(i = 1, 2)$ as above. We suppose $\partial_+ M_1 \cong \partial_- M_2$ and fix a diffeomorphism and its lift to a bundle isomorphism $\mathcal{P}_{\partial M_1} \to \mathcal{P}_{\partial M_2}$. We glue M_1 and M_2 along $\partial_+ M_1 \cong \partial_- M_2$ by this diffeomorphism to obtain $M_3 = M_1 \cup_{\partial_+ = \partial_-} M_2$ and an $SO(3)$ bundle $\mathcal{P}_{M_3}$ on it. Note

$$\partial_- M_3 = \partial_- M_1, \quad \partial_+ M_3 = \partial_+ M_2.$$

---

[9] When M has a connected component which does not intersect with boundary, we require that the $SO(3)$ bundle $\mathcal{P}_M$ is non-trivial on such a component.

Theorem 8.3. *The filtered A_∞ functor $\mathcal{HF}_{(M_3,\mathcal{P}_{M_3})}$: $\mathfrak{Fuk}(\partial_- M_1) \rightarrow \mathfrak{Fuk}(\partial_+ M_2)$ associated to $(M_3, \mathcal{P}_{M_3})$ by Definition 8.2 is homotopy equivalent to the composition*

$$\mathcal{HF}_{(M_2,\mathcal{P}_{M_2})} \circ \mathcal{HF}_{(M_1,\mathcal{P}_{M_1})}.$$

Here $\mathcal{HF}_{(M_2,\mathcal{P}_{M_2})}$, $\mathcal{HF}_{(M_1,\mathcal{P}_{M_1})}$ are filtered A_∞ functors associated to $(M_2, \mathcal{P}_{M_2})$ and $(M_1, \mathcal{P}_{M_1})$ by Definition 8.2 and their composition is defined by Theorem 7.6 (1).

Remark 8.4. Let $\mathbb{M}(\Sigma)$ be the mapping class group, that is, π_0 of the group of oriented self-diffeomorphisms of a surface Σ. We include the homotopy class of bundle isomorphisms $\mathcal{P}_\Sigma \rightarrow \mathcal{P}_\Sigma$ which covers an element of $\mathbb{M}(\Sigma)$ to define $\mathbb{M}^+(\Sigma)$. The functoriality of $\Sigma \mapsto \mathfrak{Fuk}(\Sigma)$ implies that the group $\mathbb{M}^+(\Sigma)$ acts on $\mathfrak{Fuk}(\Sigma)$ as the homotopy class of automorphisms of filtered A_∞ category. Note a pair of an oriented self-diffeomorphism of a surface Σ and its lift to the bundle induces a symplectic diffeomorphism of $R(\Sigma, \mathcal{P}_\Sigma)$. The action of $\mathbb{M}^+(\Sigma)$ is induced by this symplectic diffeomorphism.

If we change the choice of the diffeomorphism $\partial_+ M_1 \cong \partial_- M_2$ then the manifold $M_3 = M_1 \cup_{\partial_+=\partial_-} M_2$ changes. The composition $\mathcal{HF}_{(M_2,\mathcal{P}_{M_2})} \circ \mathcal{HF}_{(M_1,\mathcal{P}_{M_1})}$ also changes by the action of the group $\mathbb{M}^+(\partial_+ M_1)$ inserted between $\mathcal{HF}_{(M_2,\mathcal{P}_{M_2})}$ and $\mathcal{HF}_{(M_1,\mathcal{P}_{M_1})}$.

Definition 8.5. *Suppose $\Sigma_- = \emptyset, \Sigma_+ \neq \emptyset$. Then we define $\mathcal{HF}_{(M,\mathcal{P}_M)}$ as the object $(R(M, \mathcal{P}_M), b_M)$ of $\mathfrak{Fuk}(\Sigma_+)$.*

Suppose $\Sigma_- \neq \emptyset, \Sigma_+ = \emptyset$. Then we define $\mathcal{HF}_{(M,\mathcal{P}_M)}$ as the filtered A_∞ functor : $\mathfrak{Fuk}(\Sigma_-) \rightarrow \mathcal{CH}$ which is represented by the object $(R(-M, \mathcal{P}_{-M}), b_{-M})$ of $\mathfrak{Fuk}(\Sigma_-)^{\mathrm{op}}$.

