

Mackey's theory of τ -conjugate representations for finite groups

Tullio Ceccherini-Silberstein · Fabio Scarabotti · Filippo Tolli

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Dedicated to our mentors and friends

Toni Machì on his 75th birthday and Pierre de la Harpe on his 70th birthday

Abstract. The aim of the present paper is to expose two contributions of Mackey, together with a more recent result of Kawanaka and Matsuyama, generalized by Bump and Ginzburg, on the representation theory of a finite group equipped with an involutory anti-automorphism (e.g. the anti-automorphism $g \mapsto g^{-1}$). Mackey's first contribution is a detailed version of the so-called Gelfand criterion for weakly symmetric Gelfand pairs. Mackey's second contribution is a characterization of simply reducible groups (a notion introduced by Wigner). The other result is a twisted version of the Frobenius–Schur theorem, where “twisted” refers to the above-mentioned involutory anti-automorphism.

Keywords and phrases: representation theory of finite groups, Gelfand pair, Kronecker product, simply reducible group, Clifford groups, Frobenius–Schur theorem

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T. CECCHERINI-SILBERSTEIN

Dipartimento di Ingegneria, Università del Sannio, C.so Garibaldi 107, 82100 Benevento, Italy
(e-mail: tceccher@mat.uniroma3.it)

F. SCARABOTTI

Dipartimento SBAI, Sapienza Università di Roma, via A. Scarpa 8, 00161 Roma, Italy
(e-mail: fabio.scarabotti@sbai.uniroma1.it)

F. TOLLI

Dipartimento di Matematica e Fisica, Università Roma TRE, L. San Leonardo Murialdo 1, 00146 Roma, Italy
(e-mail: tolli@mat.uniroma3.it)

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1. Introduction

Finite Gelfand pairs not only constitute a useful tool for analyzing a wide range of problems ranging from combinatorics to orthogonal polynomials and to stochastic processes, but may also be used to shed light into theoretical problems of representation theory. The simplest example is provided by the possibility to recast the decomposition of the group algebra of a given finite group G , together with the associated harmonic analysis, by using the action on G of the direct product $G \times G$. Another example comes from the application of Gelfand pairs in the theory of multiplicity free groups, a key tool in the recent approach of Okounkov and Vershik to the representation theory of the symmetric groups [62, 75] (see also [16]).

Let G be a finite group. Recall that the *conjugate* of a (unitary) representation (ρ, V) of G , is the G -representation (ρ', V') where V' is the dual of V and $[\rho'(g)v'](v) = v'[\rho(g^{-1})v]$ for all $g \in G$, $v \in V$, and $v' \in V'$. One then says that ρ is *self-conjugate* provided $\rho \sim \rho'$; this is in turn equivalent to the associated character χ_ρ being real-valued. When ρ is not self-conjugate, one says that it is *complex*. The class of self-conjugate G -representations splits into two subclasses according to the associated matrix coefficients of the representation ρ being real-valued or not: in the first case, one says that ρ is *real*, in the second case ρ is termed *quaternionic*.

Now let $K \leq G$ be a subgroup and denote by $X = G/K$ the corresponding homogeneous space of left cosets of K in G . Setting $L(X) = \{f : X \rightarrow \mathbb{C}\}$, denote by $(\lambda, L(X))$ the corresponding *permutation representation* defined by $[\lambda(g)f](x) = f(g^{-1}x)$ for all $g \in G$ and $f \in L(X)$. Recall that (G, K) is a *Gelfand pair* provided the permutation representation λ decomposes

multiplicity-freely, that is,

$$(1.1) \quad \lambda = \bigoplus_{i \in I} \rho_i$$

with $\rho_i \not\cong \rho_j$ for $i \neq j$. It is well known that if $g^{-1} \in KgK$ for all $g \in G$, then (G, K) is a Gelfand pair; in this case all representations ρ_i in (1.1) are real, and one then says that (G, K) is *symmetric*. This last terminology is due to the fact that the G -orbits on $X \times X$ under the diagonal action are symmetric (with respect to the flip $(x_1, x_2) \mapsto (x_2, x_1)$, $x_1, x_2 \in X$).

A remarkable classical problem in representation theory is to determine the decomposition of the tensor product of two (irreducible) representations. In particular, one says that G is *simply reducible* if (i) $\rho_1 \otimes \rho_2$ decomposes multiplicity free for all irreducible G -representations ρ_1 and ρ_2 and (ii) every irreducible G -representation is self-conjugate.

The class of simply reducible groups was introduced by E. Wigner [78] in his research on group representations and quantum mechanics. This notion is quite useful since many of the symmetry groups one encounters in atomic and molecular systems are simply reducible, and algebraic manipulations of tensor operators become much easier for such groups. Wigner wrote: “The groups of most eigenvalue problems occurring in quantum theory are S.R.” (where “S.R.” stands for “simply reducible”) having in mind the study of “small perturbations” of the “united system” of two eigenvalue problems invariant under some group G of symmetries. Then simple reducibility guarantees that the characteristic functions of the eigenvalues into which the united system splits can be determined in “first approximation” by the invariance of the eigenvalue problem under G . This is the case, for instance, for the *angular momentum* in quantum mechanics. We mention that the multiplicity-freeness of the representations in the definition of simply reducible groups is the condition for the validity of the well known *Eckart–Wigner theorem* in quantum mechanics. Also, an important task in spectroscopy is to calculate matrix elements in order to determine energy spectra and transition intensities. One way to incorporate symmetry considerations connected to a group G or rather a pair (G, H) of groups, where $H \leq G$, is to use the *Wigner–Racah* calculus associated with the inclusion under consideration: this is generally understood as the set of algebraic manipulations concerning the coupling and the coupling coefficients for the group G . The Wigner–Racah calculus was originally developed for simply reducible groups [64, 65, 79, 80] and, later, for some other groups of interest in nuclear, atomic, molecular, condensed matter physics [36, Chapter 5] as well as in quantum chemistry [34].

Returning back to purely representation theory, Wigner [78] listed the following examples of simply reducible groups: the symmetric groups S_3 ($\cong D_3$) and S_4 ($\cong T_h$), the quaternion group Q_8 and the *rotational groups* $O(3)$, $SO(3)$ or $SU(2)$. More generally, it is nowadays known (*cf.* [68, Appendix 3.A]) that

most of the *molecular symmetry groups* such as (using Schoenflies notation) $D_{\infty h}$, $C_{\infty v}$, C_{2v} , C_{3v} , C_{2h} , D_{3h} , D_{3d} , D_{6h} , T_d and O_h are simply reducible. On the other hand, the *icosahedral group* I_h is not simply reducible, although it possesses only real characters.

In the automorphic setting, Prasad [63] implicitly showed that if k is a local field then the (infinite) group $G = \mathrm{GL}(2, k)$ is simply reducible. Indeed, he proved that the number of G -invariant linear forms on the tensor product of three admissible representations of G is at most one (up to scalars). This is also discussed in Sect. 10 of the survey article by Gross (Prasad's advisor) [35], on Gelfand pairs and their applications to number theory.

We also mention that simply reducible groups are of some interest also in the theory of *association schemes* (see [3, Chapter 2]).

As pointed out by A.I. Kostrikin in [53], there is no complete description of all simply reducible groups. Strunkov investigated simple reducibility in [73] and suggested (*cf.* [54, Problem 11.94] in the Kourovka notebook) that the simply reducible groups must be solvable. After some partial results by Kazarin and Yanishevskii [49], this conjecture was settled by Kazarin and Chankov [48].

Wigner [78] gave a curious criterion for simply reducibility. He showed that, denoting by $v(g) = |\{h \in G : hg = gh\}|$ (resp. $\zeta(g) = |\{h \in G : h^2 = g\}|$) the cardinality of the centralizer (resp. the number of square roots) of an element $g \in G$, then the equality

$$(1.2) \quad \sum_{g \in G} \zeta(g)^3 = \sum_{g \in G} v(g)^2$$

holds if and only if G is simply reducible.

A fundamental theorem of Frobenius and Schur [28] provides a criterion for determining the type of a given irreducible representation ρ , namely

$$(1.3) \quad \frac{1}{|G|} \sum_{g \in G} \chi_{\rho}(g^2) = \begin{cases} 1 & \text{if } \rho \text{ is real,} \\ -1 & \text{if } \rho \text{ is quaternionic,} \\ 0 & \text{if } \rho \text{ is complex} \end{cases}$$

see, for instance, [13, Theorem 9.7.7]. Moreover, the number h of pairwise inequivalent irreducible self-conjugate G -representations is given by

$$(1.4) \quad h = \frac{1}{|G|} \sum_{g \in G} \zeta(g)^2$$

(*cf.* [13, Theorem 9.7.10]).

In this research-expository paper, following Mackey [59], Kawanaka and Matsuyama [47], and Bump and Ginzburg [11], we endow G with an involutory anti-automorphism $\tau : G \rightarrow G$. Mackey in [58] originally analyzed only

the case when τ is the anti-automorphism $g \mapsto g^{-1}$ and then, in [59], generalized his results by considering any involutory anti-automorphism. The proofs are even simpler but heavily rely on [58] (the reader cannot read the second paper without having at hand the first one). Here we give a complete and self-contained treatment of all principal results in [59], providing more details and using modern notation.

We then present in Theorem 8.2 (Twisted Frobenius–Schur theorem) the main result of Kawanaka and Matsuyama in [47]. Our proof follows the lines indicated in Bump's monograph [7, Exercise 4.5.1] but also heavily uses the powerful machinery of A.H. Clifford theory specialized for subgroups of index two (see, for instance, [15, Sect. 3]). Note that Bump and Ginzburg [11] consider further generalizations involving anti-automorphisms of finite order (i.e., not necessarily involutive).

Let $\tau : G \rightarrow G$ be an involutory anti-automorphism.

Given a G -representation (ρ, V) , we then define its τ -conjugate as the G -representation (ρ^τ, V') defined by setting $[\rho^\tau(g)v'](v) = v'[\rho(\tau(g))v]$ for all $g \in G$, $v \in V$, and $v' \in V'$. Then we introduce (cf. [47]) the associated τ -Frobenius–Schur number (or τ -Frobenius–Schur indicator) $C_\tau(\rho)$ defined by

$$C_\tau(\rho) = \dim \operatorname{Hom}_G^{\operatorname{Sym}}(\rho^\tau, \rho) - \dim \operatorname{Hom}_G^{\operatorname{Skew}}(\rho^\tau, \rho),$$

where $\operatorname{Hom}_G^{\operatorname{Sym}}$ (resp. $\operatorname{Hom}_G^{\operatorname{Skew}}$) denotes the space of symmetric (resp. antisymmetric) intertwining operators, and show that, if ρ is irreducible, it may take only the three values 1, -1 , and 0.

Given a subgroup K , we consider the τ -conjugate λ^τ of the associated permutation representation. Suppose that $\lambda^\tau \sim \lambda$ (note that this is always the case if K is τ -invariant, i.e., $\tau(K) = K$), then we present a characterization (the Mackey–Gelfand criterion, see Theorem 4.5) of the corresponding analogue of “symmetric Gelfand pair” that we recover as a particular case.

We say that G is τ -simply reducible provided (i) $\rho_1 \otimes \rho_2$ is multiplicity-free and (ii) $\rho^\tau \sim \rho$, for all irreducible G -representations ρ_1, ρ_2 and ρ . We then present the Mackey criterion (Theorem 5.3) and the Mackey–Wigner criterion (Corollary 6.6) for τ -simple reducibility, a generalization of Wigner's original criterion we alluded to above (cf. (1.2)); the latter is expressed in terms of the equality

$$\sum_{g \in G} \zeta_\tau(g)^3 = \sum_{g \in G} v(g)^2,$$

where $\zeta_\tau(g)$ denotes the number of elements $h \in G$ such that $\tau(h^{-1})h = g$.

As an application of both the Mackey criterion and the Mackey–Wigner criterion, we present new examples of τ -simply reducible groups: in Sect. 7 we show that the W.K. Clifford groups $\mathbb{C}\mathbb{L}(n)$ are τ -simply reducible (where the

involutive anti-automorphism τ of $\mathbb{C}\mathbb{L}(n)$ is suitably defined according to the congruence class of n modulo 4).

Generalizing the characterization (1.3), we show (the Kawanaka and Matsuyama theorem (Theorem 8.2)) that

$$C_\tau(\rho) = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(\tau(g)^{-1}g).$$

Finally, in the last section, we present a twisted Frobenius–Schur type theorem in the context of Gelfand pairs (Theorem 9.1). This result, together with the ones on τ -simple reducibility of the Clifford groups we alluded to above, constitutes our original contribution to the theory.

2. Preliminaries and notation

2.1. Linear algebra

In order to fix notation, we begin by recalling some elementary notions of linear algebra. Let V, W be finite dimensional complex vector spaces and denote by V', W' their duals. We denote by $\text{Hom}(V, W)$ the space of all linear operators $A: V \rightarrow W$.

If $A \in \text{Hom}(V, W)$ its *transpose* is the linear operator $A^T : W' \rightarrow V'$ defined by setting

$$(A^T w')(v) = w'(Av)$$

for all $w' \in W'$ and $v \in V$. Let Z be another finite dimensional complex vector space and suppose that $B \in \text{Hom}(V, W)$ and $A \in \text{Hom}(W, Z)$. Then, it is immediate to check that $(AB)^T = B^T A^T$. Moreover, modulo the canonical identification of V and its bidual $V'' = (V')'$ (this is given by $v \leftrightarrow v''$ where $v'' \in V''$ is defined by $v''(v') = v'(v)$ for all $v' \in V'$), we have $(A^T)^T = A$. Given a basis $\{v_1, v_2, \dots, v_n\}$ in V , we denote by $\{v'_1, v'_2, \dots, v'_n\}$ the corresponding dual basis of V' which is defined by the conditions $v'_j(v_i) = \delta_{i,j}$ for $i, j = 1, 2, \dots, n$. Let now $\{w_1, w_2, \dots, w_m\}$ be a basis for W . Let $M_A = (a_{ki})_{k=1,2,\dots,m, i=1,2,\dots,n}$ the matrix associated with the linear operator A , that is, $Av_i = \sum_{k=1}^m a_{ki} w_k$, for all $i = 1, 2, \dots, n$. Then $a_{ki} = w'_k(Av_i)$ and $A^T w'_k = \sum_{i=1}^n a_{ki} v'_i$; in other words, the matrix M_{A^T} associated with the transpose operator A^T equals the transpose $(M_A)^T = (a_{ik})_{i=1,2,\dots,n, k=1,2,\dots,m}$ of the matrix M_A associated with A .

Suppose now that V is endowed with a hermitian scalar product denoted $\langle \cdot, \cdot \rangle_V$. The associated *Riesz map* is the antilinear bijective map $\theta_V : V \rightarrow V'$ defined by setting

$$(\theta_V v)(u) = \langle u, v \rangle_V$$

for all $u, v \in V$. Moreover, the *adjoint* of $A \in \text{Hom}(V, W)$ is the (unique) linear operator $A^* \in \text{Hom}(W, V)$ such that

$$\langle Av, w \rangle_W = \langle v, A^*w \rangle_V$$

for all $v \in V$ and $w \in W$. Observe that $(A^*)^* = A$. Also, the matrix M_{A^*} associated with the adjoint operator A^* equals the adjoint $(M_A)^* = (\bar{a}_{ik})_{\substack{i=1,2,\dots,n \\ k=1,2,\dots,m}}$ of the matrix M_A associated with A . Moreover, we say that A is *unitary* if $A^*A = I_V$ and $AA^* = I_W$, where $I_V \in \text{Hom}(V, V)$ denotes the identity map (note that a necessary condition for A to be unitary is that $\dim(V) = \dim(W)$, i.e., $n = m$).

Lemma 2.1. *Let $A \in \text{Hom}(V, W)$. Then $A^T \theta_W = \theta_V A^*$.*

Proof. For all $v \in V$ and $w \in W$ we have:

$$(A^T \theta_W w)(v) = (\theta_W w)(Av) = \langle Av, w \rangle_W = \langle v, A^*w \rangle_V = (\theta_V A^*w)(v).$$

□

We now define the *conjugate* of $A \in \text{Hom}(V, W)$ as the linear operator $\bar{A} = (A^*)^T \in \text{Hom}(V', W')$. Then, the associated matrix $M_{\bar{A}}$ equals the conjugate $\overline{M_A} = (\bar{a}_{ki})_{\substack{k=1,2,\dots,m \\ i=1,2,\dots,n}}$ of the matrix associated with A . Note that $\overline{\bar{A}} = A$ and $A^* = \bar{A}^T$ (here we implicitly use the canonical identification of V with its bidual V''). As a consequence, A is unitary if and only if $A^T \bar{A} = I_{V'}$ and $\bar{A} A^T = I_{W'}$.

Suppose now that $A \in \text{Hom}(V', V)$. Then, again modulo the canonical identification of V and V'' , we have

$$A^T \in \text{Hom}(V', V) \quad \text{and} \quad u'(A^T v') = v'(Au')$$

for all $u', v' \in V'$. We then say that $A \in \text{Hom}(V', V)$ is *symmetric* (resp. *antisymmetric* or *skew-symmetric*) if $A^T = A$ (resp. $A^T = -A$). We denote by $\text{Hom}^{\text{Sym}}(V', V)$ (resp. $\text{Hom}^{\text{Skew}}(V', V)$) the space of all symmetric (resp. antisymmetric) operators in $\text{Hom}(V', V)$. We have the elementary identity

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2},$$

where $\frac{A+A^T}{2}$ is symmetric and $\frac{A-A^T}{2}$ is antisymmetric: note that this is the unique decomposition of A as a sum of a symmetric operator and an antisymmetric operator. This yields the direct sum decomposition

$$(2.1) \quad \text{Hom}(V', V) = \text{Hom}^{\text{Sym}}(V', V) \oplus \text{Hom}^{\text{Skew}}(V', V).$$

Let now $A \in \text{Hom}[(V \oplus W)', V \oplus W]$. Then there exist $A_1 \in \text{Hom}(V', V)$, $A_2 \in \text{Hom}(W', W)$, $A_3 \in \text{Hom}(W', V)$ and $A_4 \in \text{Hom}(V', W)$ such that

$$(2.2) \quad A(v' + w') = (A_1 v' + A_3 w') + (A_4 v' + A_2 w'),$$

for all $v' \in V'$, $w' \in W'$. In other words, identifying A with the operator matrix $\begin{pmatrix} A_1 & A_3 \\ A_4 & A_2 \end{pmatrix}$, we may express (2.2) in matrix form

$$(2.3) \quad \begin{pmatrix} A_1 & A_3 \\ A_4 & A_2 \end{pmatrix} \begin{pmatrix} v' \\ w' \end{pmatrix} = \begin{pmatrix} A_1 v' + A_3 w' \\ A_4 v' + A_2 w' \end{pmatrix}.$$

Since

$$\begin{pmatrix} A_1 & A_3 \\ A_4 & A_2 \end{pmatrix}^T = \begin{pmatrix} A_1^T & A_4^T \\ A_3^T & A_2^T \end{pmatrix}$$

we have that $A = A^T$ if and only if $A_1 = A_1^T$, $A_2 = A_2^T$, $A_3 = A_4^T$ and $A_4 = A_3^T$ and this proves the first statement of the following lemma (the proof of the second statement is similar).

