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On the mathematics of emergence*

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Abstract. We describe a setting where convergence to consensus in a population of autonomous agents can be formally addressed and prove some general results establishing conditions under which such convergence occurs. Both continuous and discrete time are considered and a number of particular examples, notably the way in which a population of animals move together, are considered as particular instances of our setting.

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1. A basic example: flocking

A common situation occurring in a number of disciplines is that in which a number of autonomous agents reach a consensus without a central direction. An example of this is the emergence of a common belief in a price system when activity takes place in a given market. Another example is the emergence of common languages in primitive societies, or the dawn of vowel systems.

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Yet a third example is the way in which populations of animals move together (referred as “schooling”, “flocking”, or “herding” depending on the considered animals).

As a motivating example in this introduction we consider a population, say of birds, whose members are moving in $\mathbb{E} = \mathbb{R}^3$. This situation has been recently studied in [6] and in what follows we freely draw from this paper.

It has been observed that under some initial conditions, for example on the positions and velocities of the birds, the state of the flock converges to one in which all birds fly with the same velocity. A way to justify this observation is to postulate a model for the evolution of the flock and exhibit conditions on the initial state under which a convergence as above is established. In case these conditions are not satisfied, dispersion of the flock may occur.

The model proposed in [6] postulates the following behavior: every bird adjusts its velocity by adding to it a weighted average of the differences of its velocity with those of the other birds. That is, at time $t \in \mathbb{N}$, and for bird i,

$$v_i(t+h) - v_i(t) = h \sum_{j=1}^{k} a_{ij}(v_j(t) - v_i(t)). \tag{1}$$

Here h is the magnitude of the time step and the weights $\{a_{ij}\}$ quantify the way the birds influence each other. It is reasonable to assume that this influence is a function of the distance between birds. This assumption is given form via a non-increasing function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that the *adjacency matrix* A_x has entries

$$a_{ij} = \frac{H}{(1 + \|x_i - x_j\|^2)^\beta} \tag{2}$$

for some fixed $H > 0$ and $\beta \geq 0$.

We can write the set of equalities (1) in a more concise form. Let A_x be the $k \times k$ matrix with entries a_{ij}, D_x be the $k \times k$ diagonal matrix whose ith diagonal entry is $d_i = \sum_{j \leq k} a_{ij}$ and $L_x = D_x - A_x$ the Laplacian of A_x (a matrix increasingly considered in the study of graphs and weighted graphs [3, 12]). Then

$$\begin{aligned} v_i(t+h) - v_i(t) &= -h \sum_{j=1}^{n} a_{ij}(v_i(t) - v_j(t)) \\ &= -h\Big(\sum_{j=1}^{n} a_{ij}\Big) v_i(t) + h \sum_{j=1}^{n} a_{ij} v_j(t) \\ &= -h[D_x v(t)]_i + h[A_x v(t)]_i \\ &= -h[L_x v(t)]_i. \end{aligned}$$

Note that the matrix notation $A_x v(t)$ does not have the usual meaning of a $k \times k$ matrix acting on $\mathbb{R}^k$. Instead, the matrix A_x is acting on $\mathbb{E}^k$ by mapping $(v_1, \ldots, v_k)$ to $(a_{i1}v_1 + \cdots + a_{ik}v_k)_{i \leq k}$. The same applies to L_x.

Adding a natural equation for the change of positions we obtain the system

$$\begin{aligned} x(t+h) &= x(t) + hv(t) \\ v(t+h) &= (\mathrm{Id} - hL_x)\, v(t). \end{aligned} \tag{3}$$

We will consider evolution for continuous time. That is, $x, v \in \mathbb{E}$ are functions of $t \in \mathbb{R}$. The corresponding model is obtained by letting h tend to zero and is given by the system of differential equations

$$\begin{aligned} x' &= v \\ v' &= -L_x v. \end{aligned} \tag{4}$$

One of the main results in [6] (Theorem 2 there) can be restated as follows (and is a particular case of one of the main results in this paper, Theorem 2 below).

Theorem 1. *Let $x_0, v_0 \in \mathbb{E}$. Then, there exists a unique solution $(x(t), v(t))$ of (4), defined for all $t \in \mathbb{R}$, with initial conditions $x(0) = x_0$ and $v(0) = v_0$. If $\beta < 1/2$ then, when $t \to \infty$ the velocities $v_i(t)$ tend to a common limit $\widehat{v} \in \mathbb{E}$ and the vectors $x_i - x_j$ tend to a limit vector $\widehat{x_{ij}}$, for all $i, j \leq k$. The same happens if $\beta \geq 1/2$ provided the initial values x_0 and v_0 satisfy a given, explicit, relation.*

2. A more general setting

In what follows we extend the situation considered in Section 1 to a more general setting.

We consider two variables: $\boldsymbol{v}$ —describing the object whose emergence is of interest— and x —describing other features of the agents— both varying with time. We assume the existence of two inner product spaces X and $\mathbb{F}$ such that $x \in X$ and $\boldsymbol{v} \in \mathbb{F}$.

When we talk of convergence of $(\boldsymbol{v}_1(t), \dots, \boldsymbol{v}_k(t))$ to a common value we mean the existence of a point $\widehat{\boldsymbol{v}} \in \mathbb{F}$ such that, when $t \to \infty$, $(\boldsymbol{v}_1(t), \dots, \boldsymbol{v}_k(t)) \to (\widehat{\boldsymbol{v}}, \dots, \widehat{\boldsymbol{v}})$.

Let $\Delta_{\mathbb{F}}$ denote the diagonal of $\mathbb{F}^k$, i.e.,

$$\Delta_{\mathbb{F}} = \{(\boldsymbol{v}, \dots, \boldsymbol{v}) \mid \boldsymbol{v} \in \mathbb{F}\}.$$

Then, convergence to a common value means convergence to the diagonal or, if we let $V = \mathbb{F}^k / \Delta_{\mathbb{F}}$, convergence to 0 in this quotient space. To establish such a convergence we need a norm in V. In the following, we will fix an inner product $\langle\ ,\ \rangle$ in V and we will consider its induced norm $\|\ \|$. We will often write $\Lambda(v)$ for $\|v\|^2$ and $\Gamma(x)$ for $\|x\|^2$.

We next give an extension of system (4) in Section 1. We will assume a (Lipschitz or $\mathscr{C}^1$) function

$$F : X \times V \to V$$

satisfying, for some $\mathbf{C}, \delta > 0$ and $0 \le \gamma < 1$, that

$$\text{for } x \in X,\ v \in V,\ \|F(x,v)\| \le \mathbf{C}(1+\|x\|^2)^{\frac{\gamma}{2}}\|v\|^{\delta}. \tag{5}$$

Let $\mathscr{M}(k \times k)$ be the space of $k \times k$ real matrices. We consider (Lipschitz or $\mathscr{C}^1$) maps

$$\begin{aligned} L : X &\to \mathscr{M}(k \times k) \\ x &\mapsto L_x \end{aligned}$$

satisfying that $L_x(1,1,\dots,1) = 0$. Note that any such L_x induces a linear operator on V which, abusing language, we will also denote by L_x.

To a pair (F, L) as above, we will associate the system of differential equations

$$\begin{aligned} x' &= F(x,v) \\ v' &= -L_x v. \end{aligned} \tag{6}$$

Our first main result deals with convergence to consensus for (6). To state it, we need to impose a hypothesis on L. We next describe this hypothesis.

For $x \in X$ define

$$\xi_x = \min_{\substack{v \in V \\ v \neq 0}} \frac{\langle L_x v, v\rangle}{\|v\|^2}. \tag{7}$$

In what follows we fix a map L satisfying that there exists $K, \beta > 0$ such that,

$$\text{for all } x \in X, \quad \xi_x \ge \frac{K}{(1+\Gamma(x))^{\beta}}. \tag{8}$$

For a solution (x, v) of (6), at a time $t \in \mathbb{R}_+$, $x(t)$ and $v(t)$ are elements in X and V, respectively. In particular, $x(t)$ determines a matrix $L_{x(t)}$ and an associated real $\xi_{x(t)}$. For notational simplicity we will denote them by L_t and ξ_t. Similarly, we will write $\Lambda(t)$ and $\Gamma(t)$ for the values of Λ and Γ, at $v(t)$ and $x(t)$, respectively. Finally, we will write Γ_0 for $\Gamma(0)$ and similarly for Λ_0.

The following quantities, independent of t, will repeatedly occur in our exposition:

$$\alpha = \frac{2\beta}{1-\gamma}, \qquad \boldsymbol{a} = 2^{\frac{1+\gamma}{1-\gamma}} \frac{((1-\gamma)\mathbf{C})^{\frac{2}{1-\gamma}} \Lambda_0^{\frac{\delta}{1-\gamma}}}{(\delta K)^{\frac{2}{1-\gamma}}}, \quad \text{and} \quad \boldsymbol{b} = 2^{\frac{1+\gamma}{1-\gamma}}(1+\Gamma_0).$$

Note that α varies in $(0, +\infty)$.

Theorem 2. *Assume that F satisfies condition (5) and that L satisfies condition (8). Let $x_0 \in X$ and $v_0 \in V$. Then, there exists a unique solution $(x(t), v(t))$ of (6) for all $t \in \mathbb{R}$. Furthermore, assume that one of the three following hypotheses hold:*

(i) $\beta < \frac{1-\gamma}{2}$,
(ii) $\beta = \frac{1-\gamma}{2}$ *and* $\Lambda_0^\delta < \frac{(\delta K)^2}{2^{1+\gamma}((1-\gamma)\mathbf{C})^2}$,
(iii) $\beta > \frac{1-\gamma}{2}$ *and*

$$\left(\frac{1}{\boldsymbol{a}}\right)^{\frac{1}{\alpha-1}}\left[\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\right] > \boldsymbol{b}$$

Then there exists a constant B_0 (independent of t, made explicit in the proof of each of the three cases) such that $\Gamma(t) \le B_0$ for all $t \in \mathbb{R}_+$. In addition, for all $t \ge 0$,

$$\Lambda(t) \le \Lambda_0 e^{-2\frac{K}{B_0^\beta}t}$$

and therefore $\Lambda(t) \to 0$ when $t \to \infty$. Finally, there exists $\widehat{x} \in X$ such that $x(t) \to \widehat{x}$ when $t \to \infty$ and there exists $B_1 > 0$ such that, for every $t \ge 0$,

$$\|x(t) - \widehat{x}\| \le B_1 e^{-2\frac{K}{B_0^\beta}t}.$$

Remark 1. (i) Write

$$\boldsymbol{a}' = 2^{\frac{1+\gamma}{1-\gamma}} \frac{((1-\gamma)\mathbf{C})^{\frac{2}{1-\gamma}}}{(\delta K)^{\frac{2\beta}{1-\gamma}}}$$

so that $\boldsymbol{a} = \boldsymbol{a}'\Lambda_0^{\frac{\delta}{1-\gamma}}$, and let

$$\mathbf{A} = 2^{\frac{1+\gamma}{\gamma-1}}\left(\frac{1}{\boldsymbol{a}'}\right)^{\frac{1}{\alpha-1}}\left[\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\right].$$

In contrast with Γ_0 and Λ_0, which describe the initial state of the system, the constant $\mathbf{A}$ is a constant associated to the model (i.e., to (6) only). The condition in Theorem 2(iii) can now be written as

$$\Lambda_0^{-\frac{\delta}{(2\beta+\gamma)-1}}\mathbf{A} > 1 + \Gamma_0.$$

This expression exhibits a neat trade-off between the initial values $v(0)$ and $x(0)$. If Γ_0 is large (i.e., under the assumption that x represents positions, if the population is dispersed) then Λ_0 needs to be small to ensure convergence to consensus (i.e., the original k values of $\boldsymbol{v}$ need to be clustered). Conversely, if Λ_0 is large then Γ_0 needs to be small but note that there is a critical value for

Λ_0, namely $\mathbf{A}^{\frac{(2\beta+\gamma)-1}{\delta}}$, above which convergence to consensus is not guaranteed for any initial value of $\boldsymbol{x}$.

