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On the excursion theory for linear diffusions

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Abstract. We present a number of important identities related to the excursion theory of linear diffusions. In particular, excursions straddling an independent exponential time are studied in detail. Letting the parameter of the exponential time tend to zero it is seen that these results connect to the corresponding results for excursions of stationary diffusions (in stationary state). We characterize also the laws of the diffusion prior and posterior to the last zero before the exponential time. It is proved using Krein's representations that, e.g. the law of the length of the excursion straddling an exponential time is infinitely divisible. As an illustration of the results we discuss the Ornstein–Uhlenbeck processes.

Keywords and phrases: Brownian motion, last exit decomposition, local time, infinite divisibility, spectral representation, Ornstein–Uhlenbeck process

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1. Introduction and preliminaries

1.1 Throughout this paper, we shall assume that X is a linear regular recurrent diffusion taking values in $\mathbf{R}_+$ with 0 an instantaneously reflecting boundary.

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Let $\mathbf{P}_x$ and $\mathbf{E}_x$ denote, respectively, the probability measure and the expectation associated with X when started from $x \geq 0$. We assume that X is defined in the canonical space C of continuous functions $\omega : \mathbf{R}_+ \mapsto \mathbf{R}_+$. Let

$$\mathscr{C}_t := \sigma\{\omega(s) : s \leq t\}$$

denote the smallest σ-algebra making the co-ordinate mappings up to time t measurable and take $\mathscr{C}$ to be the smallest σ-algebra including all σ-algebras $\mathscr{C}_t$, $t \geq 0$.

The excursion space for excursions from 0 to 0 associated with X is a subset of C, denoted by E, and given by

$$\begin{aligned} E := \{\varepsilon \in C : \varepsilon(0) = 0,\ \exists\, \zeta(\varepsilon) > 0 \text{ such that } \varepsilon(t) > 0\ \forall\, t \in (0, \zeta(\varepsilon)) \\ \text{and } \varepsilon(t) = 0\ \forall\, t \geq \zeta(\varepsilon)\}. \end{aligned}$$

The notation $\mathscr{E}_t$ is used for the trace of $\mathscr{C}_t$ on E.

As indicated in the title of the paper our aim is to gather a number of fundamental results concerning the excursion theory for the diffusion X. In Section 2 the classical descriptions, the first one due to Itô and McKean and the second one due to Williams, are presented. In Section 3 the stationary excursions are discussed and, in particular, the description due to Bismut is reviewed. After this, in Section 4, we proceed by analyzing excursions straddling an exponential time. The paper is concluded with an example on Ornstein–Uhlenbeck processes.

Our motivation for this work arose from different origins:

- First, we would like to contribute to Professor Itô's being awarded the 1st Gauss prize, by offering some discussion and illustration of K. Itô's excursion theory, see [21], when specialized to linear diffusions. The present paper also illustrates Pitman and Yor's discussion (see [42] in this volume) of K. Itô's general theory of excursions for a Markov process.
- In the literature there seems to be lacking a detailed discussion on the excursion theory of linear diffusions. Information available has a very scattered character, see, e.g. Williams [52], Walsh [51], Pitman and Yor [38], [39], [40], [41], Rogers [46], Salminen [49]. The general theory of excursions has been developed in Itô [21], Meyer [35], Getoor [13], Getoor and Sharpe [15], [14], [16], [17], and Blumenthal [4]. Although the case with Brownian motion is well studied and understood, for textbook treatments see, e.g. Revuz and Yor [43] and Rogers and Williams [47], we find it important to highlight the main formulas for more general diffusions using the traditional Fellerian terminology and language.
- To generalize some recent results (see Winkel [53] and Bertoin, Fujita, Roynette and Yor [2]) on infinite divisibility of the distribution of the length of the excursion of a diffusion straddling an independent exponential time.

- The Ornstein–Uhlenbeck process is one of the most essential diffusions. To present in detail formulae for its excursions is important per se. One of the key tools hereby is the distribution of the first hitting time H_y of the point y from which the excursions are observed. For $y = 0$ this distribution can be derived via Doob's transform (see Doob [8]) which connects the Ornstein–Uhlenbeck process with standard Brownian motion (see Sato [50], and Göing–Jaeschke and Yor [19]). For arbitrary y the distribution is very complicated; for explicit expressions via series expansions, see Ricciardi and Sato [44], Linetsky [32] and Alili, Patie and Pedersen [1]. We will focus on excursions from 0 to 0 and relate our work to earlier papers by Hawkes and Truman [20], Pitman and Yor [40], and Salminen [48]. Due to the symmetry of the Ornstein–Uhlenbeck process around 0, it is sufficient for our purposes to consider only positive excursions— the treatment of negative ones is similar— and view the process with values in $\mathbf{R}_+$ and 0 being a reflecting boundary.

1.2 In this subsection we introduce the basic notation and facts concerning linear diffusions needed in the sequel. A main source of information remains Itô and McKean [22], see also Rogers and Williams [47], and Borodin and Salminen [6].

(i) Speed measure m associated with X is a measure on $\mathbf{R}_+$ which satisfies for all $0 < a < b < \infty$

$$0 < m((a,b)) < \infty.$$

For simplicity, it is assumed that m does not have atoms. An important fact is that X has a jointly continuous transition density $p(t;x,y)$ with respect to m, i.e.,

$$\mathbf{P}_x(X_t \in A) = \int_A p(t;x,y)\, m(dy),$$

where A is a Borel subset of $\mathbf{R}_+$. Moreover, p is symmetric in x and y, that is, $p(t;x,y) = p(t;y,x)$. The Green or the resolvent kernel of X is defined for $\lambda > 0$ as

$$R_\lambda(x,y) = \int_0^\infty dt\, \mathrm{e}^{-\lambda t}\, p(t;x,y).$$

(ii) Scale function S is an increasing and continuous function which can be defined via the identity

$$\mathbf{P}_x(H_a < H_b) = \frac{S(b) - S(x)}{S(b) - S(a)}, \quad 0 \le a < x < b, \tag{1}$$

where $H_.$ denotes the first hitting time, i.e.,

$$H_y := \inf\{t : X_t = y\}, \quad y \ge 0.$$

We normalize by setting $S(0) = 0$. Due to the recurrence assumption it holds $S(+\infty) = +\infty$. Recall that $\{S(X_{t\wedge H_0}) : t \geq 0\}$ is a continuous local $\mathbf{P}_x$-martingale for every $x \geq 0$ (see, e.g. Rogers and Williams [47] p. 276). It is easily proved that $S(X) = \{S(X_t) : t \geq 0\}$ is a (recurrent) diffusion taking values in $\mathbf{R}_+$. The scale function associated with $S(X)$ is the identity mapping $x \mapsto x$, $x \geq 0$, and we say that $S(X)$ is in natural scale. Clearly, also for $S(X)$ the boundary point 0 is instantaneously reflecting. Using the Skorokhod reflection equation it is seen that $S(X)$ is a $\mathbf{P}_x$-submartingale (*cf.* Meleard [34] Proposition 1.4 where the semimartingale decomposition is given in case there are two reflecting boundaries).

(iii) The infinitesimal generator of X can be expressed as the generalized differential operator

$$\mathscr{G} = \frac{d}{dm}\frac{d}{dS}$$

acting on functions f belonging to the appropriately defined domain $\mathscr{D}(\mathscr{G})$ of $\mathscr{G}$ (see Itô and McKean [22], Freedman [12], Borodin and Salminen [6]). In particular, since 0 is assumed to be reflecting then $f \in \mathscr{D}(\mathscr{G})$ implies that

$$f^+(0) := \lim_{x\uparrow 0} \frac{f(x) - f(0)}{S(x) - S(0)} = 0.$$

(iv) The distribution of the first hitting time of a point $y > 0$ has a $\mathbf{P}_x$-density

$$\mathbf{P}_x(H_y \in dt) = f_{xy}(t)\,dt.$$

This density can be connected with the derivative of the transition density of a killed diffusion obtained from X. To explain this, introduce the sample paths

$$\widehat{X}_t^{(y)} := \begin{cases} X_t, & t < H_y, \\ \partial, & t \geq H_y, \end{cases}$$

where ∂ is a point isolated from $\mathbf{R}_+$ (a "cemetary" point). Then $\{\widehat{X}_t^{(y)} : t \geq 0\}$ is a diffusion with the same scale and speed as X. Let $\hat{p}$ denote the transition density of $\widehat{X}^{(y)}$ with respect to m. Then, e.g. for $x > y$

$$f_{xy}(t) = \lim_{z\downarrow y} \frac{\hat{p}(t;x,z)}{S(z) - S(y)}. \tag{2}$$

For a fixed x and y, the mapping $t \mapsto f_{xy}(t)$ is continuous, as follows from the eigen-differential expansions and discussion in Itô and McKean p. 153 and 217 (see also Kent [24], [25]). Recall also the following formula for the Laplace transform of H_y

$$\mathbf{E}_x\left(\mathrm{e}^{-\alpha H_y}\right) = \frac{R_\alpha(x,y)}{R_\alpha(y,y)}, \tag{3}$$

which leads to

$$\int_0^\infty m(dx)\, \mathbf{E}_x\left(\mathrm{e}^{-\alpha H_y}\right) = \frac{1}{\alpha R_\alpha(y,y)}.$$

(v) There exists a jointly continuous family of local times

$$\{L_t^{(y)} : t \geq 0, y \geq 0\}$$

such that X satisfies the occupation time formula

$$\int_0^t ds\, h(X_s) = \int_0^\infty h(y) L_t^{(y)} m(dy),$$

where h is a nonnegative measurable function (see, e.g. Rogers and Williams [47] 49.1 Theorem p. 289). Consequently,

$$L_t^{(y)} = \lim_{\delta \downarrow 0} \frac{1}{m((y-\delta, y+\delta))} \int_0^t \mathbf{1}_{(y-\delta,y+\delta)}(X_s)\, ds.$$

For a fixed y introduce the inverse of $L^{(y)}$ via

$$\tau_\ell^{(y)} := \inf\{s : L_s^{(y)} > \ell\}.$$

Then $\tau^{(y)} = \{\tau_\ell^{(y)} : \ell \geq 0\}$ is an increasing Lévy process, in other words, a subordinator and its Lévy exponent is given by

$$\begin{aligned}\mathbf{E}_y\left(\exp(-\lambda \tau_\ell^{(y)})\right) &= \exp\left(-\ell / R_\lambda(y,y)\right) \\ &= \exp(-\ell \int_0^\infty \nu^{(y)}(dv)(1-\mathrm{e}^{-\lambda v})), \qquad (4)\end{aligned}$$

where $\nu^{(y)}$ is the Lévy measure of $\tau^{(y)}$. The assumption that the speed measure does not have atoms implies that $\tau^{(y)}$ does not have a drift. In case $y=0$ we write simply L, τ and ν.

