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Towards a Lie theory of locally convex groups

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Abstract. In this survey, we report on the state of the art of some of the fundamental problems in the Lie theory of Lie groups modeled on locally convex spaces, such as integrability of Lie algebras, integrability of Lie subalgebras to Lie subgroups, and integrability of Lie algebra extensions to Lie group extensions. We further describe how regularity or local exponentiality of a Lie group can be used to obtain quite satisfactory answers to some of the fundamental problems. These results are illustrated by specialization to some specific classes of Lie groups, such as direct limit groups, linear Lie groups, groups of smooth maps and groups of diffeomorphisms.

Keywords and phrases: infinite-dimensional Lie group, infinite-dimensional Lie algebra, continuous inverse algebra, diffeomorphism group, gauge group, pro-Lie group, BCH–Lie group, exponential function, Maurer–Cartan equation, Lie functor, integrable Lie algebra

Mathematics Subject Classification (2000): 22E65, 22E15

Introduction

Symmetries play a decisive role in the natural sciences and throughout mathematics. Infinite-dimensional Lie theory deals with symmetries depending on infinitely many parameters. Such symmetries may be studied on an infinitesimal, local or global level, which amounts to studying Lie algebras, local Lie groups and global Lie groups, respectively. Here the passage from the infinitesimal to the local level requires a smooth structure on the symmetry group (such as a Lie group structure as defined below), whereas the passage from the local to the global level is purely topological.

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Finite-dimensional Lie theory was created about 130 years ago by Marius Sophus Lie and Friedrich Engel, who showed that in finite dimensions the local and the infinitesimal theory are essentially equivalent ([Lie80/95]). The differential geometric approach to finite-dimensional global Lie groups (as smooth or analytic manifolds) is naturally complemented by the theory of algebraic groups with which it interacts most fruitfully. A crucial point of the finite-dimensional theory is that finiteness conditions permit to develop a powerful structure theory of finite-dimensional Lie groups in terms of the Levi splitting and the fine structure of semisimple Lie groups ([Ho65], [Wa72]).

A substantial part of the literature on infinite-dimensional Lie theory exclusively deals with the level of Lie algebras, their structure, and their representations (*cf.* [Ka90], [Neh96], [Su97], [AABGP97], [DiPe99], [ABG00]). However, only special classes of groups, such as Kac–Moody groups, can be approached with success by purely algebraic methods ([KP83], [Ka85]); see also [Rod89] for an analytic approach to Kac–Moody groups. In mathematical physics, the infinitesimal approach dealing mainly with Lie algebras and their representations is convenient for calculations, but a global analytic perspective is required to understand global phenomena (*cf.* [AI95], [Ot95], [CVLL98], [EMi99], [Go04], [Sch04]). A similar statement applies to non-commutative geometry, throughout of which derivations and covariant derivatives are used. It would be interesting to understand how global symmetry groups and the associated geometry fit into the picture ([Co94], [GVF01]).

In infinite dimensions, the passage from the infinitesimal to the local and from there to the global level is not possible in general, whence Lie theory splits into three properly distinct levels. It is a central point of this survey to explain some of the concepts and the results that can be used to translate between these three levels.

We shall use a Lie group concept which is both simple and very general: A Lie group is a manifold, endowed with a group structure such that multiplication and inversion are smooth maps. The main difference, compared to the finite-dimensional theory, concerns the notion of a manifold: The manifolds we consider are modeled on arbitrary locally convex spaces. As we shall see later, it is natural to approach Lie groups from such a general perspective, because it leads to a unified treatment of the basic aspects of the theory without unnatural restrictions on model spaces or the notion of a Lie group. Although we shall simply call them *Lie groups*, a more specific terminology is *locally convex Lie groups*. Depending on the type of the model spaces, we obtain in particular the classes of finite-dimensional, Banach–, Fréchet–, LF– and Silva–Lie groups.

The fundamental problems of Lie theory

As in finite dimensions, the Lie algebra $\mathbf{L}(G)$ of a Lie group G is identified with the tangent space $T_{\mathbf{1}}(G)$, where the Lie bracket it obtained by identification with the space of left invariant vector fields. This turns $\mathbf{L}(G)$ into a locally convex (topological) Lie algebra. Associating, furthermore, to a morphism φ of Lie groups its tangent map $\mathbf{L}(\varphi) := T_{\mathbf{1}}(\varphi)$, we obtain the *Lie functor* from the category of (locally convex) Lie groups to the category of locally convex topological Lie algebras. The core of Lie theory now consists in determining how much information the Lie functor forgets and how much can be reconstructed. This leads to several integration problems such as:

(FP1) When does a continuous homomorphism $\mathbf{L}(G) \to \mathbf{L}(H)$ between Lie algebras of connected Lie groups integrate to a (local) group homomorphism $G \to H$?

(FP2) **Integrability Problem for subalgebras:** Which Lie subalgebras $\mathfrak{h}$ of the Lie algebra $\mathbf{L}(G)$ of a Lie group G correspond to Lie group morphisms $H \to G$ with $\mathbf{L}(H) = \mathfrak{h}$?

(FP3) **Integrability Problem for Lie algebras (LIE III):** For which locally convex Lie algebras $\mathfrak{g}$ does a local/global Lie group G with $\mathbf{L}(G) = \mathfrak{g}$ exist?

(FP4) **Integrability Problem for extensions:** When does an extension of the Lie algebra $\mathbf{L}(G)$ of a Lie group G by the Lie algebra $\mathbf{L}(N)$ of a Lie group N integrate to a Lie group extension of G by N?

(FP5) **Subgroup Problem:** Which subgroups of a Lie group G carry natural Lie group structures?

(FP6) When does a Lie group have an exponential map $\exp_G \colon \mathbf{L}(G) \to G$?

(FP7) **Integrability Problem for smooth actions:** When does a homomorphism $\mathfrak{g} \to \mathcal{V}(M)$ into the Lie algebra $\mathcal{V}(M)$ of smooth vector fields on a manifold M integrate to a smooth action of a corresponding Lie group?

(FP8) **Small Subgroup Problem:** Which Lie groups have identity neighborhoods containing no non-trivial subgroup?

(FP9) **Locally Compact Subgroup Problem:** For which Lie groups are locally compact subgroups (finite-dimensional) Lie groups?

(FP10) **Automatic Smoothness Problem:** When are continuous homomorphisms between Lie groups smooth?

An important tool in the finite-dimensional and Banach context is the exponential map, but as vector fields on locally convex manifolds need not possess integral curves, there is no general theorem that guarantees the existence of a (smooth) exponential map, i.e., a smooth function

$$\exp_G \colon \mathbf{L}(G) \to G,$$

for which the curves $\gamma_x(t) := \exp_G(tx)$ are homomorphisms $(\mathbb{R},+) \to G$ with $\gamma_x'(0) = x$. Therefore the existence of an exponential function has to be treated as an additional requirement (*cf.* (FP6)). Even stronger is the requirement of *regularity*, meaning that, for each smooth curve $\xi\colon [0,1] \to \mathbf{L}(G)$, the initial value problem

$$\gamma'(t) = \gamma(t).\xi(t) := T_{\mathbf{1}}(\lambda_{\gamma(t)})\xi(t), \quad \gamma(0) = \mathbf{1}$$

has a solution $\gamma_\xi\colon [0,1] \to G$ and that $\gamma_\xi(1)$ depends smoothly on ξ. Regularity is a natural assumption that provides a good deal of methods to pass from the infinitesimal to the global level. This regularity concept is due to Milnor ([Mil84]). It weakens the μ-regularity (in our terminology) introduced by Omori et al. in [OMYK82/83a] (see [KYMO85] for a survey), but it is still strong enough for the essential Lie theoretic applications. Presently, we do not know of any Lie group modeled on a complete space which is not regular. For all major concrete classes discussed below, one can prove regularity, but there is no general theorem saying that each locally convex Lie group with a complete model space is regular or even that it has an exponential function. To prove or disprove such a theorem is another fundamental open problem of the theory. An assumption of a different nature than regularity, and which can be used to develop a profound Lie theory, is that G is *locally exponential* in the sense that it has an exponential function which is a local diffeomorphism in 0. Even stronger is the assumption that, in addition, G is analytic and that the exponential function is an analytic local diffeomorphism in 0. Groups with this property are called *BCH–Lie groups*, because the local multiplication in canonical local coordinates is given by the Baker–Campbell–Hausdorff series. This class contains in particular all Banach–Lie groups.

Important classes of infinite-dimensional Lie groups

Each general theory lives from the concrete classes of objects it can be applied to. Therefore it is good to have certain major classes of Lie groups in mind to which the general theory should apply. Here we briefly describe four such classes:

Linear Lie groups: Loosely speaking, linear Lie groups are Lie groups of operators on locally convex spaces, but this has to be made more precise.

Let E be a locally convex space and $\mathcal{L}(E)$ the unital associative algebra of all continuous linear endomorphisms of E. Its unit group is the general linear group $\mathrm{GL}(E)$ of E. If E is not normable, there is no vector topology on $\mathcal{L}(E)$, for which the composition map is continuous ([Mais63, Satz 2]). In general, the group $\mathrm{GL}(E)$ is open for no vector topology on $\mathcal{L}(E)$, as follows from the observation that if the spectrum of the operator D is unbounded, then $\mathbf{1} + tD$ is not invertible for arbitrarily small values of t. Therefore we need a class of associative algebras which are better behaved than $\mathscr{L}(E)$ to define a natural class of linear Lie groups.

The most natural class of associative algebras for infinite-dimensional Lie theory are *continuous inverse algebras* (CIAs for short), introduced in [Wae54a/b/c] by Waelbroeck in the context of commutative spectral theory. A CIA is a unital associative locally convex algebra A with continuous multiplication, for which the unit group $A^\times$ is open and the inversion $A^\times \to A, a \mapsto a^{-1}$ is a continuous map. As this implies the smoothness of the inversion map, $A^\times$ carries a natural Lie group structure. It is not hard to see that the CIA property is inherited by matrix algebras $M_n(A)$ over A ([Swa62]), so that $\mathrm{GL}_n(A) = M_n(A)^\times$ also is a Lie group, and under mild completeness assumptions (sequential completeness) on A, the Lie group $A^\times$ is regular and locally exponential ([Gl02b], [GN06]). This leads to natural classes of Lie subgroups of CIAs and hence to a natural concept of a *linear Lie group*.

Mapping groups and gauge groups: There are many natural classes of groups of maps with values in Lie groups which can be endowed with Lie group structures. The most important cases are the following: If M is a compact manifold and K a Lie group (possibly infinite-dimensional) with Lie algebra $\mathbf{L}(K) = \mathfrak{k}$, then the group $C^\infty(M,K)$ always carries a natural Lie group structure such that $C^\infty(M,\mathfrak{k})$, endowed with the pointwise bracket, is its Lie algebra ([GG61], [Mil82/84]; see also [Mi80]). A prominent class of such groups are the smooth loop groups $C^\infty(\mathbb{S}^1,K)$, which, for finite-dimensional simple groups K, are closely related to Kac–Moody groups ([PS86], [Mick87/89]).