(ii) We note that Theorem 1 follows from Theorem 2. We will see this in §3.2 below.

3. Some examples

In this section we give substance to the general setting we just developed by describing how it applies in several examples (and, in particular, how assumptions (5) and (8) are checked). Before doing so, we describe some features of the matrix L_x we associated to A_x in Section 1.

3.1. On Laplacians of non-negative, symmetric matrices

The most interesting instances of the map L are those for which, as in Section 1, L_x depends on an adjacency matrix. We define *adjacency functions* to be smooth functions

$$\begin{aligned} A : X &\to \mathscr{M}(k \times k) \\ x &\mapsto A_x. \end{aligned}$$

Given a $k \times k$ matrix A_x the *Laplacian* L_x of A_x is defined to be

$$L_x = D_x - A_x$$

where $D_x = \mathsf{diag}(d_1, \dots, d_k)$ and $d_\ell = \sum_{j=1}^{k} a_{\ell j}$. Note that L_x does not depend on the diagonal entries of A_x. Note also that, in contrast with the contents of Section 1, we did not require here that A_x is symmetric, or even non-negative, let alone be defined by a distance function in Euclidean 3-space. Our use of the word "Laplacian" in this context follows the one in [1] rather than the one in [4].

The Laplacian of a non-negative, symmetric matrix A_x, however, has a number of properties which deserve attention. Therefore, for some time to come, we assume that A_x is non-negative and symmetric. In this context the Laplacian has its origins in graph theory where the matrix A_x is the adjacency matrix of a (possibly weighted) graph G and many of the properties of G can be read out from L_x (see [12]).

The space $\mathbb{F}^k$ inherits an inner product from that of $\mathbb{F}$. Moreover, the Laplacian L_x acts on $\mathbb{F}^k$ and satisfies the following:

(a) For all $\boldsymbol{v} \in \mathbb{F}$, $L_x(\boldsymbol{v}, \dots, \boldsymbol{v}) = 0$.
(b) If $\lambda_1, \dots, \lambda_k$ are the eigenvalues of L_x then

$$0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k = \|L_x\| \le k.$$

(c) For all $\boldsymbol{v} \in \mathbb{F}^k$,

$$\langle L_x \boldsymbol{v}, \boldsymbol{v} \rangle = \frac{1}{2} \sum_{i,j=1}^{k} a_{ij} \|\boldsymbol{v}_i - \boldsymbol{v}_j\|^2.$$

A proof for (c) can be found in [3]. The other two properties are easy to prove (but see [12, Theorem 2.2(c)] for the inequality $\lambda_k \le k$). Note that (b) implies L_x is positive semidefinite.

The second eigenvalue λ_2 of L_x is called the *Fiedler number* of A_x. We denote the Fiedler number of A_x by ϕ_x. Note that, since $L_x(\Delta_{\mathbb{F}}) = 0$, L_x induces an endomorphism on V which we will also denote by L_x. In addition, since L_x is positive semidefinite it follows that

$$\phi_x = \min_{\substack{v \in V \\ v \neq 0}} \frac{\langle L_x v, v \rangle}{\|v\|^2}. \tag{9}$$

That is (when A_x is non-negative and symmetric), ϕ_x coincides with ξ_x as defined in (7). Note also that since ϕ_x is the second smallest eigenvalue of L_x (the smallest being 0) one has that

$$\phi_x = \min_{\substack{\boldsymbol{v} \in \Delta_F^{\perp} \\ \boldsymbol{v} \neq 0}} \frac{\langle L_x \boldsymbol{v}, \boldsymbol{v} \rangle}{\|\boldsymbol{v}\|^2} \tag{10}$$

where now $\langle\ ,\ \rangle$ is the inner product in $\mathbb{F}^k$ induced by that on $\mathbb{F}$ and $\Delta_{\mathbb{F}}^{\perp}$ is the orthogonal subspace to Δ_F (and hence the eigenspace of $\lambda_2, \ldots, \lambda_k$).

In all the examples in this section, the matrix L_x is obtained, given an adjacency function A, by taking the Laplacian of A_x. We have just seen that when A_x is non-negative and symmetric all the eigenvalues of L_x (as an element in $\mathscr{M}(k \times k)$) are non-negative, and that the number ξ_x is the second smallest of these eigenvalues, the smallest being 0. Therefore, in this case, to check condition (8) amounts to prove a lower bound on the Fiedler number. This is not necessarily true when A_x is non-symmetric.

3.2. The basic flocking situation

We refer here to the flocking situation considered in Section 1. In this case we have spaces $\mathbb{E} = \mathbb{F} = \mathbb{R}^3$ and the space X is defined as V, by letting $X = \mathbb{E}^k/\Delta_{\mathbb{E}}$. We next need to define inner products in X and V. While we could use those induced by the inner products of $\mathbb{E}$ and $\mathbb{F}$, respectively, it appears to be more convenient to proceed differently.

We let $Q_{\mathbb{F}} : \mathbb{F}^k \times \mathbb{F}^k \to \mathbb{R}$ defined by

$$Q_{\mathbb{F}}(\boldsymbol{u}, \boldsymbol{v}) = \frac{1}{2} \sum_{i,j=1}^{k} \langle \boldsymbol{u}_i - \boldsymbol{u}_j, \boldsymbol{v}_i - \boldsymbol{v}_j \rangle.$$

Then $Q_{\mathbb{F}}$ is bilinear, symmetric, and, when restricted to $V \times V$, positive definite. It follows that it defines an inner product $\langle\ ,\ \rangle_V$ on V. In a similar way we define $Q_{\mathbb{E}}$ and its induced inner product $\langle\ ,\ \rangle_X$ on X. Since the norms $\|\ \|_V$ and $\|\ \|_X$ are the only ones we consider on V and X, respectively, we will drop the subscript in what follows.

Recall, given $\boldsymbol{x} \in \mathbb{E}^k$, the matrix A_x has entries

$$a_{ij} = \frac{H}{(1 + \|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2)^\beta}$$

for some $H > 0$ and $\beta \geq 0$. Note that the dependence of the elements a_{ij} on $\boldsymbol{x}$ is only through the class x of $\boldsymbol{x}$ in X. Hence the notation A_x. A key feature of this matrix is that it is non-negative and symmetric. We are therefore interested in a lower bound on ϕ_x. The following result, which we will prove in §3.3, provides such a bound.

Proposition 1. *Let A be a nonnegative, symmetric matrix, $L = D - A$ its Laplacian, ϕ its Fiedler number, and*

$$\mu = \min_{i \neq j} a_{ij}.$$

Then $\phi \geq k\mu$.

The following corollary shows that condition (8) holds in the basic flocking situation.

Corollary 1. *If $a_{ij} = \frac{H}{(1+\|x_i - x_j\|^2)^\beta}$ then $\phi_x \geq \frac{kH}{(1+\Gamma(x))^\beta}$.*

Proof. We apply Proposition 1 and use that $\mu \geq \frac{H}{(1+\Gamma(x))^\beta}$. □

Corollary 1 shows that condition (8) holds with $K = kH$. Also, $F : X \times V \to V$ is given by $F(x, v) = v$ which satisfies condition (5) with $\mathbf{C} = 1$, $\gamma = 0$, and $\delta = 1$ due to the choice of norms in V and X. In this case $\alpha = 2\beta$ and

$$\mathbf{A} = \left(\frac{K}{2}\right)^{\frac{\alpha}{\alpha-1}} \left[\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\right].$$

We can therefore apply Theorem 2 to obtain, as a special case, Theorem 1.

3.3. Flocking with unrelated pairs: I

We next extend the basic flocking situation to one where some pairs of birds do not communicate. We do require, however, that there is "sufficient connection" between the birds. To do so, we recall a few relations between graphs and matrices. Any $k \times k$ symmetric matrix S with entries in $\{0,1\}$ induces a (undirected) graph $G(S)$ with vertex set $\{1,\dots,k\}$ and edge set $E = \{(i,j) \mid a_{ij} = 1\}$. Conversely, any graph G with k vertices induces a $k \times k$ symmetric matrix $S(G)$ with entries in $\{0,1\}$. Furthermore, these constructions are inverse to each other in the sense that $G(S(G)) = G$ and $S(G(S)) = S$. One can extend these considerations to nonnegative matrices. A non-negative symmetric matrix A induces a matrix $\overline{A}$ over $\{0,1\}$ by replacing each non-zero entry of A by a 1, and hence, a graph $G(\overline{A})$.

Fix a $k \times k$ symmetric matrix $M = (m_{ij})$ over $\{0,1\}$ such that $G(M)$ is connected. Given $\boldsymbol{x} \in \mathbb{E}^k$ we take the adjacency matrix A_x given by

$$a_{ij} = m_{ij} \frac{H}{(1 + \|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2)^{\beta}}. \tag{11}$$

Since the actual value of the diagonal elements a_{ii} of A_x are irrelevant (they do not affect L_x) in the sequel we assume, without loss of generality, that $m_{ii} = 0$ for all $1 \le i \le k$, and therefore, that $a_{ii} = 0$ as well. The matrix A_x thus defined is symmetric and $G(\overline{A_x})$ coincides with $G(M)$. In particular, $G(\overline{A_x})$ is connected.