Suppose $\Sigma_- = \Sigma_+ = \emptyset$. Then we define $\mathcal{HF}_{(M,\mathcal{P}_M)}$ as its Floer homology group as in Definition 1.8.

We can generalize Theorem 8.3 appropriately including the situation of Definition 8.5. Since this generalization is straightforward we omit it.

The proof of Theorem 8.3 uses the following Fig. 17. X in the figure is a 4 manifold. X has 3 ends and 3 boundary components. The ends are identified with

$$(8.4) \qquad M_1 \times (-\infty, 0], \quad M_2 \times (-\infty, 0], \quad M_3 \times [0, +\infty).$$

The boundary (which are drawn by dotted lines) are identifies with

$$\partial_- M_1 \times \mathbb{R} = \partial_- M_2 \times \mathbb{R}, \quad \partial_+ M_1 \times \mathbb{R} = \partial_- M_2 \times \mathbb{R}, \quad \partial_+ M_2 \times \mathbb{R} = \partial_- M_3 \times \mathbb{R}.$$

The free domains Ω_1, Ω_2, Ω_3 of $\mathbb{C}$ are attached to each of such boundary components. We consider an Anti-Self-Dual connection A on X and holomorphic maps $u_1 : \Omega_1 \rightarrow R(\partial_- M_1), u_2 : \Omega_2 \rightarrow R(\partial_- M_2), u_3 : \Omega_3 \rightarrow R(\partial_+ M_3)$.

Along three dotted lines we require appropriate matching conditions similar to those in [Fu6], [Ly1]. We also require u_1, u_2, u_3 satisfy appropriate boundary conditions on $\partial\Omega_i \setminus$ dotted lines, formulated by using Lagrangian submanifolds $R(M_1; \mathcal{P}_{M_1})$, $R(M_2; \mathcal{P}_{M_2})$, $R(M_3; \mathcal{P}_{M_4})$, respectively.

We consider the moduli space of the such triples (A, u_1, u_2, u_3). (We also include boundary marked points on the $\partial\Omega_i \setminus$ dotted lines and require certain asymptotic boundary conditions on the three ends.)

We observe the sliding ends of this moduli space corresponding to the three ends in (8.4) coincide with the moduli spaces we use to obtain bounding cochain $b_{M_1}, b_{M_2}, b_{M_3}$, respectively.

Using the moduli space of the triples (A, u_1, u_2, u_3) (plus marked points), we can show that $b_{M_1}, b_{M_2}, b_{M_3}$ satisfies certain equalities which is the one we need to prove Theorem 8.3.

In this article we restrict ourselves to the case of $SO(3)$ bundles $\mathcal{P}$ with non-trivial $w_2(\mathcal{P})$. The research to include the case when E is a trivial bundle is now in progress, as a joint work of the author with A. Daemi [DF].
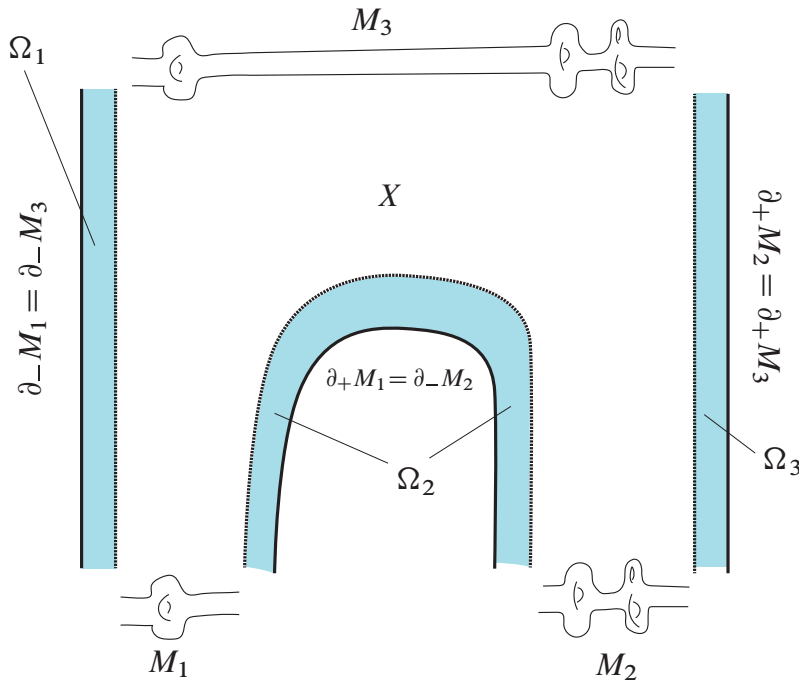

Fig. 17. Proof of Theorem 8.3

References

[AFOOO]   M. Abouzaid, K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, Quantum cohomology and split generation in Lagrangian Floer theory, in preparation.