1.3 Assuming that X is started from 0 we define for $t > 0$

$$G_t := \sup\{s \leq t : X_s = 0\} \quad \text{and} \quad D_t := \inf\{s \geq t : X_s = 0\}. \qquad (5)$$

The *last exit decomposition* at a fixed time t states that for $u < t < v$

$$\begin{aligned}&\mathbf{P}_0(G_t \in du, X_t \in dy, D_t \in dv) \\ &\qquad = p(u;0,0)\, f_{y0}(t-u)\, f_{y0}(v-t)\, du\, dv\, m(dy). \qquad (6)\end{aligned}$$

In fact, this trivariate distribution is only the skeleton of a more complete body of processes:

$$\{X_u : u \leq G_t\}, \quad \{X_{G_t+v} : v \leq t - G_t\}, \quad \text{and} \quad \{X_{t+v} : v \leq D_t - t\} \qquad (7)$$

the distributions of which we now characterize following Salminen [49]. For general approaches, see Getoor and Sharpe [15], [14], and Maisonneuve [33].

Let $x, y \in \mathbf{R}_+$ and $u > 0$ be given. Denote by $(X^{x,u,y}, \mathbf{P}_{x,u,y})$ the diffusion bridge from x to y of length u constructed from X, i.e., the measure $\mathbf{P}_{x,u,y}$ governing $X^{x,u,y}$ is the conditional measure associated with X started from x and conditioned to be at y at time u. The bridge $X^{x,u,y}$ is a strong non-time-homogeneous Markov process defined on the time axis $[0,u]$. For the first component in (7), we have conditionally on $G_t = u$

$$\{X_s : 0 \le s << G_t\} \stackrel{d}{=} \{X_s^{0,u,0} : 0 \le s \le u\}. \tag{8}$$

For the second component in (7), consider the process $\widehat{X}^{(y)}$ as introduced in (iv) above with $y = 0$. We write simply $\widehat{X}$ instead of $\widehat{X}^{(0)}$. For positive x and y let $\widehat{X}^{x,u,y}$ denote the bridge from x to y of length u constructed, as above, from $\widehat{X}$. The measure $\widehat{\mathbf{P}}_{x,u,y}$ governing $\widehat{X}^{x,u,y}$ can be extended by taking (in the weak sense)

$$\widehat{\mathbf{P}}_{0,u,y} := \lim_{x \downarrow 0} \widehat{\mathbf{P}}_{x,u,y}.$$

We let $\widehat{X}^{0,u,y}$ denote the process associated with $\widehat{\mathbf{P}}_{0,u,y}$. Then, conditionally on $G_t = u$ and $X_t = y$,

$$\{X_{G_t+s} : 0 \le s \le t - G_t\} \stackrel{d}{=} \{\widehat{X}_s^{0,t-u,y} : 0 \le s \le t-u\}. \tag{9}$$

For the final part in (7), by the Markov property, we have conditionally on $X_t = y$

$$\{X_{t+s} : s \le D_t - t\} \stackrel{d}{=} \{X_s : s \le H_0\}, \tag{10}$$

where $X_0 = y$.

Remark 1. Diffusion bridges can be seen as Doob's h-transforms of the underlying diffusion. Indeed, for the nonnegative functional Ψ on the path space we have for $v < u$

$$\widehat{\mathbf{E}}_{x,u,y}\left(\Psi\left(\omega_s : s \le v\right)\right) = \widehat{\mathbf{E}}_x\left(\Psi(\omega_s : s \le v)\,\frac{\hat{p}(u-v;\omega_v,y)}{\hat{p}(u;x,y)}\right).$$

Letting here $y \downarrow 0$ and using (2) we may describe the measure associated with the bridge $\widehat{X}^{x,u,0}$ via

$$\widehat{\mathbf{E}}_{x,u,0}\left(\Psi\left(\omega_s : s \le v\right)\right) = \widehat{\mathbf{E}}_x\left(\Psi(\omega_s : s \le v)\,\frac{f_{\omega_v 0}(u-v)}{f_{x0}(u)}\right). \tag{11}$$

2. Two descriptions of the Itô measure

2.1. Description due to Itô and McKean

We discuss the description of the Itô measure $\mathbf{n}$ where the excursions are studied by conditioning with respect to their lifetimes. Let $\widehat{X}$ be as in Section 1.3 and $\hat{p}(t;x,y)$ its transition density with respect to the speed measure; in other words,

$$\mathbf{P}_x(\widehat{X}_t \in dy) = \mathbf{P}_x(X_t \in dy; t < H_0) = \hat{p}(t;x,y)\, m(dy).$$

The Lévy measure ν of τ is absolutely continuous with respect to the Lebesgue measure, and the density— which we also denote by ν— is given by

$$\nu(v) := \nu(dv)/dv = \lim_{x\downarrow 0}\lim_{y\downarrow 0} \frac{\hat{p}(v;x,y)}{S(x)S(y)} =: p^{\uparrow}(v;0,0). \tag{12}$$

In Section 1.3 we have introduced the bridge $\widehat{X}^{x,t,y}$ and the measure $\widehat{\mathbf{P}}_{x,t,y}$ associated with it. The family of probability measures $\{\widehat{\mathbf{P}}_{x,t,y} : x > 0, y > 0\}$ is weakly convergent as $y \downarrow 0$ thus defining $\widehat{\mathbf{P}}_{x,t,0}$ for all $x > 0$. Intuitively, this is the process $\widehat{X}$ conditioned to hit 0 at time t, *cf.* (11). Moreover, letting now $x \downarrow 0$ we obtain a measure which we denote by $\widehat{\mathbf{P}}_{0,t,0}$ which governs a non-time homogeneous Markov process $\widehat{X}^{0,t,0}$ starting from 0, staying positive on the time interval $(0,t)$ and ending at 0 at time t.

Theorem 2. *(i) The law of the excursion life time ζ under the Itô excursion measure $\mathbf{n}$ is equal to the Lévy measure of the subordinator $\{\tau_\ell\}_{\ell\geq 0}$ and is given by*

$$\mathbf{n}(\zeta \in dv) = \nu(dv) = p^{\uparrow}(v;0,0)\, dv. \tag{13}$$

(ii) The Itô measure can be represented as the following integral

$$\mathbf{n}(d\varepsilon) = \int_0^\infty \mathbf{n}(\zeta \in dv)\, \widehat{\mathbf{P}}_{0,v,0}(d\varepsilon). \tag{14}$$

Moreover, the finite dimensional distributions of the excursion are characterized for $0 < t_1 < t_2 < \cdots < t_n$ and $x_i > 0,\ i = 1,2,\dots,n$ by

$$\begin{aligned}\mathbf{n}(\varepsilon_{t_1} \in dx_1, \varepsilon_{t_2} \in dx_2, \dots, \varepsilon_{t_n} \in dx_n)& \\ = m(dx_1)\, f_{x_1 0}(t_1)\, \hat{p}(t_2 - t_1; x_1, x_2)\, m(dx_2)& \\ \times \cdots \hat{p}(t_n - t_{n-1}; x_{n-1}, x_n)\, m(dx_n).&\end{aligned} \tag{15}$$

In particular, the excursion entrance law is given by

$$\mathbf{n}(\varepsilon_t \in dx) = m(dx)\, f_{x0}(t),$$

and it holds

$$\mathbf{n}(\zeta > t) = \int_0^\infty \mathbf{n}(\varepsilon_t \in dx) = \int_0^\infty m(dx)\, f_{x0}(t). \tag{16}$$

Combining the formulas (13) and (14) with the last exit decomposition (6) leads to a curious relation between the transition densities p and $p^{\uparrow}$.

Proposition 3. *The functions $p(t;0,0)$ and $p^{\uparrow}(t;0,0)$ satisfy the identity*

$$\int_0^t du\, p(u;0,0) \int_{t-u}^{\infty} dv\, p^{\uparrow}(v;0,0) = 1. \tag{17}$$

Proof. From (13) and (16) we may write

$$\int_t^{\infty} dv\, p^{\uparrow}(v;0,0) = \mathbf{n}(\zeta > t) = \int_0^{\infty} m(dx)\, f_{x0}(t).$$

Consequently, identity (17) can be rewritten as

$$\int_0^t du\, p(u;0,0) \int_0^{\infty} m(dx)\, f_{x0}(t-u) = 1, \tag{18}$$

but, in view of the last exit decomposition (6), identity (18) states that the last exit from 0 when starting from 0 takes place with probability 1 before t, in other words,

$$\mathbf{P}_0(G_t \le t) = 1,$$

which, of course, is trivially true. □

Remark 4. For another approach to (17), notice that (4) and (12) yield

$$\frac{1}{R_\lambda(0,0)} = \int_0^{\infty} dv\, p^{\uparrow}(v;0,0) \left(1 - \mathrm{e}^{-\lambda v}\right).$$

Hence, from the definition of the Green kernel,

$$1 = \int_0^{\infty} du\, \mathrm{e}^{-\lambda u}\, p(u;0,0) \int_0^{\infty} dv\, p^{\uparrow}(v;0,0)(1 - \mathrm{e}^{-\lambda v}). \tag{19}$$

Consequently,

$$\begin{aligned} \frac{1}{\lambda} &= \int_0^{\infty} du\, \mathrm{e}^{-\lambda u}\, p(u;0,0) \int_0^{\infty} dv\, \mathrm{e}^{-\lambda v} \int_v^{\infty} ds\, p^{\uparrow}(s;0,0) \\ &= \int_0^{\infty} du \int_0^{\infty} dv \mathrm{e}^{-\lambda(u+v)}\, p(u;0,0) \int_v^{\infty} ds\, p^{\uparrow}(s;0,0), \end{aligned}$$

from which (17) is easily deduced.