Lemma 2.2. (1) *The map*

$$\begin{pmatrix} A_1 & A_3 \\ A_3^T & A_2 \end{pmatrix} \mapsto (A_1, A_2, A_3^T)$$

yields the isomorphism

$$\begin{aligned} & \text{Hom}^{\text{Sym}}[(V \oplus W)', V \oplus W] \\ & \cong \text{Hom}^{\text{Sym}}(V', V) \oplus \text{Hom}^{\text{Sym}}(W', W) \oplus \text{Hom}(V', W). \end{aligned}$$

(2) *The map*

$$\begin{pmatrix} A_1 & A_3 \\ -A_3^T & A_2 \end{pmatrix} \mapsto (A_1, A_2, A_3^T)$$

yields the isomorphism

$$\begin{aligned} & \text{Hom}^{\text{Skew}}[(V \oplus W)', V \oplus W] \\ & \cong \text{Hom}^{\text{Skew}}(V', V) \oplus \text{Hom}^{\text{Skew}}(W', W) \oplus \text{Hom}(V', W). \end{aligned}$$

2.2. Representation theory of finite groups

We now recall some notions from the representation theory of finite groups. We refer to our monographs [13, 16] for a complete exposition and detailed proofs.

Let G be a finite group. We always suppose that all G -representations (ρ, W) are *unitary*: the representation space W is finite dimensional hermitian and $\rho(g) \in \text{Hom}(W, W)$ is unitary for every $g \in G$ (it is well known that every representation of a finite group over a complex vector space is unitarizable (cf. [13, Proposition 3.3.1])). We denote by \widehat{G} a complete set of pairwise inequivalent irreducible G -representations.

Given two G -representations (ρ, W) and (σ, V) , we denote by $\text{Hom}_G(W, V)$ (sometimes we shall also use the notation $\text{Hom}_G(\rho, \sigma)$) the space of all linear operators $A: W \rightarrow V$, called *intertwiners* of ρ and σ , such that $A\rho(g) = \sigma(g)A$ for all $g \in G$.

Let (ρ, W) be a G -representation. We denote by $\chi_\rho: G \rightarrow \mathbb{C}$ its character and, in the notation from Sect. 2.1, we denote by (ρ', W') the *conjugate representation* defined by setting

$$(2.4) \quad \rho'(g) = \rho(g^{-1})^T$$

for all $g \in G$. We have $\chi_{\rho'} = \overline{\chi_\rho}$ (complex conjugation). Suppose now that (ρ, W) is irreducible. Then ρ is said to be *complex* if $\chi_\rho \neq \chi_{\rho'}$, that is, ρ and ρ' are not (unitarily) equivalent; on the other hand, ρ is called *self-conjugate* if $\chi_\rho = \chi_{\rho'}$, that is, ρ and ρ' are (unitarily) equivalent. Clearly, ρ is self-conjugate if and only if χ_ρ is real valued. The class of self-conjugate representations in turn may be splitted into two subclasses. Let (ρ, W) be a self-conjugate G -representation and suppose that there exists an orthonormal basis $\{w_1, w_2, \dots, w_d\}$ in W such that the corresponding matrix coefficients are real valued: $u_{i,j}(g) = \langle \rho(g)w_j, w_i \rangle \in \mathbb{R}$ for all $g \in G$ and $i, j = 1, 2, \dots, d$. Then ρ is termed *real*. Otherwise, ρ is said to be *quaternionic*.

Lemma 2.3. *Let (ρ, W) be an irreducible, self-conjugate G -representation and let $A \in \text{Hom}_G(W, W')$ be a unitary operator. Then, if ρ is real one has $A\bar{A} = I_{W'}$ (equivalently, $A = A^T$), while if ρ is quaternionic one has $A\bar{A} = -I_{W'}$ (equivalently, $A = -A^T$).*

Proof. See [13, Lemma 9.7.6]. □

Let n be a positive integer, and consider the diagonal subgroup $\widetilde{G}^n = \{(g, g, \dots, g) : g \in G\}$ of $G^n = \underbrace{G \times G \times \dots \times G}_{n \text{ times}}$. Given G -representations (ρ_i, V_i) ,

$i = 1, 2, \dots, n$, following our monograph, we denote by $(\rho_1 \boxtimes \rho_2 \boxtimes \dots \boxtimes \rho_n, V_1 \otimes V_2 \otimes \dots \otimes V_n)$ their *external tensor product* which is a G^n -representation. Moreover we denote by $(\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_n, V_1 \otimes V_2 \otimes \dots \otimes V_n)$ the *Kronecker product* of the ρ_i 's, that is the G -representation defined by $\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_n = \text{Res}_{G^n}^{G^n}(\rho_1 \boxtimes \rho_2 \boxtimes \dots \boxtimes \rho_n)$.

3. The τ -Frobenius–Schur number

In what follows, $\tau: G \rightarrow G$ is an *involutory anti-automorphism* of G , that is a bijection such that

$$\tau(g_1 g_2) = \tau(g_2) \tau(g_1) \quad \text{and} \quad \tau^2(g) = g$$

for all $g_1, g_2, g \in G$. In particular, $\tau(1_G) = 1_G$ and $\tau(g^{-1}) = \tau(g)^{-1}$, for all $g \in G$. Let (ρ, W) be a G -representation. Then the associated τ -conjugate representation is the G -representation (ρ^τ, W') defined by setting

$$(3.1) \quad \rho^\tau(g) = \rho[\tau(g)]^T,$$

that is

$$(3.2) \quad [\rho^\tau(g)w'](w) = w'[\rho(\tau(g))w],$$

for all $g \in G$, $w' \in W'$ and $w \in W$. Note that if $\tau_{\text{inv}}: G \rightarrow G$ is the involutory anti-automorphism of G defined by $\tau_{\text{inv}}(g) = g^{-1}$ for all $g \in G$, then $\rho^{\tau_{\text{inv}}} = \rho'$ (cf. (2.4)).

Remark 3.1. Let $g_0 \in G$ and denote by τ_{g_0} the *inner* involutory anti-automorphism of G given by composing conjugation by g_0 and τ_{inv} , that is, $\tau_{g_0}(g) = g_0 g^{-1} g_0^{-1}$ for all $g \in G$. Then, given a G -representation (ρ, W) we have, for all $g \in G$, $w' \in W'$ and $w \in W$,

$$\begin{aligned} [\rho^{\tau_{g_0}}(g)w'](w) &= w'[\rho(\tau_{g_0}(g))w] = w'[\rho(g_0 g^{-1} g_0^{-1})w] \\ &= w'[\rho(g_0)\rho(g^{-1})\rho(g_0)^{-1}w] \\ &= [(\rho(g_0)\rho(g^{-1})\rho(g_0)^{-1})^T w'](w) \\ &= [\rho(g_0^{-1})^T \rho^{\tau_{\text{inv}}}(g)\rho(g_0)^T w'](w) \end{aligned}$$

yielding $\rho^{\tau_{g_0}}(g) = (\rho(g_0)^T)^{-1} \rho^{\tau_{\text{inv}}}(g) \rho(g_0)^T$, so that

$$\rho^{\tau_{g_0}} \sim \rho^{\tau_{\text{inv}}}.$$

Proposition 3.2. (1) *The G -representation (ρ^τ, W') is irreducible if and only if (ρ, W) is irreducible.*

(2) *If $A \in \text{Hom}_G(\rho^\tau, \rho)$ then also $A^T \in \text{Hom}_G(\rho^\tau, \rho)$ and we have the direct sum decomposition*

$$\text{Hom}_G(\rho^\tau, \rho) = \text{Hom}_G^{\text{Sym}}(\rho^\tau, \rho) \oplus \text{Hom}_G^{\text{Skew}}(\rho^\tau, \rho),$$

where $\text{Hom}_G^{\text{Sym}} = \text{Hom}^{\text{Sym}} \cap \text{Hom}_G$ and $\text{Hom}_G^{\text{Skew}} = \text{Hom}^{\text{Skew}} \cap \text{Hom}_G$ (compare with (2.1)).

Proof. (1) Suppose first that ρ is reducible and let $U \leq W$ be a nontrivial ρ -invariant subspace. Then $Z = \{w' \in W' : w'(u) = 0 \text{ for all } u \in U\} \leq W'$ is nontrivial and ρ^τ -invariant, thus showing that ρ^τ is also reducible. Since $(\rho^\tau)^\tau = \rho$, by applying the previous argument we also deduce the converse.

(2) Let $A \in \text{Hom}_G(\rho^\tau, \rho)$ and $g \in G$. Then, by transposing the identity $A\rho^\tau(g) = \rho(g)A$ we get $\rho^\tau(g)^T A^T = A^T \rho(g)^T$ which, by (3.1), becomes $\rho[\tau(g)]A^T = A^T \rho^\tau[\tau(g)]$. Since τ is bijective, by replacing $\tau(g)$ with g , we finally obtain $\rho(g)A^T = A^T \rho^\tau(g)$, thus showing that $A^T \in \text{Hom}_G(\rho^\tau, \rho)$. The direct sum decomposition is obvious. \square

Lemma 3.3. *Let (ρ, W) and (σ, V) be two G -representations. Then the following isomorphisms hold:*

$$(3.3) \quad \text{Hom}_G(\sigma^\tau, \rho) \cong \text{Hom}_G(\rho^\tau, \sigma),$$

$$(3.4) \quad \begin{aligned} & \text{Hom}_G^{\text{Sym}}[(\sigma \oplus \rho)^\tau, \sigma \oplus \rho] \\ & \cong \text{Hom}_G^{\text{Sym}}(\sigma^\tau, \sigma) \oplus \text{Hom}_G^{\text{Sym}}(\rho^\tau, \rho) \oplus \text{Hom}_G(\sigma^\tau, \rho) \end{aligned}$$

and

$$(3.5) \quad \begin{aligned} & \text{Hom}_G^{\text{Skew}}[(\sigma \oplus \rho)^\tau, \sigma \oplus \rho] \\ & \cong \text{Hom}_G^{\text{Skew}}(\sigma^\tau, \sigma) \oplus \text{Hom}_G^{\text{Skew}}(\rho^\tau, \rho) \oplus \text{Hom}_G(\sigma^\tau, \rho). \end{aligned}$$

Proof. The isomorphism (3.3) is realized by the map $A \mapsto A^T$. The isomorphism (3.4) (resp. (3.5)) is realized by the map in Lemma 2.2 (1) (resp. Lemma 2.2 (2)), keeping into account that in the matrix notation (2.3) A is an intertwining operator if and only if A_1, A_2, A_3, A_4 are intertwining operators. \square

Definition 3.4. *The τ -Frobenius–Schur number of a G representation (ρ, W) is the integer number $C_\tau(\rho)$ defined by*

$$C_\tau(\rho) = \dim \text{Hom}_G^{\text{Sym}}(\rho^\tau, \rho) - \dim \text{Hom}_G^{\text{Skew}}(\rho^\tau, \rho).$$

We also set $C(\rho) = \dim \text{Hom}_G^{\text{Sym}}(\rho', \rho) - \dim \text{Hom}_G^{\text{Skew}}(\rho', \rho)$, that is $C(\rho) = C_{\tau_{\text{inv}}}(\rho)$. We start by examining $C_\tau(\rho)$ and $C(\rho)$ when ρ is irreducible.

Theorem 3.5. *Suppose that ρ is irreducible. Then*

- (1) $C_\tau(\rho) \in \{-1, 0, 1\}$. Moreover, $C_\tau(\rho) = 0$ (resp. $C_\tau(\rho) = \pm 1$) if and only if $\rho^\tau \not\sim \rho$ (resp. $\rho^\tau \sim \rho$).
- (2) In particular,

$$C(\rho) = \begin{cases} 1 & \text{if } \rho \text{ is real,} \\ 0 & \text{if } \rho \text{ is complex,} \\ -1 & \text{if } \rho \text{ is quaternionic.} \end{cases}$$

Proof. (1) If $\rho^\tau \not\sim \rho$ then $\dim \operatorname{Hom}_G(\rho^\tau, \rho) = 0$ and therefore $C_\tau(\rho) = 0$. Now suppose that $\rho \sim \rho^\tau$. If $A \in \operatorname{Hom}_G(\rho^\tau, \rho)$, $A \neq 0$, then also $A^T \in \operatorname{Hom}_G(\rho^\tau, \rho)$ and therefore, by Proposition 3.2 and Schur's lemma, there exists $\lambda \in \mathbb{C}$ such that $A^T = \lambda A$. By transposing we get $A = \lambda A^T = \lambda^2 A$, which implies that $\lambda = \pm 1$. If $\lambda = 1$ then A is symmetric and $C_\tau(\rho) = \dim \operatorname{Hom}_G^{\operatorname{Sym}}(\rho^\tau, \rho) - \dim \operatorname{Hom}_G^{\operatorname{Skew}}(\rho^\tau, \rho) = 1 - 0 = 1$; similarly, if $\lambda = -1$ then $C_\tau(\rho) = -1$.

(2) If ρ is complex then $\rho' \not\sim \rho$ and therefore $\dim \operatorname{Hom}_G(\rho', \rho) = 0$ and $C(\rho) = 0$. If ρ is self-adjoint then $\dim \operatorname{Hom}_G(\rho', \rho) = 1$ and this space is spanned by a unitary matrix A as in Lemma 2.3, which is symmetric if ρ is real and antisymmetric if ρ is quaternionic. \square

We now examine the behaviour of C_τ with respect to direct sums and tensor products.

Proposition 3.6. *Let (ρ, W) and (σ, V) be two G -representations. Then*

$$C_\tau(\sigma \oplus \rho) = C_\tau(\sigma) + C_\tau(\rho).$$

Proof. It is an immediate consequence of (3.4) and (3.5). \square

Proposition 3.7. *Suppose that $G = G_1 \times G_2$ and that τ satisfies $\tau(G_1 \times \{1_{G_2}\}) = G_1 \times \{1_{G_2}\}$ and $\tau(\{1_{G_1}\} \times G_2) = \{1_{G_1}\} \times G_2$. Let (ρ_i, W_i) be a G_i -representation for $i = 1, 2$. Then*

$$(3.6) \quad C_\tau(\rho_1 \boxtimes \rho_2) = C_\tau(\rho_1)C_\tau(\rho_2).$$

Proof. We first prove (3.6) under the assumption that both ρ_1 and ρ_2 are irreducible. The representation $(\rho_1 \boxtimes \rho_2)^\tau \sim \rho_1^\tau \boxtimes \rho_2^\tau$ is equivalent to $\rho_1 \boxtimes \rho_2$ if and only if $\rho_1 \sim \rho_1^\tau$ and $\rho_2 \sim \rho_2^\tau$. Therefore $C_\tau(\rho_1 \boxtimes \rho_2) = 0$ if and only if $C_\tau(\rho_1) = 0$ or $C_\tau(\rho_2) = 0$. On the other hand, if $\rho_1 \sim \rho_1^\tau$, $\rho_2 \sim \rho_2^\tau$ and A_i spans $\operatorname{Hom}_{G_i}(\rho_i^\tau, \rho_i)$, for $i = 1, 2$, then $\operatorname{Hom}_G[(\rho_1 \boxtimes \rho_2)^\tau, \rho_1 \boxtimes \rho_2]$ is spanned by $A_1 \otimes A_2$. It is easy to check that $(A_1 \otimes A_2)^T = A_1^T \otimes A_2^T$ so that $A_1 \otimes A_2$ is symmetric if and only if A_1 and A_2 are both symmetric or antisymmetric, while $A_1 \otimes A_2$ is antisymmetric if and only if one of the operators A_1 and A_2 is symmetric and the other is antisymmetric. In both cases, (3.6) follows.

Now we remove the irreducibility assumption and we suppose that

$$\rho_1 = \bigoplus_{i=1}^n m_i \sigma_i \quad \text{and} \quad \rho_2 = \bigoplus_{j=1}^k h_j \theta_j$$

are the decompositions of ρ_1 and ρ_2 into irreducible representations. Then

$$\begin{aligned}
C_\tau(\rho_1 \boxtimes \rho_2) &= C_\tau \left[\bigoplus_{i=1}^n \bigoplus_{j=1}^k m_i h_j (\sigma_i \boxtimes \theta_j) \right] \\
&= \sum_{i=1}^n \sum_{j=1}^k m_i h_j C_\tau(\sigma_i \boxtimes \theta_j) \quad (\text{by Proposition 3.6}) \\
&= \sum_{i=1}^n \sum_{j=1}^k m_i h_j C_\tau(\sigma_i) C_\tau(\theta_j) \quad (\text{by the first part of the proof}) \\
&= \left[\sum_{i=1}^n m_i C_\tau(\sigma_i) \right] \left[\sum_{j=1}^k h_j C_\tau(\theta_j) \right] \\
&= C_\tau(\rho_1) C_\tau(\rho_2) \quad (\text{again by Proposition 3.6}).
\end{aligned}$$

□

Note that when $\tau = \tau_{\text{inv}}$ the first part of the proof of the preceding proposition may be also deduced from Theorem 3.5 (2).

Let now $\omega: G \rightarrow G$ be another involutory anti-automorphism of G commuting with τ , that is $\omega\tau = \tau\omega$. Clearly, the composition $\omega\tau$ is now an (involutory) automorphism of G . Moreover we have

$$(\rho^\tau)^\omega(g) = (\rho^\tau(\omega(g)))^T = \rho(\tau(\omega(g))) = \rho(\omega(\tau(g))) = (\rho^\omega)^\tau(g)$$

for all $g \in G$, that is,

$$(3.7) \quad (\rho^\tau)^\omega = (\rho^\omega)^\tau.$$

Lemma 3.8. *Let ω, τ and (ρ, W) (not necessarily irreducible) be as above. Then*

$$C_\tau(\rho^\omega) = C_\tau(\rho).$$

Proof. By virtue of Proposition 3.6 it suffices to examine the case when ρ is irreducible. If $C_\tau(\rho) = 0$ then $\rho^\tau \not\sim \rho$ and therefore $(\rho^\omega)^\tau = (\rho^\tau)^\omega \not\sim \rho^\omega$ (recall that ω is involutory). We deduce that $C_\tau(\rho^\omega) = 0$ as well.

Suppose now that $\rho^\tau \sim \rho$ and let $A \in \text{Hom}_G(\rho^\tau, \rho)$ be a nontrivial unitary intertwiner. Then, for all $g \in G$ we have $\rho(g)A = A\rho^\tau(g)$ so that

$$\begin{aligned}
(\rho^\tau)^\omega(g)A^T &= (\rho^\tau(\omega(g)))^T A^T = (A\rho^\tau(\omega(g)))^T = (\rho(\omega(g))A)^T \\
&= A^T \rho^\omega(g).
\end{aligned}$$

This shows that $\text{Hom}_G(\rho^\omega, (\rho^\omega)^\tau) \equiv \text{Hom}_G(\rho^\omega, (\rho^\tau)^\omega)$ is spanned by A^T , so that $\text{Hom}_G((\rho^\omega)^\tau, \rho^\omega)$ is spanned by $(A^T)^{-1} \equiv (A^T)^* = \overline{A}$. Thus since \overline{A}

is symmetric (resp. antisymmetric) if and only if A is symmetric (resp. antisymmetric), we deduce that $C_\tau(\rho^\omega) = 1$ (resp. $C_\tau(\rho^\omega) = -1$) if and only if $C_\tau(\rho) = 1$ (resp. $C_\tau(\rho) = -1$). \square

By taking $\omega = \tau$ we deduce the following:

Corollary 3.9. $C_\tau(\rho^\tau) = C_\tau(\rho)$.