[Ak]   M. Akaho, Intersection theory for Lagrangian immersions, Math. Res. Lett., **12** (2005), 543–550.

[AJ]   M. Akaho and D. Joyce, Immersed Lagrangian Floer theory, J. Differential Geom., **86** (2010), 381–500.

[At]   M.F. Atiyah, New invariants of 3- and 4-dimensional manifolds, In: The Mathematical Heritage of Hermann Weyl, Proc. Sympos. Pure Math., **48**, Amer. Math. Soc., Providence, RI, 1988, pp. 285–299.

[AB]   M.F. Atiyah and R. Bott, The Yang–Mills equations over Riemann surfaces, Philos. Trans. Roy. Soc. London Ser. A, **308** (1983), 523–615.

[Bo]   N. Bottman, Pseudoholomorphic quilts with figure eight singularity, preprint, arXiv:1410.3834.

[BW]   N. Bottman and K. Wehrheim, Gromov compactness for squiggly strip shrinking in pseudoholomorphic quilts, preprint, arXiv:1503.03486.

[BD]   P.J. Braam and S.K. Donaldson, Floer's work on instanton homology, knots, and surgery, In: The Floer Memorial Volume, (eds. H. Hofer, C.H. Taubes, A. Weinstein and E. Zehnder), Progr. Math., **133**, Birkhäuser, Basel, 1995, pp. 195–256.

[DF]   A. Daemi and K. Fukaya, Atiyah–Floer conjecture: a formulation, a strategy of prove and generalizations, preprint, arXiv:1707.03924.

[DFL]   A. Daemi, K. Fukaya and M. Lipyanskiy, Lagrangians, $SO(3)$-instantons, and an analogue of Atiyah–Floer conjecture, in preparation.

[D1]   S.K. Donaldson, Connections, cohomology and the intersection forms of 4-manifolds, J. Differential Geom., **24** (1986), 275–341.

[D2]   S.K. Donaldson, The orientation of Yang–Mills moduli spaces and 4-manifold topology, J. Differential Geom., **26** (1987), 397–428.

[D3]   S.K. Donaldson, Polynomial invariants for smooth four-manifolds, Topology, **29** (1990), 257–315.

[D4]   S.K. Donaldson, Gauge theory for non-athlete, Lecture at Univ. of Warwick, 1992.

[D5]   S.K. Donaldson, Floer Homology Groups in Yang–Mills Theory. With the Assistance of M. Furuta and D. Kotschick, Cambridge Tracts in Math., **147**, Cambridge Univ. Press, Cambridge, 2002.

[DS]   S. Dostoglou and D.A. Salamon, Self-dual instantons and holomorphic curves, Ann. of Math., **139** (1994), 581–640.

[Fl1]   A. Floer, Morse theory for Lagrangian intersections, J. Differential Geom., **28** (1988), 513–547.

[Fl2]   A. Floer, An instanton invariant for 3-manifolds, Comm. Math. Phys., **118** (1988), 215–240.

[Fl3]   A. Floer, Instanton homology and Dehn surgery, In: The Floer Memorial Volume, (eds. H. Hofer, C.H. Taubes, A. Weinstein and E. Zehnder), Progr. Math., **133**, Birkhäuser, Basel, 1995, pp. 77–97.

[Fl4]   A. Floer, Instanton homology and Dehn surgery, In: The Floer Memorial Volume, (eds. by H. Hofer, C.H. Taubes, A. Weinstein and E. Zehnder), Progr. Math., **133**, Birkhäuser, Basel, 1995, pp. 77–97.

[Fu1]   K. Fukaya, Morse homotopy, A^∞-category, and Floer homologies, In: Proceedings of GARC Workshop on Geometry and Topology, (ed. H.J. Kim), Lecture Notes Ser., **18**, Seoul Nat. Univ., 1994, pp. 1–102.

[Fu2]   K. Fukaya, Floer homology for 3-manifolds with boundary, In: Topology Geometry and Field Theory, (eds. K. Fukaya, M. Furuta, T. Kohno and D. Kotschick), World Sci. Publ., River Edge, NJ, 1994, pp. 1–22.