If, more generally, $q\colon P \to M$ is a smooth principal bundle over a compact manifold M with locally exponential structure group K, then its *gauge group*

$$\mathrm{Gau}(P) := \{\varphi \in \mathrm{Diff}(P)\colon q \circ \varphi = q, (\forall p \in P, k \in K)\ \varphi(p.k) = \varphi(p).k\}$$

also carries a natural Lie group structure. For trivial bundles, we obtain the mapping groups $C^\infty(M,K)$ as special cases. Natural generalizations are the groups $C^\infty_c(M,K)$ of compactly supported smooth maps on a σ-compact finite-dimensional manifold ([Mi80], [Gl02c]) and also Sobolev completions of the groups $C^\infty(M,K)$ ([Sch04]).

Direct limit groups: A quite natural method to obtain infinite-dimensional groups from finite-dimensional ones is to consider a sequence $(G_n)_{n\in\mathbb{N}}$ of finite-dimensional Lie groups and morphisms $\varphi_n\colon G_n \to G_{n+1}$, so that we can define a direct limit group $G := \varinjlim G_n$ whose representations correspond to compatible sequences of representations of the groups G_n. According to a recent theorem of Glöckner ([Gl05]), generalizing previous work of J. Wolf and his coauthors ([NRW91/93]), the direct limit group G can always be endowed with a natural Lie group structure. Its Lie algebra $\mathbf{L}(G)$ is the countably dimensional direct limit space $\varinjlim \mathbf{L}(G_n)$, endowed with the finest locally convex topology. This provides an interesting class of infinite-dimensional Lie groups which is still quite close to finite-dimensional groups and has a very rich representation theory ([DiPe99], [NRW99], [Wol05]).

Groups of diffeomorphisms: In a similar fashion as linear Lie groups arise as symmetry groups of linear structures, such as bilinear forms on modules of CIAs, Lie groups of diffeomorphisms arise as symmetry groups of geometric structures on manifolds, such as symplectic structures, contact structures or volume forms.

A basic result is that, for any compact manifold M, the group $\mathrm{Diff}(M)$ can be turned into a Lie group modeled on the Fréchet space $\mathscr{V}(M)$ of smooth vector fields on M (*cf.* [Les67], [Omo70], [EM69/70], [Gu77], [Mi80], [Ham82]).

If M is non-compact and finite-dimensional, but σ-compact, then there is no natural Lie group structure on $\mathrm{Diff}(M)$ such that smooth actions of Lie groups G on M correspond to Lie group homomorphisms $G \to \mathrm{Diff}(M)$. Nevertheless, it is possible to turn $\mathrm{Diff}(M)$ into a Lie group with Lie algebra $\mathscr{V}_c(M)$, the Lie algebra of all smooth vector fields with compact support, endowed with the natural test function topology, turning it into an LF space (*cf.* [Mi80], [Mil82], [Gl02d]). If M is compact, this yields the aforementioned Lie group structure on $\mathrm{Diff}(M)$, but if M is not compact, then the corresponding topology on $\mathrm{Diff}(M)$ is so fine that the global flow generated by a vector field whose support is not compact does not lead to a continuous homomorphism $\mathbb{R} \to \mathrm{Diff}(M)$. For this Lie group structure, the normal subgroup $\mathrm{Diff}_c(M)$ of all diffeomorphisms which coincide with id_M outside a compact set is an open subgroup.

By *groups of diffeomorphisms* we mean groups of the type $\mathrm{Diff}_c(M)$, as well as natural subgroups defined as symmetry groups of geometric structures, such as groups of symplectomorphisms, groups of contact transformations and groups of volume preserving diffeomorphisms. Of a different nature, but also locally convex Lie groups, are groups of formal diffeomorphism as studied by Lewis ([Lew39]), Sternberg ([St61]) and Kuranishi ([Kur59]), groups of germs of smooth and analytic diffeomorphisms of $\mathbb{R}^n$ fixing 0 ([RK97], [Rob02]), and also germs of biholomorphic maps of $\mathbb{C}^n$ fixing 0 ([Pis76/77/79]).

As the discussion of these classes of examples shows, the concept of a locally convex Lie group subsumes quite different classes of Lie groups: Banach–Lie groups, groups of diffeomorphisms (modeled on Fréchet and LF spaces), groups of germs (modeled on Silva spaces) and formal groups (modeled on Fréchet spaces such as $\mathbb{R}^{\mathbb{N}}$).

In this survey article, we present our personal view of the current state of several aspects of the Lie theory of locally convex Lie groups. We shall focus on the general structures and concepts related to the fundamental problems (FP1)-(FP10) and on what can be said for the classes of Lie groups mentioned above.

Due to limited space and time, we had to make choices, and as a result, we could not take up many interesting directions such as the modern theory of symmetries of differential equations as exposed in Olver's beautiful book

[Olv93] and the fine structure and the geometry of specific groups of diffeomorphisms, such as the group $\mathrm{Diff}(M, \omega)$ of symplectomorphisms of a symplectic manifold (M, ω) ([Ban97], [MDS98], [Pol01] are recent textbooks on this topic). We do not go into (unitary) representation theory (*cf.* [AHMTT93], [Is96], [DP03], [Pic00a/b], [Ki05]) and connections to physics, which are nicely described in the recent surveys of Goldin [Go04] and Schmid [Sch04]. Other topics are only mentioned very briefly, such as the ILB and ILH-theory of Lie groups of diffeomorphisms which plays an important role in geometric analysis (*cf.* [AK98], [EMi99]) and the group of invertible Fourier integral operators of order zero, whose Lie group property was the main goal of the series of papers [OMY80/81], [OMYK81/82/83a/b], completed in [MOKY85]. An alternative approach to these groups is described in [ARS84,86a/b]. More recently, very interesting results concerning diffeomorphism groups and Fourier integral operators on non-compact manifolds (with bounded geometry) have been obtained by Eichhorn and Schmid ([ES96/01]).

Some history

To put the Lie theory of locally convex groups into proper perspective, we take a brief look at the historical development of infinite-dimensional Lie theory. Infinite-dimensional Lie algebras, such as Lie algebras of vector fields, where present in Lie theory right from the beginning, when Sophus Lie started to study (local) Lie groups as groups "generated" by finite-dimensional Lie algebras of vector fields ([Lie80]). The general global theory of finite-dimensional Lie groups started to develop in the late 19th century, driven substantially by É. Cartan's work on symmetric spaces ([CaE98]). The first exposition of a global theory, including the description of all connected groups with a given Lie algebra and analytic subgroups, was given by Mayer and Thomas ([MaTh35]). After the combination with the structure theory of Lie algebras, the theory reached its mature form in the middle of the 20th century, which is exposed in the fundamental books of Chevalley ([Ch46]) and Hochschild ([Ho65]) (see also [Po39] for an early textbook situated on the borderline between topological groups and Lie groups).

Already in the work of S. Lie infinite-dimensional groups show up as groups of (local) diffeomorphisms of open domains in $\mathbb{R}^n$ (*cf.* [Lie95]) and later É. Cartan undertook a more systematic study of certain types of infinite-dimensional Lie algebras, resp., groups of diffeomorphisms preserving geometric structures on a manifold, such a symplectic, contact or volume forms (*cf.* [CaE04]). On the other hand, the advent of Quantum Mechanics in the 1920s created a need to understand the structure of groups of operators on Hilbert spaces, which is a quite different class of infinite-dimensional groups (*cf.* [De32]).

The first attempt to deal with infinite-dimensional groups as Lie groups, i.e., as smooth manifolds, was undertaken by Birkhoff in [Bir36/38], where he developed the local Lie theory of Banach–Lie groups, resp., Banach–Lie algebras (see also [MiE37] for first steps in extending Lie's theory of local transformation groups to the Banach setting). In particular, he proved that (locally) C^1-Banach–Lie groups admit exponential coordinates which leads to analytic Lie group structures, that continuous homomorphisms are analytic and that, for every Banach–Lie algebra, the BCH series defines an analytic local group structure. He also defines the Lie algebra of such a local group, derives its functoriality properties and establishes the correspondence between closed subalgebras/ideals and the corresponding local subgroups. Even product integrals, which play a central role in the modern theory, appear in his work as solutions of left invariant differential equations. The local theory of Banach–Lie groups was continued by Dynkin ([Dy47/53]) who developed the algebraic theory of the BCH series further and by Laugwitz ([Lau55/56]) who developed a differential geometric perspective, which is quite close in spirit to the theory of locally exponential Lie groups described in Section IV below. Put in modern terms, he uses the Maurer–Cartan form and integrability conditions on (partial) differential equations on Banach space, developed by Michal and Elconin ([MiE37], [MicA48]), to derive the existence of the local group structure from the Maurer–Cartan form, which in turn is obtained from the Lie bracket. In the finite-dimensional case, this strategy is due to F. Schur ([SchF90a]) and quite close to Lie's original approach. In [Lau55], Laugwitz shows in particular that the center and any locally compact subgroup of a Banach–Lie group is a Banach–Lie subgroup. Formal Lie groups of infinitely many parameters were introduced by Ritt a few years earlier ([Ri50]).

The global theory of Banach–Lie groups started in the early 1960s with Maissen's paper [Mais62] which contains the first basic results on the Lie functor on the global level, such as the existence of integral subgroups for closed Lie subalgebras and the integrability of Lie algebra homomorphisms for simply connected groups. Later van Est and Korthagen studied the integrability problem for Banach–Lie algebras and found the first example of a non-integrable Banach–Lie algebra ([EK64]). Based on Kuiper's Theorem that the unitary group of an infinite-dimensional Hilbert space is contractible ([Ku65]), simpler examples were given later by Douady and Lazard ([DL66]). Chapters 2 and 3 in Bourbaki's "Lie groups and Lie algebras" contain in particular the basic local theory of Banach–Lie groups and Lie algebras and also some global aspects ([Bou89]). Although most of the material in Hofmann's Tulane Lecture Notes ([Hof68]), approaching the subject from a topological group perspective, was never published until recently ([HoMo98]), it was an important source of information for many people working on Banach–Lie theory (see also [Hof72/75]).

In the early 1970s, de la Harpe extended É. Cartan's classification of Riemannian symmetric spaces to Hilbert manifolds associated to a certain class of Hilbert–Lie algebras, called L^*-algebras, and studied different classes of operator groups related to Schatten ideals. Another context where a structure-theoretic approach leads quite far is the theory of bounded symmetric domains in Banach spaces and the related theory of (normed) symmetric spaces, developed by Kaup and Upmeier (*cf.* [Ka81/83a/b], [Up85]). For a more general approach to Banach symmetric spaces in the sense of Loos ([Lo69]), extending the class of all finite-dimensional symmetric spaces, not only Riemannian ones, we refer to [Ne02c] (*cf.* also [La99] for the corresponding basic differential geometry). In the context of symplectic geometry, resp., Hamiltonian flows, Banach manifolds were introduced by Marsden ([Mar67]), and Weinstein obtained a Darboux Theorem in this context ([Wei69]). Schmid's monograph [Sch87] provides a nice introduction to infinite-dimensional Hamiltonian systems. For more recent results on Banach–Kähler manifolds and their connections to representation theory, we refer to ([Ne04b], [Bel06]) and for Banach–Poisson manifolds to the recent work of Ratiu, Odzijewicz and Beltita ([RO03/04], [BR05a/b]).