Let now $K \leq G$ be a τ -invariant (that is $\tau(K) = K$) subgroup. It is clear that if (ρ, W) is a G -representation, then $\text{Res}_K^G(\rho^\tau) = (\text{Res}_K^G \rho)^\tau$. Conversely, suppose now that (σ, V) is a K -representation and let us show that $\text{Ind}_K^G(\sigma^\tau) \sim (\text{Ind}_K^G \sigma)^\tau$. We set $\text{Ind}_K^G V = \{F \in V^G : F(gk) = \sigma(k^{-1})F(g), \text{ for all } g \in G, k \in K\}$, $\rho = \text{Ind}_K^G \sigma$, so that (cf. [16, Definition 1.6.1]) $[\rho(g_0)F](g) = F(g_0^{-1}g)$ for all $g, g_0 \in G$ and $F \in \text{Ind}_K^G V$, and $\text{Ind}_K^G V' = \{F' \in (V')^G : F'(gk) = \sigma^\tau(k^{-1})F'(g), \text{ for all } g \in G, k \in K\}$, $\tilde{\rho} = \text{Ind}_K^G \sigma^\tau$, so that $[\tilde{\rho}(g_0)F'](g) = F'(g_0^{-1}g)$ for all $g, g_0 \in G$ and $F' \in \text{Ind}_K^G V'$.

Lemma 3.10. *The linear map $\xi : \text{Ind}_K^G V' \rightarrow (\text{Ind}_K^G V)'$ defined by setting*

$$(\xi F')(F) = \frac{1}{|K|} \sum_{g \in G} F'(g)(F(\tau(g^{-1})))$$

for all $F \in \text{Ind}_K^G V$, $F' \in \text{Ind}_K^G V'$, yields an isomorphism between $\text{Ind}_K^G \sigma^\tau$ and $(\text{Ind}_K^G \sigma)^\tau$.

Proof. Let $S \subseteq G$ be a complete system of representatives for the set G/K of left cosets of K in G , so that $G = \coprod_{s \in S} sK$. On the one hand:

$$\begin{aligned} (\xi F')(F) &= \frac{1}{|K|} \sum_{g \in G} F'(g)(F(\tau(g^{-1}))) \\ &= \frac{1}{|K|} \sum_{s \in S} \sum_{k \in K} F'(sk)(F(\tau((sk)^{-1}))) \\ &= \frac{1}{|K|} \sum_{s \in S} \sum_{k \in K} [\sigma^\tau(k^{-1})F'(s)](\sigma(\tau(k))F(\tau(s)^{-1})) \\ &= \frac{1}{|K|} \sum_{s \in S} \sum_{k \in K} F'(s)(\sigma(\tau(k^{-1}))\sigma(\tau(k))F(\tau(s)^{-1})) \quad (\text{by (3.2)}) \\ &= \sum_{s \in S} F'(s)(F(\tau(s)^{-1})). \end{aligned}$$

Since F' is uniquely determined by $(F'(s))_{s \in S}$, we deduce that ξ is injective. Moreover, as $\dim \text{Ind}_K^G V' = \dim(\text{Ind}_K^G V)'$ we deduce that ξ is indeed bijective.

On the other hand, for $g_0 \in G$, we have

$$\begin{aligned}
[\rho^\tau(g_0)\xi F'](F) &= [\xi F'](\rho(\tau(g_0))F) \quad (\text{by (3.2)}) \\
&= \frac{1}{|K|} \sum_{g \in G} F'(g)(F(\tau(g_0)^{-1}\tau(g)^{-1})) \\
&= \frac{1}{|K|} \sum_{g_1 \in G} F'(g_0^{-1}g_1)(F(\tau(g_1)^{-1})) \quad (\text{setting } g_1 = g_0g) \\
&= \frac{1}{|K|} \sum_{g_1 \in G} \{[\tilde{\rho}(g_0)F'](g_1)\}(F(\tau(g_1)^{-1})) \\
&= \{\xi[\tilde{\rho}(g_0)F']\}(F).
\end{aligned}$$

This shows that $\rho^\tau(g_0)\xi = \xi\tilde{\rho}(g_0)$ for all $g_0 \in G$ so that $\xi \in \text{Hom}_G(\text{Ind}_K^G \sigma^\tau, (\text{Ind}_K^G \sigma)^\tau)$, completing the proof. \square

Theorem 3.11. *Let $K \leq G$ be a τ -invariant subgroup. Let also (ρ, W) be an irreducible G -representation whose restriction $\text{Res}_K^G \rho$ is multiplicity-free, that is, $\text{Res}_K^G(\rho, W) = \bigoplus_{i=1}^m (\sigma_i, V_i)$, with σ_i irreducible and $\sigma_i \not\sim \sigma_j$ for $1 \leq i \neq j \leq m$. Suppose that $\rho^\tau \sim \rho$ and $\sigma_i^\tau \sim \sigma_i$ for $i = 1, 2, \dots, m$. Then*

$$C_\tau(\sigma_i) = C_\tau(\rho)$$

for all $i = 1, 2, \dots, m$.

Proof. Let us set, for $i = 1, 2, \dots, m$,

$$W'_i = \left\{ w' \in W' : \ker w' = \left(\bigoplus_{j=1}^{i-1} V_j \right) \oplus \left(\bigoplus_{j=i+1}^m V_j \right) \right\} \cong V'_i.$$

If we identify V'_i with W'_i then $(\text{Res}_K^G \rho^\tau)|_{W'_i} = \sigma_i^\tau$: indeed

$$[\rho^\tau(k)w'](v) = w'[\rho(\tau(k))v] = w'[\sigma_i(\tau(k))v] = [\sigma_i^\tau(k)w'](v),$$

for all $w' \in W'_i$, $v \in V_i$, and $k \in K$ (clearly W'_i is K -invariant).

Now, if $A \in \text{Hom}_G(\rho^\tau, \rho)$, that is, $A\rho^\tau(g) = \rho(g)A$ for all $g \in G$, we deduce that

$$A\sigma_i^\tau(k)w' = A\rho^\tau(k)w' = \rho(k)Aw'$$

for all $w' \in W'_i$ and $k \in K$. It follows that $A|_{W'_i}: W'_i \rightarrow V_i$ (recall that $\sigma_i^\tau \sim \sigma_i$ and $\sigma_i^\tau \not\sim \sigma_j$ for $1 \leq i \neq j \leq m$). Thus, setting $A_i = A|_{W'_i}$ we have $A = A_1 \oplus A_2 \oplus \dots \oplus A_m$, $A^T = A_1^T \oplus A_2^T \oplus \dots \oplus A_m^T$, and A is symmetric (resp. antisymmetric) if and only if A_1, A_2, \dots, A_m are all symmetric (resp. antisymmetric).

Lemma 3.12. *Let ρ be a G -representation and denote by $\rho = \bigoplus_{j=1}^n \rho_j$ a decomposition into irreducibles (now the sub-representations ρ_j need not be pairwise inequivalent). Then $\dim \text{Hom}_G^{\text{Skew}}(\rho^\tau, \rho) = 0$ if and only if for every $j = 1, 2, \dots, n$ one of the following conditions holds:*

- (i) $C_\tau(\rho_j) = 1$ and $\rho_j \not\sim \rho_k$ for all $k \neq j$;
- (ii) $C_\tau(\rho_j) = 0$ and $\rho_j^\tau \not\sim \rho_k$ for all $k \neq j$.

Different j 's may satisfy different conditions.

Proof. Let us set $\sigma_i = \bigoplus_{j=i}^n \rho_j$ for $i = 1, 2, \dots, n$. By repeatedly applying Lemma 3.3, we deduce that

$$\begin{aligned} & \dim \text{Hom}_G^{\text{Skew}}(\rho^\tau, \rho) \\ &= \dim \text{Hom}_G^{\text{Skew}}((\rho_1 \oplus \sigma_2)^\tau, \rho_1 \oplus \sigma_2) \\ &= \dim \text{Hom}_G^{\text{Skew}}(\rho_1^\tau, \rho_1) + \dim \text{Hom}_G^{\text{Skew}}(\sigma_2^\tau, \sigma_2) + \dim \text{Hom}_G(\rho_1^\tau, \sigma_2) \\ & \quad \dots \\ &= \sum_{j=1}^n \dim \text{Hom}_G^{\text{Skew}}(\rho_j^\tau, \rho_j) + \sum_{j=1}^{n-1} \dim \text{Hom}_G(\rho_j^\tau, \sigma_{j+1}). \end{aligned}$$

Thus $\dim \text{Hom}_G^{\text{Skew}}(\rho^\tau, \rho) = 0$ if and only if $\dim \text{Hom}_G^{\text{Skew}}(\rho_j^\tau, \rho_j) = 0$ for all $j = 1, 2, \dots, n$ and $\dim \text{Hom}_G(\rho_j^\tau, \sigma_{j+1}) = 0$ for all $j = 1, 2, \dots, n-1$. It follows that if $\rho_j^\tau \sim \rho_j$ we necessarily have $C_\tau(\rho_j) = 1$ and $\rho_j \not\sim \rho_k$ for all $j < k \leq n$, while if $\rho_j^\tau \not\sim \rho_j$ we necessarily have $C_\tau(\rho_j) = 0$ and $\rho_j^\tau \not\sim \rho_k$ for all $j < k \leq n$. Now, in both cases, the condition $k > j$ can be replaced by $k \neq j$: since the order in $\rho = \bigoplus_{j=1}^n \rho_j$ is arbitrary, we may always suppose $j = 1$.

4. Multiplicity-free permutation representations: the Mackey–Gelfand criterion

Let G be a finite group and suppose we are given a transitive action $\pi : G \rightarrow \text{Sym}(X)$ of G on a (finite) set X . Fix $x_0 \in X$ and denote by $K = \text{Stab}_G^\pi(x_0) = \{g \in G : \pi(g)x_0 = x_0\} \leq G$ its G -stabilizer. Then we identify the homogeneous space X with the set G/K of left cosets of K in G . This way, the action is given by $\pi(g)x = (gg')K$ for all $g \in G$ and $x = g'K \in X$ (note that, in particular, $x_0 \equiv K$). We also denote by $\mathcal{O}_G^\pi(X)$ the corresponding set of G -orbits in X .

Let also $\tau : G \rightarrow G$ be an involutory anti-automorphism of G which does not necessarily preserve K . Let $Y = G/\tau(K)$ denote the corresponding homogeneous space and by $y_0 = \tau(K) \in Y$ the corresponding $\tau(K)$ -fixed point.

We denote by $L(X)$ the vector space of all functions $f : X \rightarrow \mathbb{C}$ and denote by $(\lambda_\pi, L(X))$ the *permutation representation* associated with the action π , that is, the G -representation defined by

$$[\lambda_\pi(g)f](x) = f(\pi(g^{-1})x)$$

for all $g \in G$, $f \in L(X)$ and $x \in X$. Also, we denote by $\gamma : G \rightarrow \text{Sym}(Y)$ the action of G on Y : $\gamma(g)y = gg'\tau(K)$ for all $g \in G$ and $y = g'\tau(K) \in Y$. We then define a map $\theta : X \rightarrow Y$ by setting, for every $x \in X$,

$$(4.1) \quad \theta(x) = \gamma(\tau(g^{-1}))y_0, \quad \text{where } g \in G \text{ satisfies } \pi(g)x_0 = x.$$

Note that the map is well-defined: if $g_1, g_2 \in G$ satisfy $\pi(g_1)x_0 = \pi(g_2)x_0$, then there exists $k \in K$ such that $g_2 = g_1k$ and therefore

$$\gamma(\tau(g_2^{-1}))y_0 = \gamma(\tau(k^{-1}g_1^{-1}))y_0 = \gamma(\tau(g_1^{-1}))\gamma(\tau(k^{-1}))y_0 = \gamma(\tau(g_1^{-1}))y_0.$$

It is clear that θ is a bijection and that $\theta(x_0) = y_0$.

We now define a second action $\pi^\tau : G \rightarrow \text{Sym}(X)$ by setting

$$\pi^\tau(g)x = \pi(\tau(g^{-1}))x$$

for all $g \in G$ and $x \in X$. The associated permutation representation $(\lambda_{\pi^\tau}, L(X))$ is then given by

$$[\lambda_{\pi^\tau}(g)f](x) = f(\pi(\tau(g))x)$$

for all $g \in G$, $f \in L(X)$ and $x \in X$.

The τ -conjugate representation (cf. Sect. 3) $(\lambda_\pi^\tau, L(X)')$ of λ_π is then given by $[\lambda_\pi^\tau(g)\varphi'](f) = \varphi'(\lambda_\pi(\tau(g))f)$ for all $g \in G$, $\varphi' \in L(X)'$ and $f \in L(X)$. In the following, we identify the dual $L(X)'$ with $L(X)$ via the bijective linear map

$$(4.2) \quad \begin{array}{ccc} L(X) & \longrightarrow & L(X)' \\ \varphi & \longmapsto & \varphi' \end{array}$$

where

$$\varphi'(f) = \sum_{x \in X} \varphi(x)f(x)$$

for all $f \in L(X)$.

Finally, we fix $S \subseteq G$ a complete set of representatives of the double $\tau(K) \backslash G / K$ -cosets, so that

$$G = \coprod_{s \in S} \tau(K)sK.$$

Observe that $\tau(\tau(K)sK) = \tau(K)\tau(s)K$ and therefore $\tau(S)$ is also a system of representatives of the double cosets. Let also $\pi^\tau \times \pi : G \rightarrow \text{Sym}(X \times X)$ be the action defined by

$$(\pi^\tau \times \pi)(g)(x_1, x_2) = (\pi^\tau(g)x_1, \pi(g)x_2)$$

for all $g \in G$ and $x_1, x_2 \in X$. Also, we denote by $\flat \in \text{Sym}(X \times X)$ the involution defined by $(x_1, x_2)^\flat = (x_2, x_1)$ for all $x_1, x_2 \in X$.

Lemma 4.1. (1) *The bijective map θ in (4.1) satisfies $\theta\pi^\tau(g) = \gamma(g)\theta$ for all $g \in G$. Thus $\text{Stab}_G^{\pi^\tau}(x_0) = \tau(K)$.*

(2) *We have*

$$(4.3) \quad (\lambda_{\pi^\tau}^\tau, L(X)') \sim (\lambda_{\pi^\tau}, L(X))$$

via the bijective map (4.2).

(3) *The maps*

$$\Psi_S: \begin{array}{l} S \longrightarrow \mathcal{O}_{\tau(K)}^\pi(X) \equiv \mathcal{O}_K^{\pi^\tau}(X), \\ s \longmapsto \{\pi(\tau(k)s)x_0 : k \in K\} \end{array}$$

and

$$\Xi_S: \begin{array}{l} S \longrightarrow \mathcal{O}_G^{\pi^\tau \times \pi}(X \times X), \\ s \longmapsto \{(\pi^\tau(g)x_0, \pi(gs)x_0) : g \in G\} \end{array}$$

are bijective. Moreover, $\Xi_{\tau(S)}(\tau(s)) = (\Xi_S(s))^\flat$ for all $s \in S$, where $\Xi_{\tau(S)}(\tau(s)) = \{(\pi^\tau(g)x_0, \pi(g\tau(s))x_0) : g \in G\}$.

Proof. (1) Let $x \in X$ and $g_0 \in G$ be such that $x = \pi(g_0)x_0$. Then for all $g \in G$ we have

$$\begin{aligned} \theta(\pi^\tau(g)x) &= \theta(\pi(\tau(g^{-1}))\pi(g_0)x_0) \\ &= \theta(\pi(\tau(g^{-1})g_0)x_0) \\ &= \gamma(g\tau(g_0^{-1}))y_0 \quad (\text{by (4.1)}) \\ &= \gamma(g)\gamma(\tau(g_0^{-1}))y_0 \quad (\text{since } \gamma \text{ is an action}) \\ &= \gamma(g)\theta(\pi(g_0)x_0) \quad (\text{again by (4.1)}) \\ &= \gamma(g)\theta(x). \end{aligned}$$

(2) Let now $\varphi, f \in L(X)$ and $g \in G$. Then we have

$$\begin{aligned}
[\lambda_{\pi}^{\tau}(g)\varphi'](f) &= \varphi'(\lambda_{\pi}(\tau(g))f) \\
&= \sum_{x \in X} \varphi(x) f(\pi(\tau(g^{-1}))x) \\
&= \sum_{z \in X} \varphi(\pi(\tau(g))z) f(z) \quad (\text{setting } z = \pi(\tau(g^{-1}))x) \\
&= \sum_{z \in X} [\lambda_{\pi^{\tau}}(g)\varphi](z) f(z) \\
&= (\lambda_{\pi^{\tau}}(g)\varphi)'(f),
\end{aligned}$$

so that $\lambda_{\pi}^{\tau}(g)\varphi' = (\lambda_{\pi^{\tau}}(g)\varphi)'$. In other words, the bijective map (4.2) yields the equivalence (4.3).

(3) By definition of S , Ψ_S is well-defined and bijective. As for Ξ_S , let $x_1, x_2 \in X$. Since the action π^{τ} (resp. π) is transitive, we can find $g_1 \in G$ ($g_2 \in G$) such that $\pi^{\tau}(g_1)x_0 = x_1$ (resp. $\pi(g_2)x_0 = x_2$). Let $k_1, k_2 \in K$ and $s \in S$ be such that $\tau(k_1)sk_2 = g_1^{-1}g_2$. Setting $g = g_1\tau(k_1)$, we then have

$$(\pi^{\tau}(g)x_0, \pi(gs)x_0) = (\pi^{\tau}(g_1)\pi(k_1^{-1})x_0, \pi(g_1)\pi(\tau(k_1)sk_2)x_0) = (x_1, x_2).$$

This shows that Ξ_S is surjective. Now we show that it is injective: if $\Xi_S(s_1) = \Xi_S(s_2)$ for $s_1, s_2 \in S$ then there exists $g \in G$ such that $(\pi^{\tau}(g)x_0, \pi(gs_2)x_0) = (x_0, \pi(s_1)x_0)$, and this implies that $g \in \tau(K)$ and $s_1^{-1}gs_2 \in K$, so that $s_1 \in \tau(K)s_2K$ and necessarily $s_1 = s_2$. Finally, let $s \in S$ and $g \in G$ and set $g' = \tau(g^{-1})s^{-1}$. It is then immediate to check that

$$(\pi^{\tau}(g)x_0, \pi(g\tau(s))x_0) = (\pi(g's)x_0, \pi^{\tau}(g')x_0) = (\pi^{\tau}(g')x_0, \pi(g's)x_0)^{\flat}.$$

□

Corollary 4.2. *Suppose that $K = \tau(K)$. Then $\pi \equiv \gamma$, so that $\theta\pi^{\tau}(g) = \pi(g)\theta$ for all $g \in G$ and $\lambda_{\pi}^{\tau} \sim \lambda_{\pi}$.*

Proof. Just note that defining $T: L(X) \rightarrow L(X)$ by setting $(Tf)(x) = f(\theta(x))$ for all $f \in L(X)$, $x \in X$, we get a linear bijection such that $T\lambda_{\pi}(g) = \lambda_{\pi^{\tau}}(g)T$, for all $g \in G$. □

Remark 4.3. Lemma 4.1 generalizes the well known facts that the maps $s \mapsto \{\pi(k_s)x_0, k \in K\}$ and $s \mapsto \{(\pi(g)x_0, \pi(gs)x_0) : g \in G\}$ are bijections respectively between S and $\mathcal{O}_K^{\pi}(X)$ and between S and $\mathcal{O}_G^{\pi \times \pi}(X \times X)$. See [13, Sect. 3.13] and [16, Sect. 1.5.3].