[Fu3]   K. Fukaya, Gauge Theory and Topology. (Japanese), Springer-Verlag, 1995.

[Fu4]   K. Fukaya, Floer homology of connected sum of homology 3-spheres, Topology, **35** (1996), 89–136.

[Fu5]   K. Fukaya, Floer homology for 3-manifolds with boundary I, 1997, never to be published but is available at http://www.math.kyoto-u.ac.jp/~fukaya/fukaya.html.

[Fu6]   K. Fukaya, Anti-self-dual equation on 4-manifolds with degenerate metric, Geom. Funct. Anal., **8** (1998), 466–528.

[Fu7]   K. Fukaya, Floer homology and mirror symmetry II, In: Minimal Surfaces, Geometric Analysis and Symplectic Geometry, Baltimore, MD, 1999, Adv. Stud. Pure Math., **34**, Math. Soc. Japan, Tokyo, 2002, pp. 31–127.

[Fu8]   K. Fukaya, Floer homology of Lagrangian submanifolds, Sugaku Expositions, **26** (2013), 99–127.

[Fu9]   K. Fukaya, $SO(3)$-Floer homology of 3-manifolds with boundary 1, preprint, arXiv:1506.01435.

[Fu10]   K. Fukaya, Unobstructed immersed Lagrangian correspondence and filtered A_∞ functor, preprint, arXiv:1706.02131.

[FOOO1]   K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, Lagrangian Intersection Floer Theory: Anomaly and Obstruction. Part I, AMS/IP Stud. Adv. Math., **46.1**, Amer. Math. Soc., Providence, RI; International Press, Someville, MA, 2009.

[FOOO2]   K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, Lagrangian Intersection Floer Theory: Anomaly and Obstruction. Part II, AMS/IP Stud. Adv. Math., **46.2**, Amer. Math. Soc., Providence, RI; International Press, Someville, MA, 2009.

[FOOO3]   K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, Anchored Lagrangian submanifolds and their Floer theory, In: Mirror Symmetry and Tropical Geometry, Contemp. Math., **527**, Amer. Math. Soc., Providence, RI, 2010, pp. 15–54.

[FOOO4]   K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, Lagrangian Floer theory over integers: spherically positive symplectic manifolds, Pure Appl. Math. Q., **9** (2013), 189–289.

[FOOO5]   K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, Kuranishi structure, pseudo-holomorphic curve, and virtual fundamental chain: Part 1, preprint, arXiv:1503.07631.

[FOOO6]   K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, Kuranishi structure, pseudo-holomorphic curve, and virtual fundamental chain: Part 2, preprint, arXiv:1704.01848.

[FOn]   K. Fukaya and K. Ono, Arnold conjecture and Gromov–Witten invariant, Topology, **38** (1999), 933–1048.

[Go]   W.M. Goldman, The symplectic nature of fundamental groups of surfaces, Adv. in Math., **54** (1984), 200–225.

[He]   C.M. Herald, Legendrian cobordism and Chern–Simons theory on 3-manifolds with boundary, Comm. Anal. Geom., **2** (1994), 337–413.

[Hu]   J. Huebchmann, Symplectic and Poisson structures of certain moduli spaces. I, Duke Math. J., **80** (1995), 737–756.

[HL]   D. Huybrechts and M. Lehn, Stable pairs on curves and surfaces, J. Algebraic Geom., **4** (1995), 67–104.

[Je]   L.C. Jeffrey, Extended moduli spaces of flat connections on Riemann surfaces, Math. Ann., **298** (1994), 667–692.

[Ke]   B. Keller, Introduction to A-infinity algebra and modules, Homology Homotopy Appl., **3** (2001), 1–35.

[LLW]   R. Lee and W. Li, Floer homologies for Lagrangian intersections and instantons, preprint.

[Le]   K. Lefèvre-Hasegawa, Sur les A_∞ catégories, Thése Doctorat, Univ. Paris 7, 2003.

[LL]   Y. Lekili and M. Lipyanskiy, Geometric composition in quilted Floer theory, Adv. Math., **236** (2013), 1–23.

[Li]   W. Li, Floer homology for connected sums of homology 3-spheres, J. Differential Geom., **40** (1994), 129–154.