2.2. Description due to Williams

In the approach via the lengths of the excursions the focus is first on the time axis. In Williams' description (see Williams [52], and Rogers [45], [46]) the starting point of the analysis is on the space axis since the basic conditioning is with respect to the maximum of an excursion. To formulate the result, let for $\varepsilon \in E$

$$M(\varepsilon) := \sup\{\varepsilon_t : 0 < t < \zeta(\varepsilon)\}.$$

The key element in Williams' description is the diffusion $X^{\uparrow}$ obtained by conditioning $\widehat{X}$ not to hit 0. We use the notation $\mathbf{P}^{\uparrow}_x$ and $\mathbf{E}^{\uparrow}_x$ for the measure and the expectation associated with $X^{\uparrow}$ when started from x. To define this process rigorously set for a bounded $F_t \in \mathscr{C}_t$, $t > 0$,

$$\begin{aligned}\mathbf{E}^{\uparrow}_x(F_t) &:= \lim_{a\uparrow+\infty} \mathbf{E}_x(F_t\,; t < H_a \,|\, H_a < H_0) \\ &= \lim_{a\uparrow+\infty} \frac{\mathbf{E}_x(F_t\,; t < H_a < H_0)}{\mathbf{P}_x(H_a < H_0)} \\ &= \lim_{a\uparrow+\infty} \frac{\mathbf{E}_x\left(F_t\, S(X_t)\,; t < H_a \wedge H_0\right)}{S(x)},\end{aligned}$$

where the Markov property and formula (1) for the scale function are applied. The monotone convergence theorem yields

$$\mathbf{E}^{\uparrow}_x(F_t) = \frac{1}{S(x)}\,\mathbf{E}_x\left(F_t\, S(X_t)\,; t < H_0\right),$$

in other words, the desired conditioning is realized as Doob's h-transform of $\widehat{X}$ by taking h to be the scale function of X. It is easily deduced that the transition density and the speed measure associated with $X^{\uparrow}$ are given by

$$p^{\uparrow}(t;x,y) := \frac{\hat{p}(t;x,y)}{S(y)S(x)}, \quad m^{\uparrow}(dy) := S(y)^2\, m(dy).$$

We remark that the boundary point 0 is entrance-not-exit for $X^{\uparrow}$ and, therefore, $X^{\uparrow}$ can be started from 0 after which it immediately enters $(0,\infty)$ and never hits 0.

Theorem 5. *(i) The law of the excursion maximum M under the Itô excursion measure* $\mathbf{n}$ *is given by*

$$\mathbf{n}(M \geq a) = \frac{1}{S(a)}.$$

(ii) The Itô excursion measure $\mathbf{n}$ *can be represented via*

$$\mathbf{n}(d\varepsilon) = \int_0^{\infty} \mathbf{n}(M \in da)\, \mathbf{Q}^{(*,a)}(d\varepsilon),$$

where $\mathbf{Q}^{(*,a)}$ *is the distribution of two independent* $X^{\uparrow}$ *processes put back to back and run from 0 until they first hit level a.*

As an illustration, we give the following formula

$$\mathbf{n}(1-\exp(-\int_0^{\zeta} ds V(\varepsilon_s))) \\ = \int_0^{\infty} \mathbf{n}(M \in da) \left(1 - \left(\mathbf{E}_0^{\uparrow}(\exp(-\int_0^{H_a} du V(\omega_u)))\right)^2\right).$$

If $V \geq 0$, this quantity is the Lévy exponent of the subordinator

$$\left\{\int_0^{\tau_\ell} ds V(X_s) : \ell \geq 0\right\},$$

that is,

$$\mathbf{E}\left\{\exp\left(-\alpha \int_0^{\tau_\ell} ds V(X_s)\right)\right\} \\ = \exp\left\{-\ell \mathbf{n}\left(1-\exp\left(-\alpha \int_0^{\zeta} ds V(\varepsilon_s)\right)\right)\right\}.$$

Comparing the descriptions of the Itô excursion measure in Theorem 2 (in particular formula (15)) and in Theorem 5 hints that the processes $\widehat{X}$ and $X^{\uparrow}$ have, in addition to conditioning relationship, also a time reversal relationship. This is due to Williams [52], who particularized to the case of diffusions the general time reversal result, obtained by Nagasawa [36]. See also [43] p. 313, and [6] p. 35.

Proposition 6. *Let for a given* $x > 0$

$$\Lambda_x := \sup\{t : \omega(t) = x\}$$

denote the last exit time from x*. Then*

$$\{\widehat{X}_s : 0 \leq s \leq H_0\} \stackrel{d}{=} \{X^{\uparrow}_{\Lambda_x - s} : 0 \leq s \leq \Lambda_x\}, \tag{20}$$

where $\widehat{X}_0 = x$ *and* $X^{\uparrow}_0 = 0$.

3. Stationary excursions; Bismut's description

Consider the diffusion X with the time parameter t taking values in the whole of $\mathbf{R}$. In the case $m(\mathbf{R}_+) < \infty$ the measure governing X can be normalized to be a probability measure. Indeed, in this case the distribution of X_t is for every $t \in \mathbf{R}$ defined to be

$$\mathbf{P}(X_t \in dx) = m(dx)/m(\mathbf{R}_+) =: \widehat{m}(dx).$$

Recall from (5) the definitions of G_t and D_t, and introduce also $\Delta_t := D_t - G_t$.

Theorem 7. *Assume that $m(\mathbf{R}_+) < \infty$. Then the joint distribution of $t - G_t$ and $D_t - t$ is given by*

$$\begin{aligned}\mathbf{P}(t - G_t \in du, D_t - t \in dv)/dudv &= \int_0^\infty \widehat{m}(dy)\, f_{y0}(u)\, f_{y0}(v) \\ &= \nu(u+v)/m(\mathbf{R}_+).\end{aligned}$$

Consequently, for Δ_t it holds

$$\mathbf{P}(\Delta_t \in du)/du = u\,\nu(u)/m(\mathbf{R}_+). \tag{21}$$

Moreover, the law of the process $\{X_{G_t+v} : v \le \Delta_t\}$ is given by

$$\zeta(\varepsilon)\,\mathbf{n}(d\varepsilon)/m(\mathbf{R}_+), \tag{22}$$

where $\mathbf{n}(d\varepsilon)$ is the Itô measure, as introduced in Theorems 2 and 5, and ζ denotes the length of an excursion.

Proof. The density of $(t - G_t, D_t - t)$ is derived using the time reversibility of the diffusion X, i.e.,

$$\{X_t : t \in \mathbf{R}\} \overset{d}{=} \{X_{-t} : t \in \mathbf{R}\},$$

and the conditional independence given X_t. The fact that the density can be expressed via the density of the Lévy measure is stated (and proved) in Proposition 14 below, see formulas (32) and (33). To compute the distribution of Δ_t is elementary from the joint distribution of $t - G_t$ and $D_t - t$. For these results, we refer also to Kozlova and Salminen [28]. The statement concerning the law of $\{X_{G_t+v} : v \le \Delta_t\}$ has been proved in Pitman [37] (see Theorem p. 290 point (iii) and the formulation for excursions on p. 293 and 294)— all that remains for us to do is to find the right normalization constant, but this is fairly obvious, e.g. from the density of Δ_t. □

If $m(\mathbf{R}_+) = \infty$ the measure associated with X is still well-defined but "only" σ-finite. In this case, the distribution of X_t is plainly taken to be m. From (22) it is seen that we are faced with a representation of the Itô measure via stationary excursions valid in both cases $m(\mathbf{R}_+) < \infty$ and $m(\mathbf{R}_+) = \infty$. We focus now on this representation as displayed in (23) below, and present a proof of the representation using the diffusion theory (this provides, of course, also a proof of (22)). We remark that in Pitman [37] a more general case concerning homogeneous random sets is proved, and, hence, it seems worthwhile to give a "direct" proof in the diffusion case.

Theorem 8. *Let F be a measurable non-negative functional defined in the excursion space E. Then up to a normalization*

$$\mathbf{n}(F(\varepsilon)) = \mathbf{E}\left(\frac{1}{\Delta_t}\, F\left(X_{G_t+s} : 0 \le s \le \Delta_t\right)\right). \tag{23}$$

In particular, the process $\{X_{G_t+s} : 0 \le s \le \Delta_t\}$ conditionally on $\Delta_t = v$ is identical in law with the excursion bridge $\widehat{X}^{0,v,0}$, as introduced in Section 2.1.

Proof. Without loss of generality, we take $t = 0$. From (21) we have

$$\mathbf{n}(f(\zeta)) = \int_0^\infty f(a)\,\nu(a)\,da = \mathbf{E}\left(\frac{1}{\Delta_0}\, f(\Delta_0)\right),$$

where f is a measurable nonnegative function on $(0,\infty)$. Therefore, it is enough (*cf.* Theorem 2) to prove that

$$\mathbf{n}(F(\varepsilon)\,|\,\zeta = u) = \mathbf{E}\left(F\left(X_{G_0+s} : 0 \le s \le \Delta_0\right)\,|\,\Delta_0 = u\right). \tag{24}$$

Define for $0 \le s_1 < s_2 < \cdots < s_n \le u$

$$A_{1,n} := \{X_{G_0+s_1} \in dy_1, \ldots, X_{G_0+s_n} \in dy_n\},$$

and consider

$$\begin{aligned}
\mathbf{P}\left(A_{1,n}\,|\,\Delta_0 = u\right) &= \int_{y=0}^\infty \int_{v=0}^u \mathbf{P}\left(A_{1,n}, -G_0 \in dv, X_0 \in dy\,|\,\Delta_0 = u\right) \\
&= \int_{y=0}^\infty \int_{v=0}^u \mathbf{P}\left(A_{1,n}\,|\,\Delta_0 = u, G_0 = -v, X_0 = y\right) \\
&\qquad\qquad \times \mathbf{P}\left(-G_0 \in dv, X_0 \in dy\,|\,\Delta_0 = u\right).
\end{aligned}$$

From the description of the process X, the conditional independence and the equality of the laws of the past and future given X_0, and by formula (21) we obtain

$$\mathbf{P}\left(-G_0 \in dv, X_0 \in dy\,|\,\Delta_0 = u\right) = \frac{1}{u\,\nu(u)}\, f_{y,0}(v)\, f_{y,0}(u-v)\, m(dy)dv. \tag{25}$$