Now let A be a nonnegative symmetric matrix. Denote by ϕ its Fiedler number and by $\phi_{\overline{A}}$ the Fiedler number of $\overline{A}$. It is well known [12, Theorem 2.1(c)] that $G(\overline{A})$ is connected if and only if the Fiedler number $\phi_{\overline{A}}$ of $\overline{A}$ is positive. This gives context to the next result (extending Proposition 1).

Proposition 2. *Let A be a nonnegative, symmetric matrix, $L = D - A$ its Laplacian, ϕ its Fiedler number, and*

$$\mu = \min_{\substack{a_{ij} \neq 0 \\ i \neq j}} a_{ij}.$$

Then $\phi \ge \phi_{\overline{A}} \mu$.

Proof. For all $\boldsymbol{v} \in \mathbb{F}^k$, using (c) above twice,

$$\langle L\boldsymbol{v}, \boldsymbol{v} \rangle = \frac{1}{2} \sum_{\substack{a_{ij} \neq 0 \\ i \neq j}} a_{ij} \|\boldsymbol{v}_i - \boldsymbol{v}_j\|^2 \ge \frac{1}{2} \mu \sum_{\substack{a_{ij} \neq 0 \\ i \neq j}} \|\boldsymbol{v}_i - \boldsymbol{v}_j\|^2 = \mu \langle \overline{L}\boldsymbol{v}, \boldsymbol{v} \rangle$$

with $\overline{L}$ the Laplacian of $\overline{A}$. Therefore, using (10) twice,

$$\phi = \min_{\substack{\boldsymbol{v} \in \Delta_F^{\perp} \\ \boldsymbol{v} \neq 0}} \frac{\langle L\boldsymbol{v}, \boldsymbol{v} \rangle}{\|\boldsymbol{v}\|^2} \ge \mu \min_{\substack{\boldsymbol{v} \in \Delta_F^{\perp} \\ \boldsymbol{v} \neq 0}} \frac{\langle \overline{L}\boldsymbol{v}, \boldsymbol{v} \rangle}{\|\boldsymbol{v}\|^2} = \mu \phi_{\overline{A}}.$$

□

Note that Proposition 2 factors a lower bound for the Fiedler number ϕ_x of A_x as a product of two quantities, one related with the size of the non-zero entries in A_x and the other to "how much connected" is $G(\overline{A_x}) = G(M)$. Proposition 1 can be now easily proved.

Proof of Proposition 1. It is well known (take, e.g. G to be the graph with k isolated points in [12, Theorem 3.6]) that the Fiedler number of the complete graph K_k with k vertices is k. Now apply Proposition 2. □

Let ϕ_M be the Fiedler number of M, which coincides with that of $\overline{A_x}$ for all $x \in X$. We can also use Proposition 2 to deduce that condition (8) holds with $K = \phi_M H$, since, for all $x \in X$,

$$\min_{v \neq 0} \frac{\langle \overline{L_x} v, v\rangle}{\|v\|^2} = \phi_x \geq \phi_{\overline{L}} \min_{a_{ij} \neq 0} a_{ij} \geq \phi_M \frac{H}{(1+\Gamma(x))^{\beta}}.$$

Remark 2. One could define

$$\Gamma_M(x) = \frac{1}{2} \sum_{m_{ij} \neq 0} \|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2$$

and the reasoning above would yield $\min_{v \neq 0} \frac{\langle \overline{L_x} v, v\rangle}{\|v\|^2} \geq \phi_M \frac{H}{(1+\Gamma_M(x))^{\beta}}$, an inequality stronger than (8).

We have seen that condition (8) holds with $K = \phi_M H$. As in §3.2, $F(x,v) = v$ which satisfies condition (5) with $\mathbf{C} = 1$, $\gamma = 0$, and $\delta = 1$. The following result thus follows from Theorem 2.

Proposition 3. *Let M be a $k \times k$ symmetric matrix such that $G(M)$ is connected and, for $\boldsymbol{x} \in \mathbb{E}^k$, let A_x be given by*

$$a_{ij} = m_{ij} \frac{H}{(1 + \|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2)^{\beta}}.$$

Let $x_0 \in X$ and $v_0 \in V$ and $(x(t), v(t))$ be the unique solution of

$$\begin{aligned} x' &= v \\ v' &= -L_x v \end{aligned}$$

where L_x is the Laplacian of A_x. Furthermore, assume that one of the three following hypotheses hold:

(i) $\beta < \frac{1}{2}$,
(ii) $\beta = \frac{1}{2}$ *and* $\Lambda_0 < \frac{(\phi_M H)^2}{2\mathbf{C}^2}$,
(iii) $\beta > \frac{1}{2}$ *and*

$$\left(\frac{1}{\boldsymbol{a}}\right)^{\frac{1}{\alpha-1}} \left[\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \right] > \boldsymbol{b}$$

where

$$\alpha = 2\beta, \qquad \boldsymbol{a} = \frac{2\mathbf{C}^2\Lambda_0}{(\phi_M H)^2}, \quad \textit{and} \quad \boldsymbol{b} = 2(1+\Gamma_0).$$

Then there exists a constant B_0 such that $\Gamma(t) \leq B_0$ for all $t \in \mathbb{R}_+$. In addition, for all $t \geq 0$,

$$\Lambda(t) \leq \Lambda_0 e^{-2\frac{\phi_M H}{B_0^\beta}t}$$

and therefore $\Lambda(t) \to 0$ when $t \to \infty$. Finally, there exists $\widehat{x} \in X$ such that $x(t) \to \widehat{x}$ when $t \to \infty$ and there exists $B_1 > 0$ such that, for every $t \geq 0$,

$$\|x(t) - \widehat{x}\| \leq B_1 e^{-2\frac{\phi_M H}{B_0^\beta}t}.$$

Proposition 3 provides an extension of Theorem 1 which models a situation in which not all birds necessarily communicate, no matter what their distance is.

3.4. Flocking with unrelated pairs: II

We can further extend the contents of §3.3 to model a situation where the flock remains connected all time but the connection pattern (which birds communicate with which others) changes.

Proposition 4. *Let $A : \mathbb{E}^k \to \mathscr{M}(k \times k)$ be an adjacency matrix, $H > 0$, $\beta \geq 0$, and $S : \mathbb{E}^k \to \{0,1\}^{k\times k}$ given by*

$$s_{ij}(\boldsymbol{x}) = \begin{cases} 1 & \textit{if } a_{ij}(\boldsymbol{x}) \geq \dfrac{H}{(1+\|\boldsymbol{x}_i(t)-\boldsymbol{x}_j(t)\|^2)^\beta} \\ 0 & \textit{otherwise.} \end{cases}$$

Let $x_0 \in X$ and $v_0 \in V$ and $(x(t), v(t))$ be the unique solution of

$$\begin{aligned} x' &= v \\ v' &= -L_x v \end{aligned}$$

where L_x is the Laplacian of A_x and assume that the graph $G(S_{x(t)})$ is connected for all $t \geq 0$. Furthermore, assume that one of the three following hypotheses holds:

(i) $\beta < \frac{1}{2}$,

(ii) $\beta = \frac{1}{2}$ *and* $\Lambda_0 < \frac{(4H)^2}{2(k(k-1)\mathbf{C})^2}$,

(iii) $\beta > \frac{1}{2}$ *and*

$$\left(\frac{1}{\boldsymbol{a}}\right)^{\frac{1}{\alpha-1}} \left[\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\right] > \boldsymbol{b}$$

where

$$\alpha = 2\beta, \qquad \boldsymbol{a} = \frac{2(k(k-1)\mathbf{C})^2\Lambda_0}{(4H)^2}, \quad \textit{and} \quad \boldsymbol{b} = 2(1+\Gamma_0).$$

Then there exists a constant B_0 such that $\Gamma(t) \le B_0$ for all $t \in \mathbb{R}_+$. In addition, for all $t \ge 0$,

$$\Lambda(t) \le \Lambda_0 e^{-2\frac{4H}{k(k-1)B_0^\beta}t}$$

and therefore $\Lambda(t) \to 0$ when $t \to \infty$. Finally, there exists $\widehat{x} \in X$ such that $x(t) \to \widehat{x}$ when $t \to \infty$ and there exists $B_1 > 0$ such that, for every $t \ge 0$,

$$\|x(t) - \widehat{x}\| \le B_1 e^{-2\frac{4H}{k(k-1)B_0^\beta}t}.$$

Proof. For $\boldsymbol{x} \in \mathbb{E}^k$ let $B \in \mathscr{M}(k \times k)$ be defined by

$$b_{ij}(\boldsymbol{x}) = \begin{cases} a_{ij} & \text{if } a_{ij}(\boldsymbol{x}) \ge \dfrac{H}{(1+\|\boldsymbol{x}_i(t)-\boldsymbol{x}_j(t)\|^2)^\beta} \\ 0 & \text{otherwise} \end{cases}$$

and $R = A - B$. Then, B and R are both non-negative and symmetric and their Laplacians L_x^B and L_x^R satisfy $L_x = L_x^B + L_x^R$ (here L_x is the Laplacian of A_x). Therefore,

$$\begin{aligned} \xi_x &= \min_{\substack{v\in V\\ v\neq 0}} \frac{\langle L_x v, v\rangle}{\|v\|^2} = \min_{\substack{v\in V\\ v\neq 0}} \frac{\langle L_x^B v, v\rangle + \langle L_x^R v, v\rangle}{\|v\|^2} \\ &\ge \min_{\substack{v\in V\\ v\neq 0}} \frac{\langle L_x^B v, v\rangle}{\|v\|^2} + \min_{\substack{v\in V\\ v\neq 0}} \frac{\langle L_x^R v, v\rangle}{\|v\|^2} \ge \phi_B \ge \frac{4H}{k(k-1)(1+\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2)^\beta} \end{aligned}$$

where the last inequality is derived as follows. One has $G(\overline{B}) = G(S)$ which is connected. Therefore, its diameter is at most $k-1$ and hence (see [12, (6.10)]), $\phi_S \ge \frac{4}{k(k-1)}$. Now apply Proposition 2.

Condition (8) therefore holds with $K = \frac{4H}{k(k-1)}$ and we can apply Theorem 2 to obtain convergence to the diagonal. □

Proposition 4 extends both Theorem 1 and (a continuous time version of) Theorem 1 in [10]. In the former, birds influence one each other always but their influence decreases with their distance. In the latter, non-zero influences have a lower bound of 1 (in the adjacency matrix) but zero influences are allowed as long as the associated graph $G(A)$ remains connected.