Although Birkhoff was already aware of the fact that his theory covered groups of operators on Banach spaces, but not groups of diffeomorphisms, it took 30 years until infinite-dimensional Lie groups modeled on (complete) locally convex spaces occurred for the first time, as an attempt to understand the Lie structure of the group $\mathrm{Diff}(M)$ of diffeomorphisms of a compact manifold M, in the work of Leslie ([Les67]) and Omori ([Omo70]). This theory was developed further by Omori in the context of strong ILB–Lie groups (*cf.* [Omo74]). A large part of [Omo74] is devoted to the construction of a strong ILB–Lie group structure on various types of groups of diffeomorphisms. In the 1980s, this theory was refined substantially by imposing and proving additional regularity conditions for such groups ([OMYK82/83a], [KYMO85]). A different type of Lie group was studied by Pisanelli in [Pis76/77/79], namely the group $\mathrm{Gh}_n(\mathbb{C})$ of germs of biholomorphic maps of $\mathbb{C}^n$ fixing 0. This group carries the structure of a Silva–Lie group and is one of the first non-Fréchet–Lie groups studied systematically in a Lie theoretic context. In [BCR81], Boseck, Czichowski and Rudolph approach infinite-dimensional Lie groups from a topological group perspective. They use the same concept of an infinite-dimensional manifold as we do here, but a stronger Lie group concept. This leads them to a natural setting for mapping groups on non-compact manifolds modeled on spaces of rapidly decreasing functions.

In his lecture notes [Mil84], Milnor undertook the first attempt to develop a general theory of Lie groups modeled on complete locally convex spaces, which already contained important cornerstones of the theory. This paper and the earlier preprint [Mil82] had a strong influence on the development of the theory. Both contain precise formulations of several problems, some of which have been solved in the meantime and some of which are still open, as we shall see in more detail below (see also [Gl06b] for the state of the art on some of these problems).

In the middle of the 1980s, groups of smooth maps, and in particular groups of smooth loops became popular because of their intimate connection with Kac–Moody theory and topology (*cf.* [PS86], [Mick87/89]). The interest in direct limits of finite-dimensional Lie groups grew in the 1990s (*cf.* [NRW91/93/94/99]). They show up naturally in the structure and representation theory of Lie algebras (*cf.* [Ne98/01b], [DiPe99], [NRW99], [NS01], [Wol05]). The general Lie theory of these groups was put into its definitive form by Glöckner in [Gl05].

There are other, weaker, concepts of Lie groups, resp., infinite-dimensional manifolds. One is based on the "convenient setting" for global analysis developed by Frölicher, Kriegl and Michor ([FB66], [Mi84], [FK88] and [KM97]). In the context of Fréchet manifolds, this setting leads to Milnor's concept of a regular Lie group, but for more general model spaces, it provides a concept of a smooth map which does not imply continuity, hence leads to Lie groups which are not topological groups. Another approach, due to Souriau, is based on the concept of a diffeological space ([So84/85], [DoIg85], [Los92]; see [HeMa02] for applications to diffeomorphism groups) which can be used to study spaces like quotients of $\mathbb{R}$ by non-discrete subgroups in a differential geometric context. On the one hand, it has the advantage that the category of diffeological spaces is cartesian closed and that any quotient of a diffeological space carries a natural diffeology. But on the other hand, this incredible freedom makes it harder to distinguish "regular" objects from "non-regular" ones. Our discussion of smoothness of maps with values in diffeomorphism groups of (possibly infinite-dimensional) manifolds is inspired by the diffeological approach. We shall see in particular, that, to some extent, one can use differential methods to deal with groups with no Lie group structure, such as groups of diffeomorphisms of non-compact manifolds or groups of linear automorphisms of locally convex spaces, and that this provides a natural framework for a Lie theory of smooth actions on manifolds and smooth linear representations.

There are other purposes, for which a Lie group structure on an infinite-dimensional group G is indispensable. The most crucial one is that without the manifold structure, there is simply not enough structure available to pass from the infinitesimal level to the global level. For instance, to integrate abelian or central extensions of Lie algebras to corresponding group extensions, the manifold structure on the group is of crucial importance (*cf.* (FP4)). To deal with these extension problems, one is naturally lead to certain classes of closed differential 2-forms on Lie groups, which in turn leads to infinite-dimensional symplectic geometry and Hamiltonian group actions. Although we do not know which coadjoint orbits of an infinite-dimensional Lie group carry manifold structures, for any such orbit, we have a natural Hamiltonian action of the group G

on itself with respect to a closed invariant 2-form which in general is degenerate; so the reduction of free actions of infinite-dimensional Lie groups causes similar difficulties as singular reduction does in finite-dimensions; but still all the geometry is visible in the non-reduced system. It is our hope that this kind of symplectic geometry will ultimately lead to a more systematic “orbit method” for infinite-dimensional Lie groups, in the sense that it paves the way to a better understanding of the unitary representations of infinite-dimensional Lie groups, based on symplectic geometry and Hamiltonian group actions (*cf.* [Ki05] for a recent survey on various aspects of the orbit method).

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Contents

The structure of this paper is as follows. Sections I-VI deal with the general theory and Sections VII-X explain how it applies to several classes of Lie groups. After explaining the basic concepts and some of the pitfalls of infinite-dimensional calculus in Section I, we discuss the basic concepts of Lie group theory in Section II. Section III is devoted to the concept of regularity, its relatives, and its applications to the fundamental problems. Section IV on locally exponential Lie groups is the longest one. This class of groups still displays many features of finite-dimensional, resp., Banach–Lie groups. It is also general enough to give a good impression of the difficulties arising beyond Banach spaces and for smooth Lie groups without any analytic structure. In this section, one also finds several results concerning Lie group structures on closed subgroups and quotients by closed normal subgroups. Beyond the class of locally exponential Lie groups, hardly anything is known in this direction.

In Sections V and VI, which are closely related, we discuss the integrability problem for abelian Lie algebra extensions (FP4) and the integrability problem for Lie algebras (FP3). In particular, we give a complete characterization of the integrable locally exponential Lie algebras and discuss integrability results for various other Lie algebras. The remaining sections are relatively short: Section VII describes some specifics of direct limits of Lie groups, Section VIII explains how the theory applies to linear Lie groups and Section IX presents some results concerning infinite-dimensional Lie groups acting on finite-dimensional manifolds. We conclude this paper with a discussion of projective limits, a construction that leads beyond the class of Lie groups. It is amazing that projective limits of finite-dimensional Lie groups still permit a powerful structure theory

([HoMo06]) and it would be of some interest to develop a theory of projective limits of infinite-dimensional Lie groups.

Notation

We write $\mathbb{N} := \{1,2,\ldots\}$ for the natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Throughout this paper all vector spaces, algebras and Lie algebras are defined over the field $\mathbb{K}$, which is $\mathbb{R}$ or $\mathbb{C}$. If E is a real vector space, we write $E_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} E$ for its complexification, considered as a complex vector space. For two topological vector spaces E, F, we write $\mathscr{L}(E,F)$ for the space of continuous linear operators $E \to F$ and put $\mathscr{L}(E) := \mathscr{L}(E,E)$. For $F = \mathbb{K}$, we write $E' := \mathscr{L}(E,\mathbb{K})$ for the *topological dual space of* E.

If G is a group, we denote the identity element by $\mathbf{1}$, and for $g \in G$, we write
$\lambda_g \colon G \to G, x \mapsto gx$ for the *left multiplication* by g,
$\rho_g \colon G \to G, x \mapsto xg$ for the *right multiplication* by g,
$m_G \colon G \times G \to G, (x,y) \mapsto xy$ for the *multiplication map*, and
$\eta_G \colon G \to G, x \mapsto x^{-1}$ for the *inversion*.

We always write G_0 for the connected component of the identity and, if G is connected, we write $q_G \colon \widetilde{G} \to G$ for the universal covering group.

We call a manifold M 1-*connected* if it is connected and simply connected.

Index of notation and concepts

I. Locally convex manifolds

In this section, we briefly explain the natural setup for manifolds modeled on locally convex spaces, vector fields and differential forms on these manifolds. An essential difference to the finite-dimensional, resp., the Banach setting is that we use a C^k-concept which on Banach spaces is slightly weaker than Fréchet differentiability, but implies C^{k-1} in the Fréchet sense, so that we obtain the same class of smooth functions. The main point is that, for a non-normable locally convex space E, the space $\mathscr{L}(E,F)$ of continuous linear maps to some locally convex space F does not carry any vector topology for which the evaluation map is continuous ([Mais63]). Therefore it is more natural to develop calculus independently of any topology on spaces of linear maps and thus to deal instead with the differential of a function as a function of two arguments, not as an operator-valued function of one variable. One readily observes that once the Fundamental Theorem of Calculus is available, which is not in general the case beyond locally convex spaces, most basic calculus results can simply be reduced to the familiar finite-dimensional situation. This is done by restricting to finite-dimensional subspaces and composing with linear functionals, which separate the points due to the Hahn–Banach Theorems.

The first steps towards a calculus on locally convex spaces have been taken by Michal (*cf.* [MicA38/40]), whose work was developed further by Bastiani in [Bas64], so that the calculus we present below is named after Michal–Bastiani, and the C^k-concept is denoted $C^k_{\rm MB}$ accordingly (if there is any need to distinguish it from other C^k-concepts). Keller's comparative discussion of various notions of differentiability on topological vector spaces ([Ke74]) shows that the Michal–Bastiani calculus is the most natural one since it does not rely on convergence structures or topologies on spaces of linear maps. Streamlined discussions of the basic results of calculus, as we use it, can be found in [Mi80] and [Ham82]. In [Gl02a], Glöckner treats real and complex analytic functions over not necessarily complete spaces, which presents some subtle difficulties. Beyond Fréchet spaces, it is more convenient to work with locally convex spaces which are not necessarily complete because quotients of complete non-metrizable locally convex spaces need not be complete (*cf.* [Kö69], §31.6). Finally we mention that the MB-calculus can even be developed for topological vector spaces over general non-discrete topological fields (see [BGN04] for details).

One of the earliest references for smooth manifolds modeled on (complete) locally convex spaces is Eell's paper [Ee58], but he uses a different smoothness concept, based on the topology of bounded convergence on the space of linear maps (*cf.* also [Bas64] and [FB66]). Lie groups in the context of MB-calculus show up for the first time in Leslie's paper on diffeomorphism groups of compact manifolds ([Les67]).