Letting k be such that $-v + s_k < 0 < -v + s_{k+1}$, if any, we write applying again the conditional independence

$$\mathbf{P}\left(A_{1,n}\,|\,\Delta_0 = u, G_0 = -v, X_0 = y\right)$$

$$
\begin{aligned}
&= \mathbf{P}\left(A_{1,k}A_{k+1,n}\,|\,\Delta_0 = u,\, G_0 = -v,\, X_0 = y\right) \\
&= \mathbf{P}\left(B_{1,k}\,|\,X_0 = y\right)\mathbf{P}\left(B_{k+1,n}\,|\,D_0 = u-v,\, X_0 = y\right),
\end{aligned}
$$

where for $1 \le i < j \le n$

$$
B_{i,j} := \{X_{-v+s_i} \in dy_i, \dots, X_{-v+s_j} \in dy_j\}.
$$

Recall from Introduction Section 1.2 (iv) the notation $\widehat{X}$ for the diffusion X killed when it hits 0, and from Remark 1 the definition of the bridge $\widehat{X}^{y,v,0}$. With these notations we have

$$
\begin{aligned}
&\mathbf{P}\left(A_{1,k}\,|\,G_0 = -v,\, X_0 = y\right) \\
&\quad = \widehat{\mathbf{P}}_{y,v,0}\left(\omega_{v-s_k} \in dy_k, \dots, \omega_{v-s_1} \in dy_1\right) \\
&\quad = \frac{1}{f_{y0}(v)}\,\widehat{p}(v-s_k; y, y_k)\,m(dy_k)\,\widehat{p}(s_k - s_{k-1}; y_k, y_{k-1})\,m(dy_{k-1}) \\
&\qquad\qquad \times \dots\, \widehat{p}(s_2 - s_1; y_2, y_1)\,m(dy_1)\,f_{y_1 0}(s_1)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbf{P}\left(B_{k+1,n}\,|\,D_0 = u-v,\, X_0 = y\right) \\
&\quad = \widehat{\mathbf{P}}_{y,u-v,0}\left(\omega_{s_{k+1}-v} \in dy_{k+1}, \dots, \omega_{s_n - v} \in dy_n\right) \\
&\quad = \frac{1}{f_{y0}(u-v)}\,\widehat{p}(s_{k+1}-v; y, y_{k+1})\,m(dy_{k+1}) \\
&\qquad \times \widehat{p}(s_{k+2} - s_{k+1}; y_{k+1}, y_{k+2})\,m(dy_{k+2}) \\
&\qquad\qquad \times \dots\, \widehat{p}(s_n - s_{n-1}; y_{n-1}, y_n)\,m(dy_n)\,f_{y_n 0}(u - s_n).
\end{aligned}
$$

Using now (25) and formulas above we have after some rearranging and after applying the symmetry of the transition density $\widehat{p}$

$$
\begin{aligned}
&\mathbf{P}\left(A_{1,n}\,|\,\Delta_0 = u\right) \\
&\quad = \frac{1}{u\,\nu(u)}\,m(dy_1)\,f_{y_1 0}(s_1)\widehat{p}(s_2 - s_1; y_1, y_2)\,m(dy_2) \\
&\qquad \times \dots \int_0^u dv \int_0^\infty m(dy)\,\widehat{p}(v - s_k; y_k, y)\,\widehat{p}(s_{k+1} - v; y, y_{k+1}) \\
&\qquad\qquad \times \dots\, \widehat{p}(s_n - s_{n-1}; y_{n-1}, y_n)\,m(dy_n)\,f_{y_n 0}(u - s_n).
\end{aligned}
$$

Performing the integration yields

$$
\begin{aligned}
&\mathbf{P}\left(A_{1,n}\,|\,\Delta_0 = u\right) \\
&\quad = \frac{1}{\nu(u)}\,m(dy_1)\,f_{y_1 0}(s_1)\widehat{p}(s_2 - s_1; y_1, y_2)\,m(dy_2) \\
&\qquad \times \dots\, \widehat{p}(s_n - s_{n-1}; y_{n-1}, y_n)\,m(dy_n)\,f_{y_n 0}(u - s_n).
\end{aligned}
$$

Consequently, (24) holds, and the proof is complete. □

Remark 9. The formula (23) was derived for Brownian motion by Bismut [3]. The connection with the Palm measure and stationary processes is discussed in Pitman [37]. In fact, Bismut describes in the Brownian case the law of the process $\{X_{G_t+s} : 0 \le s \le \Delta_t\}$ in terms of two independent 3-dimensional Bessel processes started from 0 and killed at the last exit time from an independent level distributed according to the Lebesgue measure (see [3] and [43] for details).

4. On the excursion straddling an independent exponential time

In the literature one can find several papers devoted to the properties of excursions straddling a fixed time t; first of all, Lévy's fundamental paper [31], which contains a lot about the zero set of Brownian motion, its (inverse) local time, excursions, and so on. See also Chung [7] starting from Lévy's paper [31], Durrett and Iglehart [9], and Getoor and Sharpe [16], [17]. In fact, the last exit decomposition (6) lies in the heart of these studies (see Getoor and Sharpe [15], [14]). However, it seems to us that excursions straddling an exponential time are not so much analyzed. Here we make some remarks on this subject.

Let T be an exponentially distributed random variable with parameter $\alpha > 0$, independent of X, and define

$$G_T := \sup\{s \le T : X_s = 0\}, \qquad D_T := \inf\{s \ge T : X_s = 0\},$$

and

$$\Delta_T := D_T - G_T.$$

The Lévy exponent of the inverse local time at 0 is denoted by Φ, in other words,

$$\mathbf{E}_0\left(\exp(-\lambda \tau_\ell)\right) = \exp\left(-\ell\,\Phi(\lambda)\right)$$

Recall the relation (*cf.* (4) with $y = 0$)

$$\Phi(\lambda) R_\lambda(0,0) = 1. \tag{26}$$

4.1. Last exit decomposition at T

In this subsection we study the distributions of different path segments of the diffusion X killed at the first hitting time of 0 after the exponential time T.

Theorem 10. *(i) The processes*

$$\{X_u : u \le G_T\} \quad \text{and} \quad \{X_{G_T+v} : v \le \Delta_T\}$$

are independent.
(ii) The law of $\{X_u : u \le G_T\}$ may be described as follows:

(a) $L_T := L_{G_T}$ *is exponentially distributed with mean* $1/\Phi(\alpha)$.
(b) The process $\{X_u : u \le G_T\}$ *conditionally on* $L_T = \ell$ *is distributed as* $\{X_u : u \le \tau_\ell\}$ *under the probability*

$$\exp(-\alpha\,\tau_\ell + \ell\,\Phi(\alpha))\,\mathbf{P}_0.$$

(iii) The law of the process $\{X_{G_T+v} : v \le \Delta_T\}$ *is given by*

$$\frac{1}{\Phi(\alpha)}\left(1-\mathrm{e}^{-\alpha\zeta(\varepsilon)}\right)\mathbf{n}(d\varepsilon). \tag{27}$$

where $\mathbf{n}(d\varepsilon)$ *is the Itô measure associated with the excursions away from 0 for* X *and* ζ *denotes the length of an excursion.*

Proof. Let F_1 and F_2 be two nonnegative functionals on the path space and consider

$$\begin{aligned}
&\mathbf{E}_0\left(F_1(X_u : u \le G_T)\,F_2(X_{G_T+v} : v \le \Delta_T)\right)\\
&\quad = \alpha\int_0^\infty dt\;\mathrm{e}^{-\alpha t}\mathbf{E}_0\left(F_1(X_u : u \le G_t)\,F_2(X_{G_t+v} : v \le \Delta_t)\right)\\
&\quad = \alpha\mathbf{E}_0\left(\sum_\ell\int_{\tau_{\ell-}}^{\tau_\ell} dt\;\mathrm{e}^{-\alpha t}F_1(X_u : u \le \tau_{\ell-})F_2(X_{\tau_{\ell-}+v} : v \le \tau_\ell - \tau_{\ell-})\right)\\
&\quad = \mathbf{E}_0\left(\int_0^\infty d\ell\;\mathrm{e}^{-\alpha\,\tau_\ell}F_1(X_u : u \le \tau_\ell)\right)\\
&\qquad\qquad\times\int\mathbf{n}(d\varepsilon)\left(1-\mathrm{e}^{-\alpha\,\zeta(\varepsilon)}\right)F_2(\varepsilon_s : s \le \zeta(\varepsilon))\\
&\quad = \Phi(\alpha)\mathbf{E}_0\left(\int_0^\infty d\ell\;\mathrm{e}^{-\alpha\,\tau_\ell}F_1(X_u : u \le \tau_\ell)\right)\\
&\qquad\qquad\times\int\mathbf{n}(d\varepsilon)\left(\frac{1-\mathrm{e}^{-\alpha\,\zeta(\varepsilon)}}{\Phi(\alpha)}\right)F_2(\varepsilon_s : s \le \zeta(\varepsilon)),
\end{aligned}$$

where the third equality is based on the properties of the Poisson random measure associated with the excursions (see Revuz and Yor [43] Master Formula p. 475). □

Remark 11. Notice that letting $\alpha \to 0$ in (27) and using

$$\lim_{\alpha\to 0}\frac{\alpha}{\Phi(\alpha)} = \lim_{\alpha\to 0}\alpha\,R_\alpha(0,0) = 1/m(\mathbf{R}_+) \tag{28}$$

yield the probability law of the excursion straddling a fixed time in the stationary setting, *cf.* (22) in Theorem 7.

As a corollary of Theorem 10 we have the following results which show that after conditioning the quantities do not depend on α. The formulas should be compared with (8), (9), and (10). The distributions of G_T and Δ_T are given, respectively, in (42) and (37) below.