We shall say that a $(\pi^{\tau} \times \pi)$ -orbit of G on $X \times X$ is τ -symmetric (resp. τ -antisymmetric) provided it is (resp. is not) invariant under the flip $\flat: (x_1, x_2) \mapsto$

(x_2, x_1) . We then denote by m_1 (resp. m_2) the number of such τ -symmetric (resp. τ -antisymmetric) orbits. From Lemma 4.1 (3) we then have m_1 (resp. m_2) equals the number of $s \in S$ such that $\tau(s) \in \tau(K)sK$ (resp. $\tau(s) \notin \tau(K)sK$). Note that $m_1 + m_2 = |S|$ and that m_2 is even.

Theorem 4.4. *We have*

$$(4.4) \quad \dim \operatorname{Hom}_G^{\operatorname{Sym}}(\lambda_\pi^\tau, \lambda_\pi) = m_1 + \frac{1}{2}m_2;$$

$$(4.5) \quad \dim \operatorname{Hom}_G^{\operatorname{Skew}}(\lambda_\pi^\tau, \lambda_\pi) = \frac{1}{2}m_2.$$

Proof. Denoting, as in (4.2), by $\varphi \mapsto \varphi'$ the identification of $L(X)$ and its dual $L(X)'$ and recalling that $\lambda_\pi^\tau \sim \lambda_{\pi^\tau}$ (cf. (4.3)), with every $A \in \operatorname{Hom}_G(\lambda_\pi^\tau, \lambda_\pi) \cong \operatorname{Hom}_G(\lambda_{\pi^\tau}, \lambda_\pi)$ we associate a complex matrix $(a(x_1, x_2))_{x_1, x_2 \in X}$ such that

$$[A\varphi](x_2) = \sum_{x_1 \in X} a(x_1, x_2)\varphi(x_1)$$

for all $x_1 \in X$. Then A is an intertwiner if and only if

$$a(\pi^\tau(g)x_1, \pi(g)x_2) = a(x_1, x_2)$$

for all $g \in G$ and $x_1, x_2 \in X$, that is, if and only if $a(x_1, x_2)$ is constant on the $(\pi^\tau \times \pi)$ -orbits of G on $X \times X$. Now, if A is symmetric, $a(x_1, x_2)$ must assume the same values on coupled antisymmetric orbits and therefore (4.4) follows. On the other hand, if A is antisymmetric, $a(x_1, x_2)$ must vanish on all symmetric orbits and assume opposite values on coupled antisymmetric orbits. Thus (4.5) follows as well. \square

Theorem 4.5 (Mackey–Gelfand criterion). *Suppose that $\lambda_\pi^\tau \sim \lambda_\pi$. Then the following conditions are equivalent.*

- (a) $\dim \operatorname{Hom}_G^{\operatorname{Skew}}(\lambda_\pi^\tau, \lambda_\pi) = 0$;
- (b) every $(\pi^\tau \times \pi)$ -orbit of G on $X \times X$ is symmetric;
- (c) every double coset $\tau(K)sK$ is τ -invariant;
- (d) (G, K) is a Gelfand pair and $C_\tau(\sigma) = 1$ for every irreducible representation σ contained in λ_π .

Proof. The equivalences (a) \Leftrightarrow (b) and (b) \Leftrightarrow (c) immediately follow from Theorem 4.4 and Lemma 4.1, respectively. Finally, the equivalence (a) \Leftrightarrow (d) follows from Lemma 3.12: indeed, the hypothesis $\lambda_\pi^\tau \sim \lambda_\pi$ guarantees that condition (2) therein cannot hold. \square

Corollary 4.6. *If $\tau(K) = K$ then the conditions (a)–(d) in Theorem 4.5 are all equivalent.*

Proof. This follows immediately from Corollary 4.2 and Theorem 4.5 since if K is τ -invariant then $\lambda_\pi^\tau \sim \lambda_\pi$. \square

In particular, if $\tau = \tau_{\text{inv}}$, then one has $\pi^\tau = \pi$ and the following result due to A. Garsia [29] (see also [13, Sect. 4.8] and [12, Lemma 2.3]) is an immediate consequence.

Corollary 4.7 (Symmetric Gelfand pairs: Garsia's criterion). *The following conditions are equivalent:*

- (a) every $(\pi \times \pi)$ -orbit of G on $X \times X$ is symmetric;
- (b) every double coset KsK is τ_{inv} -invariant;
- (c) (G, K) is a Gelfand pair and every irreducible subrepresentation of λ_π is real.

Corollary 4.8 (Weakly symmetric Gelfand pairs). *Suppose that*

$$(4.6) \quad g \in K\tau(g)K$$

for all $g \in G$. Then $\tau(K) = K$, $\lambda_\pi^\tau \sim \lambda_\pi$ and the conditions (a), (b), (c), and (d) in Theorem 4.5 are verified.

Proof. For $k \in K$, (4.6) becomes $k \in K\tau(k)K$, which implies $\tau(k) \in K$ and that K is τ -invariant. Then $\lambda_\pi^\tau \sim \lambda_\pi$ by Corollary 4.2 and (4.6) yields (c) in Theorem 4.5. \square

5. Simply reducible groups I: Mackey's criterion

Let G be a finite group. We recall that for $n \in \mathbb{N}$ we denote by $\widetilde{G}^n = \{(g, g, \dots, g) : g \in G\}$ the diagonal subgroup of $G^n = \underbrace{G \times G \times \dots \times G}_{n \text{ times}}$. Let $\tau : G \rightarrow$

G be an involutive anti-automorphism, as before. We extend it to an involutive anti-automorphism $\tau_n : G^n \rightarrow G^n$ in the obvious way, namely by setting $\tau_n(g_1, g_2, \dots, g_n) = (\tau(g_1), \tau(g_2), \dots, \tau(g_n))$ for all $g_1, g_2, \dots, g_n \in G$. Observe that $\tau_n(\widetilde{G}^n) = \widetilde{G}^n$.

Lemma 5.1. *Let (σ_i, V_i) , $i = 1, 2$, be G -representations and denote by (ι_G, \mathbb{C}) the trivial representation of G . For $T : V_1 \rightarrow V_2'$ define $\widetilde{T} : V_1 \otimes V_2 \rightarrow \mathbb{C}$ by setting $\widetilde{T}(v_1 \otimes v_2) = T(v_1)(v_2)$ for all $v_i \in V_i$, $i = 1, 2$. Then the map*

$$\begin{array}{ccc} \text{Hom}_G(\sigma_1, \sigma_2') & \longrightarrow & \text{Hom}_G(\sigma_1 \otimes \sigma_2, \iota_G), \\ T & \longmapsto & \widetilde{T} \end{array}$$

is a linear isomorphism. In particular,

$$\dim \text{Hom}_G(\sigma_1, \sigma_2') = \dim \text{Hom}_G(\sigma_1 \otimes \sigma_2, \iota_G)$$

so that, if σ_2 is irreducible, the multiplicity of σ_2' in σ_1 equals the multiplicity of ι_G in $\sigma_1 \otimes \sigma_2$.

Proof. We leave the simple proof to the reader. \square

Definition 5.2 (Mackey–Wigner). *One says that G is τ -simply reducible provided the following two conditions are satisfied:*

- (i) $\rho_1 \otimes \rho_2$ is multiplicity-free for all $\rho_1, \rho_2 \in \widehat{G}$;
- (ii) $\rho^\tau \sim \rho$ for all $\rho \in \widehat{G}$.

When $\tau = \tau_{\text{inv}}$ condition (ii) becomes

- (ii') $\rho' \sim \rho$ for all $\rho \in \widehat{G}$

and, provided that condition (i) and (ii') are both satisfied, one simply says that G is simply reducible.

Theorem 5.3 (Mackey's criterion for τ -simply reducible groups). *G is τ -simply reducible if and only if every double coset of \widetilde{G}^3 in G^3 is τ_3 -invariant. In particular, G is simply reducible if and only if every double coset of \widetilde{G}^3 in G^3 is invariant under the inverse involution $(\tau_{\text{inv}})_3$.*

Proof. We use the Mackey–Gelfand criterion (Theorem 4.5) with G^3 (resp. \widetilde{G}^3) in place of G (resp. K). Actually, we may apply Corollary 4.6 because $\tau_3(\widetilde{G}^3) = \widetilde{G}^3$. We now show that the present theorem is a particular case of the equivalence between (c) and (d) in the Mackey–Gelfand criterion. First of all, observe that condition (c) of Theorem 4.5, in the present setting, reads that every double coset of \widetilde{G}^3 in G^3 is τ_3 -invariant. Similarly, denoting by $\iota_{\widetilde{G}^3}$ the trivial representation of \widetilde{G}^3 , the first part of the equivalent condition (d) of the same theorem, reads that $\text{Ind}_{\widetilde{G}^3}^{G^3}(\iota_{\widetilde{G}^3})$, which (cf. [16, Example 1.6.4]) coincides with the permutation representation λ of G^3 on $L(G^3/\widetilde{G}^3)$, is multiplicity free. Let then $\rho_1, \rho_2, \rho_3 \in \widehat{G}$. Consider the representation $\rho_1 \boxtimes \rho_2 \boxtimes \rho_3 \in \widehat{G}^3$. By virtue of Frobenius' reciprocity (cf. [16, Theorem 1.6.11]) the multiplicity of $\rho_1 \boxtimes \rho_2 \boxtimes \rho_3$ in $\text{Ind}_{\widetilde{G}^3}^{G^3}(\iota_{\widetilde{G}^3})$ equals the multiplicity of $\iota_{\widetilde{G}^3}$ in $\rho_1 \otimes \rho_2 \otimes \rho_3 = \text{Res}_{\widetilde{G}^3}^{G^3}(\rho_1 \boxtimes \rho_2 \boxtimes \rho_3)$. By virtue of Lemma 5.1 (with σ_1 (resp. σ_2) now replaced by $\rho_1 \otimes \rho_2$ (resp. ρ_3)) the latter equals the multiplicity of ρ'_3 in $\rho_1 \otimes \rho_2$. Therefore $\text{Ind}_{\widetilde{G}^3}^{G^3}(\iota_{\widetilde{G}^3})$ is multiplicity free if and only if $\rho_1 \otimes \rho_2$ is multiplicity free for all $\rho_1, \rho_2 \in \widehat{G}$. This shows that condition (i) in Definition 5.2 is equivalent to the first part of (d).

Since $C_\tau(\rho_1 \boxtimes \rho_2 \boxtimes \rho_3) = C_\tau(\rho_1)C_\tau(\rho_2)C_\tau(\rho_3)$ (cf. Proposition 3.7), the second part of condition (d) in Theorem 4.5 holds if and only if $C_\tau(\rho_1)C_\tau(\rho_2)C_\tau(\rho_3) = 1$ whenever ρ'_3 is contained in $\rho_1 \otimes \rho_2$ (in particular, by Theorem 3.5 (1), we also have $\rho \sim \rho^\tau$ for all $\rho \in \widehat{G}$).

Now, if $\rho_1 \otimes \rho_2 = \text{Res}_{\widetilde{G}^2}^{G^2}(\rho_1 \boxtimes \rho_2)$ is multiplicity free, then by virtue of Theorem 3.11 we have that $C_\tau(\rho_1)C_\tau(\rho_2) = C_\tau(\rho_1 \boxtimes \rho_2)$ equals $C_\tau(\rho'_3)$

whenever ρ'_3 is contained in $\rho_1 \otimes \rho_2$. Since by Lemma 3.8 $C_\tau(\rho_3) = C_\tau(\rho'_3)$, we deduce that the condition

$$C_\tau(\rho_1 \boxtimes \rho_2 \boxtimes \rho_3) = 1 \quad \text{whenever} \quad \rho'_3 \preceq \rho_1 \otimes \rho_2$$

is equivalent to

$$C_\tau(\rho_3)^2 = 1 \quad \text{whenever} \quad \rho_3 \preceq \rho_1 \otimes \rho_2.$$

Now, $C_\tau(\rho_3)^2 = 1$ if and only if $C_\tau(\rho_3) = \pm 1$ which in turn is equivalent to the condition $\rho_3^\tau \sim \rho_3$, by virtue of Theorem 3.5 (1). Since the latter is nothing but condition (ii) in Definition 5.2, this ends the proof. \square

6. Simply reducible groups II: Mackey's generalizations of Wigner's criterion

Let G be a finite group and $\pi: G \rightarrow \text{Sym}(X)$ a (not necessarily transitive) action of G on a finite set X . As usual, we denote by $\mathcal{O}_G^\pi(X)$ the set of all G -orbits of X . Let $\alpha \in \text{Sym}(X)$ and suppose that $\mathcal{O}_G^\pi(X)$ is α -invariant, that is, $\alpha(\Omega) \in \mathcal{O}_G^\pi(X)$ for all $\Omega \in \mathcal{O}_G^\pi(X)$. Note that this condition is always satisfied whenever α commutes with π , namely, $\alpha\pi(g) = \pi(g)\alpha$ for all $g \in G$. Indeed, in this case, denoting by $\Omega_x = \{\pi(g)x : g \in G\} \in \mathcal{O}_G^\pi(X)$ the orbit of a point $x \in X$, we have $\alpha(\Omega_x) = \Omega_{\alpha x}$. We denote by $\mathcal{O}_G^\pi(X)^\alpha = \{\Omega \in \mathcal{O}_G^\pi(X) : \alpha(\Omega) = \Omega\}$ the set of orbits which are (globally) fixed by α .

The following generalization of the classical Cauchy–Frobenius–Burnside lemma (*cf.* [13, Lemma 3.11.1]) is due to Mackey.

Lemma 6.1. *Setting $p(g) = |\{x \in X : \pi(g)x = \alpha x\}|$ for all $g \in G$, we have*

$$\frac{1}{|G|} \sum_{g \in G} p(g) = |\mathcal{O}_G^\pi(X)^\alpha|.$$

Proof. Let $x \in X$ and set $q(x) = |\{g \in G : \pi(g)x = \alpha x\}|$. Note that if $\Omega_x \notin \mathcal{O}_G^\pi(X)^\alpha$, then $q(x) = 0$. Indeed, since $\alpha(\Omega_x) \cap \Omega_x = \emptyset$, there is no $g \in G$ such that $\pi(g)x \in \Omega_x$ equals $\alpha x \in \alpha(\Omega_x)$. On the other hand, suppose that $\Omega_x \in \mathcal{O}_G^\pi(X)^\alpha$. If $g \in G$ satisfies $\pi(g)x = \alpha x$, then gk satisfies the same condition for all $k \in \text{Stab}_G^\pi(x)$; also if $g_1, g_2 \in G$ satisfy $\pi(g_1)x = \alpha x = \pi(g_2)x$ we deduce that $g_1^{-1}g_2 \in \text{Stab}_G^\pi(x)$. This shows that $q(x) = |\text{Stab}_G^\pi(x)| = \frac{|G|}{|\Omega_x|}$. We then have

$$\begin{aligned}
\sum_{g \in G} p(g) &= |\{(x, g) \in X \times G : \pi(g) = \alpha x\}| \\
&= \sum_{x \in X} q(x) \\
&= \sum_{\Omega \in \mathcal{O}_G^\pi(X)^\alpha} \sum_{x \in \Omega} \frac{|G|}{|\Omega|} \\
&= |G| \cdot |\mathcal{O}_G^\pi(X)^\alpha|.
\end{aligned}$$

□

The classical Cauchy–Frobenius–Burnside lemma corresponds to the case when α is the identity map (so that, in this case, $p(g)$, $g \in G$, equals the number of fixed points of $\pi(g)$).

Let $n \geq 1$ and denote by $\pi_{n+1}: G^{n+1} \rightarrow \text{Sym}(G^n)$ the action defined by

$$\begin{aligned}
&\pi_{n+1}(g_1, g_2, \dots, g_n, g_{n+1})(h_1, h_2, \dots, h_n) \\
&= (g_1 h_1 g_{n+1}^{-1}, g_2 h_2 g_{n+1}^{-1}, \dots, g_n h_n g_{n+1}^{-1})
\end{aligned}$$

for all $g_1, g_2, \dots, g_n, g_{n+1}, h_1, h_2, \dots, h_n \in G$. Note that π_{n+1} is transitive and that the stabilizer of $(1_G, 1_G, \dots, 1_G) \in G^{n+1}$ is given by

$$\text{Stab}_{G^{n+1}}^{\pi_{n+1}}(1_G, 1_G, \dots, 1_G) = \widetilde{G}^{n+1}.$$

It follows that

$$(6.1) \quad G^{n+1} / \widetilde{G}^{n+1} \cong G^n$$

as homogeneous spaces. We also denote by $\gamma_n: G \rightarrow \text{Sym}(G^n)$ the conjugacy action of G on G^n given by

$$\gamma_n(g)(g_1, g_2, \dots, g_n) = (g g_1 g^{-1}, g g_2 g^{-1}, \dots, g g_n g^{-1})$$

for all $g, g_1, g_2, \dots, g_n \in G$. Denoting by $\varepsilon_{n+1}: G \rightarrow \widetilde{G}^{n+1}$ the natural bijection given by $\varepsilon_{n+1}(g) = (g, g, \dots, g)$, we have

$$\gamma_n = \pi_{n+1}|_{\widetilde{G}^{n+1}} \circ \varepsilon_{n+1}.$$

In other words, γ_n coincides with the action of \widetilde{G}^{n+1} on G^n and therefore (see Remark 4.3) the map

$$(6.2) \quad \begin{array}{ccc} \widetilde{G}^{n+1} \backslash G^{n+1} / \widetilde{G}^{n+1} & \longrightarrow & \mathcal{O}_G^{\gamma_n}(G^n), \\ \widetilde{G}^{n+1}(g_1, g_2, \dots, g_n, g_{n+1}) \widetilde{G}^{n+1} & \longmapsto & \Omega_{(g_1 g_{n+1}^{-1}, g_2 g_{n+1}^{-1}, \dots, g_n g_{n+1}^{-1})} \end{array}$$

is well-defined and bijective. Indeed, the orbit corresponding to the double coset of $(g_1, g_2, \dots, g_n, g_{n+1}) \in G^{n+1}$ contains the element

$$\begin{aligned} & \pi_{n+1}(g_1, g_2, \dots, g_{n+1})(1_G, 1_G, \dots, 1_G) \\ &= (g_1 g_{n+1}^{-1}, g_2 g_{n+1}^{-1}, \dots, g_n g_{n+1}^{-1}). \end{aligned}$$

For every $g \in G$, we now denote by $v(g)$ the cardinality of the centralizer of g in G , that is, the number of elements $h \in G$ such that $hg = gh$. We then have

Theorem 6.2. *Let $n \geq 1$. The following quantities are all equal:*

- (a) $\frac{1}{|G|} \sum_{g \in G} v(g)^n$;
- (b) *the number of double \widetilde{G}^{n+1} cosets in G^{n+1} ;*
- (c) *the number of G -orbits on G^n with respect to the action γ_n .*

Proof. We first observe that the cardinality of the centralizer of $(g, g, \dots, g) \in \widetilde{G}^n$ in G^n is $v(g)^n$. Moreover, this is equal to the cardinality of the set of fixed points of $\gamma_n(g)$ in G^n . Then, applying Lemma 6.1 with $X = G^n$, $G = \widetilde{G}^n$, $\pi = \gamma_n$, and $\alpha = \text{Id}_{G^n}$, we deduce the equality between (a) and (c) (actually, we have used the classical Cauchy–Frobenius–Burnside formula). Finally, the equality between (b) and (c) directly follows from the bijection (6.2). \square

Let τ be, as usual, an involutory anti-automorphism of G . For every $g \in G$, we denote by $\zeta_\tau(g)$ the number of elements $h \in G$ such that $\tau(h^{-1})h = g$; in formulae

$$(6.3) \quad \zeta_\tau(g) = |\{h \in G : \tau(h^{-1})h = g\}|.$$

Note that if $\tau = \tau_{\text{inv}}$ then, simply writing $\zeta(g)$ instead of $\zeta_{\tau_{\text{inv}}}(g)$, we have that $\zeta(g) = |\{h \in G : h^2 = g\}|$. As before, we denote by $\tau_n: G^n \rightarrow G^n$ its natural extension. Since \widetilde{G}^n is τ_n -invariant, the set $\widetilde{G}^n \backslash G^n / \widetilde{G}^n$ of double \widetilde{G}^n cosets in G^n is also τ_n -invariant. Indeed, $\tau_n(\widetilde{G}^n(g_1, g_2, \dots, g_n)\widetilde{G}^n) = \widetilde{G}^n(\tau(g_1), \tau(g_2), \dots, \tau(g_n))\widetilde{G}^n$, for all $g_1, g_2, \dots, g_n \in G$. As a consequence, τ_n induces a permutation of the double \widetilde{G}^n cosets in G^n ; we shall call the corresponding fixed points τ_n -invariant double cosets. Similarly, the set $\mathcal{O}_G^{\gamma_n}(G^n)$ of all γ_n orbits of $G \cong \widetilde{G}^n$ on G^n is τ_n -invariant. Therefore, τ_n induces a permutation of such orbits whose fixed points we shall call τ_n -invariant γ_n -orbits.