I.1. Locally convex spaces

Definition I.1.1. *A topological vector space E is said to be* locally convex *if each* 0*-neighborhood in E contains a convex one. Throughout, topological vector spaces E are assumed to be Hausdorff.* □

It is a standard result in functional analysis that local convexity is equivalent to the embeddability of E into a product of normed spaces. This holds if and only if the topology can be defined by a family $(p_i)_{i\in I}$ of seminorms in the sense that a subset U of E is a 0-neighborhood if and only if it contains a finite intersection of sets of the form

$$V(p_i,\varepsilon_i) := \{x \in E\colon\ p_i(x) < \varepsilon_i\}, \quad i \in I, \varepsilon_i > 0.$$

Definition I.1.2. *(a) A locally convex space E is called a* Fréchet space *if there exists a sequence $\{p_n\colon n \in \mathbb{N}\}$ of seminorms on E, such that the topology on E is induced by the metric*

$$d(x,y) := \sum_{n\in\mathbb{N}} 2^{-n} \frac{p_n(x-y)}{1+p_n(x-y)},$$

and the metric space (E,d) is complete. Important examples of Fréchet spaces are Banach spaces, which are the ones where the topology is defined by a single (semi-)norm.

(b) Let E be a vector space which can be written as $E = \bigcup_{n=1}^{\infty} E_n$, where $E_n \subseteq E_{n+1}$ are subspaces of E, endowed with structures of locally convex spaces in such a way that the inclusion mappings $E_n \to E_{n+1}$ are continuous.

Then we obtain a locally convex vector topology on E by defining a seminorm p on E to be continuous if and only if its restriction to all the subspaces E_n is continuous. We call E the inductive limit *(or* direct limit*) of the spaces $(E_n)_{n\in\mathbb{N}}$.*

If all maps $E_n \hookrightarrow E_{n+1}$ are embeddings, we speak of a strict inductive limit*. If, in addition, the spaces E_n are Fréchet spaces, then each E_n is closed in E_{n+1} and E is called an* LF space*. If the spaces E_n are Banach spaces and the inclusion maps $E_n \to E_{n+1}$ are compact, then E is called a* Silva space*.* □

Examples I.1.3. To give an impression of the different types of locally convex spaces occurring below, we take a brief look at function spaces on the real line.

(a) For $r \in \mathbb{N}_0$ and $a < b$, the spaces $C^r([a,b],\mathbb{R})$ of r-times continuously differentiable functions on $[a,b]$ form a Banach space with respect to the norms

$$\|f\|_r := \sum_{k=0}^{r} \|f^{(k)}\|_\infty$$

and $C^\infty([a,b],\mathbb{R})$ is a Fréchet space with respect to the topology defined by the sequence $(\|\cdot\|_r)_{r\in\mathbb{N}_0}$ of norms.

(b) For each fixed $r \in \mathbb{N}_0$, the space $C^r(\mathbb{R},\mathbb{R})$ is a Fréchet space with respect to the sequence of seminorms $p_n(f) := \|f|_{[-n,n]}\|_r$. It is the projective limit of the Banach spaces $C^r([-n,n],\mathbb{R})$. On $C^\infty(\mathbb{R},\mathbb{R})$ we also obtain a Fréchet space structure defined by the countable family of seminorms $p_{n,r}(f) := \|f|_{[-n,n]}\|_r$, $n,r \in \mathbb{N}$.

(c) For each $r \in \mathbb{N}_0 \cup \{\infty\}$, the space $C^r_c(\mathbb{R},\mathbb{R})$ of compactly supported C^r-functions on $\mathbb{R}$ is the union of the subspaces $C^r_{[-n,n]}(\mathbb{R},\mathbb{R})$ of all those functions supported by the interval $[-n,n]$. As a closed subspace of the Fréchet space $C^r([-n,n],\mathbb{R})$, $C^r_{[-n,n]}(\mathbb{R},\mathbb{R})$ inherits a Fréchet space structure, so that we obtain on $C^r_c(\mathbb{R},\mathbb{R})$ the structure of an LF space.

(d) For each $r > 0$, the space of all sequences $(a_n)_{n\in\mathbb{N}_0}$ for which $\sum_{n=0}^\infty |a_n| r^n$ converges can be identified with the space E_r of all functions $f\colon [-r,r] \to \mathbb{R}$ which can be represented by a power series, uniformly convergent on $[-r,r]$. This is a Banach space with respect to the norm $\|f\|_r := \sum_{n=0}^\infty \frac{|f^{(n)}(0)|}{n!} r^n$. The direct limit space $E := \bigcup_{r>0}^\infty E_r$ is the space of germs of analytic function in 0. Since the inclusion maps $E_{\frac{1}{n}} \to E_{\frac{1}{n+1}}$ are compact operators, E carries a natural Silva space structure. Note that the subspaces E_r, $r > 0$, are dense and not closed, so that E is *not* an LF space. ☐

The natural completeness requirement for calculus on locally convex spaces is the following:

Definition I.1.4. *A locally convex space E is said to be* Mackey complete *if for each smooth curve $\xi\colon [0,1] \to E$ there exists a smooth curve $\eta\colon [0,1] \to E$ with $\eta' = \xi$.* ☐

For each continuous linear functional $\lambda\colon E \to \mathbb{R}$ on a locally convex space E and each continuous curve $\xi\colon [0,1] \to E$, we have a continuous real-valued function $\lambda \circ \xi\colon [0,1] \to \mathbb{R}$ which we may integrate to obtain a linear functional

$$I_\xi\colon E' \to \mathbb{R}, \quad \lambda \mapsto \int_0^1 \lambda(\xi(t))\,dt,$$

called the *weak integral of* ξ. On the other hand, we have a natural embedding

$$\eta_E\colon E \to (E')^*, \quad \eta_E(x)(\lambda) := \lambda(x)$$

which is injective, because E' separates the points of E by the Hahn–Banach Theorem. Therefore Mackey completeness means that for each smooth curve ξ the weak integral I_ξ is represented by an element of E, i.e., contained in $\eta_E(E)$. If this is the case, we simply write $\int_0^1 \xi(t)\,dt$ for the representing element of E. The curve $\eta(s) := \int_0^s \xi(t)\,dt$ then is differentiable and satisfies $\eta' = \xi$.

For a more detailed discussion of Mackey completeness and equivalent conditions, we refer to [KM97, Th. 2.14], where it is shown in particular that integrals exist for Lipschitz curves and in particular for each $\eta \in C^1([0,1],E)$.

I.2. Calculus on locally convex spaces

The following notion of C^k-maps is also known as C^k_{MB} (C^k in the Michal–Bastiani sense) ([MicA38/40], [Bas64]) or Keller's C^k_c-maps ([Ke74]). Its main advantage is that it does not refer to any topology on spaces of linear maps or any quasi-topology (*cf.* [Bas64]).

Definition I.2.1. *(a) Let E and F be locally convex spaces, $U \subseteq E$ open and $f\colon U \to F$ a map. Then the* derivative of f at x in the direction h *is defined as*

$$df(x)(h) := (D_h f)(x) := \frac{d}{dt}\Big|_{t=0} f(x+th) = \lim_{t\to 0}\frac{1}{t}(f(x+th)-f(x))$$

whenever it exists. The function f is called differentiable at x *if $df(x)(h)$ exists for all $h \in E$. It is called* continuously differentiable*, if it is differentiable at all points of U and*

$$df\colon U\times E \to F, \quad (x,h)\mapsto df(x)(h)$$

is a continuous map. It is called a C^k-map*, $k \in \mathbb{N}\cup\{\infty\}$, if it is continuous, the iterated directional derivatives*

$$d^j f(x)(h_1,\ldots,h_j) := (D_{h_j}\cdots D_{h_1} f)(x)$$

exist for all integers $j \leq k$, $x \in U$ and $h_1,\ldots,h_j \in E$, and all maps $d^j f\colon U\times E^j \to F$ are continuous. As usual, C^∞-maps are called smooth*.*

(b) If E and F are complex vector spaces, then a map f is called complex analytic *if it is continuous and for each $x \in U$ there exists a* 0*-neighborhood V with $x+V \subseteq U$ and continuous homogeneous polynomials $\beta_k\colon E \to F$ of degree k such that for each $h \in V$ we have*

$$f(x+h) = \sum_{k=0}^{\infty} \beta_k(h),$$

as a pointwise limit ([BoSi71]).

If E and F are real locally convex spaces, then we call f real analytic*, resp., C^ω, if for each point $x \in U$ there exists an open neighborhood $V \subseteq E_{\mathbb{C}}$ and a holomorphic map $f_{\mathbb{C}}\colon V \to F_{\mathbb{C}}$ with $f_{\mathbb{C}}|_{U\cap V} = f|_{U\cap V}$ (cf. [Mil84]). The advantage of this definition, which differs from the one in [BoSi71], is that it works nicely for non-complete spaces, any analytic map is smooth, and the corresponding chain rule holds without any condition on the underlying spaces, which is the key to the definition of analytic manifolds (see [Gl02a] for details).*

The map f is called holomorphic *if it is C^1 and for each $x \in U$ the map $df(x)\colon E \to F$ is complex linear (cf. [Mil84, p.1027]). If F is sequentially complete, then f is holomorphic if and only if it is complex analytic (cf. [Gl02a], [BoSi71, Ths. 3.1, 6.4], [Mil82, Lemma 2.11]).* □

Remark I.2.2. If E and F are Banach spaces, then the Michal–Bastiani C^1_{MB}-concept from above is weaker than continuous Fréchet differentiability, which requires that the map $x \mapsto df(x)$ is continuous with respect to the operator norm (*cf.* [Mil82, Ex. 6.8]). Nevertheless, one can show that C^{k+1}_{MB} implies C^k in the sense of Fréchet differentiability, which in turn implies C^k_{MB}. Therefore the different C^k-concepts lead to the same class of smooth functions (*cf.* [Mil82, Lemma 2.10], [Ne01a, I.6 and I.7]). □

After clarifying the C^k-concept, we recall the precise statements of the most fundamental facts from calculus on locally convex spaces.

Proposition I.2.3. *Let E and F be locally convex spaces, $U \subseteq E$ an open subset, and $f\colon U \to F$ a continuously differentiable function.*

(i) *For any $x \in U$, the map $df(x)\colon E \to F$ is real linear and continuous.*

(ii) (Fundamental Theorem of Calculus) *If $x + [0,1]h \subseteq U$, then*

$$f(x+h) = f(x) + \int_0^1 df(x+th)(h)\,dt.$$

In particular, f is locally constant if and only if $df = 0$.

(iii) *f is continuous.*

(iv) *If f is C^n, $n \geq 2$, then the functions $d^n f(x)$, $x \in U$, are symmetric n-linear maps.*

(v) *If $x + [0,1]h \subseteq U$ and f is C^n, then we have the Taylor Formula*

$$\begin{aligned} f(x+h) &= f(x) + df(x)(h) + \cdots + \frac{1}{(n-1)!} d^{n-1} f(x)(h,\ldots,h) \\ &\quad + \frac{1}{(n-1)!} \int_0^1 (1-t)^{n-1} d^n f(x+th)(h,\ldots,h)\,dt. \end{aligned}$$

(vi) (Chain Rule) *If, in addition, Z is a locally convex space, $V \subseteq F$ is open, and $f_1\colon U \to V$, $f_2\colon V \to Z$ are C^1, then $f_2 \circ f_1\colon U \to Z$ is C^1 with*

$$d(f_2 \circ f_1)(x) = df_2\big(f_1(x)\big) \circ df_1(x) \quad \text{for} \quad x \in U.$$

If f_1 and f_2 are C^k, $k \in \mathbb{N} \cup \{\infty\}$, the Chain Rule implies that $f_2 \circ f_1$ is also C^k. □

Remark I.2.4. A continuous k-linear map $m\colon E_1 \times \cdots \times E_k \to F$ is continuously differentiable with

$$dm(x)(h_1,\ldots,h_k) = m(h_1,x_2,\ldots,x_k) + \cdots + m(x_1,\ldots,x_{k-1},h_k).$$

Inductively, one obtains that m is smooth with $d^{k+1}m = 0$. □

Example I.2.5. The following example shows that local convexity is crucial for the validity of the Fundamental Theorem of Calculus.