Corollary 12. *For any three nonnegative functionals F_1, F_2, and F_3 on the path space it holds*

$$\begin{aligned}\mathbf{E}_0\left(F_1(X_u : u \le G_T)\,|\,G_T = g\right) &= \mathbf{E}_0\left(F_1(X_u : u \le g)\,|\,X_g = 0\right)\\ &= \mathbf{E}_{0,g,0}\left(F_1(\omega_u : u \le g)\right),\end{aligned} \tag{29}$$

$$\begin{aligned}&\mathbf{E}_0\left(F_2(X_{G_T+u} : u \le T - G_T)\,|\,T - G_T = d, X_T = y\right)\\ &\qquad = \mathbf{E}_0^{\uparrow}\left(F_2(X_u : u \le d)\,|\,X_d = y\right)\\ &\qquad = \widehat{\mathbf{E}}_{0,d,y}(F_2(\omega_s : s \le d)),\end{aligned} \tag{30}$$

and

$$\begin{aligned}\mathbf{E}_0\left(F_3(X_{G_T+v} : v \le \Delta_T)\,|\,\Delta_T = h\right) &= \mathbf{E}_0^{\uparrow}\left(F_3(X_v : v \le h)\,|\,X_h = 0\right)\\ &= \widehat{\mathbf{E}}_{0,h,0}(F_3(\omega_s : s \le h)).\end{aligned} \tag{31}$$

Proof. The statements (29) and (30) can be obtained from the corresponding result for fixed time as presented in (8) and (9), respectively. Also (31) can be derived from the fixed time result but we prefer to present here a proof based on the Master Formula. For this consider for $0 < a < b$

$$\begin{aligned}&\mathbf{E}_0\left(F_2(X_{G_T+v} : v \le \Delta_T)\,\mathbf{1}_{\{a \le \Delta_T < b\}}\right)\\ &\quad = \int \mathbf{n}(d\varepsilon)\left(\frac{1 - \mathrm{e}^{-\alpha\,\zeta(\varepsilon)}}{\Phi(\alpha)}\right) F_2(\varepsilon_s : s \le \zeta(\varepsilon))\,\mathbf{1}_{\{a \le \zeta(\varepsilon) < b\}}.\end{aligned}$$

Using the description (14) of the Itô excursion law we obtain

$$\begin{aligned}&\mathbf{E}_0\left(F_2(X_{G_T+v} : v \le \Delta_T)\,\mathbf{1}_{\{a \le \Delta_T < b\}}\right)\\ &\quad = \int_a^b du\,\nu(u)\left(\frac{1 - \mathrm{e}^{-\alpha u}}{\Phi(\alpha)}\right)\widehat{\mathbf{E}}_{0,u,0}(F_2(\omega_s : s \le u)).\end{aligned}$$

Recognizing here the density of the distribution of Δ_T as given in (37), we obtain (31). □

In the next proposition we discuss some properties of the processes $\{X_{G_T+u} : u \le T - G_T\}$ and $\{X_{T+u} : u \le D_T - T\}$. In particular, it is interesting that the time reversal of $\{X_{G_T+u} : u \le T - G_T\}$ has a clean description, as presented in point (iii) below.

Proposition 13. *(i) Conditionally on X_T, the processes*

$$\{X_{G_T+u} : u \leq T - G_T\} \quad \text{and} \quad \{X_{T+u} : u \leq D_T - T\}$$

are independent.
(ii) Given $X_T = y$, the process $\{X_{T+u} : u \leq D_T - T\}$ is distributed as the process $\{X_u : u \leq H_0\}$ when started from y.
(iii) Given $X_T = y$, the process $\{X_{T-u} : u \leq T - G_T\}$ is distributed as the process $\{X_u : u \leq H_0\}$ conditioned by the event $\{H_0 < T\}$ and started from y.

Proof. The first two claims are easy consequences of the Markov property. To prove the third claim we compute first the joint distribution of $T - G_T$ and X_T. Letting $(u,v) \mapsto \varphi(u,v)$ be a nonnegative measurable function formula (6) yields

$$\mathbf{E}_0\left(\varphi(T-G_T, X_T)\right) = \alpha \int_{\mathbf{R}_+^4} \varphi(u,y)\, \mathbf{1}_{\{u<t<v\}}\, \mathrm{e}^{-\alpha t}\, p(t-u;0,0) \\ \times f_{y0}(u)\, f_{y0}(v-t)\, m(dy)\, du\, dv\, dt.$$

Integrating with respect to v and t yields

$$\mathbf{E}_0\left(\varphi(T-G_T, X_T)\right) = \alpha R_\alpha(0,0) \int_{\mathbf{R}_+^2} \varphi(u,y)\, \mathrm{e}^{-\alpha u}\, f_{y0}(u)\, m(dy)\, du.$$

For any pair (Ψ, ψ), which consists of a nonnegative functional on path space and a Borel function, we consider for $u_o > 0$

$$\begin{aligned} Q(\Psi,\psi) &:= \mathbf{E}_0\left(\Psi\left(X_{T-u} : u \leq u_o\right) \mathbf{1}_{\{u_o < T-G_T\}}\, \psi(X_T)\right) \\ &= \alpha \int_0^\infty \mathrm{e}^{-\alpha t}\, \mathbf{E}_0\left(\Psi\left(X_{t-u} : u \leq u_o\right) \mathbf{1}_{\{u_o < t-G_t\}}\, \psi(X_t)\right) dt \\ &= \alpha \int_0^\infty dt\, \mathrm{e}^{-\alpha t} \int_{v=0}^t \int_{y=0}^\infty \mathbf{P}_0(t-G_t \in dv, X_t \in dy)\, \psi(y) \\ &\qquad \times \mathbf{E}_0\left(\Psi\left(X_{t-u} : u \leq u_o\right) \mathbf{1}_{\{u_o < t-G_t\}} \,|\, t-G_t = v, X_t = y\right). \end{aligned}$$

From (9),

$$\mathbf{E}_0\left(\Psi\left(X_{t-u} : u \leq u_o\right) \mathbf{1}_{\{u_o < t-G_t\}} \,|\, t-G_t = v, X_t = y\right) \\ = \widehat{\mathbf{E}}_{0,v,y}\left(\Psi\left(\omega_{v-u} : u \leq u_o\right)\right), \quad u_o < v.$$

By the time reversal property of diffusion bridges (see [49] Proposition 1) it holds for all $u_o \leq v$

$$\widehat{\mathbf{E}}_{0,v,y}\left(\Psi\left(\omega_{v-u} : u \leq u_o\right)\right) = \widehat{\mathbf{E}}_{y,v,0}\left(\Psi\left(\omega_u : u \leq u_o\right)\right).$$

From (11) we have for $u_o < v$

$$\widehat{\mathbf{E}}_{y,v,0}\left(\Psi\left(\omega_u : u \leq u_o\right)\right) = \widehat{\mathbf{E}}_y\left(\Psi\left(X_u : u \leq u_o\right) f_{X_{u_o}0}(v-u_o)\right) / f_{y0}(v)$$

and, therefore,

$$\begin{aligned} Q(\Psi,\psi) &= \int_{\mathbf{R}_+^2} \mathbf{P}_0(T-G_T\in dv, X_T\in dy)\,\psi(y)\,\widehat{\mathbf{E}}_{y,v,0}\left(\Psi\left(\omega_u : u\le u_o\right)\right) \\ &= \alpha\, R_\alpha(0,0)\int_{\mathbf{R}_+^2} dv\, m(dy)\,\psi(y) \\ &\qquad\qquad \times \mathrm{e}^{-\alpha v}\,\widehat{\mathbf{E}}_y\left(\Psi\left(X_u : u\le u_o\right) f_{X_{u_o}0}(v-u_o)\right) \\ &= \alpha\, R_\alpha(0,0)\int_{\mathbf{R}_+} m(dy)\,\psi(y) \\ &\qquad\qquad \times \mathbf{E}_y\left(\Psi\left(X_u : u\le u_o\right); u_o<H_0<T\right) \end{aligned}$$

by the Markov property. Consequently,

$$\begin{aligned} Q(\Psi,\psi) = \alpha\, R_\alpha(0,0)\int_{\mathbf{R}_+} & m(dy)\,\psi(y)\,\mathbf{P}_y(H_0<T) \\ & \times \mathbf{E}_y\left(\Psi\left(X_u : u\le u_o\right)\mathbf{1}_{\{u_o<H_0\}}\,|\, H_0<T\right), \end{aligned}$$

and this proves the claim since we may identify the distribution of X_T by taking $\Psi\equiv 1$ and letting $u_o\downarrow 0$. □

4.2. On the distribution of (G_T, D_T)

In this subsection the distributions of $T-G_T$, D_T-T and $\Delta_T := D_T-G_T$ are studied in detail.

Proposition 14. *The joint distribution of* $T-G_T$ *and* D_T-T *is given by*

$$\begin{aligned} &\mathbf{P}_0(T-G_T\in du, D_T-T\in dv) \\ &\qquad = du\,dv\,\alpha\, R_\alpha(0,0)\,\mathrm{e}^{-\alpha u}\int_0^\infty m(dy)\, f_{y0}(u)\, f_{y0}(v). && (32) \\ &\qquad = \frac{\alpha\,\mathrm{e}^{-\alpha u}\,\nu(u+v)}{\Phi(\alpha)}\,du\,dv. && (33) \end{aligned}$$