Theorem 6.3. *Let $n \geq 1$. The following quantities are all equal:*

- (a) $\frac{1}{|G|} \sum_{g \in G} \zeta_\tau(g)^{n+1}$;
- (b) *the number of τ_n -invariant double \widetilde{G}^{n+1} cosets in G^{n+1} ;*
- (c) *the number of τ_n -invariant γ_n -orbits on G^n .*

Proof. We start by proving the equality between (b) and (c). It suffices to show that the bijective correspondence (6.2) transforms τ_n -invariant double cosets into τ_n -invariant orbits. Now, the double coset containing the element (g_1, g_2, \dots, g_n) is τ_n -invariant if and only if

$$(6.4) \quad \exists h_1, h_2 \in G \text{ such that } \tau(g_i) = h_1 g_i h_2 \text{ for all } i = 1, 2, \dots, n+1$$

while, the orbit containing $(g_1 g_{n+1}^{-1}, g_2 g_{n+1}^{-1}, \dots, g_n g_{n+1}^{-1})$ is τ_n -invariant if and only if

$$(6.5) \quad \exists h_3 \in G \text{ such that } \tau(g_{n+1})^{-1} \tau(g_i) = h_3 g_i g_{n+1}^{-1} h_3^{-1} \text{ for all } i = 1, 2, \dots, n.$$

Let us show that conditions (6.4) and (6.5) are both equivalent to

$$(6.6) \quad \exists h_4 \in G \text{ such that } \tau(g_{n+1})^{-1} \tau(g_i) = h_4 g_{n+1}^{-1} g_i h_4^{-1} \text{ for all } i = 1, 2, \dots, n.$$

Indeed, (6.4) implies (6.6) by taking $h_4 = h_2^{-1}$, while if (6.6) holds we have

$$\tau(g_1) h_4 g_1^{-1} = \tau(g_2) h_4 g_2^{-1} = \dots = \tau(g_{n+1}) h_4 g_{n+1}^{-1}$$

and therefore (6.4) holds with $h_1 = \tau(g_{n+1}) h_4 g_{n+1}^{-1}$ and $h_2 = h_4^{-1}$. Finally the equivalence between (6.5) and (6.6) trivially follows from the identity

$$h_4 g_{n+1}^{-1} g_i h_4^{-1} = (h_4 g_{n+1}^{-1}) g_i g_{n+1}^{-1} (h_4 g_{n+1}^{-1})^{-1}$$

which shows that the relation between h_3 and h_4 is simply given by $h_3 = h_4 g_{n+1}^{-1}$. This completes the proof of the equality of (b) and (c).

We now turn to prove that (a) equals (c). For $g \in G$ let us set

$$p_n(g)$$

$$= |\{(g_1, g_2, \dots, g_n) \in G^n : \gamma_n(g)(g_1, g_2, \dots, g_n) = \tau_n(g_1, g_2, \dots, g_n)\}|$$

so that, in particular, $p_1(g) = |\{h \in G : ghg^{-1} = \tau(h)\}|$. It is obvious that $p_n(g) = p_1(g)^n$. Moreover, we have $ghg^{-1} = \tau(h)$ if and only if $\tau(g)^{-1}g = \tau(gh^{-1})^{-1}gh^{-1}$ and the number of elements $h \in G$ satisfying the latter identity equals the number of elements $u \in G$ such that $\tau(g)^{-1}g = \tau(u)^{-1}u$ (just take $u = gh^{-1}$). In other words, we have $p_1(g) = \zeta_\tau(\tau(g)^{-1}g)$. Thus

$$\begin{aligned} \sum_{g \in G} p_n(g) &= \sum_{g \in G} p_1(g)^n \\ &= \sum_{g \in G} \zeta_\tau(\tau(g)^{-1}g)^n \\ &= \sum_{t \in G} \zeta_\tau(t)^n |\{g \in G : \tau(g)^{-1}g = t\}| \\ &= \sum_{t \in G} \zeta_\tau(t)^{n+1}. \end{aligned}$$

Then the equality of (a) and (c) follows from Lemma 6.1 with $X = G^n$, $G = \widetilde{G}^n$, $\pi = \gamma_n$, $\alpha = \tau_n$ and, obviously, $p = p_n$. \square

Recall that a conjugacy class is *ambivalent* when it is invariant with respect to τ_{inv} . Moreover the group is *ambivalent* when every conjugacy class is ambivalent, equivalently, every element is conjugate to its inverse. Then with the above notation, when $n = 1$ we have:

Corollary 6.4. $\frac{1}{|G|} \sum_{g \in G} \zeta_\tau(g)^2$ equals the number of τ -invariant conjugacy classes of G . In particular (when $\tau = \tau_{\text{inv}}$), $\frac{1}{|G|} \sum_{g \in G} \zeta(g)^2$ equals the number of ambivalent conjugacy classes of G .

A more complete formulation of Corollary 6.4 will be given in Theorem 8.4. Also, from Theorem 6.2 and Theorem 6.3 we deduce:

Corollary 6.5. *We have*

$$\sum_{g \in G} \zeta_\tau(g)^{n+1} \leq \sum_{g \in G} v(g)^n.$$

Moreover equality holds if and only if every double \widetilde{G}^{n+1} coset in G^{n+1} is τ_n -invariant (equivalently, if and only if every γ_n -orbit of G on G^n is τ_n -invariant).

When $n = 2$ Theorem 5.3 yields the following remarkable criterion.

Corollary 6.6 (Mackey–Wigner criterion for simple reducibility). *The group G is τ -simply reducible if and only if*

$$(6.7) \quad \sum_{g \in G} \zeta_\tau(g)^3 = \sum_{g \in G} v(g)^2.$$

In particular (Wigner's criterion), G is simply reducible if and only if

$$\sum_{g \in G} \zeta(g)^3 = \sum_{g \in G} v(g)^2.$$

We now examine in detail the case $n = 1$.

Theorem 6.7. *The following conditions are equivalent:*

- (a) $\sum_{g \in G} \zeta_\tau(g)^2 = \sum_{g \in G} v(g)$;
- (b) every conjugacy class of G is τ -invariant;
- (c) $\rho \sim \rho^\tau$ for every irreducible representation ρ of G .

Proof. The equivalence (a) \Leftrightarrow (b) is a particular case of Corollary 6.4 since $\frac{1}{|G|} \sum_{g \in G} v(g)$ equals the number of conjugacy classes of G (cf. Theorem 6.2).

Recall that the permutation representation $L(G) = L(G^2/\widetilde{G}^2)$ equals the induced representation $\text{Ind}_{\widetilde{G}^2}^{G^2} \iota_{\widetilde{G}^2}$. Moreover this representation is multiplicity free (i.e., (G^2, \widetilde{G}^2) is a Gelfand pair) and its decomposition into irreducibles is:

$$(6.8) \quad \text{Ind}_{\widetilde{G}^2}^{G^2} \iota_{\widetilde{G}^2} \sim \bigoplus_{\rho \in \widehat{G}} (\rho' \boxtimes \rho);$$

see [13, Sect. 9.5] and [16, Corollary 2.16]. We now observe that $(\rho')^\tau = (\rho^\tau)'$: indeed $\rho' = \rho^{\tau_{\text{inv}}}$ and since τ and τ_{inv} commute, (3.7) holds. Moreover since $\widehat{G} = \{\rho^\tau : \rho \in \widehat{G}\}$, from (6.8) we deduce

$$(6.9) \quad \text{Ind}_{\widetilde{G}^2}^{G^2} \iota_{\widetilde{G}^2} \sim \bigoplus_{\rho \in \widehat{G}} ((\rho')^\tau \boxtimes \rho^\tau).$$

By virtue of Proposition 3.7 and Lemma 3.8 we have

$$C_\tau((\rho')^\tau \boxtimes \rho^\tau) = C_\tau((\rho')^\tau) C_\tau(\rho^\tau) = C_\tau(\rho)^2.$$

Since by Theorem 3.5 $\rho \sim \rho^\tau$ if and only if $C_\tau(\rho) = \pm 1$, we deduce that this holds if and only if $C_\tau((\rho')^\tau \boxtimes \rho^\tau) = 1$. As a consequence, the equivalence (b) \Leftrightarrow (c) follows from the Mackey–Gelfand criterion (Theorem 4.5) applied to (6.9), also taking into account the equality of the quantities (b) and (c) in Theorem 6.3. \square

The remaining part of this section is devoted to the analysis of the consequences when equality occurs in Corollary 6.5 for $n \geq 3$. We need two auxiliary lemmas.

Lemma 6.8. *If there exists a positive integer n_0 such that $\sum_{g \in G} \zeta_\tau(g)^{n_0+1} = \sum_{g \in G} v(g)^{n_0}$ then we also have $\sum_{g \in G} \zeta_\tau(g)^{n+1} = \sum_{g \in G} v(g)^n$ for all $n \leq n_0$.*

Proof. By Theorem 6.2 and Theorem 6.3 we have $\sum_{g \in G} \zeta_\tau(g)^{n_0+1} = \sum_{g \in G} v(g)^{n_0}$ if and only if every γ_{n_0} -orbit on G^{n_0} is τ_{n_0} -invariant. This is equivalent to saying that for each choice of $g_1, g_2, \dots, g_{n_0} \in G$ there exists $g \in G$ such that $\tau(g_i) = gg_i g^{-1}$, $i = 1, 2, \dots, n_0$. But this implies the τ_n -invariance of the γ_n -orbits also for all $n \leq n_0$ and another application of the two above mentioned theorems completes the proof. \square

Lemma 6.9. *Let σ be a representation. Suppose that $\sigma \otimes \sigma'$ contains the trivial representation exactly once. Then σ is irreducible.*

Proof. Let $\sigma = \bigoplus_{i=1}^m \rho_i$ denote the decomposition into irreducibles of σ . By applying Frobenius reciprocity to (6.8) we deduce that each $\rho_i \otimes \rho'_i$ contains the trivial representation exactly once. Then $\sigma \otimes \sigma'$, which contains $\rho_i \otimes \rho'_i$ for all $i = 1, 2, \dots, m$, also contains at least m copies of the trivial representation. By our assumptions, this forces $m = 1$, yielding the irreducibility of σ . \square

Theorem 6.10. *The following conditions are equivalent:*

- (a) *There exists an integer $n \geq 3$ such that $\sum_{g \in G} \zeta_\tau(g)^{n+1} = \sum_{g \in G} v(g)^n$ for all $g \in G$;*
- (b) *For all $n \geq 1$ and $g \in G$ we have $\sum_{g \in G} \zeta_\tau(g)^{n+1} = \sum_{g \in G} v(g)^n$;*
- (c) *The group G is abelian and τ is the identity.*

Proof. Clearly, (b) implies (a) and, by Corollary 6.5, (c) implies (b). Now assume (a). By Lemma 6.8 the identity is verified also for $n = 3$ and therefore Corollary 6.5 ensures that the double \widetilde{G}^4 cosets in G^4 is τ_3 -invariant. Then we may apply the Mackey–Gelfand criterion (Theorem 5.3) deducing that (G^4, \widetilde{G}^4) is a Gelfand pair. It follows that if $\rho, \sigma, \theta, \xi \in \widehat{G}$ then $\rho \boxtimes \sigma \boxtimes \theta \boxtimes \xi$ is G^4 -irreducible and its multiplicity in $\text{Ind}_{G^4}^{G^4} \iota_{\widetilde{G}^4}$ is either 0 or 1. In particular, the representation

$$(6.10) \quad \text{Res}_{\widetilde{G}^4}^{G^4} [(\rho \boxtimes \sigma) \boxtimes (\rho' \boxtimes \sigma')]$$

(which, modulo the identification of \widetilde{G}^4 and G , is equivalent to the G -representation $\rho \otimes \sigma \otimes \rho' \otimes \sigma'$) contains the trivial representation at most once. Therefore, by Lemma 6.9, $\rho \otimes \sigma$ is G -irreducible. Note that this holds for all $\rho, \sigma \in \widehat{G}$. In particular, $\rho \otimes \rho'$ is irreducible and contains the trivial representation. Thus $\rho \otimes \rho' \sim \iota_G$ forcing ρ to be one-dimensional. It follows that G is abelian (see [13, Exercise 3.9.11 or Sect. 9.2]).

Moreover, by (6.8) (with G^2 in place of G , so that $(\widetilde{G}^2)^2 = \{(g, h, g, h) : g, h \in G\}$), the representation

$$\text{Res}_{(\widetilde{G}^2)^2}^{G^4} [(\rho \boxtimes \sigma) \boxtimes (\rho' \boxtimes \sigma')]$$

contains the trivial $(\widetilde{G}^2)^2$ -representation exactly once. By further restricting to the subgroup \widetilde{G}^4 we deduce that (6.10) contains the trivial \widetilde{G}^4 -representation exactly once.

By Frobenius reciprocity, this implies that $(\rho \boxtimes \sigma) \boxtimes (\rho' \boxtimes \sigma')$ is contained in $\text{Ind}_{\widetilde{G}^4}^{G^4} \iota_{\widetilde{G}^4}$ (with multiplicity one). Then Theorem 4.5 (d) ensures that

$$1 = C_\tau((\rho \boxtimes \sigma) \boxtimes (\rho' \boxtimes \sigma')) = C_\tau(\rho)^2 C_\tau(\sigma)^2,$$

where the second equality follows from Proposition 3.7 and Corollary 3.9. This implies $C_\tau(\rho) = \pm 1$ so that Theorem 3.5 (1) ensures the equivalence $\rho \sim \rho^\tau$,

for all $\rho \in \widehat{G}$. Since all these representations are one-dimensional, the latter means exactly that $\rho(g) = \rho(\tau(g))$ for all $g \in G$ and $\rho \in \widehat{G}$. Since G is abelian (so that \widehat{G} separates the elements in G), this in turn gives $\tau = \text{Id}_G$. \square

Note that if in the preceding theorem τ is the inversion, then G must be a direct product of cyclic groups of order two.

7. An example: the Clifford groups

In this section, as an application of Mackey's criterion (Theorem 5.3) and, independently, of the Mackey–Wigner criterion (Corollary 6.6), we show that the Clifford groups $\mathbb{C}\mathbb{L}(n)$ are τ -simply reducible (where the involutive anti-automorphism τ of $\mathbb{C}\mathbb{L}(n)$ is suitably defined according to the congruence class of n modulo 4).

The *Clifford group* $\mathbb{C}\mathbb{L}(n)$, $n \geq 1$, is the group generated by the elements $\varepsilon, \gamma_1, \gamma_2, \dots, \gamma_n$ with defining relations (called *Clifford relations*)

$$(7.1) \quad \begin{aligned} \varepsilon^2 &= 1, \\ \gamma_i^2 &= 1, \\ \gamma_i \gamma_j &= \varepsilon \gamma_j \gamma_i \end{aligned}$$

for all $i, j = 1, 2, \dots, n$ such that $i \neq j$. For $n = 1$ one should also add the relation

$$\varepsilon \gamma_1 = \gamma_1 \varepsilon$$

(Note that for $n \geq 2$ the relations $\varepsilon \gamma_i = \gamma_i \varepsilon$, $i = 1, 2, \dots, n$, are easily deduced from (7.1)).

The *Clifford–Littlewood–Eckmann group* $G_{s,t}$, $s, t \in \mathbb{N}$, is the group with generators $\varepsilon, a_1, a_2, \dots, a_s, b_1, b_2, \dots, b_t$ and the following defining relations: $\varepsilon^2 = 1$; $a_i^2 = \varepsilon$, $b_j^2 = 1$, $b_j \varepsilon = \varepsilon b_j$, and $a_i b_j = \varepsilon b_j a_i$ for all i, j ; and $a_i a_j = \varepsilon a_j a_i$ and $b_i b_j = \varepsilon b_j b_i$ for all $i \neq j$ (see [55]). It can be easily shown that $G_{s,t}$ is a finite group of order 2^{s+t} . Note also that $G_{0,n} = \mathbb{C}\mathbb{L}(n)$ for all $n \in \mathbb{N}$.

The groups $G_{s,t}$ are implicit in W.K. Clifford's work on "geometric algebra" [19]. Indeed, $G_{s,t}$ appears naturally as a subgroup of the group of units of the Clifford algebra $C(\varphi_{s,t})$ of the quadratic form $\varphi_{s,t} := s\langle -1 \rangle \perp t\langle 1 \rangle$ over any field of characteristic $\neq 2$. They were explicitly defined by D.E. Littlewood [56] in 1934. These groups are of great interest to theoretical physicists. For example, $G_{0,3} = \mathbb{C}\mathbb{L}(3)$ is the group generated by the three (Hermitian) *Pauli spin matrices* (coming from the commutation relations between angular momentum operators in the study of the spin of the electron) and $G_{0,4} = \mathbb{C}\mathbb{L}(4)$ is the *Dirac group*, generated by the four (Hermitian) *Dirac matrices* (defined by Dirac [22] in his study of the relativistic wave equation). More generally,

the groups $G_{0,2n} = \mathbb{C}\mathbb{L}(2n)$, $n \geq 1$, arise naturally in quantum field theory (e.g. in the theory of Fermion fields): originally they were introduced by Jordan and Wigner in their paper on *Pauli's Exclusion Principle* [45]. Using Frobenius–Burnside theory of finite group representations (cf. [13, Sect. 3.11]) they determined all irreducible representations of these groups: apart the 2^n one-dimensional representations, $G_{0,2n}$ has only one irreducible representation (of dimension 2^n).