Let E denote the space of measurable functions $f\colon [0,1] \to \mathbb{R}$ for which

$$|f| := \int_0^1 |f(x)|^{\frac{1}{2}}\, dx$$

is finite and identify functions that coincide on a set whose complement has measure zero. Then $d(f,g) := |f-g|$ defines a metric on E. We thus obtain a metric topological vector space (E,d).

For a subset $S \subseteq [0,1]$, let χ_S denote its characteristic function. Consider the curve

$$\gamma\colon [0,1] \to E, \quad \gamma(t) := \chi_{[0,t]}.$$

Then $|h^{-1}\big(\gamma(t+h)-\gamma(t)\big)| = |h|^{-\frac{1}{2}}|h| \to 0$ for each $t \in [0,1]$ as $h \to 0$. Hence γ is continuously differentiable with $\gamma' = 0$. Since γ is not constant, the Fundamental Theorem of Calculus does not hold in E.

The defect in this example is caused by the non-local convexity of E. In fact, one can even show that all continuous linear functionals on E vanish. □

The preceding phenomenon could also be excluded by requiring that the topological vector spaces under consideration have the property that the continuous linear functionals separate the points, which is automatic for locally convex spaces. Another reason for working with locally convex spaces is that local convexity is also crucial for approximation arguments, more specifically to approximate continuous maps by smooth ones in the same homotopy class (*cf.* [Ne04c], [Wo05a]). Local convexity it also crucial for the continuous parameter-dependence of integrals which in turn goes into the proof of the Chain Rule.

One frequently encounters situations where it is convenient to describe multilinear maps $m\colon E_1 \times \cdots \times E_k \to F$ as continuous linear maps on the tensor product space $E_1 \otimes \cdots \otimes E_k$, endowed with a suitable topology. For locally convex spaces, there is a natural such topology, the *projective tensor topology*, and it has the nice property that projective tensor products are associative. That this is no longer true for more general topological vector spaces is one more reason to work in the locally convex setting (*cf.* [Gl04a]).

Remark I.2.6. (Inverse Function Theorems) In the context of Banach spaces, one has an Inverse Function Theorem and also an Implicit Function Theorem (*cf.* [La99]). Such results cannot be expected in general for Fréchet spaces. One of the simplest examples demonstrating this fact arises from the algebra $A := C(\mathbb{R},\mathbb{R})$ of all continuous functions on $\mathbb{R}$, endowed with the topology of uniform convergence on compact subsets, turning A into a Fréchet space on which the algebra multiplication is continuous. We have a smooth exponential map

$$\exp_A \colon A \to A, \quad f \mapsto e^f$$

with $T_0(\exp_A) = \mathrm{id}_A$. Since the range of $\exp_A$ lies in the unit group $A^\times = C(\mathbb{R},\mathbb{R}^\times)$, which apparently is not a neighborhood of the constant function $\mathbf{1}$, the Inverse Function Theorem fails in this case (*cf.* [Ee66, p.761]).

In Example II.5.9 below, we shall even encounter examples of exponential functions of Lie groups which, in spite of $T_{\mathbf{1}}(\exp_G) = \mathrm{id}_{\mathbf{L}(G)}$, are not a local diffeomorphism in 0. In view of these examples, the usual Inverse Function Theorem cannot be generalized in any straightforward manner to arbitrary Fréchet spaces.

Nevertheless, Glöckner ([Gl03a]) obtained a quite useful Implicit Function Theorem for maps of the type $f \colon E \times F \to F$, where F is a Banach space and E is locally convex. These results have many interesting applications, even in the case where F is finite-dimensional. Similar results have been achieved by Hiltunen in [Hi99], but he uses a different notion of smoothness.

A complementary Inverse Function Theorem is due to Nash and Moser (*cf.* [Mo61] and [Ham82] for a nice exposition). This is a variant that can be applied to Fréchet spaces endowed with an additional structure, called a grading, and to smooth maps which are "tame" in the sense that they are compatible with the grading.

Another variant based on compatibility with a projective limit of Banach spaces is the ILB-Implicit Function Theorem to be found in Omori's book ([Omo97]). □

Remark I.2.7. (Non-complemented subspaces) Another serious pathology occurring already for Banach spaces is that a closed subspace F of a locally convex space E need not have a closed complement. A simple example is the subspace $F := c_0(\mathbb{N},\mathbb{R})$ of the Banach space $E := \ell^\infty(\mathbb{N},\mathbb{R})$ (*cf.* [Wer95, Satz IV.6.5] for an elementary proof).

This implies that if $q \colon E \to E/F$ is a quotient map of locally convex spaces, there need not be any continuous linear map $\sigma \colon E/F \to E$ with $q \circ \sigma = \mathrm{id}_{E/F}$. If such a map σ exists, then

$$F \times E/F \to E, \quad (x,y) \mapsto x + \sigma(y)$$

is a linear isomorphism of topological vector spaces, which implies that $\sigma(E/F)$ is a closed complement of F in E. We then call the quotient map q, resp., the subspace F, *topologically split*. If E is a Fréchet space, then the Open Mapping

Theorem implies that the existence of a closed complement for F is equivalent to the existence of a splitting map σ.

For Fréchet spaces, it is quite easy to find natural examples of non-splitting quotient maps: Let $E := C^\infty([0,1],\mathbb{R})$ be the Fréchet space of smooth functions on the unit interval and

$$q\colon E \to \mathbb{R}^{\mathbb{N}}, \quad q(f) = (f^{(n)}(0))_{n\in\mathbb{N}}.$$

In view of E. Borel's Theorem, this map is surjective, hence a quotient map by the Open Mapping Theorem. Since every 0-neighborhood in $\mathbb{R}^{\mathbb{N}}$ contains a non-trivial subspace, there is no continuous norm on $\mathbb{R}^{\mathbb{N}}$, hence there is no continuous linear cross section $\sigma\colon \mathbb{R}^{\mathbb{N}} \to E$ for q because the topology on E is defined by a sequence of norms.

If a continuous linear cross section σ does not exist, then q has no smooth local sections either, because for any such section $\sigma\colon U \to E$, U open in E/F, the differential of σ in any point would be a continuous linear section of q. If E is Fréchet, then q has a continuous global section by Michael's Selection Theorem ([MicE59], [Bou87]), and the preceding argument shows that no such section is continuously differentiable.

For more detailed information on splitting conditions for extensions of Fréchet spaces, we refer to [Pala71] and [Vo87]. □

I.3. Smooth manifolds

Since the Chain Rule is valid for smooth maps between open subsets of locally convex spaces, we can define smooth manifolds as in the finite-dimensional case (see [Ee58] for one of the first occurrences of manifolds modeled on complete locally convex spaces).

Definition I.3.1. *Let M be a Hausdorff space and E a locally convex space. An E-*chart *of M is a pair (φ,U) of an open subset $U \subseteq M$ and a homeomorphism $\varphi\colon U \to \varphi(U) \subseteq E$ onto an open subset $\varphi(U)$ of E. For $k \in \mathbb{N}_0 \cup \{\infty,\omega\}$, two E-charts (φ,U) and (ψ,V) are said to be C^k-*compatible *if the maps*

$$\psi\circ\varphi^{-1}|_{\varphi(U\cap V)}\colon \varphi(U\cap V) \to \psi(U\cap V)$$

*and $\varphi\circ\psi^{-1}$ are C^k, where $k=\omega$ stands for analyticity. Since compositions of C^k-maps are C^k-maps, C^k-compatibility of E-charts is an equivalence relation. An E-*atlas of class C^k of *M is a set $\mathscr{A} := \{(\varphi_i,U_i)\colon i\in I\}$ of pairwise compatible E-charts of M with $\bigcup_i U_i = M$. A* smooth/analytic E-structure on *M is a maximal E-atlas of class C^∞/C^ω, and a* smooth/analytic E-manifold *is a pair $(M,\mathscr{A})$, where $\mathscr{A}$ is a maximal smooth/analytic E-atlas on M.*

We call a manifold modeled on a locally convex, resp., Fréchet, resp., Banach space a locally convex*, resp.,* Fréchet*, resp.,* Banach manifold. □

We do not make any further assumptions on the topology of smooth locally convex manifolds, such as regularity (as in [Mil84]) or paracompactness. But we impose the Hausdorff condition, an assumption not made in some textbooks (*cf.* [La99], [Pa57]). We refer to Example 6.9 in [Mil82] for a non-regular manifold.

Remark I.3.2. If $M_1,\ldots,M_n$ are smooth manifolds modeled on the spaces E_i, $i=1,\ldots,n$, then the product set $M := M_1 \times \cdots \times M_n$ carries a natural manifold structure with model space $E = \prod_{i=1}^n E_i$. □

Smooth maps between smooth manifolds are defined as usual.

Definition I.3.3. *For $p \in M$, tangent vectors $v \in T_p(M)$ are defined as equivalence classes of smooth curves $\gamma\colon]-\varepsilon,\varepsilon[\to M$ with $\varepsilon > 0$ and $\gamma(0) = p$, where the equivalence relation is given by $\gamma_1 \sim \gamma_2$ if $(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$ holds for a chart (φ,U) with $p \in U$. Then $T_p(M)$ carries a natural vector space structure such that for any E-chart (φ,U), the map $T_p(M) \to E, [\gamma] \mapsto (\varphi \circ \gamma)'(0)$ is a linear isomorphism. We write $T(M) := \bigcup_{p\in M} T_p(M)$ for the* tangent bundle *of M. The map $\pi_{TM}\colon T(M) \to M$ mapping elements of $T_p(M)$ to p is called the* bundle projection.

If $f\colon M \to N$ is a smooth map between smooth manifolds, we obtain for each $p \in M$ a linear tangent map

$$T_p(f)\colon T_p(M) \to T_{f(p)}(N), \quad [\gamma] \mapsto [f \circ \gamma],$$

and these maps combine to the tangent map $T(f)\colon T(M) \to T(N)$. On the tangent bundle $T(M)$ we obtain for each E-chart (φ,U) of M an $E \times E$-chart by

$$T(\varphi)\colon T(U) := \bigcup_{p\in U} T_p(M) \to T(\varphi(U)) \cong \varphi(U) \times E.$$

Endowing $T(M)$ with the topology for which $O \subseteq T(M)$ is open if and only if for each E-chart (φ,U) of M the set $T(\varphi)(O \cap T(U))$ is open in $\varphi(U) \times E$, we obtain on $T(M)$ the structure of an $E \times E$-manifold defined by the charts $(T(\varphi),T(U))$, obtained from E-charts (φ,U) of M. This leads to an endofunctor T on the category of smooth manifolds, preserving finite products (cf. Remark I.3.2).