In particular,

$$\nu(u+v) = \int_0^\infty m(dy)\, f_{y0}(u)\, f_{y0}(v). \qquad (34)$$

Proof. From (6),

$$\begin{aligned} &\mathbf{P}_0(t-G_t\in du, D_t-t\in dv) \\ &\qquad = du\,dv\, p(t-u;0,0)\,\mathbf{1}_{\{u\le t, v\ge 0\}}\int_0^\infty m(dy)\, f_{y0}(u)\, f_{y0}(v), \end{aligned}$$

and, hence,

$$\mathbf{P}_0(T-G_T\in du, D_T-T\in dv)$$

$$= dudv\,\alpha \int_u^\infty dt\, \mathrm{e}^{-\alpha t}\, p(t-u;0,0) \int_0^\infty m(dy)\, f_{y0}(u)\, f_{y0}(v),$$

from which (32) follows. To derive (33), we apply again the Master Formula (see Revuz and Yor [43] p. 475). For this, let $(u,v) \mapsto \varphi(u,v)$ be a non-negative and Borel measurable function and define

$$Q(\varphi) := \mathbf{E}_0(\varphi(T-G_T, D_T - T)).$$

Letting τ_ℓ denote the inverse of the local time L at 0 we have

$$\begin{aligned} Q(\varphi) &= \mathbf{E}_0\left(\sum_{\ell \ge 0} \varphi(T-\tau_{\ell-}, \tau_\ell - T)\, \mathbf{1}_{\{\tau_{\ell-}<T<\tau_\ell\}}\right) \\ &= \mathbf{E}_0\left(\int_{\mathbf{R}_+^2} \varphi(T-\tau_\ell, z+\tau_\ell - T)\, \mathbf{1}_{\{\tau_\ell<T<\tau_\ell+z\}} \nu(z)\, dz d\ell\right), \end{aligned}$$

since $\{(\ell,\tau_\ell) : \ell \ge 0\}$ is a Poisson point process with Lévy measure $d\ell\, d\nu$, and T is independent of $\{\tau_\ell : \ell \ge 0\}$. Apply next that T is exponentially distributed to obtain

$$\begin{aligned} Q(\varphi) &= \mathbf{E}_0\left(\int_{\mathbf{R}_+^2} \nu(z) dz d\ell \int_{\tau_\ell}^{\tau_\ell+z} dt\, \alpha\, \mathrm{e}^{-\alpha t}\, \varphi(t-\tau_\ell, z+\tau_\ell - t)\right) \\ &= \alpha \mathbf{E}_0\left(\int_{\mathbf{R}_+^3} \nu(z) \mathrm{e}^{-\alpha(x+\tau_\ell)}\, \varphi(x, z-x)\, \mathbf{1}_{\{x \le z\}}\, dx dz d\ell\right), \end{aligned}$$

where we have substituted $x = t - \tau_\ell$. Furthermore, setting $y = z - x$ yields

$$\begin{aligned} Q(\varphi) &= \alpha \mathbf{E}_0\left(\int_{\mathbf{R}_+^3} \nu(y+x) \mathrm{e}^{-\alpha(x+\tau_\ell)}\, \varphi(x,y)\, dx dy d\ell\right) \\ &= \alpha \mathbf{E}_0\left(\int_0^\infty \mathrm{e}^{-\alpha\, \tau_\ell}\, d\ell\right) \int_{\mathbf{R}_+^2} \varphi(x,y)\, \mathrm{e}^{-\alpha\, x}\, \nu(y+x)\, dx dy, \end{aligned}$$

and (33) follows now easily from (4). The equality (34) is an immediate consequence of (32) and (33). □

Corollary 15. *(i) The densities for $T - G_T$, $D_T - T$, and Δ_T are given, respectively, by*

$$\mathbf{P}_0(T - G_T \in du)/du = \frac{\alpha}{\Phi(\alpha)}\, \mathrm{e}^{-\alpha u} \int_u^\infty \nu(z) dz, \tag{35}$$

$$\mathbf{P}_0(D_T - T \in dv)/dv = \frac{\alpha}{\Phi(\alpha)}\, \mathrm{e}^{\alpha v} \int_v^\infty \mathrm{e}^{-\alpha z} \nu(z) dz, \tag{36}$$

and

$$\mathbf{P}_0(\Delta_T \in da)/da = \frac{(1-\mathrm{e}^{-\alpha a})\nu(a)}{\Phi(\alpha)}. \tag{37}$$

(ii) The joint density of $T - G_T$ and Δ_T is

$$\mathbf{P}_0(T - G_T \in du, \Delta_T \in da)/du\,da = \frac{\alpha}{\Phi(\alpha)}\,\mathrm{e}^{-\alpha u}\,\nu(a), \quad u \le a. \tag{38}$$

(iii) The density of $T - G_T$ conditionally on $\Delta_T = a$ is

$$\mathbf{P}_0(T - G_T \in du \,|\, \Delta_T = a)/du = \frac{\alpha}{1 - \mathrm{e}^{-\alpha a}}\,\mathrm{e}^{-\alpha u}, \quad u \le a. \tag{39}$$

Proposition 16. *The joint Laplace transform of G_T and D_T is given by*

$$\mathbf{E}_0\left(\exp\left(-\gamma_1 G_T - \gamma_2 D_T\right)\right) = \frac{\Phi(\gamma_2 + \alpha) - \Phi(\gamma_2)}{\Phi(\gamma_1 + \gamma_2 + \alpha)}. \tag{40}$$

In particular,

$$\mathbf{E}_0\left(\mathrm{e}^{-\gamma \Delta_T}\right) = \frac{\Phi(\gamma + \alpha) - \Phi(\gamma)}{\Phi(\alpha)}, \tag{41}$$

and the random variables G_T and Δ_T are independent. The density of G_T is given by

$$\mathbf{P}_0(G_T \in du)/du = \Phi(\alpha)\,\mathrm{e}^{-\alpha u}\,p(u;0,0). \tag{42}$$

Proof. The formula (42) for the density of G_T is obtained from (6) by integrating. The independence of G_T and Δ_T follows easily from (40). To derive the joint Laplace transform of G_T and D_T consider

$$\begin{aligned}
&\mathbf{E}_0\left(\exp\left(-\gamma_1 G_T - \gamma_2 D_T\right)\right) \\
&\quad = \int_0^\infty dt\,\alpha\,\mathrm{e}^{-\alpha t} \int_{u<t<v} \mathrm{e}^{-\gamma_1 u - \gamma_2 v}\,\mathbf{P}_0(G_t \in du, D_t \in dv).
\end{aligned}$$

Applying the last exit decomposition formula (6) yields

$$\begin{aligned}
&\mathbf{E}_0\left(\exp\left(-\gamma_1 G_T - \gamma_2 D_T\right)\right) \\
&\quad = \int_0^\infty du\,\mathrm{e}^{-\gamma_1 u}\,p(u;0,0) \int_u^\infty dv\,\mathrm{e}^{-\gamma_2 v} \int_u^v dt\,\alpha\,\mathrm{e}^{-\alpha t} \\
&\qquad\qquad \times \int_0^\infty m(dy)\,f_{y0}(t-u)\,f_{y0}(v-t) \\
&\quad = \int_0^\infty du\,\mathrm{e}^{-\gamma_1 u}\,p(u;0,0) \int_0^\infty da\,\mathrm{e}^{-\gamma_2 (a+u)} \int_u^{a+u} dt\,\alpha\,\mathrm{e}^{-\alpha t}\,p(u;0,0) \\
&\qquad\qquad \times \int_0^\infty m(dy)\,f_{y0}(t-u)\,f_{y0}(a+u-t) \\
&\quad = \int_0^\infty du\,\mathrm{e}^{-(\gamma_1+\gamma_2)u}\,p(u;0,0) \int_0^\infty da\,\mathrm{e}^{-\gamma_2 a} \int_0^a db\,\alpha\,\mathrm{e}^{-\alpha(b+u)}\,p(u;0,0) \\
&\qquad\qquad \times \int_0^\infty m(dy)\,f_{y0}(b)\,f_{y0}(a-b)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha \int_0^\infty du\, \mathrm{e}^{-(\gamma_1+\gamma_2+\alpha)u} p(u;0,0) \int_0^\infty m(dy) \int_0^\infty da\, \mathrm{e}^{-\gamma_2 a} \\
&\qquad\qquad\qquad \times \int_0^a db\, \mathrm{e}^{-\alpha b} f_{y0}(b)\, f_{y0}(a-b) \\
&= \alpha R_{\gamma_1+\gamma_2+\alpha}(0,0) \int_0^\infty m(dy)\, \mathbf{E}_y\left(\mathrm{e}^{-(\gamma_2+\alpha)H_0}\right) \mathbf{E}_y\left(\mathrm{e}^{-\gamma_2 H_0}\right).
\end{aligned}
$$

To proceed, we have

$$
\begin{aligned}
&\int_0^\infty m(dy)\, \mathbf{E}_y\left(\mathrm{e}^{-(\gamma_2+\alpha)H_0}\right) \mathbf{E}_y\left(\mathrm{e}^{-\gamma_2 H_0}\right) \\
&\qquad = \frac{1}{R_{\gamma_2+\alpha}(0,0) R_{\gamma_2}(0,0)} \int_0^\infty m(dy)\, R_{\gamma_2+\alpha}(y,0) R_{\gamma_2}(y,0).
\end{aligned}
$$

The integral term in this expression can be evaluated:

$$
\begin{aligned}
&\int_0^\infty m(dy)\, R_{\alpha+\gamma_2}(y,0) R_{\gamma_2}(y,0) \\
&\qquad = \int_0^\infty m(dy) \int_0^\infty dt\, \mathrm{e}^{-(\alpha+\gamma_2)t} p(t;y,0) \int_0^\infty ds\, \mathrm{e}^{-\gamma_2 s} p(s;y,0) \\
&\qquad = \int_0^\infty dt\, \mathrm{e}^{-(\alpha+\gamma_2)t} \int_0^\infty ds\, \mathrm{e}^{-\gamma_2 s} p(t+s;0,0) \\
&\qquad = \int_0^\infty dt\, \mathrm{e}^{-(\alpha+\gamma_2)t} \int_t^\infty du\, \mathrm{e}^{-\gamma_2 (u-t)} p(u;0,0) \\
&\qquad = \int_0^\infty du\, \mathrm{e}^{-\gamma_2 u} \frac{1-\mathrm{e}^{-\alpha u}}{\alpha} p(u;0,0), \\
&\qquad = \frac{1}{\alpha}\left(R_{\gamma_2}(0,0) - R_{\gamma_2+\alpha}(0,0)\right).
\end{aligned}
$$

where the Chapman–Kolmogorov equation and the symmetry of the transition density p is applied, and, by (26), this completes the proof. □

Remark 17. (i) From Proposition 14 it is seen that the density of Δ_T can also be written in the form

$$
\mathbf{P}_0(\Delta_T \in da)/da = \frac{\alpha}{\Phi(\alpha)} \int_0^\infty m(dy) \int_0^a db\, \mathrm{e}^{-\alpha b} f_{y0}(b)\, f_{y0}(a-b),
$$

which taking into account (37) leads to the identity

$$
\frac{(1-\mathrm{e}^{-\alpha a})}{\alpha}\, \nu(a) = \int_0^\infty m(dy) \int_0^a db\, \mathrm{e}^{-\alpha b} f_{y0}(b)\, f_{y0}(a-b).
$$

Let here $\alpha \to 0$ to obtain

$$
\nu(a) = \int_0^\infty m(dy) \int_0^a \frac{db}{a} f_{y0}(b)\, f_{y0}(a-b). \tag{43}
$$

It is interesting to compare this expression with the following one obtained from (34)

$$\nu(a) = \int_0^\infty m(dy)\, f_{y0}(b)\, f_{y0}(a-b). \tag{44}$$

The fact that the right hand sides of (43) and (44) do not depend on b can also be explained via the Chapman–Kolmogorov equation.
(ii) We may study distributions associated with G_t, D_t and Δ_t in the stationary case, i.e., if $m(\mathbf{R}_+) < \infty$, by letting $\alpha \to 0$, as observed in Remark 11. From Proposition 14 and Corollary 15 we deduce the following results:

$$\mathbf{P}(-G_t \in du, D_t \in dv)/dudv = \frac{1}{m(\mathbf{R}_+)}\, \nu(u+v).$$

$$\mathbf{P}(-G_t \in du)/du = \mathbf{P}(D_t \in du)/du = \frac{1}{m(\mathbf{R}_+)} \int_u^\infty \nu(v)\, dv,$$

$$\mathbf{P}(\Delta_t \in da)/da = \frac{1}{m(\mathbf{R}_+)}\, a\, \nu(a).$$

Moreover, letting $Z_T := (T - G_T)/\Delta_T$ then (Z_T, Δ_T) converges in distribution as $\alpha \to 0$ to (U, Δ), where U and Δ are independent with U uniformly distributed on $(0,1)$ and Δ is distributed as Δ_t (*cf.* Theorem 7).