For the sake of completeness, we mention that the groups $G_{s,0}$, $s \geq 1$, are also important to physicists. For instance, they were studied by Jordan, von Neumann and Wigner [44] in connection with their algebraic formalism for the mathematical foundations of quantum mechanics. Eddington [24, 25], in his studies in astrophysics, considered sets of anticommuting matrices and complex representations of the group $G_{5,0}$. Note also that the groups $G_{s,0}$ play an important role in connection with the *Hurwitz problem* of composition of quadratic forms [37, 38, 67]. Indeed, Eckmann [23] rediscovered these groups and observed that a set of solutions to the Hurwitz equations over a field \mathbb{F} corresponds to an n -dimensional orthogonal representation ρ of $G_{s,0}$ satisfying $\rho(\varepsilon) = -I_n$. Then, Eckmann determined all irreducible orthogonal representations of $G_{s,0}$ over the real field \mathbb{R} and deduced a purely group theoretical proof of the Hurwitz–Radon theorem on the composition of sums of squares.

Returning back to our investigations, we shall make use of the following alternative description of the Clifford groups (cf. [72, Chapter 4]). Setting $X = \{1, 2, \dots, n\}$, we have $\mathbb{C}\mathbb{L}(n) = \{\pm\gamma_A : A \subseteq X\}$ with multiplication given by

$$(7.2) \quad \varepsilon_1 \gamma_A \cdot \varepsilon_2 \gamma_B = \varepsilon_1 \varepsilon_2 (-1)^{\xi(A,B)} \gamma_{A \Delta B}$$

where Δ denotes the symmetric difference of two sets and $\xi(A, B)$ equals the number of elements $(a, b) \in A \times B$ such that $a > b$, for all $\varepsilon_1, \varepsilon_2 \in \{1, -1\}$ and $A, B \subseteq X$. Notice that the identity element is given by $1_{\mathbb{C}\mathbb{L}(n)} = \gamma_\emptyset$ and that $(\varepsilon \gamma_A)^{-1} = \varepsilon (-1)^{\frac{|A|(|A|-1)}{2}} \gamma_A$ for all $\varepsilon = \pm 1$ and $A \subseteq X$.

Consider now the map $\tau' : \mathbb{C}\mathbb{L}(n) \rightarrow \mathbb{C}\mathbb{L}(n)$ defined by

$$(7.3) \quad \tau'(\varepsilon \gamma_A) = \varepsilon (-1)^{\frac{|A|(|A|+1)}{2}} \gamma_A$$

and note that $\tau' = \tau_{\text{inv}} \circ \tau''$ where

$$\tau''(\varepsilon \gamma_A) = \varepsilon (-1)^{|A|} \gamma_A$$

for all $\varepsilon = \pm 1$ and $A \subseteq X$. It is straightforward to check that τ'' is an involutive automorphism of $\mathbb{C}\mathbb{L}(n)$ so that τ' is an involutive anti-automorphism of $\mathbb{C}\mathbb{L}(n)$. We then set

$$\tau = \begin{cases} \tau' & \text{if } n \equiv 3 \pmod{4}, \\ \tau_{\text{inv}} & \text{otherwise.} \end{cases}$$

Theorem 7.1. *The group $\mathbb{C}\mathbb{L}(n)$ is τ -simply reducible.*

Proof. We present two different proofs: the first one making use of the Mackey criterion (Theorem 5.3) and the second one based on the Mackey–Wigner criterion (Corollary 6.6).

First proof. Set $G = \mathbb{C}\mathbb{L}(n)$ and let us check that every double coset of \widetilde{G}^3 in G^3 is τ_3 -invariant. Consider the action π of G^3 on G^2 given by

$$\pi(g_1, g_2, g_3)(h_1, h_2) = (g_1 h_1 g_3^{-1}, g_2 h_2 g_3^{-1})$$

for all $g_1, g_2, g_3, h_1, h_2 \in G$. Then the stabilizer of the element $(1_G, 1_G) \in G^2$ is exactly \widetilde{G}^3 and the double cosets of \widetilde{G}^3 in G^3 coincide with the \widetilde{G}^3 -orbits of G^2 under the action π . Let $A, C \subseteq X$. Then

$$\begin{aligned} \gamma_C^{-1} \gamma_A \gamma_C &= (-1)^{\frac{|C|(|C|-1)}{2} + \xi(A, C) + \xi(C, A \Delta C)} \gamma_{C \Delta (A \Delta C)} \\ (7.4) \quad &= (-1)^{\frac{|C|(|C|-1)}{2} + \xi(A, C) + \xi(C, A) + \xi(C, C)} \gamma_A \\ &= {}_* (-1)^{|A||C| - |A \cap C|} \gamma_A \end{aligned}$$

where $=_*$ follows from the fact that $\xi(C, C) = \frac{|C|(|C|-1)}{2}$. From (7.4), a case-by-case analysis yields the τ_3 -invariance of all \widetilde{G}^3 -orbits. As an example, consider the \widetilde{G}^3 -orbit of the element (γ_X, γ_X) . Suppose first that n is odd. Then

$$\pi(\gamma_C, \gamma_C, \gamma_C)(\gamma_X, \gamma_X) = ((-1)^{(n-1)|C|} \gamma_X, (-1)^{(n-1)|C|} \gamma_X) = (\gamma_X, \gamma_X)$$

for all $C \subseteq X$. Therefore the \widetilde{G}^3 -orbit of (γ_X, γ_X) reduces to one point. Now, if $n \equiv 1 \pmod{4}$ we have $\tau(\gamma_X) = \gamma_X^{-1} = \gamma_X$, while if $n \equiv 3 \pmod{4}$ we have $\tau(\gamma_X) = -\gamma_X^{-1} = \gamma_X$. Similarly, when n is even we have that the \widetilde{G}^3 -orbit of (γ_X, γ_X) is $\{(\gamma_X, \gamma_X), (-\gamma_X, -\gamma_X)\}$ and $\tau(\gamma_X) = \gamma_X^{-1} = (-1)^{\frac{n(n-1)}{2}} \gamma_X$.

Thus, since all double cosets of \widetilde{G}^3 in G^3 are τ_3 -invariant, from the Mackey criterion (Theorem 5.3) we deduce that $\mathbb{C}\mathbb{L}(n)$ is τ -simply reducible.

Second proof. We start by computing the Right Hand Side in (6.7) where $G = \mathbb{C}\mathbb{L}(n)$. First note that for $A \subseteq X$ we have,

$$(7.5) \quad v(\gamma_A) = 2^n.$$

Indeed, by virtue of (7.4), we have

$$v(\gamma_A) = |\{\varepsilon \gamma_C : \varepsilon = \pm 1, |C \cap A| \text{ is even}\}| = 2 \cdot 2^{|A|-1} \cdot 2^{n-|A|} = 2^n$$

if $|A|$ is even, and

$$\begin{aligned} v(\gamma_A) &= |\{\varepsilon\gamma_C : \varepsilon = \pm 1, |C \cap A| \text{ is even and } |C| \text{ is even}\}| \\ &\quad + |\{\varepsilon\gamma_C : \varepsilon = \pm 1, |C \cap A| \text{ is odd and } |C| \text{ is odd}\}| \\ &= 2 \cdot 2^{|A|-1} \cdot 2^{n-|A|-1} + 2 \cdot 2^{|A|-1} \cdot 2^{n-|A|-1} \\ &= 2^n \end{aligned}$$

if $|A|$ is odd. We then have

$$\begin{aligned} \sum_{g \in G} v(g)^2 &= \sum_{A \subseteq X} v(\gamma_A)^2 + \sum_{A \subseteq X} v(-\gamma_A)^2 \\ &= 2 \sum_{A \subseteq X} v(\gamma_A)^2 \quad (\text{since } v(-\gamma_A) = v(\gamma_A)) \\ (7.6) \quad &= 2 \sum_{A \subseteq X} 2^{2n} \quad (\text{by (7.5)}) \\ &= 2^{3n+1}. \end{aligned}$$

We now compute the Left Hand Side in (6.7). First observe that by (7.2) we have that $\zeta_\tau(g) = 0$ if (and only if) $g \neq \pm 1_G$. Suppose first that $n \equiv 3 \pmod{4}$. Then we have

$$\begin{aligned} \zeta_\tau(1_G) &= |\{\varepsilon\gamma_C : \varepsilon = \pm 1 \text{ and } \tau(\varepsilon\gamma_C)^{-1}\varepsilon\gamma_C = 1_G\}| \\ &= 2|\{\gamma_C : \tau(\gamma_C)^{-1}\gamma_C = 1_G\}| \\ &= 2|\{\gamma_C : (-1)^{|C|}\gamma_C\gamma_C = 1_G\}| \\ &= 2|\{\gamma_C : (-1)^{\frac{|C|(|C|+1)}{2}} 1_G = 1_G\}| \\ &= 2|\{\gamma_C : |C|(|C|+1) \equiv 0, 3 \pmod{4}\}| \\ &= 2 \cdot 2^{n-1} = 2^n. \end{aligned}$$

Analogously, one has

$$\zeta_\tau(-1_G) = 2|\{\gamma_C : |C|(|C|+1) \equiv 1, 2 \pmod{4}\}| = 2^n$$

so that, all together,

$$(7.7) \quad \sum_{g \in G} \zeta_\tau(g)^3 = \zeta_\tau(1_G)^3 + \zeta_\tau(-1_G)^3 = 2^{3n+1}.$$

On the other hand, if $n \equiv 0, 1, 2 \pmod{4}$ we have

$$\zeta_\tau(1_G) = 2|\{\gamma_C : |C|(|C|-1) \equiv 0, 1 \pmod{4}\}| = 2^n$$

and

$$\zeta_\tau(-1_G) = 2|\{\gamma_C : |C|(|C|-1) \equiv 2, 3 \pmod{4}\}| = 2^n$$

thus showing that (7.7) holds also in this case. Comparing (7.6) and (7.7), from the Mackey–Wigner criterion (Corollary 6.6) we deduce that $\mathbb{C}\mathbb{L}(n)$ is τ -simply reducible. \square

Remark 7.2. Let $\rho, \sigma \in \widehat{\mathbb{C}\mathbb{L}(n)}$. In [18] we give an explicit decomposition of the tensor product $\rho \otimes \sigma$. According to Theorem 7.1, this is multiplicity free and, moreover, $\rho \sim \rho^\tau$.

8. The twisted Frobenius–Schur theorem

Let N be a finite group, $\tau: N \rightarrow N$ an involutory anti-automorphism, and denote by $\alpha \in \text{Aut}(N)$ the involutory automorphism defined by $\alpha(n) = \tau(n^{-1})$ for all $n \in N$. Consider the semi-direct product

$$(8.1) \quad G = N \rtimes_{\alpha} \langle \alpha \rangle.$$

In other words, $G = \{(n, \alpha^\varepsilon) : n \in N, \varepsilon \in \{0, 1\}\}$ and

$$(n, \alpha^\varepsilon)(n', \alpha^{\varepsilon'}) = (n\alpha^\varepsilon(n'), \alpha^{\varepsilon+\varepsilon'})$$

for all $n, n' \in N$ and $\varepsilon, \varepsilon' \in \{0, 1\}$. If we identify N with the normal subgroup $\{(n, \alpha^0) : n \in N\}$ and we set $h = (1_N, \alpha)$, then G is generated by N and h and the following relations hold: $h^2 = 1$ and $hnh = \tau(n)^{-1}$, for all $n \in N$. We then have the coset decomposition $G = N \amalg hN$. Moreover, we can define the *alternating representation* of G (with respect to N) as the one-dimensional representation $(\varepsilon, \mathbb{C})$ defined by

$$\varepsilon(g) = \begin{cases} 1 & \text{if } g \in N, \\ -1 & \text{otherwise.} \end{cases}$$

We define two actions of $C_2 = \{1, -1\}$ on \widehat{N} and \widehat{G} as follows: 1 acts trivially in both cases; -1 acts on \widehat{N} by $\widehat{N} \ni \sigma \mapsto {}^h\sigma \in \widehat{N}$ where

$${}^h\sigma(n) = \sigma(h^{-1}nh)$$

for all $n \in N$; finally, -1 acts on \widehat{G} by $\widehat{G} \ni \theta \mapsto \theta \otimes \varepsilon \in \widehat{G}$. Clearly, both \widehat{N} and \widehat{G} are partitioned into their C_2 -orbits. Moreover every such an orbit consists of one or two representations. Let also

$$I_G(\sigma) = \{g \in G : {}^g\sigma \sim \sigma\}$$

be the inærtia group of $\sigma \in \widehat{N}$ with respect to G (again, ${}^g\sigma(n) = \sigma(g^{-1}ng)$) and

$$\widehat{G}(\sigma) = \{\theta \in \widehat{G} : \sigma \leq \text{Res}_N^G \theta\} \equiv \{\theta \in \widehat{G} : \theta \leq \text{Ind}_N^G \sigma\}.$$

The following theorem yields a very natural bijection between the orbits of C_2 on \widehat{N} and those on \widehat{G} . For the proof we refer to [15, Theorem 3.1] and [72, Sect. III.11].

Theorem 8.1. (1) If $I_G(\sigma) = N$, then $\theta := \text{Ind}_N^G \sigma \in \widehat{G}$, $\theta \otimes \varepsilon = \theta$ and $\text{Res}_N^G \theta = \sigma \oplus {}^h \sigma$, with σ and ${}^h \sigma$ not equivalent.

(2) If $I_G(\sigma) = G$, then, taking $\theta \in \widehat{G}(\sigma)$ we have $\text{Ind}_N^G(\sigma) = \theta \oplus (\theta \otimes \varepsilon)$ with $\theta \not\sim \theta \otimes \varepsilon$ and $\text{Res}_N^G \theta = \text{Res}_N^G(\theta \otimes \varepsilon) = \sigma$.

(3) The map

$$\{\sigma, {}^h \sigma\} \mapsto \widehat{G}(\sigma) \quad \text{when } I_G(\sigma) = N$$

and

$$\{\sigma\} \mapsto \widehat{G}(\sigma) \quad \text{when } I_G(\sigma) = G$$

yields a one-to-one correspondence between the C_2 -orbits on \widehat{N} and on \widehat{G} . In particular, to each single-element orbit on \widehat{N} (resp. on \widehat{G}) there corresponds a two-elements orbit on \widehat{G} (resp. on \widehat{N}).

In the following, given an irreducible representation σ of N , we denote by σ^τ and $C_\tau(\sigma)$ the associated τ -conjugate representation and the τ -Frobenius–Schur indicator of σ (as in (3.1) and Definition 3.4; recall that $C = C_{\tau_{\text{inv}}}$). As remarked in the Introduction, the following result goes back to Kawanaka and Matsuyama [47] but the proof follows the lines in [7, Exercise 4.5.1].

Theorem 8.2 (Twisted Frobenius–Schur theorem). Let σ be an irreducible representation of N and denote by χ_σ its character. Then

$$(8.2) \quad \frac{1}{|N|} \sum_{n \in N} \chi_\sigma(\tau(n)^{-1}n) = C_\tau(\sigma).$$

Proof. Let G be as in (8.1). We distinguish two cases.

Case 1: ${}^h \sigma \sim \sigma$. In this case, for $\theta \in \widehat{G}(\sigma)$, by Theorem 8.1 (2) we have $\theta \not\sim \theta \otimes \varepsilon$ and $\text{Res}_N^G \theta = \sigma$, so that $\chi_\theta(n) = \chi_\sigma(n)$ for all $n \in N$. Since $g^2 \in N$ for all $g \in G$, we have

$$\begin{aligned} \sum_{g \in G} \chi_\theta(g^2) &= \sum_{n \in N} \chi_\sigma(n^2) + \sum_{n \in N} \chi_\sigma((hn)^2) \\ &= \sum_{n \in N} \chi_\sigma(n^2) + \sum_{n \in N} \chi_\sigma(\tau(n)^{-1}n). \end{aligned}$$

By the classical Frobenius–Schur theorem [13, Theorem 9.7.7], we have

$$C(\theta) = \frac{1}{|G|} \sum_{g \in G} \chi_\theta(g^2)$$

and

$$C(\sigma) = \frac{1}{|N|} \sum_{n \in N} \chi_\sigma(n^2).$$

Therefore, since $|G| = 2|N|$, we deduce that

$$(8.3) \quad C(\theta) = \frac{1}{2}C(\sigma) + \frac{1}{2|N|} \sum_{n \in N} \chi_{\sigma}(\tau(n)^{-1}n).$$

Suppose that σ is self-conjugate.

Denote by $A(n)$ (resp. $A^{\tau}(n)$), with $n \in N$, a matrix realization of σ (resp. σ^{τ}). Note that ${}^hA(n) := A(h^{-1}nh)$, with $n \in N$, is a matrix realization of ${}^h\sigma$ (cf. [15, Lemma 3.1]). Moreover,

$$(8.4) \quad A^{\tau}(n) = \overline{{}^hA(n)}$$

for all $n \in N$. Indeed, for all $n \in N$ we have ${}^h\sigma(n) = \sigma(\tau(n)^{-1})$ and therefore

$$(8.5) \quad \sigma^{\tau}(n) = \sigma[\tau(n)]^T = {}^h\sigma(n^{-1})^T = ({}^h\sigma)'(n)$$

so that $A^{\tau}(n) = {}^hA(n^{-1})^T = \overline{{}^hA(n)}$. Let also $M(g)$, $g \in G$, denote a matrix realization of θ such that $M(n) = A(n)$ for all $n \in N$. Then, the unitary matrix $V = M(h)$ satisfies $M(nh) = A(n)V$ for all $n \in N$, $V^2 = M(h^2) = M(1_G) = I$ so that

$$(8.6) \quad V^* = V \quad \text{and} \quad V^T = \overline{V}$$

and

$$(8.7) \quad {}^hA(n) = V^*A(n)V = VA(n)V$$

for all $n \in N$. Since σ is self-conjugate, we can find a unitary matrix W such that

$$(8.8) \quad \overline{{}^hA(n)} = WA(n)W^*$$

for all $n \in N$. We therefore obtain

$$\begin{aligned} A^{\tau}(n) &= \overline{{}^hA(n)} \quad (\text{by (8.4)}) \\ &= \overline{VA(n)V} \quad (\text{by (8.7)}) \\ &= V^T \overline{{}^hA(n)} V^T \quad (\text{by (8.6)}) \\ &= V^T WA(n)W^* V^T \quad (\text{by (8.8)}) \end{aligned}$$

that is,

$$(8.9) \quad W^* V^T A^{\tau}(n) = A(n) W^* V^T$$

for all $n \in N$. Since $W^* \overline{V} W = \pm V$ (cf. [15, Theorem 3.4]), and applying also (8.6), we get

$$(8.10) \quad (W^* V^T)^T = V \overline{W} = \pm \overline{W} V = \pm \overline{W} V^T.$$

From Lemma 2.3, it follows that $W\overline{W} = \pm I$. More precisely, $W\overline{W} = I$ (resp. $W\overline{W} = -I$) if σ is real (resp. quaternionic). Combining with Theorem 3.5, this is equivalent to

$$(8.11) \quad \overline{W} = C(\sigma)W^*.$$

From (8.10) and (8.11) we obtain

$$(8.12) \quad (W^*V^T)^T = \pm C(\sigma)W^*V^T$$

where the sign is the same as in $W^*\overline{V}W = \pm V$. Thus, from (8.9) (W^*V^T intertwines A^τ and A) and (8.12) (W^*V^T is symmetric/antisymmetric) with the same sign therein, we obtain

$$(8.13) \quad C_\tau(\sigma) = \pm C(\sigma).$$

Now, if in (8.13) the sign is $+$, then by [15, Theorem 3.4 (2)] we have $C(\theta) = C(\sigma)$ and therefore from (8.3) and (8.13) we deduce that (8.2) is satisfied. On the other hand, if in (8.13) the sign is $-$, then θ is complex, $C(\theta) = 0$ and $C_\tau(\sigma) = -C(\sigma)$ and from (8.3) we again deduce (8.2). This completes the proof in the case σ is self-conjugate.