3.5. *Other extensions with symmetric A_x*

(i) In all the situations considered so far, we have taken $X = \mathbb{E}^k/\Delta_{\mathbb{E}}$ and a specific norm in X, namely, that induced by $Q_{\mathbb{E}}$. We now note that we could take as X any Euclidean space $\mathbb{R}^\ell$ and, for $x \in X$, a symmetric matrix A_x such that condition (8) is satisfied. In particular, entries a_{ij} may be functions of $x \in X$ which are not necessarily functions of the distance between birds i and j. Theorem 2 will apply provided that, for some $H > 0$ and $\beta \geq 0$, and for all $x \in X$,

$$a_{ij}(x) = a_{ji}(x) \geq \frac{H}{(1+\|x\|^2)^\beta}.$$

(ii) Also, in all the previous examples, we have considered matrices A_x which are nonnegative. For (8) to hold this need not to be the case. For instance, let

$$A_x = \begin{bmatrix} 0 & 3 & 2 \\ 3 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

for all $x \in X$. Then

$$L_x = \begin{bmatrix} 5 & -3 & -2 \\ -3 & 2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$

whose characteristic polynomial is $\chi(\lambda) = \lambda^3 - 8\lambda^2 + 3\lambda$. The roots of this polynomial are 0, $4-\sqrt{13}$, and $4+\sqrt{13}$. Therefore, condition (8) holds with $K = 4-\sqrt{13}$ and $\beta = 0$. Thus, Theorem 2 applies.

3.6. *Three birds with a leader*

Now consider a set of three birds and the *communication scheme* given by the (non-symmetric) matrix

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

This matrix models a situation in which bird 1 influences birds 2 and 3 and no other influence between different birds occur. In this situation birds 2 and 3 follow bird 1, the leader. To simplify ideas, we assume that $\mathbb{E} = \mathbb{F} = \mathbb{R}$. Given $\boldsymbol{x} \in \mathbb{E}^3$ we take as adjacency matrix

$$A_x = \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{bmatrix}$$

with $a_{ij} = \frac{H}{(1+\|\boldsymbol{x}_i-\boldsymbol{x}_j\|^2)^\beta}$. We consider again the Laplacian

$$L_x = \begin{bmatrix} 0 & 0 & 0 \\ -a_{21} & a_{21} & 0 \\ -a_{31} & 0 & a_{31} \end{bmatrix}$$

whose eigenvalues are 0, a_{21}, and a_{31}. Their corresponding eigenvectors are $w_1 = (1,1,1)$, $w_2 = (0,1,0)$, and $w_3 = (0,0,1)$. In this case we have

$$V = \mathbb{F}^3/\Delta_{\mathbb{F}} \simeq \mathsf{span}\{w_2, w_3\} = \{(0,y,z) \mid y,z \in \mathbb{F}\}.$$

In contrast with the situation in the two previous cases, we take in V the inner product induced by that in $\mathbb{F}$. Then, for $v = (0,y,z) \in V$, $Lv = (0, a_{21}y, a_{31}z)$, and $\langle Lv, v\rangle = a_{21}y^2 + a_{31}z^2$. This implies

$$\xi_x = \min_{v\neq 0} \frac{\langle Lv, v\rangle}{\|v\|^2} = \min_{(y,z)\neq 0} \frac{a_{21}y^2 + a_{31}z^2}{y^2+z^2} \geq \min\{a_{21}, a_{31}\}.$$

Now, taking the norm on X considered in the previous situations (i.e., that induced by $Q_{\mathbb{E}}$) we have

$$\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|^2 \leq \|x\|^2 = \Gamma(x)$$

and the same for $\|\boldsymbol{x}_1 - \boldsymbol{x}_3\|^2$. This shows that condition (8) holds with $K = H$. On the other hand, since $x' = v$,

$$\|x'\|^2 = Q_{\mathbb{E}}(x', x') = Q_{\mathbb{F}}(v,v) \leq 18\|v\|^2$$

which shows that condition (5) holds with $\mathbf{C} = 3\sqrt{2}$, $\gamma = 0$, and $\delta = 1$.

The situation we just considered trivially extends to an arbitrary number k of birds.

3.7. A different leadership structure

Now consider a set of three birds and the communication scheme given by the matrix

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

In this situation bird 1 influences bird 2 and bird 2 influences bird 3 but bird 1 has no direct influence over bird 3. Given $x \in X$ we take

$$A_x = \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & b & 0 \end{bmatrix}$$

with a, b functions of x satisfying

$$a \geq b \geq \frac{H}{(1+\Gamma(x))^{\beta}}.$$

We consider again the Laplacian

$$L_x = \begin{bmatrix} 0 & 0 & 0 \\ -a & a & 0 \\ 0 & -b & b \end{bmatrix}$$

whose eigenvalues are 0, a, and b, and, as in §3.6,

$$V = \mathbb{F}^3/\Delta_{\mathbb{F}} \simeq \{(0,y,z) \mid y,z \in \mathbb{F}\}$$

and we take in V the inner product induced by that in $\mathbb{F}$. Then, for $v = (0,y,z) \in V$, $Lv = (0, ay, b(z-y))$ and

$$\begin{aligned}
\langle Lv, v\rangle &= ay^2 + bz^2 - bzy \\
&\geq b(y^2 + z^2 - zy) \\
&= \frac{1}{2} b\left((y^2+z^2) + (y-z)^2\right) \\
&\geq \frac{1}{2} b(y^2+z^2).
\end{aligned}$$

This implies

$$\xi_x = \min_{v \neq 0} \frac{\langle Lv, v\rangle}{\|v\|^2} \geq \min_{(y,z)\neq 0} \frac{1}{2} b \frac{y^2+z^2}{y^2+z^2} = \frac{1}{2} b.$$

This shows that condition (8) holds with $K = \frac{H}{2}$ and Theorem 2 applies.

The situations considered here and in §3.6 extend to more general leadership structures, giving rise to triangular Laplacian matrices. Jackie Shen (in a personal communication) told us that he has proved convergence results in this extended setting.

4. Proof of Theorem 2

Denote $\Theta_t = \min_{\tau \in [0,t]} \xi_\tau$.

Proposition 5. *For all $t \geq 0$*

$$\Lambda(t) \leq \Lambda_0 e^{-2t\Theta_t}.$$

Proof. Let $\tau \in [0,t]$. Then

$$\begin{aligned}\Lambda'(\tau) &= \frac{d}{d\tau}\langle v(\tau), v(\tau)\rangle \\ &= 2\langle v'(\tau), v(\tau)\rangle \\ &= -2\langle L_\tau v(\tau), v(\tau)\rangle \\ &\leq -2\xi_\tau \Lambda(\tau).\end{aligned}$$

Using this inequality,

$$\ln(\Lambda(\tau))\Big|_0^t = \int_0^t \frac{\Lambda'(\tau)}{\Lambda(\tau)} d\tau \leq \int_0^t -2\xi_\tau d\tau \leq -2t\Theta_t$$

i.e.,

$$\ln(\Lambda(t)) - \ln(\Lambda_0) \leq -2t\Theta_t$$

from which the statement follows. □

Proposition 6. *For $T > 0$ and $\gamma < 1$*

$$\Gamma(T) \leq 2^{\frac{1+\gamma}{1-\gamma}} \left((1+\Gamma_0) + \frac{((1-\gamma)\mathbf{C})^{\frac{2}{1-\gamma}} \Lambda_0^{\frac{\delta}{1-\gamma}}}{(\delta\Theta_T)^{\frac{2}{1-\gamma}}} \right) - 1.$$

Proof. We have

$$\begin{aligned}|\Gamma'(t)| &= |2\langle F(x(t), v(t)), x(t)\rangle| \leq 2\|F(x(t), v(t))\| \|x(t)\| \\ &\leq 2\mathbf{C}(1 + \|x(t)\|^2)^{\frac{\gamma}{2}} \|v(t)\|^\delta \|x(t)\| \leq 2\mathbf{C}(1 + \|x(t)\|^2)^{\frac{1+\gamma}{2}} \|v(t)\|^\delta.\end{aligned}$$

But $1 + \|x(t)\|^2 = 1 + \Gamma(t)$ and $\|v(t)\|^2 = \Lambda(t) \leq \Lambda_0 e^{-2t\Theta_t}$, by Proposition 5. Therefore,

$$\Gamma'(t) \leq |\Gamma'(t)| \leq 2\mathbf{C}(1+\Gamma(t))^{\frac{1+\gamma}{2}} \left(\Lambda_0 e^{-2t\Theta_t}\right)^{\frac{\delta}{2}} \tag{12}$$

and, using that $t \mapsto \Theta_t$ is non-increasing,

$$\begin{aligned}\int_0^T \frac{\Gamma'(t)}{(1+\Gamma(t))^{\frac{1+\gamma}{2}}} dt &\leq 2\mathbf{C} \int_0^T \left(\Lambda_0 e^{-2t\Theta_t}\right)^{\frac{\delta}{2}} dt \\ &\leq 2\mathbf{C} \int_0^T \Lambda_0^{\delta/2} e^{-t\delta\Theta_T} dt \\ &= 2\mathbf{C}\Lambda_0^{\delta/2} \left(-\frac{1}{\delta\Theta_T}\right) e^{-t\delta\Theta_T}\Big|_0^T \leq \frac{2\mathbf{C}\Lambda_0^{\delta/2}}{\delta\Theta_T}\end{aligned}$$

which implies, since $\gamma < 1$,

$$(1+\Gamma(t))^{\frac{1-\gamma}{2}}\Big|_0^T = \frac{1-\gamma}{2} \int_0^T \frac{\Gamma'(t)}{(1+\Gamma(t))^{\frac{1+\gamma}{2}}} dt \leq \frac{(1-\gamma)\mathbf{C}\Lambda_0^{\delta/2}}{\delta\Theta_T}$$

from which it follows that

$$\Gamma(T) \leq \left((1+\Gamma_0)^{\frac{1-\gamma}{2}} + \frac{(1-\gamma)\mathbf{C}\Lambda_0^{\delta/2}}{\delta\Theta_T} \right)^{\frac{2}{1-\gamma}} - 1.$$

The statement now follows from the elementary inequality, for $s \geq 2$, $(\alpha+\beta)^s \leq 2^{s-1}(\alpha^s + \beta^s)$. □

A proof of the following lemma is in [5, Lemma 7].

Lemma 1. *Let $c_1, c_2 > 0$ and $s > q > 0$. Then the equation*

$$F(z) = z^s - c_1 z^q - c_2 = 0$$

has a unique positive zero z_*. In addition

$$z_* \leq \max\left\{ (2c_1)^{\frac{1}{s-q}}, (2c_2)^{\frac{1}{s}} \right\}$$

and $F(z) \leq 0$ for $0 \leq z \leq z^*$. □

Remark 3. Although we will not use this in what follows, we mention here that it is possible to prove a lower bound for z_* in Lemma 1 namely,

$$z_* \geq \max\left\{ c_1^{\frac{1}{s-q}}, c_2^{\frac{1}{s}} \right\}.$$

Proof of Theorem 2. The existence and uniqueness of a solution under assumptions (5) and (8) follows from [9, Chapter 8].