[LOT]   R. Lipshitz, P.S. Ozsváth and D.P. Thurston, Tour of bordered Floer theory, Proc. Natl. Acad. Sci. USA, **108** (2011), 8085–8092.

[Ly1]   M. Lipyanskiy, Gromov–Uhlenbeck compactness, preprint, arXiv:1409.1129.

[Ly2]   M. Lipyanskiy, Gromov–Uhlenbeck compactness and the Atiyah–Floer conjecture, talk at SCGP 2012-09-25, video is available at SCGP home page.

[Lu]   J. Lurie, On the classification of topological field theories, preprint, arXiv:0905.0465v1.

[Ly]   V. Lyubashenko, Category of A_∞-categories, Homology Homotopy Appl., **5** (2003), 1–48.

[MW]   C. Manolescu and C. Woodward, Floer homology on the extended moduli space, In: Perspective in Analysis, Geometry, and Topology, Progr. Math., **296**, Birkhäuser, New York, 2012, pp. 283–329.

[MWW]   S. Ma'u, K. Wehrheim and C. Woodward, A_∞ functors for Lagrangian correspondences, preprint, arXiv:1601.04919.

[Oh]   Y.-G. Oh, Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks. I, II, Comm. Pure Appl. Math., **46** (1993), 949–993; Comm. Pure Appl. Math., **46** (1993), 995–1012; Addendum, Comm. Pure Appl. Math., **48** (1995), 1299–1302.

[Sa]   D. Salamon, Lagrangian intersections, 3-manifolds with boundary, and the Atiyah–Floer conjecture, In: Proceedings of the International Congress of Mathematicians. Vol. 1, Zürich, 1994, Birkhäuser Verlag, Basel, 1995, pp. 526–536.

[SaWe]   D. Salamon and K. Wehrheim, Instanton Floer homology with Lagrangian boundary conditions, Geom. Topol., **12** (2008), 747–918.

[Se1]   G.B. Segal, The definition of conformal field theory, In: Differential Geometrical Methods in Theoretical Physics, Como, 1987, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **250**, Kluwer Acad. Publ., Dordrecht, 1988, pp. 165–171.

[Se2]   G.B. Segal, Geometric aspects of quantum field theory, In: Proceedings of the International Congress of Mathematicians. Vol. I, II, Kyoto, 1990, Math. Soc. Japan, Tokyo, 1991, pp. 1387–1396.

[Sei]   P. Seidel, Fukaya Categories and Picard–Lefschetz Theory, Zur. Lect. Adv. Math., Eur. Math. Soc. (EMS), Zürich, 2008.

[We1]   K. Wehrheim, Anti-self-dual instantons with Lagrangian boundary conditions. I: Elliptic Theory; II: Bubbling, Comm. Math. Phys., **254** (2005), 45–89; Comm. Math. Phys., **258** (2005), 275–315.

[We2]   K. Wehrheim, Lagrangian boundary conditions for anti-self-dual instantons and the Atiyah–Floer conjecture, In: Conference on Symplectic Topology, J. Symplectic Geom., **3**, International Press, Somerville, MA, 2005, pp. 703–747.

[WW1]   K. Wehrheim and C.T. Woodward, Quilted Floer cohomology, Geom. Topol., **14** (2010), 833–902.

[WW2]   K. Wehrheim and C.T. Woodward, Functoriality for Lagrangian correspondences in Floer theory, Quantum Topol., **1** (2010), 129–170.

[WW3]   K. Wehrheim and C.T. Woodward, Floer cohomology and geometric composition of Lagrangian correspondences, Adv. Math., **230** (2012), 177–228.

[WW4]   K. Wehrheim and C.T. Woodward, Pseudoholomorphic quilts, J. Symplectic Geom., **13** (2015), 849–904.

[WW5]   K. Wehrheim and C.T. Woodward, Floer field theory for tangles, preprint, arXiv:1503.07615.

[We]   A. Weinstein, Symplectic geometry, Bull. Amer. Math. Soc. (N.S.), **5** (1981), 1–13.

[Wi1]   E. Witten, Topological quantum field theory, Comm. Math. Phys., **117** (1988), 353–386.

[Wi2]   E. Witten, Quantum field theory and the Jones polynomial, Comm. Math. Phys., **121** (1989), 351–399.

[Yo]   T. Yoshida, Floer homology and holomorphic curves —Atiyah conjecture, preprint, 1992.