Suppose now that σ is complex. Then by [15, Theorem 3.4 (1)] we have that θ is complex as well. Moreover $\sigma' \not\sim \sigma$ and ${}^h\sigma \sim \sigma$ imply that $\sigma^\tau \equiv {}^h\sigma' \sim \sigma' \not\sim \sigma$ (recall (8.5)) and therefore $C_\tau(\sigma) = 0$. Again, from (8.3) we deduce (8.2). This completes the proof in the case σ is complex and, together with the previous step completes the proof for the case ${}^h\sigma \sim \sigma$.

We now discuss the remaining case.

Case 2: ${}^h\sigma \not\sim \sigma$. From Theorem 8.1 (1) we deduce that $\text{Res}_N^G \theta = \sigma \oplus {}^h\sigma$ and therefore

$$\begin{aligned} \sum_{g \in G} \chi_\theta(g^2) &= \sum_{n \in N} \chi_\theta(n^2) + \sum_{n \in N} \chi_\theta(\tau(n)^{-1}n) \\ &= \sum_{n \in N} \chi_\sigma(n^2) + \sum_{n \in N} \chi_{{}^h\sigma}(n^2) \\ &\quad + \sum_{n \in N} \chi_\sigma(\tau(n)^{-1}n) + \sum_{n \in N} \chi_{{}^h\sigma}(\tau(n)^{-1}n). \end{aligned}$$

As

$$\chi_{{}^h\sigma}(n^2) = \chi_\sigma(hn^2h) = \chi_\sigma(\tau(n)^{-2})$$

and

$$\chi_{{}^h\sigma}(\tau(n)^{-1}n) = \chi_\sigma(h\tau(n)^{-1}nh) = \chi_\sigma(\tau((\tau(n)^{-1}n)^{-1})) = \chi_\sigma(n\tau(n)^{-1})$$

from the fact that $n \mapsto \tau(n)^{-1}$ is an automorphism, we deduce that

$$(8.14) \quad C(\theta) = C(\sigma) + \frac{1}{|N|} \sum_{n \in N} \chi_\sigma(\tau(n)^{-1}n).$$

Suppose that σ is real (resp. quaterionionic). Then, by virtue of [15, Theorem 3.3 (1)] θ is real (resp. quaterionionic) as well and therefore $C(\sigma) = C(\theta)$. Now, if $\sigma \sim \sigma'$, since by hypothesis $h_\sigma \not\sim \sigma$, and (8.5) holds also in this case, we have $\sigma^\tau \equiv h_{\sigma'} \not\sim \sigma' \sim \sigma$, and therefore $C_\tau(\sigma) = 0$. Then (8.2) follows from (8.14).

Suppose now that σ is complex. If $\sigma' \not\sim h_\sigma$, from [15, Theorem 3.3 (2)] we deduce that θ is complex as well. Moreover, $\sigma^\tau \equiv h_{\sigma'} \not\sim \sigma$ and therefore $C(\sigma) = C(\theta) = C_\tau(\sigma) = 0$ and (8.2) follows again from (8.14).

Finally, if σ is complex and $\sigma' \sim h_\sigma$, then [15, Theorem 3.3 (3)] ensures that θ is self-conjugate. Moreover, $\sigma^\tau \equiv h_{\sigma'} \sim \sigma$. We then denote by U an intertwining unitary matrix such that

$$(8.15) \quad UA^\tau(n) = A(n)U$$

for all $n \in N$ ($A(n)$ is as in (8.4)). Since $h^2 = 1_G$, from [15, Theorem 3.3 (3)] we deduce that $\overline{U}U = \pm I$, that is,

$$(8.16) \quad U = \pm U^T.$$

Now, if in (8.16) the sign is $+$, [15, Theorem 3.3 (3)] ensures that θ is real, while (8.15) and (8.16) give $C_\tau(\sigma) = 1$. In other words, $C(\sigma) = 0$, $C_\tau(\sigma) = C(\theta) = 1$ and, once more, (8.2) follows from (8.14). Similarly if the sign in (8.16) is $-$. \square

Recall that $\zeta_\tau(n)$, $n \in N$, denotes the number of elements $m \in N$ such that $\tau(m^{-1})m = n$ (cf. (6.3)).

Corollary 8.3. *For all $n \in N$ we have*

$$(8.17) \quad \zeta_\tau(n) = \sum_{\sigma \in \widehat{N}} C_\tau(\sigma) \chi_\sigma(n).$$

In particular,

$$\zeta_\tau(1_N) = \sum_{\substack{\sigma \in \widehat{N}: \\ C_\tau(\sigma)=1}} d_\sigma - \sum_{\substack{\sigma \in \widehat{N}: \\ C_\tau(\sigma)=-1}} d_\sigma.$$

Proof. We observe that ζ_τ is a central function. Indeed, if $m, n, s \in N$ and $\tau(m^{-1})m = n$, then

$$sns^{-1} = s\tau(m^{-1})\tau(s)\tau(s^{-1})ms^{-1} = \tau[\tau(s^{-1})ms^{-1}]^{-1}\tau(s^{-1})ms^{-1}.$$

Therefore the map $m \mapsto \tau(s^{-1})ms^{-1}$ yields a bijection between the set of solutions of $\tau(m^{-1})m = n$ and the set of solutions of $\tau(m_s^{-1})m_s = sns^{-1}$. From the Frobenius–Schur twisted formula (8.2) we deduce

$$(8.18) \quad \frac{1}{|N|} \sum_{n \in N} \overline{\chi_\sigma(n)} \zeta_\tau(n) = C_\tau(\sigma)$$

(observe that $\zeta_\tau(n)$ and $C_\tau(\sigma)$ are both real). Since ζ_τ is central, by the orthogonality relation for characters (*cf.* [13, Equation (3.21)]) (8.17) immediately follows from (8.18). \square

We are now in position to complete Corollary 6.4 by adding a third representation theoretic quantity.

Theorem 8.4. *The following quantities are equal:*

- (a) *the number of $\sigma \in \widehat{N}$ such that $\sigma \sim \sigma^\tau$;*
- (b) *the number of τ -invariant conjugacy classes of N ;*
- (c) $\frac{1}{|N|} \sum_{n \in N} \zeta_\tau(n)^2$.

Proof. The equality between the numbers in (b) and (c) corresponds to Corollary 6.4. On the other hand, from Corollary 8.3 we also have

$$\begin{aligned} \frac{1}{|N|} \sum_{n \in N} \zeta_\tau(n)^2 &= \frac{1}{|N|} \sum_{n \in N} \zeta_\tau(n) \overline{\zeta_\tau(n)} \\ &= \sum_{\sigma, \rho \in \widehat{N}} C_\tau(\sigma) C_\tau(\rho) \frac{1}{|N|} \sum_{n \in N} \chi_\sigma(n) \overline{\chi_\rho(n)} \\ &= *_\sum_{\sigma, \rho \in \widehat{N}} C_\tau(\sigma) C_\tau(\rho) \delta_{\sigma, \rho} \\ &= |\{\sigma \in \widehat{N} : C_\tau(\sigma) \neq 0\}| \\ &\equiv |\{\sigma \in \widehat{N} : \sigma^\tau \sim \sigma\}| \end{aligned}$$

where $=_*$ follows from the orthogonality relations for the characters, and therefore we get the equality between (a) and (c). \square

9. The twisted Frobenius–Schur theorem for a Gelfand pair

In this section we specialize the results of the previous section to the context of Gelfand pairs.

Let (G, K) be a Gelfand pair, X and x_0 as in Sect. 4 and

$$L(X) = \bigoplus_{\rho \in I} V_\rho$$

the corresponding multiplicity free decomposition. By virtue of Frobenius reciprocity, for each $\rho \in I$, there exists a unique (modulo a complex factor of modulus 1) unit vector $v \in V_\rho$ such that $\rho(k)v_\rho = v_\rho$ for all $k \in K$. The *spherical function* associated with v_ρ is the complex valued function ϕ_ρ on G defined by

$$\phi_\rho(g) = \langle v_\rho, \rho(g)v_\rho \rangle_{V_\rho}$$

for all $g \in G$. We observe that the spherical function ϕ_ρ is bi- K -invariant ($\phi_\rho(k_1 g k_2) = \phi_\rho(g)$ for all $k_1, k_2 \in K$ and $g \in G$) and recall the following relations between ϕ_ρ and the corresponding character χ_ρ (cf. [13, Exercise 9.5.8]):

$$(9.1) \quad \phi_\rho(g) = \frac{1}{|K|} \sum_{k \in K} \overline{\chi_\rho(gk)}$$

and

$$(9.2) \quad \chi_\rho(g) = \frac{d_\rho}{|G|} \sum_{h \in G} \overline{\phi_\rho(h^{-1}gh)}$$

for all $g \in G$, where $d_\rho = \dim V_\rho$.

Theorem 9.1 (Twisted Frobenius–Schur for a Gelfand pair). *Let $\tau : G \rightarrow G$ be an involutory anti-automorphism. Then we have*

(1) *For every $\rho \in I$*

$$\frac{d_\rho}{|G|} \sum_{g \in G} \phi_\rho(\tau(g)^{-1}g) = C_\tau(\rho).$$

(2) *For $x \in X$ we set $\zeta_\tau(x) = |\{g \in G : \tau(g)^{-1}gx_0 = x\}|$. Then we have*

$$\frac{1}{|G|} \sum_{x \in X} \zeta_\tau(x)^2 = |K| \sum_{\substack{\rho \in I: \\ \rho \sim \rho^\tau}} \frac{1}{d_\rho}.$$

Proof. (1) From Theorem 8.2 we obtain ($C_\tau(\rho)$ is real)

$$\begin{aligned} C_\tau(\rho) &= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_\rho(\tau(g)^{-1}g)} \\ &= \frac{d_\rho}{|G|^2} \sum_{g, h \in G} \phi_\rho(h^{-1}\tau(g)^{-1}gh) \quad (\text{by (9.2)}) \\ &= \frac{d_\rho}{|G|^2} \sum_{g, h \in G} \phi_\rho(\tau(\tau(h)gh)^{-1} \cdot \tau(h)gh) \\ &= \frac{d_\rho}{|G|} \sum_{s \in G} \phi_\rho(\tau(s)^{-1}s) \quad (\text{by setting } s = \tau(h)gh). \end{aligned}$$

(2) From the previous fact (by setting $\varphi_\rho(x) = \phi_\rho(g)$ if $gx_0 = x$: note that this is well-defined by virtue of the bi- K -invariance of ϕ_ρ) we get

$$C_\tau(\rho) = \frac{d_\rho}{|G|} \sum_{x \in X} \zeta_\tau(x) \varphi_\rho(x).$$

Then the spherical Fourier inversion formula [13, Equation (4.15)] yields

$$\zeta_\tau(x) = |K| \sum_{\rho \in I} \varphi_\rho(x) C_\tau(\rho).$$

Therefore, from the orthogonality relations for spherical functions [13, Proposition 4.7.1] we have

$$\begin{aligned} \frac{1}{|G|} \sum_{x \in X} \zeta_\tau(x)^2 &= \frac{1}{|G|} \sum_{x \in X} \zeta_\tau(x) \overline{\zeta_\tau(x)} \\ &= \sum_{\rho \in I} C_\tau(\rho)^2 \frac{|K|^2}{|G|} \cdot \frac{|X|}{d_\rho} \\ &= |K| \sum_{\substack{\rho \in I: \\ \rho^\tau \sim \rho}} \frac{1}{d_\rho}. \end{aligned}$$

□

If $\tau(K) = K$ then τ induces an involution (that we keep denoting by τ) on the set $K \backslash G / K$ of double cosets and therefore on the set $K \backslash X$ of K -orbits on X . This way, if $g \in G$ and Ω_{gx_0} is the K -orbit containing gx_0 , then $\tau(\Omega_{gx_0})$ is the K -orbit containing $\tau(g)x_0$.

Theorem 9.2 (Twisted Frobenius–Schur for a Gelfand pair II). *Suppose that $\tau(K) = K$.*

- (1) *If $\rho \in I$ then also $\rho^\tau \in I$ and the number of $\rho \in I$ such that $\rho^\tau \sim \rho$ is equal to the number of τ -invariant K -orbits on X .*
- (2) *If $\rho \in I$ and $\rho^\tau \sim \rho$ then $C_\tau(\rho) = 1$.*

Proof. (1) Let us define $f_\rho \in V'_\rho$ by setting

$$(9.3) \quad f_\rho(v) = \langle v, v_\rho \rangle_{V_\rho}$$

for all $v \in V_\rho$. Then we have

$$\phi_\rho(g) = f_\rho[\rho(g^{-1})v_\rho].$$

Moreover, $\rho^\tau(k)f_\rho = f_\rho$ for all $k \in K$. Indeed, for all $k \in K$ and $v \in V_\rho$, we have:

$$\begin{aligned} [\rho^\tau(k)f_\rho]v &= f_\rho[\rho(\tau(k))v] \\ &= \langle \rho(\tau(k))v, v_\rho \rangle \\ &= \langle v, \rho(\tau(k))^{-1}v_\rho \rangle \\ &= \langle v, v_\rho \rangle \quad (\text{since } \tau(K) = K \text{ and } v_\rho \text{ is } K\text{-invariant}). \end{aligned}$$

This shows that $\rho^\tau \in I$ because f_ρ is a non-trivial K -invariant vector. Then, we equip V'_ρ with the scalar product given by duality so that $\langle f, f_\rho \rangle_{V'_\rho} = f(v_\rho)$ for all $f \in V'_\rho$ and, recalling (9.3), we deduce that the spherical function ϕ_{ρ^τ} is given by

$$\begin{aligned} \phi_{\rho^\tau}(g) &= [\rho^\tau(g^{-1})f_\rho](v_\rho) = f_\rho[\rho(\tau(g))^{-1}v_\rho] = \langle v_\rho, \rho(\tau(g))v_\rho \rangle_{V_\rho} \\ &\equiv \phi_\rho(\tau(g)) \end{aligned}$$

for all $g \in G$. Thus,

$$(9.4) \quad \begin{aligned} \frac{1}{|G|} \sum_{g \in G} \phi_\rho(g) \overline{\phi_\rho(\tau(g))} &= \frac{1}{|G|} \sum_{g \in G} \phi_\rho(g) \overline{\phi_{\rho^\tau}(g)} \\ &= \begin{cases} 0 & \text{if } \rho \not\sim \rho^\tau, \\ \frac{1}{d_\rho} & \text{if } \rho \sim \rho^\tau. \end{cases} \end{aligned}$$

On the other hand, by virtue of the dual orthogonality relations for spherical functions (cf. [13, Proposition 4.7.1]), we get

$$(9.5) \quad \sum_{\rho \in I} d_\rho \phi_\rho(g) \overline{\phi_\rho(\tau(g))} = \begin{cases} \frac{|X|}{|\Omega_{gx_0}|} & \text{if } \tau(g)x_0 \in \Omega_{gx_0}, \\ 0 & \text{otherwise.} \end{cases}$$

From (9.4) and (9.5) we deduce

$$\begin{aligned} |\{\rho \in I : \rho \sim \rho^\tau\}| &= \sum_{\rho \in I} \frac{1}{|G|} d_\rho \sum_{g \in G} \phi_\rho(g) \overline{\phi_\rho(\tau(g))} \\ &= \frac{1}{|G|} \sum_{\substack{\Omega \in K \setminus X: \\ \tau(\Omega) = \Omega}} \sum_{\substack{g \in G: \\ gx_0 \in \Omega}} \frac{|X|}{|\Omega|} \\ &= \sum_{\substack{\Omega \in K \setminus X: \\ \tau(\Omega) = \Omega}} \frac{1}{|G|} |K| \cdot |X| \\ &= |\{\Omega \in K \setminus X : \tau(\Omega) = \Omega\}|. \end{aligned}$$

(2) Consider the permutation representation $(\lambda, L(X))$. By Theorem 4.4 and Lemma 4.1 (3) we know that $C_\tau(\lambda) = m_1 = |\{\Omega \in K \setminus X : \tau(\Omega) = \Omega\}|$. By Proposition 3.6 and Theorem 3.5 (1) we have that $C_\tau(\lambda) = \sum_{\substack{\rho \in I \\ \rho \sim \rho^\tau}} C_\tau(\rho)$.

Taking into account that, by the previous facts, $C_\tau(\lambda) = |\{\rho \in I : \rho \sim \rho^\tau\}|$, we conclude that $C_\tau(\rho) = 1$ for all $\rho \in I$ such that $\rho \sim \rho^\tau$. \square

10. Examples

In this section we review some examples related to our investigations. Note that all the examples discussed below refer to involutive automorphisms of the given finite group G while our treatment concerns involutive *anti*-automorphisms. Modulo the composition with the inverse map $\tau_{\text{inv}} : g \mapsto g^{-1}$, the two approaches are clearly equivalent.

Let us recall that, given a finite group G , a *Gelfand model*, briefly a *model*, for G (a notion introduced by I.N. Bernstein, I.M. Gelfand and S.I. Gelfand in [6]) is a representation containing every irreducible representation with multiplicity one. Models for the finite symmetric groups S_n and general linear groups were described by A.A. Klyachko [50, 52]. In [40] Inglis, Richardson and Saxl presented a brief and elegant construction of an explicit *involution* model for S_n (the term “involution” refers to the fact that the model is obtained by summing up induced representations of subgroups which are centralizers of certain involutions). Their work was continued by Baddeley [2] who showed that if a finite group H has an involution model, then the wreath product $H \wr S_n$ also has an involution model for any $n \in \mathbb{N}$. As a byproduct, he obtained involution models for Weyl groups of type A_n, B_n, C_n and D_{2n+1} for all $n \in \mathbb{N}$. Note that the theorem of Frobenius–Schur (*cf.* Corollary 8.3 with $n = 1_G$ and $\tau = \tau_{\text{inv}}$) imposes an obstruction for a group G to have an involution model: G admits an involution model only if G has only real representations.

In the spirit of the present paper, Bump and Ginzburg [11] considered *generalized involution models*: these consist in replacing the involutions (*resp.* their centralizers) with twisted-involutions (*resp.* their twisted-centralizers) with respect to some involutive automorphism τ of the ambient group (thus a model for G is a generalized involution model for G with τ equal to the identity automorphism of G). In analogy with the standard involution models, we have the following obstruction: G admits a generalized involution model (with respect to τ) only if $C_\tau(\sigma) = 1$ for all $\sigma \in \widehat{G}$. We remark that the only abelian groups with involution models are $(\mathbb{Z}_2)^n$, $n \in \mathbb{N}$, but every abelian group has generalized involution models. On the other hand, a Coxeter group has an involution model if and only if it has a generalized involution model. More recently, Marberg [60] proved that if a finite group H has a generalized involution model, then, in analogy to Baddeley's main result, the wreath product $H \wr S_n$ also has a generalized

involution model for any $n \in \mathbb{N}$. As an application, it is shown that when H is abelian, then $H \wr S_n$ has a model: when $H = \mathbb{Z}_r$, $r \in \mathbb{N}$, this recovers a result previously obtained by Adin, Postnikov and Roichman [1] (see below).