Let $t > 0$ and $t^* \in [0,t]$ be the point maximizing Γ in $[0,t]$. Then, by (8),

$$\Theta_t = \min_{\tau \in [0,t]} \xi_\tau \geq \min_{\tau \in [0,t]} \frac{K}{(1+\Gamma(\tau))^\beta} = \frac{K}{(1+\Gamma(t^*))^\beta}.$$

Using this bound on Θ_t and Proposition 6 we deduce

$$\Gamma(t) \leq 2^{\frac{1+\gamma}{1-\gamma}} \left((1+\Gamma_0) + \frac{((1-\gamma)\mathbf{C})^{\frac{2}{1-\gamma}} \Lambda_0^{\frac{\delta}{1-\gamma}}}{(\delta K)^{\frac{2}{1-\gamma}}} (1+\Gamma(t^*))^{\frac{2\beta}{1-\gamma}} \right) - 1. \tag{13}$$

Since t^* maximizes Γ in $[0,t]$ it also does so in $[0,t^*]$. Thus, for $t = t^*$, (13) takes the form

$$(1+\Gamma(t^*)) - 2^{\frac{1+\gamma}{1-\gamma}} \frac{((1-\gamma)\mathbf{C})^{\frac{2}{1-\gamma}} \Lambda_0^{\frac{\delta}{1-\gamma}}}{(\delta K)^{\frac{2}{1-\gamma}}} (1+\Gamma(t^*))^{\frac{2\beta}{1-\gamma}} - 2^{\frac{1+\gamma}{1-\gamma}} (1+\Gamma_0) \leq 0. \tag{14}$$

Let $z = 1 + \Gamma(t^*)$. Then (14) can be rewritten as $F(z) \leq 0$ with

$$F(z) = z - \boldsymbol{a} z^\alpha - \boldsymbol{b}.$$

(i) Assume $\beta < \frac{1-\gamma}{2}$. By Lemma 1, $F(z) \leq 0$ implies that $z = 1+\Gamma(t^*) \leq B_0$ with

$$B_0 = \max\left\{(2\boldsymbol{a})^{\frac{1-\gamma}{1-\gamma-2\beta}}, 2\boldsymbol{b}\right\}.$$

Since B_0 is independent of t, we deduce that, for all $t \in \mathbb{R}_+$, $1+\Gamma(t) \leq B_0$. But this implies that $\xi_t \geq \frac{K}{B_0^\beta}$ for all $t \in \mathbb{R}_+$ and therefore, the same bound holds for Θ_t. By Proposition 5,

$$\Lambda(t) \leq \Lambda_0 e^{-2\frac{K}{B_0^\beta}t} \tag{15}$$

which shows that $\Lambda(t) \to 0$ when $t \to \infty$. Finally, for all $T > t$,

$$\begin{aligned}
\|x(T)-x(t)\| &= \left\|\int_t^T F(x(s),v(s))ds\right\| \leq \int_t^T \|F(x(s),v(s))\|ds \\
&\leq \int_t^T \mathbf{C}(1+\|x(s)\|^2)^{\frac{\gamma}{2}}\|v(s)\|^\delta ds = \int_t^T \mathbf{C}(1+\Gamma(s))^{\frac{\gamma}{2}}\Lambda(s)^{\frac{\delta}{2}} \\
&\leq \int_t^T \mathbf{C}B_0^{\frac{\gamma}{2}}\Lambda_0^{\delta/2}e^{-\frac{\delta K}{B_0^\beta}s}ds = \mathbf{C}B_0^{\frac{\gamma}{2}}\Lambda_0^{\delta/2}\left(-\frac{B_0^\beta}{\delta K}e^{-\frac{\delta K}{B_0^\beta}s}\right)\Bigg|_t^T \\
&= \frac{\mathbf{C}\Lambda_0^{\delta/2}B_0^{\frac{\gamma}{2}+\beta}}{\delta K}\left(e^{-\frac{K}{B_0^\beta}t} - e^{-\frac{K}{B_0^\beta}T}\right) \\
&\leq \frac{\mathbf{C}\Lambda_0^{\delta/2}B_0^{\frac{\gamma}{2}+\beta}}{\delta K}e^{-\frac{K}{B_0^\beta}t}.
\end{aligned}$$

Since the last tends to zero with t and is independent of T we deduce that there exists $\widehat{x} \in X$ such that, $x \to \widehat{x}$.

(ii) Assume now $\beta = \frac{1-\gamma}{2}$. Then (14) takes the form

$$(1+\Gamma(t^*))\left(1-2^{\frac{1+\gamma}{1-\gamma}}\frac{((1-\gamma)\mathbf{C})^{\frac{2}{1-\gamma}}\Lambda_0^{\frac{\delta}{1-\gamma}}}{(\delta K)^{\frac{2}{1-\gamma}}}\right) - 2^{\frac{1+\gamma}{1-\gamma}}(1+\Gamma_0) \leq 0.$$

Since $\Lambda_0^\delta < \frac{(\delta K)^2}{2^{1+\gamma}((1-\gamma)\mathbf{C})^2}$, the expression between parenthesis is positive, and therefore

$$1+\Gamma(t^*) \leq B_0 = \frac{2^{\frac{1+\gamma}{1-\gamma}}(1+\Gamma_0)}{\left(1-2^{\frac{1+\gamma}{1-\gamma}}\frac{((1-\gamma)\mathbf{C})^{\frac{2}{1-\gamma}}\Lambda_0^{\frac{\delta}{1-\gamma}}}{(\delta K)^{\frac{2}{1-\gamma}}}\right)}.$$

We now proceed as in case (i).

(iii) Assume finally $\beta > \frac{1-\gamma}{2}$ (i.e., that $\alpha > 1$). The derivative $F'(z) = 1 - \alpha \boldsymbol{a} z^{\alpha-1}$ has a unique zero at $z_* = \left(\frac{1}{\alpha \boldsymbol{a}}\right)^{\frac{1}{\alpha-1}}$ and

$$\begin{aligned} F(z_*) &= \left(\frac{1}{\alpha \boldsymbol{a}}\right)^{\frac{1}{\alpha-1}} - \boldsymbol{a}\left(\frac{1}{\alpha \boldsymbol{a}}\right)^{\frac{\alpha}{\alpha-1}} - \boldsymbol{b} \\ &= \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(\frac{1}{\boldsymbol{a}}\right)^{\frac{1}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\left(\frac{1}{\boldsymbol{a}}\right)^{\frac{1}{\alpha-1}} - \boldsymbol{b} \\ &= \left(\frac{1}{\boldsymbol{a}}\right)^{\frac{1}{\alpha-1}}\left[\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\right] - \boldsymbol{b} \\ &\geq 0 \end{aligned}$$

the last by our hypothesis. Since $F(0) = -\boldsymbol{b} < 0$ and $F(z) \to -\infty$ when $z \to \infty$ we deduce that the shape of F is as follows:

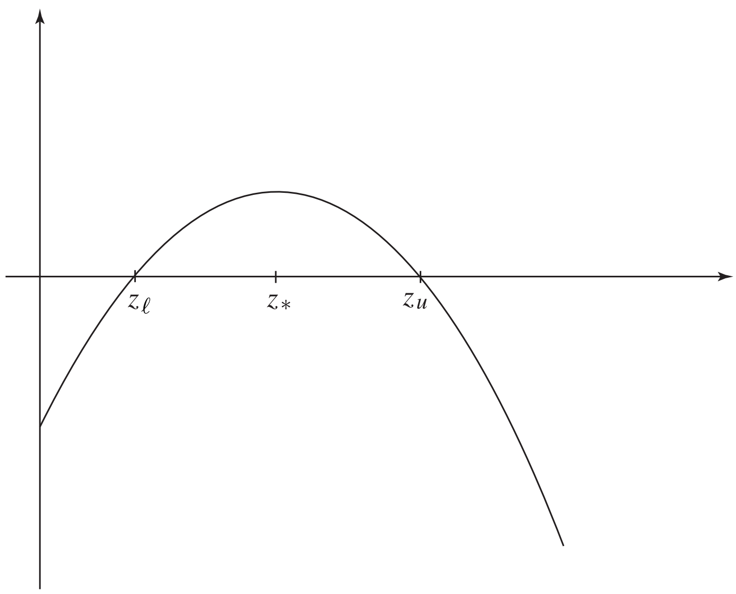

Fig. 1.

Even though t^* is not continuous as a function of t, the mapping $t \mapsto 1 + \Gamma(t^*)$ is continuous and therefore, so is the mapping $t \mapsto F(1 + \Gamma(t^*))$. This fact, together with (14), shows that, for all $t \geq 0$, $F(1 + \Gamma(t^*)) \leq 0$. In addition, when $t = 0$ we have $t^* = 0$ as well and

$$\begin{aligned} 1 + \Gamma_0 &\leq 2^{\frac{1+\gamma}{1-\gamma}}(1 + \Gamma_0) = \boldsymbol{b} \\ &< \left(\frac{1}{\boldsymbol{a}}\right)^{\frac{1}{\alpha-1}}\left[\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\right] \end{aligned}$$

$$
\begin{aligned}
&< \left(\frac{1}{\boldsymbol{a}}\right)^{\frac{1}{\alpha-1}} \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} \\
&= z_*.
\end{aligned}
$$

This implies that $1+\Gamma_0 < z_\ell$ (the latter being the smallest zero of F on $\mathbb{R}_+$, see the figure above) and the continuity of the map $t \mapsto 1+\Gamma(t^*)$ implies that, for all $t \geq 0$,

$$1+\Gamma(t^*) \leq z_\ell \leq z_*.$$

Therefore

$$1+\Gamma(t^*) \leq B_0 = \left(\frac{1}{\alpha \boldsymbol{a}}\right)^{\frac{1}{\alpha-1}}.$$

We now proceed as in case (i). □

5. Emergence with discrete time

We next extend system (3) in Section 1. Similarly to our development in Section 2, for discrete time now, we assume a map $\mathscr{S} : X \to \mathsf{End}(V)$ and consider the dynamics

$$
\begin{aligned}
x(t+h) &= x(t) + hF(x(t), v(t)) \\
v(t+h) &= \mathscr{S}(x(t))v(t).
\end{aligned}
\tag{16}
$$

We assume that $\mathscr{S}$ satisfies, for some $G > 0$ and $\beta \geq 0$, that

$$\text{for all } x \in X, \quad \|\mathscr{S}(x)\| \leq 1 - \frac{hG}{(1+\Gamma(x))^\beta}. \tag{17}$$

We also assume that $0 < h < \frac{1}{G}$ which makes sure that the quantity in the right-hand side above is in the interval $(0,1)$.