If $f\colon M \to V$ is a smooth map into a locally convex space, then $T(f)\colon T(M) \to T(V) \cong V \times V$ is smooth, and can be written as $T(f) = (f, df)$, where $df\colon T(M) \to V$ is called the differential of f. □

As a consequence of Proposition I.2.3(ii), we have:

Proposition I.3.4. *A smooth map $f\colon M \to N$ is locally constant if and only if $T(f) = 0$.* □

Definition I.3.5. *Let M be a smooth manifold modeled on the space E, and $N \subseteq M$ a subset.*

(a) N is called a submanifold *of M if there exists a closed subspace $F \subseteq E$ and for each $n \in N$ there exists an E-chart (φ, U) of M with $n \in U$ and $\varphi(U \cap N) = \varphi(U) \cap F$.*

(b) N is called a split submanifold *of M if, in addition, there exists a subspace $G \subseteq E$ for which the addition map $F \times G \to E, (f,g) \mapsto f + g$ is a topological isomorphism.* □

Definition I.3.6. *A* (smooth) vector field *X on M is a smooth section of the tangent bundle $\pi_{TM} \colon TM \to M$, i.e., a smooth map $X \colon M \to TM$ with $\pi_{TM} \circ X = \mathrm{id}_M$. We write $\mathscr{V}(M)$ for the space of all vector fields on M. If $f \in C^\infty(M,V)$ is a smooth function on M with values in some locally convex space V and $X \in \mathscr{V}(M)$, then we obtain a smooth function on M via*

$$X.f := df \circ X \colon M \to V.$$

For $X, Y \in \mathscr{V}(M)$, there exists a unique vector field $[X,Y] \in \mathscr{V}(M)$ determined by the property that on each open subset $U \subseteq M$ we have

$$[X,Y].f = X.(Y.f) - Y.(X.f) \tag{1.3.1}$$

for all $f \in C^\infty(U,\mathbb{R})$. We thus obtain on $\mathscr{V}(M)$ the structure of a Lie algebra. □

Remark I.3.7. If $M = U$ is an open subset of the locally convex space E, then $TU = U \times E$ with the bundle projection $\pi_{TU} \colon U \times E \to U, (x,v) \mapsto x$. Each smooth vector field is of the form $X(x) = (x, \widetilde{X}(x))$ for some smooth function $\widetilde{X} \colon U \to E$, and we may thus identify $\mathscr{V}(U)$ with the space $C^\infty(U,E)$. Then the Lie bracket satisfies

$$[X,Y]\widetilde{\ }(p) = d\widetilde{Y}(p)\widetilde{X}(p) - d\widetilde{X}(p)\widetilde{Y}(p) \quad \text{for each} \quad p \in U. \qquad \square$$

Definition I.3.8. *Let M be a smooth E-manifold and F a locally convex space. A* smooth vector bundle of type *F over M is a pair $(\pi, \mathbb{F}, F)$, consisting of a smooth manifold $\mathbb{F}$, a smooth map $\pi \colon \mathbb{F} \to M$ and a locally convex space F, with the following properties:*

(a) For each $m \in M$, the fiber $\mathbb{F}_m := \pi^{-1}(m)$ carries a locally convex vector space structure isomorphic to F.

(b) Each point $m \in M$ has an open neighborhood U for which there exists a diffeomorphism

$$\varphi_U \colon \pi^{-1}(U) \to U \times F$$

with $\varphi_U = (\pi|_U, g_U)$, where $g_U \colon \pi^{-1}(U) \to F$ is linear on each fiber $\mathbb{F}_m$, $m \in U$.

We then call U a trivializing subset *of M and φ_U a* bundle chart*. If φ_U and φ_V are two bundle charts and $U \cap V \neq \emptyset$, then we obtain a diffeomorphism*

$$\varphi_U \circ \varphi_V^{-1} \colon (U \cap V) \times F \to (U \cap V) \times F$$

of the form $(x,v) \mapsto (x, g_{VU}(x)v)$. This leads to a map

$$g_{UV} \colon U \cap V \to \mathrm{GL}(F)$$

for which it does not make sense to speak about smoothness because $\mathrm{GL}(F)$ is not a Lie group if F is not a Banach space. This is a major difference between the Banach and the locally convex context. Nevertheless, g_{UV} is smooth in the sense that the map

$$\widehat{g}_{UV} \colon (U \cap V) \times F \to F \times F, \quad (x,v) \mapsto (g_{UV}(x)v, g_{UV}(x)^{-1}v) = (g_{UV}(x)v, g_{VU}(x)v)$$

is smooth (cf. Definition II.3.1 below). □

Obviously, the tangent bundle $T(M)$ of a smooth (locally convex) manifold is an example of a vector bundle, but the cotangent bundle is more problematic:

Remark I.3.9. We define for each E-manifold M the cotangent bundle by $T^*(M) := \bigcup_{m \in M} T_m(M)'$ and observe that, as a set, it carries a natural structure of a vector bundle over M, but to endow it with a smooth manifold structure we need a locally convex topology on the dual space E' such that for each local diffeomorphism $f \colon U \to E$, U open in E, the map $U \times E' \to E', (x,\lambda) \mapsto \lambda \circ df(x)$ is smooth. If E is a Banach space, then the norm topology on E' has this property, but in general this property fails for non-Banach manifolds.

Indeed, let E be a locally convex space which is not normable and pick a non-zero $\alpha_0 \in E'$. We consider the smooth map

$$f \colon E \to E, \quad x \mapsto x + \alpha_0(x)x = (1 + \alpha_0(x))x.$$

Then $df(x)v = (1+\alpha_0(x))v + \alpha_0(v)x$ implies that $df(x) = (1+\alpha_0(x))\mathbf{1} + \alpha_0 \otimes x$, which is invertible for $\alpha_0(x) \not\in \{-1, -\frac{1}{2}\}$. If $\varphi \colon]-\frac{1}{4}, \infty[\to \mathbb{R}$ is the inverse function of $\psi(x) = x + x^2$ on $]-\frac{1}{2}, \infty[$, then an easy calculation gives on $\{y \in E \colon \alpha_0(y) > -\frac{1}{4}\}$ the inverse function $f^{-1}(y) = (1 + \varphi(\alpha_0(y)))^{-1} \cdot y$. We conclude that f is a local diffeomorphism on some 0-neighborhood of E.

On the other hand, the map $U \times E' \to E', (x,\lambda) \mapsto \lambda \circ df(x)$ satisfies

$$\lambda \circ df(x) = (1 + \alpha_0(x))\lambda + \lambda(x)\alpha_0.$$

Since the evaluation map $E' \times E \to \mathbb{R}$ is discontinuous in 0 for any vector topology on E' ([Mais63]), f does not induce a continuous map on $T^*(E) \cong E \times E'$ for any such topology. Hence there is no natural smooth vector bundle structure on $T^*(M)$ if E is not normable. □

In view of the difficulties caused by the cotangent bundle, we shall introduce differential forms directly, not as sections of a vector bundle.

I.4. Differential forms

Differential forms play a significant role throughout infinite-dimensional Lie theory. In the present subsection, we describe a natural approach to differential forms on manifolds modeled on locally convex spaces. A major difference to the finite-dimensional case is that in local charts there is no natural coordinate description of differential forms in terms of basic forms, that differential forms cannot be defined as the smooth sections of a natural vector bundle (Remark I.3.9), and that, even for Banach manifolds, smooth partitions of unity need not exist, so that one has to be careful with localization arguments.

In [KM97], one finds a discussion of various types of differential forms, containing in particular those introduced below, which are also used by Beggs in [Beg87].

Definition I.4.1. *(a) If M is a differentiable manifold and E a locally convex space, then an E-*valued *p-*form *ω on M is a function ω which associates to each $x \in M$ a p-linear alternating map $\omega_x \colon T_x(M)^p \to E$ such that in local coordinates the map $(x, v_1, \ldots, v_p) \mapsto \omega_x(v_1, \ldots, v_p)$ is smooth. We write $\Omega^p(M,E)$ for the space of E-valued p-forms on M and identify $\Omega^0(M,E)$ with the space $C^\infty(M,E)$ of smooth E-valued functions on M.*

(b) Let E_1, E_2, E_3 be locally convex spaces and $\beta \colon E_1 \times E_2 \to E_3$ be a continuous bilinear map. Then the wedge product

$$\Omega^p(M,E_1) \times \Omega^q(M,E_2) \to \Omega^{p+q}(M,E_3), \quad (\omega,\eta) \mapsto \omega \wedge \eta$$

is defined by $(\omega \wedge \eta)_x := \omega_x \wedge \eta_x$, where

$$(\omega_x \wedge \eta_x)(v_1, \ldots, v_{p+q}) \\ := \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \mathrm{sgn}(\sigma) \beta\big(\omega_x(v_{\sigma(1)}, \ldots, v_{\sigma(p)}), \eta_x(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)})\big).$$

Important special cases, where such wedge products are used, are:

(1) $\beta \colon \mathbb{R} \times E \to E$ is the scalar multiplication of E.
(2) $\beta \colon A \times A \to A$ is the multiplication of an associative algebra.
(3) $\beta \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the Lie bracket of a Lie algebra. In this case, we also write $[\omega, \eta] := \omega \wedge \eta$. □

The definition of the exterior differential $d \colon \Omega^p(M,E) \to \Omega^{p+1}(M,E)$ is a bit more subtle than in finite dimensions, where one usually uses local coordinates to define it in charts. Here the exterior differential is determined uniquely by the property that for each open subset $U \subseteq M$ we have for $X_0, \ldots, X_p \in \mathcal{V}(U)$ in the space $C^\infty(U,E)$ the identity

$$(d\omega)(X_0, \ldots, X_p) := \sum_{i=0}^{p} (-1)^i X_i . \omega(X_0, \ldots, \widehat{X_i}, \ldots, X_p)$$

$$+\sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_p).$$

The main point is to show that in a point $x \in U$ the right hand side only depends on the values of the vector fields X_i in x. The exterior differential has the usual properties, such as $d^2 = 0$ and the compatibility with pullbacks: $\varphi^*(d\omega) = d(\varphi^*\omega)$.

Extending d to a linear map on the space $\Omega(M,E) := \bigoplus_{p\in\mathbb{N}_0}\Omega^p(M,E)$ of all E-valued differential forms on M, the relation $d^2 = 0$ implies that the space

$$Z^p_{\rm dR}(M,E) := \ker(d\,|_{\Omega^p(M,E)})$$

of *closed p-forms* contains the space $B^p_{\rm dR}(M,E) := d(\Omega^{p-1}(M,E))$ of *exact p-forms*, so that we may define the *E-valued de Rham cohomology space* by

$$H^p_{\rm dR}(M,E) := Z^p_{\rm dR}(M,E)/B^p_{\rm dR}(M,E).$$

For finite-dimensional manifolds, one usually defines the Lie derivative of a differential form in the direction of a vector field X by using its local flow $t \mapsto \mathrm{Fl}^X_t$:

$$\mathscr{L}_X\omega := \frac{d}{dt}\Big|_{t=0}(\mathrm{Fl}^X_{-t})^*\omega.$$

Since vector fields on infinite-dimensional manifold need not have a local flow (*cf.* Example II.3.11 below), we introduce the Lie derivative more directly.