4.3. Infinite divisibility

In the paper by Bertoin et al. [2] it is proved that the distribution of Δ_T for a Bessel process with dimension $d = 2(1-\alpha), 0 < \alpha < 1$, is infinitely divisible (in fact, self-decomposable) and the Lévy measure associated with this distribution is computed. In this section we show that the distribution of Δ_T is infinitely divisible in general, i.e., for all regular and recurrent diffusions. Moreover, we also prove that the distributions of $T - G_T$ and $D_T - T$ have this property. The key to these results is the Krein representation of the density of the Lévy measure ν (see Knight [26], Kent [25], Küchler and Salminen [30], and, in general on Krein's theory of strings, Kotani and Watanabe [27], Dym and McKean [10]) according to which

$$\nu(a) = \int_0^\infty \mathrm{e}^{-az} M(dz), \tag{45}$$

where the measure M has the properties

$$\int_0^\infty \frac{M(dz)}{z(z+1)} < \infty \quad \text{and} \quad \int_0^\infty \frac{M(dz)}{z} = \infty.$$

.

Theorem 18. *The distributions of $T - G_T, D_T - T$ and Δ_T are infinitely divisible.*

Proof. As seen from (35), (36), and (37), the intrinsic term in the densities of $T - G_T$, $D_T - T$ and Δ_T is the density $\nu(a)$ of the Lévy measure of the inverse local time at 0. We consider first the distribution of $T - G_T$. Applying the Krein representation (45) in (35) yields

$$\begin{aligned}\mathbf{P}_0(T - G_T \in du)/du &= \frac{\alpha}{\Phi(\alpha)}\,\mathrm{e}^{-\alpha u}\int_u^\infty da\int_0^\infty M(dz)\,\mathrm{e}^{-az}\\ &= \frac{\alpha}{\Phi(\alpha)}\,\mathrm{e}^{-\alpha u}\int_0^\infty \frac{M(dz)}{z}\,\mathrm{e}^{-uz}\\ &= \frac{\alpha}{\Phi(\alpha)}\int_0^\infty \frac{M(dz)}{z}\,\mathrm{e}^{-(\alpha+z)u}\\ &= \int_0^\infty (\alpha+z)\,\mathrm{e}^{-(\alpha+z)u}\,\widehat{M}_\alpha(dz)\end{aligned}$$

with

$$\widehat{M}_\alpha(dz) = \frac{\alpha}{\Phi(\alpha)}\,\frac{M(dz)}{z(\alpha+z)}. \tag{46}$$

The claim of the theorem follows now from the fact that mixtures of exponential distributions are infinitely divisible (see Bondesson [5]). For $D_T - T$ we compute similarly from (36) via the Krein representation

$$\begin{aligned}\mathbf{P}_0(D_T - T \in dv)/dv &= \frac{\alpha}{\Phi(\alpha)}\,\mathrm{e}^{\alpha v}\int_v^\infty \mathrm{e}^{-\alpha a}\,\nu(a)da.\\ &= \int_0^\infty z\mathrm{e}^{-zv}\,\widehat{M}_\alpha(dz).\end{aligned}$$

To analyze the distribution of Δ_T we use the Krein representation in (37) to obtain

$$\mathbf{P}_0(\Delta_T \in da)/da = \frac{1}{\Phi(\alpha)}\int_0^\infty \left(\mathrm{e}^{-za} - \mathrm{e}^{-(\alpha+z)a}\right) M(dz). \tag{47}$$

Notice that for $a \geq 0$

$$f(a;z,\alpha) = \frac{z(\alpha+z)}{\alpha}\left(\mathrm{e}^{-za} - \mathrm{e}^{-(\alpha+z)a}\right)$$

is a probability density as a function of a. In fact, letting T_1 and T_2 be two independent exponentially distributed random variables, with respective parameters z and $\alpha + z$, then the sum $T_1 + T_2$ has the density $f(a;z,\alpha)$. In particular, the distribution of $T_1 + T_2$ is a gamma convolution (which, by definition, is the law of finite sum of independent gamma variables). Next we notice that letting

$$\Pi_{z,\alpha}(dx) := \frac{z(\alpha+z)}{\alpha}x^{-2}\,dx, \quad z < x < \alpha + z$$

we may represent the distribution of $T_1 + T_2$ as a mixture of Gamma(2)-distributions as follows

$$f(a;z,\alpha) = \int_0^\infty x^2\, a\, \mathrm{e}^{-xa}\, \Pi_{z,\alpha}(dx). \tag{48}$$

Combining the representation (48) with (47) yields

$$\begin{aligned} \mathbf{P}_0(\Delta_T \in da)/da &= \frac{\alpha}{\Phi(\alpha)} \int_0^\infty \frac{f(a;z,\alpha)}{z(\alpha+z)} M(dz) \\ &= \int_0^\infty x^2\, a\, \mathrm{e}^{-xa}\, \widehat{\Pi}_\alpha(dx), \end{aligned} \tag{49}$$

where $\widehat{\Pi}_\alpha$ is a probability measure on $\mathbf{R}_+$ given for any Borel set A in $\mathbf{R}_+$ by

$$\widehat{\Pi}_\alpha(A) = \int_0^\infty \widehat{M}_\alpha(dz) \Pi_{z,\alpha}(A). \tag{50}$$

The claim that the distribution of Δ_T is infinitely divisible follows now from (49) by evoking the result that mixtures of Gamma(2)-distributions are infinitely divisible (see Kristiansen [29]). □

Remark 19. (i) Recall from Bondesson [5] that a probability distribution F on $\mathbf{R}_+$ is called a generalized gamma convolution (GGC) if its Laplace transform can be written as

$$\int_0^\infty \mathrm{e}^{-sa} F(da) = \exp\left(-\mu s + \int_0^\infty \log\left(\frac{t}{t+s}\right) U(dt) \right), \tag{51}$$

where $\mu \geq 0$ and U is a measure on $(0,\infty)$ satisfying

$$\int_{(0,1]} |\log t| U(dt) < \infty \quad \text{and} \quad \int_{(1,\infty)} \frac{U(dt)}{t} < \infty.$$

It is known that if β is the total mass of U then the distribution F in (51) is a mixture of Gamma(β)-distributions (see [5] Theorem 4.1.1 p. 49).
(ii) The distribution of the length Δ_t of an excursion straddling a fixed time t for a stationary diffusion (with stationary probability distribution) is given in Theorem 7 (21) as

$$\mathbf{P}(\Delta_t \in da) = \frac{a\,\nu(a)}{m(\mathbf{R}_+)}\, da.$$

Also in this case the distribution of Δ_t is a mixture of Gamma(2)-distributions and, hence, it is infinitely divisible. In fact,

$$\mathbf{P}(\Delta_t \in da)/da = \int_0^\infty z^2\, a\, \mathrm{e}^{-za} \widetilde{M}(dz).$$

where the probability measure $\widetilde{M}$ is given in terms of the Krein measure M via

$$\widetilde{M}(dz) = M(dz)/(m(\mathbf{R}_+) z^2).$$

5. Case study: Ornstein–Uhlenbeck processes

In this section we give some explicit formulas for excursions from 0 to 0 associated with Ornstein–Uhlenbeck processes. It is possible to obtain such formulas due to the symmetry of the Ornstein–Uhlenbeck process around 0. Analogous results for excursions from an arbitrary point x to x are less tractable.

5.1. Basics

Let U denote the Ornstein-Uhlenbeck diffusion with parameter $\gamma > 0$, i.e., U is the solution of the SDE

$$dU_t = dB_t - \gamma U_t dt \quad \text{with} \quad U_0 = u,$$

and most of the time, but not always, we take $u = 0$. Recall that the speed measure and the scale function of U can be taken to be

$$m(dx) := 2\,\mathrm{e}^{-\gamma x^2}\,dx \quad \text{and} \quad S(x) := \int_0^x \mathrm{e}^{\gamma y^2}\,dy,$$

respectively. Moreover, see [6] p. 137, the Green kernel of Ornstein–Uhlenbeck process with respect to the speed measure is given for $x \geq y$ by

$$R_\lambda(x,y) = \frac{\Gamma(\lambda/\gamma)}{2\sqrt{\gamma\pi}} \exp\left(\frac{\gamma x^2}{2}\right) D_{-\lambda/\gamma}\left(x\sqrt{2\gamma}\right) \times \exp\left(\frac{\gamma y^2}{2}\right) D_{-\lambda/\gamma}\left(-y\sqrt{2\gamma}\right),$$

where D denotes the parabolic cylinder function. In particular, since

$$D_{-\lambda/\gamma}(0) = \sqrt{\pi}\left(2^{\lambda/(2\gamma)}\,\Gamma((\lambda+\gamma)/(2\gamma))\right)^{-1},$$

we have, after some manipulations,

$$R_\lambda(0,0) = \frac{\sqrt{\pi}\,\Gamma(\lambda/\gamma)}{2\sqrt{\gamma}}\left(2^{\lambda/(2\gamma)}\,\Gamma((\lambda+\gamma)/(2\gamma))\right)^{-2}.$$

Consequently, using the formula

$$\Gamma(x) = \frac{2^{x-1}}{\sqrt{\pi}}\Gamma((x+1)/2)\,\Gamma(x/2)$$

we obtain

$$R_\lambda(0,0) = \frac{1}{\Phi(\lambda)} = \frac{\Gamma(\lambda/(2\gamma))}{4\Gamma((\lambda+\gamma)/(2\gamma))}. \tag{52}$$

We remind also that U can be represented as the deterministic time change (Doob's transformation) of Brownian motion via

$$U_t = \mathrm{e}^{-\gamma t}\,(u + \beta_{a_t}),$$

where β is a standard Brownian motion and $a_t := (\mathrm{e}^{2\gamma t} - 1)/2\gamma$ (see Doob [8]).