Remark 4. We note that, in contrast with the contents of Section 2, for Theorem 3 below to hold, we do not require an inner product structure in neither X nor V. Only that they will be endowed with a norm. In this sense, the expression $\|\mathscr{S}(x)\|$ above referes to the operator norm of $\mathscr{S}(x)$ with respect to the norm on V.

Theorem 3. *Assume that F satisfies condition (5), $\mathscr{S}$ satisfies condition (17), and $0 < h < \frac{1}{G}$. Assume also that one of the three following hypothesis hold:*

(i) $\beta < \frac{1-\gamma}{2}$,

(ii) $\beta = \frac{1-\gamma}{2}$ *and* $\Lambda_0 < \left(\frac{G}{\mathbf{C}R(\delta)}\right)^{2/\delta}$.

(iii) $\beta > \frac{1-\gamma}{2}$,

$$\left(\frac{1}{\boldsymbol{a}}\right)^{\frac{1}{\alpha-1}}\left[\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}}-\left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\right] > \boldsymbol{b} \tag{18}$$

and

$$h < \left(\left[1-\frac{1}{\alpha}\right]-\left(\frac{1}{\alpha\boldsymbol{a}}\right)^{-\frac{2}{\alpha-1}}\boldsymbol{b}\right)\left(\frac{R(\delta)}{2R(\delta)+\boldsymbol{a}}\right)\frac{R(\delta)}{\boldsymbol{a}G}. \tag{19}$$

Here $\alpha = 2\beta+\gamma$, $R(\delta) = \max\{1, \frac{1}{\delta}\}$, *and*

$$\boldsymbol{a} = \frac{\Lambda_0^{\delta/2}\mathbf{C}R(\delta)}{G} \qquad \textit{and} \qquad \boldsymbol{b} = 1+\Gamma_0^{1/2}.$$

Then there exists a constant B_0 *(independent of* t*, made explicit in the proof of each of the three cases) such that* $\|x(t)\|^2 \le B_0$ *for all* $t \in \mathbb{N}$*. In addition,* $\|v(t)\| \to 0$ *when* $t \to \infty$*. Finally, there exists* $\widehat{x} \in X$ *such that* $x(t) \to \widehat{x}$ *when* $t \to \infty$*.*

Remark 5. Note that, besides the hypothesis $h < \frac{1}{G}$ which is assumed in all three cases, case (iii) also requires an additional condition on h. Furthermore, this extra condition depends on the initial state of the system and this dependence is clear: the larger Γ_0 and Λ_0 are the smaller h needs to be.

In the sequel we assume a solution (x,v) of (16). To simplify notation we write $v(n)$ for $v(nh)$ and the same for $x(n)$. In addition, we will write $\Gamma(n)$ instead of $\Gamma(x(n))$.

Proof of Theorem 3. First note that, for all $n \in \mathbb{N}$,

$$\|v(n)\|^\delta = \|\mathscr{S}(x(n-1))v(n)\|^\delta \le \left(1-\frac{hG}{(1+\Gamma(n))^\beta}\right)^\delta \|v(n-1)\|^\delta$$

and therefore,

$$\|v(n)\|^\delta \le \|v(0)\|^\delta \prod_{i=0}^{n-1}\left(1-\frac{hG}{(1+\Gamma(i))^\beta}\right)^\delta. \tag{20}$$

Let $n \ge 0$ and n^* be the point maximizing $\Gamma(i)$ in $\{0,h,2h,\ldots,nh\}$. Then, for all $\tau \le n$,

$$\begin{aligned}\|x(\tau)\| &\le \|x(0)\| + \sum_{j=0}^{\tau-1}\|x(j+1)-x(j)\| \\ &\le \|x(0)\| + \sum_{j=0}^{\tau-1} h\mathbf{C}(1+\|x(j)\|^2)^{\frac{\gamma}{2}}\|v(j)\|^\delta\end{aligned}$$

$$
\begin{aligned}
&\leq \|x(0)\| + \|v(0)\|^{\delta} h\mathbf{C} \sum_{j=0}^{\tau-1} (1+\Gamma(j))^{\frac{\gamma}{2}} \prod_{i=0}^{j-1} \left(1 - \frac{hG}{(1+\Gamma(i))^{\beta}}\right)^{\delta} \\
&\leq \|x(0)\| + \|v(0)\|^{\delta} h\mathbf{C} (1+\Gamma(n^*))^{\frac{\gamma}{2}} \sum_{j=0}^{\tau-1} \left(1 - \frac{hG}{(1+\Gamma(n^*))^{\beta}}\right)^{j\delta} \\
&\leq \|x(0)\| + \|v(0)\|^{\delta} h\mathbf{C} (1+\Gamma(n^*))^{\frac{\gamma}{2}} \frac{1}{1 - \left(1 - \frac{hG}{(1+\Gamma(n^*))^{\beta}}\right)^{\delta}} \\
&\leq \|x(0)\| + \|v(0)\|^{\delta} \mathbf{C} (1+\Gamma(n^*))^{\frac{\gamma}{2}} \frac{(1+\Gamma(n^*))^{\beta}}{G} R(\delta)
\end{aligned}
$$

where $R(\delta) = \max\{1, \frac{1}{\delta}\}$. For $\tau = n^*$, the inequality above takes the following equivalent form

$$
\|x(n^*)\| \leq \|x(0)\| + \|v(0)\|^{\delta} \mathbf{C} (1+\Gamma(n^*))^{\frac{\gamma}{2}} \frac{(1+\Gamma(n^*))^{\beta}}{G} R(\delta)
$$

which implies

$$
(1+\Gamma(n^*))^{\frac{1}{2}} \leq 1+\Gamma(n^*)^{\frac{1}{2}} \leq (1+\Gamma_0^{1/2}) + \Lambda_0^{\delta/2} \mathbf{C} (1+\Gamma(n^*))^{\frac{\gamma}{2}} \frac{(1+\Gamma(n^*))^{\beta}}{G} R(\delta)
$$

or yet

$$
(1+\Gamma(n^*))^{\frac{1}{2}} \leq (1+\Gamma_0^{1/2}) + \frac{\Lambda_0^{\delta/2} \mathbf{C} R(\delta)}{G} (1+\Gamma(n^*))^{\beta+\frac{\gamma}{2}}. \tag{21}
$$

Let $z = (1+\Gamma(n^*))^{1/2}$,

$$
\boldsymbol{a} = \frac{\Lambda_0^{\delta/2} \mathbf{C} R(\delta)}{G} \qquad \text{and} \qquad \boldsymbol{b} = 1 + \Gamma_0^{1/2}.
$$

Then (21) can be rewritten as $F(z) \leq 0$ with

$$
F(z) = z - \boldsymbol{a} z^{2\beta+\gamma} - \boldsymbol{b}.
$$

(i) Assume $\beta < \frac{1-\gamma}{2}$. By Lemma 1, $F(z) \leq 0$ implies that $(1+\Gamma(n^*))^{1/2} \leq U_0$ with

$$
U_0 = \max\left\{ (2\boldsymbol{a})^{\frac{1-\gamma}{1-\gamma-2\beta}}, 2\boldsymbol{b} \right\}.
$$

Since U_0 is independent of n we deduce that, for all $n \geq 0$,

$$
\|x(n)\|^2 \leq B_0 = U_0^2 - 1
$$

and therefore, using (20),

$$
\|v(n)\| \leq \|v(0)\| \left(1 - \frac{hG}{(1+B_0)^{\beta}}\right)^n
$$

and this expression tends to zero when $n \to \infty$.

(ii) Assume $\beta = \frac{1-\gamma}{2}$. Then the inequality $F(z) \le 0$ takes the form

$$z\left(1 - \frac{\Lambda_0^{\delta/2}\mathbf{C}R(\delta)}{G}\right) \le 1 + \Gamma_0^{1/2}.$$

Since, by hypothesis, the expression between parenthesis is positive we deduce that

$$z \le U_0 = \frac{1+\Gamma_0^{1/2}}{1 - \frac{\Lambda_0^{\delta/2}\mathbf{C}R(\delta)}{G}}.$$

We now proceed as in case (i).

(iii) Assume finally $\beta > \frac{1-\gamma}{2}$. Letting $\alpha = 2\beta + \gamma$ we have $F'(z) = 1 - \alpha \boldsymbol{a} z^{\alpha-1}$ and the arguments in the proof of Theorem 2 show that $F'(z)$ has a unique zero at $z_* = \left(\frac{1}{\alpha \boldsymbol{a}}\right)^{\frac{1}{\alpha-1}}$ and that $F(z_*) = \left(\frac{1}{\boldsymbol{a}}\right)^{\frac{1}{\alpha-1}}\left[\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\right] - \boldsymbol{b}$. Our hypothesis (18) then implies that $F(z_*) > 0$. Since $F(0) = -\boldsymbol{b} < 0$, we have that the graph of F is as in Fig. 1.