- R. Gow [32] considers the general linear group $G = \text{GL}(n, k)$, where k is a field, equipped with the involutory automorphism which sends each matrix $x \in G$ into its transposed inverse $(x^T)^{-1}$ (in our setting, this corresponds to the involutory anti-automorphism τ which sends each $x \in G$ into its transposed x^T) and the corresponding semi-direct product denoted G^+ . It is first shown that every element of G^+ is a product of two involutions (Theorem 1) and therefore it is conjugate to its inverse, so that G^+ is ambivalent and all its irreducible representations are self-conjugate.

Let now $k = \mathbb{F}_q$ be the field with q elements. Suppose q is odd. In Theorem 2, Gow shows that every irreducible representation of G^+ is indeed real. From this result the author deduces (Theorem 3) the formula

$$(10.1) \quad \sum_{\sigma \in \widehat{G}} \chi_\sigma(g) = \zeta_\tau(g)$$

for all $g \in G$. Comparing (10.1) and (8.17) one deduces that $C_\tau(\sigma) = 1$ for all $\sigma \in \widehat{G}$. Note that by taking $g = 1_G$, the left hand side in (10.1) gives the sum of dimensions of all irreducible representations of G , equivalently the dimension of a model of G , while the right hand side gives the number of symmetric matrices in G (Theorem 4). One may remark that Theorem 2 can be derived from Theorem 4 using Theorem 1. Moreover, Theorems 2 and 4 are both valid also for even q . In fact, Theorem 4 has been proved independently, for even as well as odd q , both by A.A. Klyachko [52, Theorem 4.1] and by I.G. Macdonald (unpublished manuscript).

- In [33], Gow considers the general linear group $G = \text{GL}(n, \mathbb{F}_{q^2})$, q a prime power, and its subgroups $U = \text{GU}(n, \mathbb{F}_{q^2})$ (the unitary group of degree n over \mathbb{F}_{q^2}) and $M = \text{GL}(n, \mathbb{F}_q)$. The *Frobenius automorphism* $c \mapsto c^q$ of \mathbb{F}_{q^2} extends to an involutory automorphism F of G (leaving U and M invariant) by raising the entries of a matrix in G to the q th power. Then also F^* , the composition of F and the transposed inverse, is an involutory automorphism of G (so that U is the G -subgroup consisting of F^* -fixed elements). By using these two automorphisms, it is then shown that (G, U) (resp. (G, M)) is a Gelfand pair and that the irreducible subrepresentations of $L(G/U)$ (resp. of $L(G/M)$) are precisely the F -fixed (resp. F^* -fixed) irreducible representations of G . We mention that the F^* -fixed representations of G were used by Kawanaka [46] to give a parameterization of the irreducible representations of U .
- Inglis, Liebeck and Saxl in [39] consider the group $G_0 = \text{PSL}(n, \mathbb{F}_q)$ (the projective special linear group of degree n over \mathbb{F}_q), with $n \geq 8$. Let G

be a group with *socle* G_0 , that is such that $G_0 \triangleleft G \leq \text{Aut}(G_0)$. Then a description of all Gelfand pairs (G, H) with H maximal subgroup of G not containing G_0 is given. Moreover, in [41] the authors find a new model of the general linear group over a finite field (this construction can also be obtained from a result of Bannai, Kawanaka and Song [4] but the methods in [41] are independent of and different from theirs).

- Vinroot [76] considers the group $G = \text{Sp}(2n, \mathbb{F}_q)$ equipped with the involutive automorphism

$$g \mapsto \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix} g \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}.$$

Let us denote by τ the composition of the above automorphism and τ_{inv} , so that

$$\tau(g) = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix} g^{-1} \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}.$$

Observe that when $q \equiv 1 \pmod{4}$ then τ is inner and every irreducible representation of G is self-conjugate. Moreover, when $q \equiv 3 \pmod{4}$ then τ is not inner, there exist irreducible representations of G which are not self-conjugate, but $C_\tau(\sigma) = 1$ for all $\sigma \in \widehat{G}$ (cf. [76, Theorem 1.3]). As a byproduct, from the analogous formula (10.1) which holds in the present setting, Vinroot determines explicitly the dimension of any model of $\text{Sp}(2n, \mathbb{F}_q)$.

In [77] Vinroot uses Klyachko's construction of a model for the irreducible complex representations of the finite general linear group $\text{GL}(n, \mathbb{F}_q)$ we alluded to above to establish, by determining the corresponding Frobenius–Schur number, whether a given irreducible self-conjugate representation of $\text{SL}(n, \mathbb{F}_q)$, the finite special linear group of degree n over \mathbb{F}_q , is real or quaternionic.

- Adin, Postnikov and Roichman [1] study Gelfand models for wreath products of the form $G = \mathbb{Z}_r \wr S_n$. Any element g of G can be expressed uniquely as $g = \sigma v$, where $\sigma \in S_n$ and $v \in \mathbb{Z}_r^n$. Consider the map $\tau: G \rightarrow G$ given by $\tau(g) = \sigma(-v)$ for all $g = v\sigma \in G$.

An element $g \in G$ is said to be an *absolute square root* of another element $h \in G$ provided $g\tau(g) = h$. Then the main result of this paper asserts that the value of the character associated to the Gelfand model of G on h equals the number $\zeta_\tau(h)$ of absolute square roots of h (cf. (6.3)). This generalizes a result concerning groups possessing only real characters, e.g. S_n (in this case, absolute square roots coincide with square roots).

- Bannai and Tanaka [5] consider a finite group G , an automorphism σ and the corresponding centralizer $K := C_G(\sigma)$ in G , i.e., the subgroup consisting of all elements fixed by σ . It is well known and easy to see that for $g, h \in G$ the double cosets KgK and KhK are equal if and only if the elements $g\sigma(g)$

and $h\sigma(h)$ are conjugate in K . Then the authors introduce the following condition:

- (\star) If the elements $g\sigma(g)$ and $h\sigma(h)$ are conjugate in G then they are conjugate in K ,

and showed (Proposition 1) that if σ is an involution and condition (\star) holds, then (G, K) is a Gelfand pair. For instance, if H is a finite group, $G = H \times H$, and $\sigma : G \rightarrow G$ is the *flip* defined by $\sigma(h_1, h_2) = (h_2, h_1)$, then (\star) is satisfied and one recovers the well known fact that (G, K) is a Gelfand pair, where $K = C_\sigma(G)$ is $\tilde{H}^2 = \{(h, h) : h \in H\}$.

Moreover, they provided a list of other interesting examples where the above condition is satisfied. In particular, when G is the symmetric group S_n , with $n \geq 4$, their list exhausts all possible examples. Other examples from the above mentioned list include some *sporadic groups* as well as some linear groups including:

- (i) $G = \text{GL}(n, \mathbb{F}_{q^2})$, $K = \text{GL}(n, \mathbb{F}_q)$;
- (ii) $G = \text{GL}(n, \mathbb{F}_{q^2})$, $K = \text{GU}(n, \mathbb{F}_{q^2})$;
- (iii) $G = \text{GL}(2n, \mathbb{F}_q)$, $K = \text{Sp}(2n, \mathbb{F}_q)$.

Moreover they leave it as an open problem to determine whether condition (\star) is satisfied in the case:

- (iv) $G = \text{GL}(2n, \mathbb{F}_q)$, $K = \text{GL}(n, \mathbb{F}_{q^2})$.

11. Open problems and further comments

Here below we indicate possible extensions and generalizations of the results discussed by listing some open problems.

Comment 11.1 (Multiplicity-free induced representations). In his work on induced representations, when looking for explicit criteria for multiplicity-freeness, Mackey basically limited his investigation to permutation representations, that is, to representations obtained by inducing the trivial representation of a subgroup. In this setting, the theory is rich and completely understood. We recall the *Gelfand–Garsia criterion* (Corollary 4.7, cf. [13, Example 4.3.2]) for a symmetric Gelfand pair and the *weak-Gelfand criterion* (Corollary 4.8, cf. [13, Exercise 4.3.3]). It is well known that there exist non-symmetric Gelfand pairs (with the cyclic groups and the alternating groups [13, Example 4.8.3]) as well as non-weakly-symmetric Gelfand pairs ([13, Sect. 9.6]).

Let G be a finite group and $K \leq G$ a subgroup. Let also τ be an involutive anti-automorphism of G .

We notice that the sufficient condition in Corollary 4.8, namely Formula (4.6), is not a necessary condition for: (i) (G, K) being a Gelfand pair and (ii) $C_\tau(\sigma) = 1$ for every irreducible representation σ contained in the permutation representation λ_π .

Bump and Ginzburg [11] gave the following *sufficient* condition (cf. (4.6)) for $\text{Ind}_K^G \theta$ being multiplicity-free, where θ is an irreducible K -representation:

- (i) $\tau(K) = K$;
- (ii) $\chi^\theta(\tau(g)) = \chi^\theta(g)$ for all $g \in G$;
- (iii) KgK is τ -invariant for all $g \in G$;
- (iv) $\tau(s) = k_1 s k_2$ for some $k_1, k_2 \in K$ such that $\chi^\theta(k_1) \chi^\theta(k_2) = 1$, for all $s \in S$,

where χ^θ denotes the character of θ and S is a suitable complete set of representatives for the double cosets of K in G .

It would be interesting to find an analogue of the Mackey–Gelfand criterion (cf. Theorem 4.5) along the lines of [11] as well as to find an example of a K -representation $\theta \neq \iota_K \theta$ such that $\text{Ind}_K^G \theta$ is multiplicity-free but does not satisfy the Bump and Ginzburg criterion above.

The first step towards the first part of the above problem, should be to examine the case $\dim \theta = 1$. In [17] we consider a *Hecke algebra* $\mathcal{H}(G, K, \theta)$ and present a sufficient condition (a Garsia-type criterion on $\mathcal{H}(G, K, \theta)$) for $\text{Ind}_K^G \theta$ being multiplicity-free and we illustrate it with the following example. Let $G = \text{GL}(2, \mathbb{F}_q)$ denote the group of invertible 2×2 matrices with coefficients in \mathbb{F}_q , the Galois field with q elements, and $K = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{F}_q \right\}$ the subgroup of *unipotent* matrices. Then for every non-trivial character χ of the (abelian) subgroup K the induced representation $\text{Ind}_K^G \chi$ is multiplicity-free. This is a particular case of the *Gelfand–Graev representation* of a simple group of Lie type obtained by inducing a non-trivial character of the maximal unipotent subgroup.

Now we list two open problems that, together with 11.7 and 11.8, suggest that the Mackey–Wigner theory should be a particular case of a more general theory.

Problem 11.2 (Harmonic analysis and tensor products). Let G be a finite group and τ an involutive anti-automorphism of G . Suppose G is τ -simply reducible. What is the relation between the spherical Fourier analysis on the homogeneous space $L(G^3/\widetilde{G}^3)$ (see [13, Sect. 4]) and the decomposition of the tensor products?

Problem 11.3 (Decomposition of tensor products). Suppose (G^3, \widetilde{G}^3) is not a Gelfand pair. Is it possible to find rules that relate the decomposition of $L(G^3/\widetilde{G}^3)$ into irreducible representations with the decomposition of tensor products of irreducible G -representations? A possible strategy could be to apply, in this context, the analysis developed in [69, 70] for permutation representations that decompose with multiplicity. Moreover, a possible application should be to shed light to one of the major open problem in the representation

theory of the symmetric group, namely the decomposition of the tensor product of two irreducible representations (usually called *Kronecker products*). See [42, Sect. 2.9] for an introduction, [30] as a classical reference, and [31] as a recent interesting paper. Explicit decompositions of tensor products are also useful in the determination of the lower bound for the rate of convergence to the stationary distribution for diffusion processes on finite groups; see [21] and [13, Sect. 10.7].

Problem 11.4 (Characterization of simply reducible groups). The major open problem in the theory of simply (or τ -simply) reducible groups is to give a nice and useful characterization of these groups. This was stated as an open problem in the famous Kourovka notebook [54]. A great advance on this problem is in the recent paper [48] where the authors show that all τ -simply reducible groups are soluble (this also was an open problem in [54, Problem 11.94], posed by Strunkov (see also [73]).

Comment 11.5 (McKay correspondence). Simple reducibility is also relevant to the McKay correspondence which we now describe.

Let G be a finite group. Given a representation σ of G , the *McKay quiver* associated with σ is the directed multi-graph defined as follows: the vertex set is \widehat{G} and, given $\rho_1, \rho_2 \in \widehat{G}$, there are $m_{\rho_1, \rho_2}^\sigma$ directed edges from ρ_1 to ρ_2 , where $m_{\rho_1, \rho_2}^\sigma$ is the multiplicity of ρ_2 in $\sigma \otimes \rho_1$.

Let now $\pi: \mathrm{SU}(2) \rightarrow \mathrm{SO}(3)$ denote the standard double cover and note that the only element of even order in $\mathrm{SU}(2)$ is the generator -1 of the kernel of π . Therefore, any finite subgroup of $\mathrm{SU}(2)$ either has even order (and is the preimage of some finite subgroup of $\mathrm{SO}(3)$) or has odd order (and is isomorphic to a finite subgroup of $\mathrm{SO}(3)$ of odd order, hence a cyclic group). Now, the finite subgroups of $\mathrm{SO}(3)$ are: the cyclic groups $\mathbb{Z}/n\mathbb{Z}$, the dihedral groups D_{2n} , the tetrahedral group T (i.e., the alternating group A_4), the octahedral group O (i.e., the symmetric group S_4), and the icosahedral rotation group I (i.e., the alternating group A_5). It follows that the finite subgroups of $\mathrm{SU}(2)$ are: the cyclic groups, the binary dihedral groups $BD_{2n} = \pi^{-1}(D_{2n})$, the binary tetrahedral group $BT = \pi^{-1}(T)$, the binary octahedral group $BO = \pi^{-1}(O)$, and the binary icosahedral rotation group $BI = \pi^{-1}(I)$.

Let G be any finite subgroup of $\mathrm{SU}(2)$ as above and let σ denote the faithful representation of G obtained from the embedding $G \hookrightarrow \mathrm{SU}(2)$. Then, one can show that the associated McKay quiver is connected and has no self-loops ($m_{\rho, \rho}^\sigma = 0$ for all $\rho \in \widehat{G}$). Moreover, simple reducibility of G implies that the McKay quiver is a simple and undirected graph (i.e., $m_{\rho_1, \rho_2}^\sigma \in \{0, 1\}$ and $m_{\rho_1, \rho_2}^\sigma = m_{\rho_2, \rho_1}^\sigma$ for all $\rho_1, \rho_2 \in \widehat{G}$).

The *McKay correspondence*, named after John McKay [61, 27], then states that the construction of McKay quivers yields a bijection between the non-trivial finite subgroups of $\mathrm{SU}(2)$ and the affine simply laced Dynkin diagrams (which

appear in the A-D-E classification of simple Lie Algebras). For an overview of the correspondence and other mathematical structures which appear in connection with solvable models (e.g. the ice-type, Potts, and spin models) in two-dimensional statistical physics, see [43, Sect. 2].

Comment 11.6 (Simple phase groups). Let G be a finite group and let (σ_i, V_i) , $i = 1, 2, \dots, n$, denote a list of all pairwise inequivalent irreducible representations of G . Let $d_i = \dim(V_i)$ and fix an orthonormal basis $\{v_s^i : s = 1, 2, \dots, d_i\}$ in V_i , for all $i = 1, 2, \dots, n$. By multiplicity freeness, given $1 \leq i, j \leq n$ we can find $1 \leq i_1 < i_2 < \dots < i_k \leq n$ such that

$$(11.1) \quad \sigma_i \otimes \sigma_j \sim \bigoplus_{t=1}^k \sigma_{i_t}.$$

Consider the vector space $V_i \otimes V_j$ and let $T \in \text{Hom}_G(\bigoplus_{t=1}^k V_{i_t}, V_i \otimes V_j)$ be an unitary intertwiner (cf. (11.1)). There are two natural orthonormal bases in $V_i \otimes V_j$, namely $\{v_s^i \otimes v_t^j : s = 1, 2, \dots, d_i, t = 1, 2, \dots, d_j\}$ and $\{T(v_{u_\ell}^{i_\ell}) : u_\ell = 1, 2, \dots, d_{i_\ell}, \ell = 1, 2, \dots, k\}$. Then we can express

$$T(v_{u_\ell}^{i_\ell}) = \sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^{d_i} \sum_{t=1}^{d_j} C_{s,t,u_\ell}^{i,j,i_\ell} v_s^i \otimes v_t^j$$

where the complex numbers

$$(11.2) \quad C_{s,t,u_\ell}^{i,j,i_\ell}$$

called the *Clebsh–Gordan coefficients* or *Wigner coefficients* constitute the unitary matrix of change of base. As Wigner [79] showed (see also [36, Chapter 5]), the Clebsh–Gordan coefficients can be chosen in such a way that their absolute values are invariant under every permutation of the i, j, i_ℓ 's and the corresponding s, t, u_ℓ 's: in other words they change only by a multiplicative phase factor. If one drops the property of ambivalence no essential new difficulties arise in the definition and in the symmetry relations of Clebsh–Gordan coefficients. However, if the multiplicity free condition is dropped, then a multiplicity index enters the Clebsh–Gordan coefficients. Derome has shown [20] that these multiplicity Clebsh–Gordan coefficients are invariant under permutations in the above sense if and only if

$$(11.3) \quad \sum_{g \in G} \chi^\sigma(g^3) = \sum_{g \in G} \chi^\sigma(g)^3$$

for all $\sigma \in \widehat{G}$. Groups for which (11.3) holds are called *simple phase groups* (see [20, 9, 10, 51, 74, 26]). In [74] van Zanten and de Vries derive several Mackey–Wigner type criteria for the existence of a real representation and then derive

analogues of some of them, giving criteria for G to fail to be a simple phase group. It would be interesting to investigate twisted versions (in terms of an involutive (anti-)automorphism τ of the group G) of their results.

Problem 11.7 (Multiplicity-free subgroups). Let G be a finite group and $H \leq G$ a subgroup. We say that H is a *multiplicity-free subgroup* of G when $\text{Res}_H^G \rho$ decomposes without multiplicity for all $\rho \in \widehat{G}$. See [82], our book [16] for its relations with the theory of Gelfand–Tsetlin basis and the Okounkov–Vershik approach to the representation theory of the symmetric group, and [70] for the not multiplicity-free case. In particular, in [16, Theorem 2.1.10] we presented a general criterion for the subgroup H being multiplicity-free in terms of commutativity of the algebra $\mathcal{C}(G, H)$ of H -conjugacy invariant functions on G and of the Gelfand pair $(G \times H, \widetilde{H})$. Also, in [16, Proposition 2.1.12] we presented the following sufficient condition: for all $g \in G$ there exists $h \in H$ such that $h^{-1}gh = g^{-1}$. We then used this criterion to show the well known fact that S_{n-1} is a multiplicity-free subgroup of S_n , the symmetric group of degree n (cf. [16, Theorem 3.2.1 and Corollary 3.2.2]).

One of the key facts of the theory is that H is multiplicity-free if and only if $(G \times H, \widetilde{H})$ is a Gelfand pair, where $\widetilde{H} = \{(h, h) : h \in H\}$. This is a generalization of (6.8). Then it should be interesting to examine pairs like $(G \times G \times H, \widetilde{H}^3)$ or $(G \times H \times H, \widetilde{H}^3)$ and their relations with the representation theory of G and H .

Problem 11.8. Theorem 8.4 gives a representation theoretical interpretation of the purely group theoretical quantities in Corollary 6.4. Is there a representation theoretical interpretation of the more general quantities in Theorem 6.3?

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