Definition I.4.2. *(a) For any smooth manifold M and each locally convex space, we have a natural representation of the Lie algebra $\mathcal{V}(M)$ on the space $\Omega^p(M,E)$ of E-valued p-forms on M, given by the* Lie derivative*, which for $Y \in \mathcal{V}(M)$ is uniquely determined by*

$$(\mathscr{L}_Y\omega)(X_1,\dots,X_p) = Y.\omega(X_1,\dots,X_p) - \sum_{j=1}^{p}\omega(X_1,\dots,[Y,X_j],\dots,X_p)$$

for $X_i \in \mathcal{V}(U)$, $U \subseteq M$ open. Again one has to verify that the value of the right hand side in some $x \in M$ only depends on the values of the vector fields X_i in x.

(b) We further obtain for each $X \in \mathcal{V}(M)$ and $p \geq 1$ a unique linear map

$$i_X\colon \Omega^p(M,E) \to \Omega^{p-1}(M,E) \quad \text{with} \quad (i_X\omega)_x = i_{X(x)}\omega_x,$$

where $(i_v\omega_x)(v_1,\dots,v_{p-1}) := \omega_x(v,v_1,\dots,v_{p-1})$. For $\omega \in \Omega^0(M,E) = C^\infty(M,E)$, we put $i_X\omega := 0$. □

Proposition I.4.3. *For $X,Y \in \mathcal{V}(M)$, we have on $\Omega(M,E)$ the Cartan formulas:*

$$[\mathcal{L}_X, i_Y] = i_{[X,Y]}, \quad \mathcal{L}_X = d \circ i_X + i_X \circ d \quad \text{and} \quad \mathcal{L}_X \circ d = d \circ \mathcal{L}_X. \qquad \square$$

Remark I.4.4. Clearly integration of differential forms $\omega \in \Omega^p(M,E)$ only makes sense if M is a p-dimensional compact oriented manifold (possibly with boundary) and E is Mackey complete (Definition I.1.4). We need the Mackey completeness to ensure that each smooth function $f\colon Q \to E$ on a cube $Q := \prod_{i=1}^p [a_i,b_i] \subseteq \mathbb{R}^p$ has an iterated integral

$$\int_Q f\,dx := \int_{a_1}^{b_1} \cdots \int_{a_p}^{b_p} f(x_1,\ldots,x_p)\,dx_p \cdots dx_1. \qquad \square$$

Remark I.4.5. (a) We call a smooth manifold M *smoothly paracompact* if every open cover has a subordinated smooth partition of unity. De Rham's Theorem holds for every smoothly paracompact manifold (*cf.* [Ee58], [KM97, Thm. 34.7], [Beg87]). Smoothly Hausdorff second countable manifolds modeled on a smoothly regular space are smoothly paracompact ([KM97, Cor. 27.4]). Typical examples of smoothly regular spaces are nuclear Fréchet spaces ([KM97, Th. 16.10]).

(b) Examples of Banach spaces which are not smoothly paracompact are $C([0,1],\mathbb{R})$ and $\ell^1(\mathbb{N},\mathbb{R})$. On these spaces, there exists no non-zero smooth function supported in the unit ball ([KM97, 14.11]). $\square$

I.5. The topology on spaces of smooth functions

In this subsection, we describe a natural topology on spaces of smooth maps which is derived from the compact open topology, the *compact open C^r-topology* (*cf.* [Mil82, Ex. 6.10] for a comparison of different topologies on spaces of smooth maps). Unfortunately, this topology has certain defects for functions on infinite-dimensional manifolds.

Definition I.5.1. *(a) If X and Y are topological Hausdorff spaces, then the* compact open topology *on the space $C(X,Y)$ is defined as the topology generated by the sets of the form*

$$W(K,U) := \{f \in C(X,Y)\colon f(K) \subseteq U\},$$

where K is a compact subset of X and U an open subset of Y. We write $C(X,Y)_c$ for the topological space obtained by endowing $C(X,Y)$ with the compact open topology.

(b) If G is a topological group and X is Hausdorff, then $C(X,G)$ is a group with respect to the pointwise product. Then the compact open topology on

$C(X,G)$ coincides with the topology of uniform convergence on compact subsets of X, for which the sets $W(K,U)$, $K \subseteq X$ compact and $U \subseteq G$ a $\mathbf{1}$-neighborhood, form a basis of $\mathbf{1}$-neighborhoods. In particular, $C(X,G)_c$ is a topological group.

(c) If Y is a locally convex space, then $C(X,Y)$ is a vector space with respect to the pointwise operations. In view of the preceding remark, the topology on $C(X,Y)_c$ is defined by the seminorms

$$p_K(f) := \sup\{p(f(x)) \colon x \in K\},$$

where $K \subseteq X$ is compact and p is a continuous seminorm on Y. It follows in particular that $C(X,Y)_c$ is a locally convex space.

(d) If M and N are smooth (possibly infinite-dimensional) manifolds, then every smooth map $f \colon M \to N$ defines a sequence of smooth maps $T^k f \colon T^k M \to T^k N$ on the iterated tangent bundles. We thus obtain for $r \in \mathbb{N}_0 \cup \{\infty\}$ an embedding

$$C^r(M,N) \hookrightarrow \prod_{k=0}^{r} C(T^k M, T^k N)_c,$$

into a topological product space, that we use to define a topology on $C^r(M,N)$, called the compact open C^r-topology. *For $r < \infty$, it suffices to consider the embedding $C^r(M,N) \hookrightarrow C(T^r(M), T^r(N))_c$. On the set $C^\infty(M,N)$, the compact open C^∞-topology is the common refinement of all C^r-topologies for $r < \infty$. Since every compact subset of M is contained in a finite union of chart domains, the topology on $C^r(M,N)$ is generated by sets of the form $W(K,U)$ in $C(T^k(M), T^k(N))$, where K lies in $T^k(U)$ for a chart (φ,U) of M.*

If E is a locally convex space, then all spaces $C(T^k M, T^k E)$ are locally convex, by (c) above. Therefore the corresponding product topology is locally convex, and hence $C^\infty(M,E)$ is a locally convex space. If M is finite-dimensional, for each chart (φ,U) of M, the topology on $C^\infty(U,E)$ coincides with the topology of uniform convergence of all partial derivatives on each compact subset of U. □

Definition I.5.2. *Since smooth vector fields are smooth functions $X \colon M \to TM$, we have a natural embedding $\mathcal{V}(M) \hookrightarrow C^\infty(M,TM)$, defining a topology on $\mathcal{V}(M)$. If (φ,U) is an E-chart of M, then $TU \cong U \times E$, and smooth vector fields on U correspond to smooth functions $U \to E$. This shows that, endowed with its natural topology, $\mathcal{V}(M)$ is a locally convex space.* □

Remark I.5.3. As a consequence of Remark I.3.7, the bracket on $\mathcal{V}(M)$ is continuous if M is finite-dimensional.

It is interesting to observe that, in general, the bracket on $\mathcal{V}(M)$ is not continuous if M is infinite-dimensional. To see this, we assume that $M = U$ is an open subset of a locally convex space E and consider the subalgebra $\mathfrak{aff}(E) \cong E \rtimes \mathfrak{gl}(E)$ of affine vector fields $X_{A,b}$ with $X_{A,b}(v) = Av + b$. It is easy to see that the natural topology on $\mathcal{V}(U)$ induces on $\mathfrak{aff}(E)$ the product topology of the original topology on E and the compact open topology on $\mathfrak{gl}(E) \cong \mathcal{L}(E)_c$. In view of

$$[X_{A,b}, X_{A',b'}] = X_{[A',A],A'b-Ab'},$$

it therefore suffices to show that the bilinear evaluation map $\mathcal{L}(E)_c \times E \to E$ is not continuous if $\dim E = \infty$. Pick $0 \neq v \in E$ and embed $E'_c \hookrightarrow \mathcal{L}(E)_c$ by assigning to $\alpha \in E'$ the operator $v \otimes \alpha \colon x \mapsto \alpha(x)v$. Hence it suffices to see that the evaluation map

$$\mathrm{ev} \colon E'_c \times E \to \mathbb{R}, \quad (\alpha, v) \mapsto \alpha(v)$$

is not continuous. Basic neighborhoods of $(0,0)$ in $E'_c \times E$ are of the form $\widehat{K} \times U_E$, where $U_E \subseteq U$ is a 0-neighborhood, $K \subseteq E$ is compact, and $\widehat{K} := \{f \in E' \colon (\forall k \in K)\, |f(k)| \leq 1\}$ is the polar set of K. On $\widehat{K} \times U_E$ the evaluation map is bounded if and only if U_E is contained in some multiple of the bipolar $\widehat{\widehat{K}}$, which, according to the Bipolar Theorem, coincides with the balanced convex closure of K, which is pre-compact ([Tr67, Prop. 7.11]). Then $\widehat{\widehat{K}}$ is a pre-compact 0-neighborhood in E, so that E is finite-dimensional (*cf.* [Ru73, proof of Th. 1.22]). A similar argument shows that, if we endow E' with the finer topology of uniform convergence on bounded subsets of E, then the evaluation map is continuous if and only if E is normable, which is equivalent to the existence of a (weakly) bounded 0-neighborhood ([Ru73]). □

Remark I.5.4. The fact that for an infinite-dimensional locally convex space E the evaluation map $\mathrm{ev} \colon E'_c \times E \to \mathbb{R}$ is not continuous also causes trouble if one wants to associate to transformation groups corresponding continuous, resp., smooth representations on function spaces.

A very simple example of a smooth group action is the translation action of E on itself. The corresponding representation of $(E,+)$ on the space of smooth functions on E is given by $(x.f)(y) := f(x+y)$. Clearly, the subspace of affine functions in $C^\infty(E,\mathbb{R})$ is isomorphic to $\mathbb{R} \times E'_c$ as a locally convex space, and on this subspace the representation of E is given by $x.(t,\alpha) = (t + \alpha(x), \alpha)$, which is discontinuous because $\mathrm{ev}(\alpha, x) = \alpha(x)$ is not continuous (Remark I.5.3). In view of [Mais63], the same pathology occurs for any locally convex topology on $C^\infty(E,\mathbb{R})$ if E is not normable. □

II. Locally convex Lie groups

In this section, we give the definition of a locally convex Lie group. We explain how its Lie algebra and the corresponding Lie functor are defined and describe some basic properties. In our discussion of Lie groups, we essentially follow

[Mil82/84], but, as for manifolds, we do not assume that the model space of a Lie group is complete ([Gl02a]). The natural strategy to endow groups with (infinite-dimensional) Lie group structures is to construct a chart around the identity in which the group operations are smooth. As we shall see in Subsection II.2, this suffices in many situations to specify a global Lie group structure.