For $n \in \mathbb{N}$ let $z(n) = (1+\Gamma(n^*))^{1/2}$. When $n = 0$ we have $n^* = 0$ as well and

$$z(0) = (1+\Gamma_0)^{1/2} \le 1 + \Gamma_0^{1/2} = \boldsymbol{b} < \left(\frac{1}{\boldsymbol{a}}\right)^{\frac{1}{\alpha-1}}\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} = z_*.$$

This actually implies that $z(0) \le z_\ell$. Assume that there exists $n \in \mathbb{N}$ such that $z(n) \ge z_u$ and let N be the first such n. Then $N = N^* \ge 1$ and, for all $n < N$

$$(1+\Gamma(n))^{1/2} \le z(N-1) \le z_\ell.$$

This shows that, for all $n < N$,

$$\Gamma(n) \le B_0 = z_\ell^2 - 1.$$

In particular,

$$\Gamma(N-1) \le z_\ell^2 - 1.$$

For N instead, we have

$$\Gamma(N) \ge z_u^2 - 1.$$

This implies

$$\Gamma(N) - \Gamma(N-1) \ge z_u^2 - z_\ell^2 \ge z_*^2 - z_\ell^2 \ge (z_* - z_\ell)z_*. \tag{22}$$

From the intermediate value theorem, there is $\zeta \in [z_\ell, z_*]$ such that $F(z_*) = F'(\zeta)(z_* - z_\ell)$. But $F'(\zeta) \ge 0$ and $F'(\zeta) = 1 - \boldsymbol{a}\alpha\zeta^{\alpha-1} \le 1$. Therefore,

$$z_* - z_\ell \ge F(z_*)$$

and it follows from (22) that

$$\Gamma(N)-\Gamma(N-1)\geq z_*F(z_*). \tag{23}$$

But

$$\begin{aligned}\|x(N)\|-\|x(N-1)\| &\leq \|x(N)-x(N-1)\|\\ &= h\mathbf{C}(1+\|x(N-1)\|^2)^{\frac{\gamma}{2}}\|v(N-1)\|^{\delta}\\ &\leq h\mathbf{C}(1+\|x(N-1)\|^2)^{\frac{\gamma}{2}}\|v(0)\|^{\delta}\prod_{i=0}^{t-1}\left(1-\frac{hG}{(1+\Gamma(i))^{\beta}}\right)^{\delta}\\ &\leq h\mathbf{C}(1+B_0)^{\frac{\gamma}{2}}\Lambda_0^{\delta/2}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|x(N)\|^2-\|x(N-1)\|^2=(\|x(N)\|+\|x(N-1)\|)(\|x(N)\|-\|x(N-1)\|)\\ &=(2\|x(N-1)\|+\|x(N)\|-\|x(N-1)\|)(\|x(N)\|-\|x(N-1)\|)\\ &\leq(2B_0^{1/2}+h\mathbf{C}(1+B_0)^{\frac{\gamma}{2}}\Lambda_0^{\delta/2})h\mathbf{C}(1+B_0)^{\frac{\gamma}{2}}\Lambda_0^{\delta/2}.\end{aligned}$$

Putting this inequality together with (23) shows that

$$z_*F(z_*)\leq(2B_0^{1/2}+h\mathbf{C}(1+B_0)^{\frac{\gamma}{2}}\Lambda_0^{\delta/2})h\mathbf{C}(1+B_0)^{\frac{\gamma}{2}}\Lambda_0^{\delta/2}$$

or equivalently, using that $B_0\leq z_*^2-1$,

$$\begin{aligned}&\left(\frac{1}{\boldsymbol{a}}\right)^{\frac{2}{\alpha-1}}\left[\left(\frac{1}{\alpha}\right)^{\frac{2}{\alpha-1}}-\left(\frac{1}{\alpha}\right)^{\frac{\alpha+1}{\alpha-1}}\right]-\boldsymbol{b}\\ &\leq(2B_0^{1/2}+h\mathbf{C}(1+B_0)^{\frac{\gamma}{2}}\Lambda_0^{\delta/2})h\mathbf{C}(1+B_0)^{\frac{\gamma}{2}}\Lambda_0^{\delta/2}\\ &\leq\left(2z_*+\frac{\mathbf{C}z_*^{\gamma}\Lambda_0^{\delta/2}}{G}\right)h\mathbf{C}z_*^{\gamma}\Lambda_0^{\delta/2}\leq\left(2+\frac{\mathbf{C}\Lambda_0^{\delta/2}}{G}\right)h\mathbf{C}z_*^2\Lambda_0^{\delta/2}\end{aligned}$$

or yet, since $z_*=\left(\frac{1}{\alpha\boldsymbol{a}}\right)^{\frac{1}{\alpha-1}}$,

$$\left(\frac{1}{\boldsymbol{a}}\right)^{\frac{2}{\alpha-1}}\left[\left(\frac{1}{\alpha}\right)^{\frac{2}{\alpha-1}}-\left(\frac{1}{\alpha}\right)^{\frac{\alpha+1}{\alpha-1}}\right]-\boldsymbol{b}\leq\left(2+\frac{\mathbf{C}\Lambda_0^{\delta/2}}{G}\right)h\mathbf{C}\Lambda_0^{\delta/2}\left(\frac{1}{\alpha\boldsymbol{a}}\right)^{\frac{2}{\alpha-1}}$$

which is equivalent to

$$\left[1-\frac{1}{\alpha}\right]-\left(\frac{1}{\alpha\boldsymbol{a}}\right)^{-\frac{2}{\alpha-1}}\boldsymbol{b}\leq\left(2+\frac{\boldsymbol{a}}{R(\delta)}\right)\frac{h\boldsymbol{a}G}{R(\delta)},$$

which contradicts hypothesis (19).

We conclude that, for all $n \in \mathbb{N}$, $z(n) \le z_\ell$ and hence, $\Gamma(n) \le B_0$. We now proceed as in case (i). □

In our central example, we take $\mathscr{S}(x) = \mathrm{Id} - hL_x$ where L_x is the Laplacian of the matrix A_x for some adjacency function A. One is then interested in conditions on A and h which would ensure that (17) is satisfied. The following result gives such conditions under assumption (8) (with $G \le \frac{K}{2}$). Note that to do that, we need to assume an inner product $\langle\ ,\ \rangle$ in V w.r.t. which the norm on V is the associated norm.

Proposition 7. *Assume that there exists $U > 0$ such that, for all $x \in X$, $\|L_x\| \le U$. Then Theorem 3 holds with hypothesis (17) replaced by condition (8) with any G satisfying $G \le \frac{K}{2}$, and the inequality*

$$h < \frac{K}{U^2(1+B_0)^\beta}$$

where B_0 is the constant in Theorem 3 (and therefore depends on the initial state $(x(0), v(0))$ of the system).

Proof. The proof follows the steps of that of Theorem 3 using an additional induction argument to show that if $\Gamma(i) \le B_0$ for all $i \le n$ then $\Gamma(n+1) \le B_0$. The only thing we need to do is to show that the new hypothesis, i.e., the bound U, (8), and the additional bound on h, imply (17).

By hypothesis, for all $i \le n$,

$$h < \frac{K}{U^2(1+B_0)^\beta} \le \frac{K}{U^2(1+\Gamma(i))^\beta}.$$

Now use Lemma 2 below with $\xi = \frac{K}{(1+\Gamma(i))^\beta}$ (note that (8) ensures that the hypothesis of the lemma are satisfied) to deduce that

$$\|\mathrm{Id} - hL\| \le 1 - h\frac{\xi}{2} = 1 - \frac{hK}{2(1+\Gamma(i))^\beta} \le 1 - \frac{hG}{(1+\Gamma(i))^\beta}$$

i.e., condition (17). □

Lemma 2. *Let $L : V \to V$ be such that $\|L\| \le U$ and*

$$\min_{\substack{v \in V \\ v \ne 0}} \frac{\langle Lv, v\rangle}{\|v^2\|} \ge \xi > 0.$$

Then, for all $h \le \frac{\xi}{U^2}$,

$$\|\mathrm{Id} - hL\| \le 1 - h\frac{\xi}{2}.$$

Proof. Take $v \in V$ such that $\|v\| = 1$ and $\|\mathrm{Id} - hL\| = \|(\mathrm{Id} - hL)v\|$. Then

$$\begin{aligned}
\|\mathrm{Id} - hL\|^2 &= \|(\mathrm{Id} - hL)v\|^2 \\
&= \langle (\mathrm{Id} - hL)v, (\mathrm{Id} - hL)v \rangle \\
&= \|v\|^2 - 2h\langle Lv, v\rangle + h^2 \langle Lv, Lv\rangle \\
&\le 1 - 2h\xi + h^2 U^2 \\
&\le 1 - 2h\xi + h\xi \\
&= 1 - h\xi
\end{aligned}$$

from which it follows that $\|\mathrm{Id} - hL\| \le 1 - \frac{h}{2}\xi$. □

6. On variations of Vicsek's model

The goal of this section is to give a proof, from our Theorem 3, of a result in [10, page 993]. Namely, we prove the following.

Proposition 8. *For $g > \frac{k}{2}$ consider the system*

$$\begin{aligned}
x(n+1) &= x(n) + v(n) \\
v(n+1) &= (\mathrm{Id} - g^{-1} L_x) v(n)
\end{aligned}$$

where L_x is the Laplacian of $A_{x(n)}$ given by

$$a_{ij} = \begin{cases} 1 & \text{if } \|\boldsymbol{x}_i(n) - \boldsymbol{x}_j(n)\| \le r \\ 0 & \text{otherwise} \end{cases}$$

for some fixed $r > 0$. If the graphs $G(A_{x(n)})$ are connected for all $n \in \mathbb{N}$ then the sequence $\{v(n)\}_{n \in \mathbb{N}}$ converges to an element in the diagonal $\Delta_{\mathbb{F}}$. □

We provide some background for Proposition 8. Very recently [10], convergence to consensus was proved in some situations for a (flocking) model proposed in [19], which we will call Vicsek's model, and some variations of it.

This is a discrete time model based upon adjacency matrices $A_x = (a_{ij})$ with $a_{ij} = \eta(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2)$ but where now

$$\eta(y) = \begin{cases} 1 & \text{if } y \le r^2 \\ 0 & \text{otherwise} \end{cases}$$

for some fixed $r > 0$. For such a matrix A_x, one considers its associated diagonal matrix D_x and Laplacian L_x. Then, matrices $\mathscr{S}(x)$ of the form

$$\mathscr{S}(x) = \mathrm{Id} - M_x^{-1} L_x$$

are considered for various choices of M_x (*cf.* [10, Equations (20) to (24)]) and convergence to consensus is proved for some choices of M_x under certain conditions on the sequence of subjacent graphs $G(A_{x(n)})$, $n \in \mathbb{N}$. The simplest of the conditions considered in [10] requires the graphs $G(A_{x(n)})$ to be connected for all $n \in \mathbb{N}$.

Vicsek's model corresponds to the choice $M_x = (\mathrm{Id} + D_x)$ (this follows immediately from [10, Equation (3)]). A variation of Vicsek's model considered in [10] (see Equation (27) therein) consists on taking $M_x = g$, with $g \in \mathbb{R}$, $g > k$, for all $x \in X$. As a simple application of Theorem 3 we next show that for this choice of M_x, if $G(A_{x(n)})$ is connected for all $n \in \mathbb{N}$, there is convergence to a common $\boldsymbol{v} \in \mathbb{F}^k$. We will actually show this with a slightly weaker condition, namely $g > \frac{k}{2}$.

Indeed, we have seen in §3.1 that the eigenvalues of L_x lie in the interval $[0,k]$ and we know that, since $G(A_{x(n)})$ is connected its eigenvalues of the restriction of $L_{x(n)}$ to V lie on the interval $\left[\frac{4}{k(k-1)}, k\right]$ (see §3.4). Therefore, those of $\frac{1}{g}L_{x(n)}$ lie on the interval $\left[\frac{4}{gk(k-1)}, \frac{k}{g}\right]$ and those of $\mathscr{S}(x(n)) = \mathrm{Id} - g^{-1}L_{x(n)}$ on the interval $\left[1-\frac{k}{g}, 1-\frac{4}{gk(k-1)}\right]$. It follows that

$$\|\mathscr{S}(x(n))\| \le W := \max\left\{\left|1-\frac{k}{g}\right|, 1-\frac{4}{gk(k-1)}\right\} < 1$$

and therefore, that condition (17) holds with $\beta = 0$ and $G = 1 - W$. Note that $G \in (0,1)$ and therefore, part (i) of Theorem 3 applies with $h = 1$ (which is the time step considered in [10]).