5.2. Killed Ornstein–Uhlenbeck processes

We consider now the Ornstein–Uhlenbeck process killed at the first hitting time of 0, and denote this process by $\widehat{U}$. Let Y be the diffusion on $\mathbf{R}_+$ satisfying the SDE

$$dY_t = dB_t + \left(\frac{1}{Y_t} - \gamma Y_t\right) dt, \quad Y_0 = y > 0.$$

Recall that Y may be described as the radial part of the three-dimensional Ornstein–Uhlenbeck process. In [6] p. 138 the basic properties of such processes are presented. In particular, we record that 0 is an entrance-not-exit boundary point and the process is positively recurrent its stationary distribution being the Maxwell distribution, i.e., the distribution with the density proportional to the speed measure of Y, that is,

$$m^Y(dx) := 2x^2\,\mathrm{e}^{-\gamma x^2}\,dx, \quad x > 0.$$

We remark that there is a misprint in [6] p. 139; the stationary distribution in the general case is not a χ^2-distribution but a generalization of the Maxwell distribution. The transition density of Y with respect to its speed measure m^Y is

$$p^Y(t;x,y) = \frac{\sqrt{\gamma}\,\mathrm{e}^{3\gamma t/2}}{\sqrt{2\pi\sinh(\gamma t)}\,xy}\exp\left(-\frac{\gamma \mathrm{e}^{-\gamma t}(x^2+y^2)}{2\sinh(\gamma t)}\right)\sinh\left(\frac{\gamma xy}{\sinh(\gamma t)}\right),$$

and can be computed from the transition density of a Bessel process using Doob's transform (for an approach via inverting the Laplace transform see Giorno et al. [18]). In Salminen [48] it is proved that

$$\begin{aligned}\mathbf{P}_x(\widehat{U}_t \in dy) &= \mathbf{P}_x(U_t \in dy, t < H_0)\\ &= \mathrm{e}^{-\gamma t}\,p^Y(t;x,y)\,\frac{\varphi_\gamma(y)}{\varphi_\gamma(x)}\,m^Y(dy),\end{aligned} \tag{53}$$

where $\varphi_\gamma(x) = 1/x$ is the unique (up to multiplicative constants) decreasing positive solution of the ODE associated with Y killed at rate γ:

$$\frac{1}{2}u''(x) + \left(\frac{1}{x} - \gamma x\right)u'(x) = \gamma u(x).$$

From (53) we obtain

Proposition 20. *The transition density (with respect to its speed measure m) of the Ornstein–Uhlenbeck killed at the first hitting time of 0 is given by*

$$\hat{p}(t;x,y) = \frac{\sqrt{\gamma}\,\mathrm{e}^{\gamma t/2}}{\sqrt{2\pi\sinh(\gamma t)}}\exp\left(-\frac{\gamma \mathrm{e}^{-\gamma t}(x^2+y^2)}{2\sinh(\gamma t)}\right)\sinh\left(\frac{\gamma xy}{\sinh(\gamma t)}\right). \tag{54}$$

Combining the expression of the transition density in (54) with formula (2) yields the distribution of H_0 (see also Sato [50] and Going–Jaeschke and Yor [19]).

Proposition 21. *The density of the first hitting time of 0 for the Ornstein–Uhlenbeck process $\{U_t\}$ is given by*

$$f_{x0}(t) = \frac{\gamma^{3/2} x \mathrm{e}^{\gamma t/2}}{\sqrt{2\pi}(\sinh(\gamma t))^{3/2}} \exp\left(-\frac{\gamma \mathrm{e}^{-\gamma t} x^2}{2\sinh(\gamma t)}\right). \tag{55}$$

5.3. Lévy measure of inverse local time and densities of $\Delta_T, T-G_T$ and $D_T - T$

The density of the Lévy measure of the inverse local time at 0 is obtained by applying formula (12) (see also Hawkes and Truman [20]). Moreover, using (52) in formula (4) leads to an explicit expression for the Bernstein function associated with the inverse local time at 0.

Proposition 22. *The density of the Lévy measure of the inverse local time at 0 is*

$$\nu(t) = \frac{\gamma^{3/2} \mathrm{e}^{\gamma t/2}}{\sqrt{2\pi}(\sinh(\gamma t))^{3/2}} = \frac{(2\gamma)^{3/2} \mathrm{e}^{2\gamma t}}{\sqrt{2\pi}\,(\mathrm{e}^{2\gamma t}-1)^{3/2}}. \tag{56}$$

Let $\{\tau_\ell : \ell \geq 0\}$ be the inverse local time at 0. Then

$$\mathbf{E}_0\left(\exp(-\lambda \tau_\ell)\right) = \exp\left(-\ell \frac{4\Gamma((\lambda+\gamma)/2\gamma)}{\Gamma(\lambda/2\gamma)}\right).$$

Next we display the distributions of $\Delta_T, T-G_T$, and $D_T - T$. Recall that these distributions are infinitely divisible and the densities are expressable via the density of the Lévy measure, as stated in Corollary 15 formulae (35) and (36), and in Theorem 18. To simply the notation, we take $\gamma = 1$.

Proposition 23. *With $\Phi(\alpha)$ as in (52), the distributions of $\Delta_T, T-G_T$ and $D_T - T$ are given, respectively, by*

$$\mathbf{P}_0(\Delta_T \in da)/da = \frac{1-\mathrm{e}^{-\alpha a}}{\Phi(\alpha)} \frac{2}{\sqrt{\pi}} \mathrm{e}^{2a} \left(\mathrm{e}^{2a}-1\right)^{-3/2}, \tag{57}$$

$$\mathbf{P}_0(T - G_T \in da)/da = \frac{\alpha\, \mathrm{e}^{-\alpha a}}{\Phi(\alpha)} \frac{2}{\sqrt{\pi}} \left(\mathrm{e}^{2a}-1\right)^{-1/2}, \tag{58}$$

and

$$\mathbf{P}_0(D_T - T \in da)/da = \frac{\alpha\, \mathrm{e}^{\alpha a}}{\Phi(\alpha)} \int_a^\infty du\, \mathrm{e}^{-\alpha u} \frac{2}{\sqrt{\pi}} \mathrm{e}^{2u} \left(\mathrm{e}^{2u}-1\right)^{-3/2}. \tag{59}$$

5.4. The Krein measure

As seen in Section 4.3, the Krein representation plays a central rôle in the proof of infinite divisibility of the distributions of $T-G_T, D_T-T$, and Δ_T. Therefore, it seems motivated to compute the measure M (*cf.* (45)) in this representation for Ornstein–Uhlenbeck processes.

To start with, we give the spectral representation of the transition density of $\hat{p}$ of the Ornstein–Uhlenbeck process killed at the first hitting time of 0. Instead of computing from scratch, we exploit the spectral representation for p^Y (with $\gamma = 1$) as presented in Karlin and Taylor [23] p. 333:

$$p^Y(t;x,y) = \sum_{n=0}^{\infty} w_{n,1/2}^{-1}\, \mathrm{e}^{-2nt}\, L_n^{(1/2)}(x^2)\, L_n^{(1/2)}(y^2), \tag{60}$$

where $\{L_n^{(1/2)} : n = 0,1,2,\dots\}$ is the family of Laguerre polynomials with parameter $1/2$ normalized via

$$\int_0^{\infty} \left(L_n^{(1/2)}(x^2)\right)^2 m^Y(dx) = \frac{\sqrt{\pi}}{2}\binom{n+\frac{1}{2}}{n} =: w_{n,1/2}. \tag{61}$$

Notice that we consider the symmetric density with respect to the speed measure m^Y. From (53) and (60) the spectral representation of $\hat{p}$ is now obtained immediately and is given by

$$\hat{p}(t;x,y) = \sum_{n=0}^{\infty} w_{n,1/2}^{-1}\, \mathrm{e}^{-(2n+1)t}\, xL_n^{(1/2)}(x^2)\, yL_n^{(1/2)}(y^2). \tag{62}$$

The normalization (61) coincides with the normalization in Erdelyi et al. [11] (see formula (2) p. 188 where the notation for the norm is h_n). Therefore, from [11] formula (13) p. 189 we have

$$L_n^{(1/2)}(0) = \binom{n+\frac{1}{2}}{n} \tag{63}$$

and, consequently (*cf.* (55)), we obtain the spectral representation for the density of the first hitting time of 0

$$\begin{aligned} f_{x0}(t) &= \sum_{n=0}^{\infty} w_{n,1/2}^{-1}\, \mathrm{e}^{-(2n+1)t}\, xL_n^{(1/2)}(x^2)\, L_n^{(1/2)}(0). \\ &= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \mathrm{e}^{-(2n+1)t}\, xL_n^{(1/2)}(x^2). \end{aligned} \tag{64}$$

To find the spectral representation for the density of the Lévy measure we apply formula (56) which yields

$$\nu(t) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \binom{n+\frac{1}{2}}{n}\, \mathrm{e}^{-(2n+1)t}. \tag{65}$$

In view of (45), we have

Proposition 24. *The measure M in the Krein representation of ν for the Ornstein–Uhlenbeck process is given by*

$$M(dz) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \binom{n+\frac{1}{2}}{n} \delta_{\{2n+1\}}(dz),$$

where $\delta_{\{a\}}$ is the Dirac measure at a.

Note that

$$\nu(t) = \frac{2}{\sqrt{\pi}} \mathrm{e}^{-t} \left(1-\mathrm{e}^{-2t}\right)^{-3/2},$$

and, hence, (65) may also be obtained from the MacLaurin expansion of $x \mapsto (1-x)^{-3/2}$ evaluated at $x = \mathrm{e}^{-2t}$.

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