7. On bounds for ξ_x and $\|\mathscr{S}(x)\|$

The inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$ and the inner product in $\mathbb{R}^k$ induce a Hermitian product $\langle\ ,\ \rangle$ and a norm $\|\ \|$ in $\mathbb{C}^k$ which we will consider in what follows.

Let $L : \mathbb{C}^{k-1} \to \mathbb{C}^{k-1}$ be a linear operator with eigenvalues $\lambda_1, \ldots, \lambda_{k-1}$. Assume that there exists a basis of eigenvectors $w_1, \ldots, w_{k-1} \in \mathbb{C}^{k-1}$ for $\lambda_1, \ldots, \lambda_{k-1}$. Without loss of generality we assume $\|w_i\| = 1$ for $i = 1, \ldots, k-1$. Let $b_{ij} = \langle w_i, w_j \rangle$, the *distortion matrix* $B = (b_{ij})$, and $\widehat{B} = B - \mathrm{Id}$. Also, let

$$\zeta = \min\{\mathrm{Re}\,\lambda \mid \lambda \in \mathsf{Spec}(L)\} > 0$$

and $|\lambda_{\max}| = \max_{i \le k-1}\{|\lambda_i|\}$.

Lemma 3. *Let $L : \mathbb{R}^{k-1} \to \mathbb{R}^{k-1}$ be a linear operator with $k-1$ different eigenvalues. Then, for all $v \in \mathbb{R}^{k-1}$,*

$$\frac{\langle v, Lv \rangle}{\langle v, v \rangle} \ge \frac{\zeta - |\lambda_{\max}|\|\widehat{B}\|}{\|B\|}.$$

Proof. Let $v \in \mathbb{R}^{k-1}$. For some $\mu_i \in \mathbb{C}$, $i = 1, \ldots, k-1$,

$$v = \sum_{i=1}^{k-1} \mu_i w_i.$$

Since the left-hand side of the inequality in the statement is homogeneous of degree 0 in v, by scaling v if necessary, we may assume $\|\mu\|^2 = \sum_{i=1}^{k-1} \mu_i \overline{\mu_i} = 1$. Then

$$Lv = \sum_{i=1}^{k-1} \lambda_i \mu_i w_i$$

and therefore

$$\langle v, Lv \rangle = \left\langle \sum_{i=1}^{k-1} \lambda_i \mu_i w_i, \sum_{j=1}^{k-1} \mu_j w_j \right\rangle = \sum_{i=1}^{k-1} \lambda_i \mu_i \overline{\mu_i} \|w_i\|^2 + \sum_{i \neq j} \lambda_i \mu_i \overline{\mu_j} \langle w_i, w_j \rangle.$$

We now take real parts in the right-hand side. For its first term we have

$$\mathrm{Re}\left(\sum_{i=1}^{k-1} \lambda_i \mu_i \overline{\mu_i} \|w_i\|^2 \right) = \sum_{i=1}^{k-1} \mathrm{Re}(\lambda_i) \mu_i \overline{\mu_i} \geq \zeta \sum_{i=1}^{k-1} \mu_i \overline{\mu_i} = \zeta.$$

For the second term,

$$\begin{aligned}
\mathrm{Re}\left(\sum_{i \neq j} \lambda_i \mu_i \overline{\mu_j} \langle w_i, w_j \rangle \right) &\leq \left| \sum_{i \neq j} \lambda_i \mu_i \overline{\mu_j} b_{ij} \right| \\
&\leq \left| \sum_{i=1}^{k-1} \lambda_i \mu_i \sum_{j \neq i} b_{ij} \overline{\mu_j} \right| \\
&= \left| \sum_{i=1}^{k-1} \lambda_i \mu_i \left(\widehat{B} \overline{\mu} \right)_i \right| \\
&\leq \|\lambda_{\max} \mu\| \left\| \widehat{B} \mu \right\| \\
&\leq |\lambda_{\max}| \left\| \widehat{B} \right\|.
\end{aligned}$$

Similarly, $\langle v, v \rangle \leq \left\langle \sum_{i=1}^{k-1} \mu_i w_i, \sum_{j=1}^{k-1} \mu_j w_j \right\rangle \leq \|B\|$. It follows that

$$\frac{\langle v, Lv \rangle}{\langle v, v \rangle} \geq \frac{\zeta - |\lambda_{\max}| \|\widehat{B}\|}{\|B\|}.$$

□

We next consider bounds for $\|\mathscr{S}(x)\|$. Let $\lambda_1, \ldots, \lambda_{k-1}$ be the eigenvalues of a linear operator $S : \mathbb{R}^{k-1} \to \mathbb{R}^{k-1}$. Assume that there exists a basis of eigenvectors $w_1, \ldots, w_{k-1} \in \mathbb{C}^{k-1}$ for $\lambda_1, \ldots, \lambda_{k-1}$. Without loss of generality we assume

$\|w_i\| = 1$ for $i = 1,\dots,k$. Let $b_{ij} = \langle w_i, w_j\rangle$, and the distortion matrix $B = (b_{ij})$. Also, let $|\lambda_{\max}| = \max_{i\le k}\{|\lambda_i|\}$.

Lemma 4. *Let $S : \mathbb{R}^{k-1} \to \mathbb{R}^{k-1}$ be a linear operator with $k-1$ different eigenvalues. Then $\|S\| \le (\|B\|\|B^{-1}\|)^{1/2}|\lambda_{\max}|$.*

Proof. Let $v \in \mathbb{R}^{k-1}$, $v \neq 0$. For some $\mu_i \in \mathbb{C}$, $i = 1,\dots,k$,

$$v = \sum_{i=1}^{k-1} \mu_i w_i.$$

Without loss of generality, we may assume $\|\mu\|^2 = \sum_{i=1}^{k-1} \mu_i \overline{\mu_i} = 1$. Then

$$Sv = \sum_{i=1}^{k-1} \lambda_i \mu_i w_i$$

and therefore

$$\begin{aligned}\langle Sv, Sv\rangle &= \left\langle \sum_{i=1}^{k-1} \lambda_i \mu_i w_i, \sum_{j=1}^{k-1} \lambda_j \mu_j w_j \right\rangle = \sum_{i,j} \lambda_i \mu_i \overline{\lambda_j}\,\overline{\mu_j} b_{ij} \\ &\le (\lambda_1\mu_1,\dots,\lambda_{k-1}\mu_{k-1})^* B(\lambda_1\mu_1,\dots,\lambda_{k-1}\mu_{k-1}) \\ &\le |\lambda_{\max}|^2 \|B\|.\end{aligned}$$

Similarly, since B is self-adjoint, $\langle v, v\rangle \ge \|B^{-1}\|^{-1}$. It follows that

$$\frac{\langle Sv, Sv\rangle}{\langle v, v\rangle} \le \|B\|\|B\|^{-1}|\lambda_{\max}|^2$$

from where the statement follows. □

Corollary 2. *For all $\ell \in \mathbb{N}$, $\|S^\ell\| \le (\|B\|\|B^{-1}\|)^{1/2}|\lambda_{\max}|^\ell$.*

Proof. The eigenvectors of S^ℓ are those of S. Therefore, S and S^ℓ have the same distortion matrix B. Also, $\lambda_{\max}(S^\ell) = (\lambda_{\max}(S))^\ell$. □

8. A last example

Let $\mathbf{A} = (\mathbf{a}_{ij})$ be a fixed $k \times k$ non-negative real matrix. Let $\mathbf{L}$ be the Laplacian of $\mathbf{A}$ and $0, \lambda_1, \dots, \lambda_{k-1}$ its eigenvalues, which we assume are different. It is known [1] that $\mathrm{Re}(\lambda_i) > 0$, for $i = 1,\dots,k-1$.

For every $g \in \mathbb{R}$, $g > 0$, the linear map $g\mathbf{L}$ has eigenvalues $0, g\lambda_1, \dots, g\lambda_{k-1}$. In addition, a basis of eigenvectors for $\mathbf{L}$ is also a basis of eigenvectors for $g\mathbf{L}$. Therefore, the distortion matrix B associated with the matrices $g\mathbf{L}$ is independent of g.

Assume that $\mathbf{A}$ satisfies $\zeta > |\lambda_{\max}|\|\widehat{B}\|$ where ζ, $\lambda_{\max}$ and $\widehat{B}$ are as in Section 7 and let

$$\mathbf{Z} = \frac{\zeta - |\lambda_{\max}|\|\widehat{B}\|}{\|B\|} > 0.$$

Note, this quantity depends only on $\mathbf{A}$. We now make g a function of $x \in X$. More precisely, for some $H > 0$ and $\beta \geq 0$, we define

$$g(x) = \frac{H}{(1+\Gamma(x))^{\beta}}.$$

We finally define the adjacency function $A : X \to \mathscr{M}(k \times k)$ given by

$$A_x = g(x)\mathbf{A}.$$

If we denote by L_x the linear operator on V induced by the Laplacian of A_x we can apply Lemma 3 to deduce that

$$\xi_x = \min_{\substack{v \in V \\ v \neq 0}} \frac{\langle v, v \rangle}{\|v\|^2} \geq g(x)\mathbf{Z}.$$

Therefore, condition (8) holds with $K = H\mathbf{Z}$ and Theorem 2 applies.

9. Relations with previous work

Literature on emergence is as scattered as the disciplines where emergence is considered. Our first interest in the subject was raised by the problem of the emergence of common languages in primitive human societies as described in [11]. Some results in this direction were worked out in [7]. The recent monograph by Niyogi [13] gives a detailed picture of the subject of language evolution and lists many of the different contributions.

Flocking, herding and schooling have been also the subject of much research in recent years. Computer simulations have been particularly adapted to test models for these problems.

Yet another area where emergence is paramount is distributed computing (see, e.g. the PhD thesis of Tsitsiklis [17]) where the reaching of consensus is of the essence.

This paper is an extension of [6], where we proposed and analyzed a model for flocking. A starting point for [6] was the model proposed in [19], which we mentioned in Section 6 as Vicsek's model. Its analytic behavior was subsequently studied in [10] (but convergence could be simply deduced from previous work [17], [18, Lemma 2.1]) and this paper, brought to our attention by Ali Jadbabaie, was helpful for us. Other works related to ours are [2, 8, 14–16, 18].

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