In Subsection II.3, we discuss a smoothness concept for maps with values in diffeomorphism groups of locally convex manifolds. This specializes in particular to maps into general linear groups of locally convex spaces. The main point of this subsection is to obtain uniqueness results for solutions of certain ordinary differential equations on locally convex manifolds. In Subsection II.4, we apply all this to smooth maps with values in Lie groups, where it shows in particular that morphisms of connected Lie groups are determined by their differential in $\mathbf{1}$.

We conclude this section with some basic results on the behavior of the exponential function (Subsection II.5), and a discussion of the concept of an initial Lie subgroup in Subsection II.6.

II.1. Infinite-dimensional Lie groups and their Lie algebras

Definition II.1.1. *A* locally convex Lie group *G is a locally convex manifold endowed with a group structure such that the multiplication map $m_G\colon G\times G\to G$ and the inversion map $\eta_G\colon G\to G$ are smooth.*

A morphism of Lie groups is a smooth group homomorphism. In the following, we call locally convex Lie groups simply Lie groups. □

Example II.1.2. (Vector groups) Each locally convex space E is an abelian Lie group with respect to addition and the obvious manifold structure. □

Vector groups $(E,+)$ form the most elementary Lie groups. The next natural class are unit groups of algebras. This leads us to the concept of a continuous inverse algebra, which came up in the 1950s (*cf.* [Wae54a/b] and [Wae71]):

Definition II.1.3. *(a) A* locally convex algebra *is a locally convex space A, endowed with an associative continuous bilinear multiplication $A\times A\to A$, $(a,b)\mapsto ab$. A unital locally convex algebra A is called a* continuous inverse algebra *(CIA for short) if its unit group $A^\times$ is open and the inversion is a continuous map $A^\times\to A, a\mapsto a^{-1}$.*

(b) If A is a locally convex algebra which is not unital, then we obtain a monoid structure on A by $x*y:=x+y+xy$ for which 0 is the identity element. In this case, we write $A^\times$ for the unit group of $(A,*)$ and say that A is a non-unital CIA *if $A^\times$ is open and the (quasi-)inversion map $\eta_A\colon A^\times\to A$ is continuous.*

If $A_+:=A\times\mathbb{K}$ is the unital locally convex algebra with the multiplication $(x,t)(x',t'):=(xx'+tx'+t'x,tt')$, then the map $(A,*)\to A\times\{1\}, a\mapsto(a,1)$ is *an isomorphism of monoids, and it is easy to see that A_+ is a CIA if and only if A is a (not necessarily unital) CIA.* ☐

Example II.1.4. Let A be a continuous inverse algebra over $\mathbb{K}$ and $A^\times$ its unit group. As an open subset of A, the group $A^\times$ carries a natural manifold structure. The multiplication on A is bilinear and continuous, hence a smooth map (Remark I.2.4). Therefore the multiplication of $A^\times$ is smooth. One can further show quite directly that the continuity of the inversion $\eta_A \colon A^\times \to A^\times$ implies that $d\eta_A(x)(y) = -x^{-1}yx^{-1}$ exists for each pair (x,y), and this formula implies inductively that η_A is smooth and hence that $A^\times$ is a Lie group.

In some cases, it is also possible to obtain a Lie group structure on the unit group $A^\times$ of a unital locally convex algebra if $A^\times$ is not open (*cf.* Remark II.2.10 below). ☐

Definition II.1.5. *A vector field X on the Lie group G is called* left invariant *if*

$$X \circ \lambda_g = T(\lambda_g) \circ X \colon G \to T(G)$$

holds for each $g \in G$, i.e., X is λ_g-related to itself for each $g \in G$. We write $\mathcal{V}(G)^l$ for the set of left invariant vector fields in $\mathcal{V}(G)$. The left invariance of a vector field X implies in particular that for each $g \in G$, we have $X(g) = g.X(\mathbf{1})$, where $G \times T(G) \to T(G), (g,v) \mapsto g.v$ denotes the smooth action of G on $T(G)$, induced by the left multiplication action of G on itself. For each $x \in \mathfrak{g}$, we have a unique left invariant vector field $x_l \in \mathcal{V}(G)^l$ defined by $x_l(g) := g.x$, and the map

$$\mathrm{ev}_{\mathbf{1}} \colon \mathcal{V}(G)^l \to T_{\mathbf{1}}(G), \quad X \mapsto X(\mathbf{1})$$

is a linear bijection. If X, Y are left invariant, then they are λ_g-related to themselves, and their Lie bracket $[X,Y]$ inherits this property. We thus obtain a unique Lie bracket $[\cdot,\cdot]$ on $T_{\mathbf{1}}(G)$ satisfying

$$[x,y]_l = [x_l, y_l] \quad \textit{for all} \quad x,y \in T_{\mathbf{1}}(G), \tag{2.1.1}$$

and from the formula for the bracket in local coordinates, it follows that it is continuous (cf. Remark II.1.8 below). ☐

Remark II.1.6. The tangent map $T(m_G) \colon T(G \times G) \cong T(G) \times T(G) \to T(G)$ defines on the tangent bundle $T(G)$ of G the structure of a Lie group with inversion map $T(\eta_G)$.

In fact, let $\varepsilon_G \colon G \to G, g \mapsto \mathbf{1}$, be the constant homomorphism. Then the group axioms for G are encoded in the relations

(1) $m_G \circ (m_G \times \mathrm{id}_G) = m_G \circ (\mathrm{id}_G \times m_G)$ (associativity),
(2) $m_G \circ (\eta_G, \mathrm{id}_G) = m_G \circ (\mathrm{id}_G, \eta_G) = \varepsilon_G$ (inversion), and
(3) $m_G \circ (\varepsilon_G, \mathrm{id}_G) = m_G \circ (\mathrm{id}_G, \varepsilon_G) = \mathrm{id}_G$ (unit element).

Applying the functor T to these relations, it follows that $T(m_G)$ defines a Lie group structure on $T(G)$ for which $T(\eta_G)$ is the inversion and $0_{\mathbf{1}} \in T_{\mathbf{1}}(G)$ is the identity. □

Definition II.1.7. (The Lie functor) *For a Lie group G, the locally convex Lie algebra* $\mathbf{L}(G) := (T_{\mathbf{1}}(G), [\cdot,\cdot])$ *is called the* Lie algebra of *G*.

To each morphism $\varphi\colon G \to H$ *of Lie groups we further associate its tangent map*

$$\mathbf{L}(\varphi) := T_{\mathbf{1}}(\varphi)\colon \mathbf{L}(G) \to \mathbf{L}(H),$$

and the usual argument with related vector fields implies that $\mathbf{L}(\varphi)$ *is a homomorphism of Lie algebras.* □

This means that the assignments $G \mapsto \mathbf{L}(G)$ and $\varphi \mapsto \mathbf{L}(\varphi)$ define a functor $\mathbf{L}$ from the category of (locally convex) Lie groups to the category of locally convex Lie algebras. Since each functor maps isomorphisms to isomorphisms, for each isomorphism $\varphi\colon G \to H$ of Lie groups, the map $\mathbf{L}(\varphi)$ is an isomorphism of locally convex Lie algebras.

The following remark describes a convenient way to calculate the Lie algebra of a given group.

Remark II.1.8. For each chart (φ, U) of G with $\mathbf{1} \in U$ and $\varphi(\mathbf{1}) = 0$, we identify $\mathfrak{g} := T_{\mathbf{1}}(G)$ via the topological isomorphism $T_{\mathbf{1}}(\varphi)$ with the corresponding model space. Then the second order Taylor expansion in $(0,0)$ of the multiplication $x * y := \varphi(\varphi^{-1}(x)\varphi^{-1}(y))$ (*cf.* Proposition I.2.3) is of the form

$$x * y = x + y + b(x,y) + \text{higher order terms},$$

where $b\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a continuous bilinear map satisfying

$$[x,y] = b(x,y) - b(y,x). \tag{2.1.2}$$

Using the chain rule for Taylor polynomials, it is easy to show that the second order Taylor polynomial of the commutator map $x * y * x^{-1} * y^{-1}$ is given by the Lie bracket:

$$x * y * x^{-1} * y^{-1} = [x,y] + \text{higher order terms}$$ □

We now take a look at the Lie algebras of the Lie groups from Examples II.1.2/4.

Examples II.1.9. (a) If G is an abelian Lie group, then the map $b\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ in Remark II.1.8 is symmetric, which implies that $\mathbf{L}(G)$ is abelian. This applies in particular to the additive Lie group $(E,+)$ of a locally convex space E.

(b) Let A be a CIA. Then the map $\varphi\colon A^\times \to A, x \mapsto x - \mathbf{1}$ is a global chart of $A^\times$, satisfying $\varphi(\mathbf{1}) = 0$. In this chart, the group multiplication is given by

$$x * y := \varphi(\varphi^{-1}(x)\varphi^{-1}(y)) = (x + \mathbf{1})(y + \mathbf{1}) - \mathbf{1} = x + y + xy.$$

In the terminology of Remark II.1.8, we then have $b(x,y) = xy$ and therefore $\mathbf{L}(A^\times) = (A, [\cdot,\cdot])$, where $[x,y] = xy - yx$ is the commutator bracket on the associative algebra A. □

We conclude this subsection with the observation that the passage from groups to Lie algebras can also be established on the local level.

Definition II.1.10. (The Lie algebra of a local Lie group) *There is a natural notion of a local Lie group. The corresponding algebraic concept is that of a local group: Let G be a set and $D \subseteq G \times G$ a subset on which we are given a map*

$$m_G\colon D \to G, \quad (x,y) \mapsto xy.$$

We say that the product xy of two elements $x, y \in G$ is defined *if $(x,y) \in D$. The quadruple $(G, D, m_G, \mathbf{1})$, where $\mathbf{1}$ is a distinguished element of G, is called a* local group *if the following conditions are satisfied:*

(1) Suppose that xy and yz are defined. If $(xy)z$ or $x(yz)$ is defined, then the other product is also defined and both are equal.
(2) For each $x \in G$, the products $x\mathbf{1}$ and $\mathbf{1}x$ are defined and equal to x.
(3) For each $x \in G$, there exists a unique element $x^{-1} \in G$ such that xx^{-1} and $x^{-1}x$ are defined and $xx^{-1} = x^{-1}x = \mathbf{1}$.
(4) If xy is defined, then $y^{-1}x^{-1}$ is defined.

If $(G, D, m_G, \mathbf{1})$ is a local group and, in addition, G has a smooth manifold structure, D is open, and the maps

$$m_G\colon D \to G \quad \text{and} \quad \eta_G\colon G \to G, x \mapsto x^{-1}$$

are smooth, then G, resp., $(G, D, m_G, \mathbf{1})$ is called a local Lie group.

Let G be a local Lie group and $T_{\mathbf{1}}(G)$ its tangent space in $\mathbf{1}$. For each $x \in T_{\mathbf{1}}(G)$, we then obtain a left invariant vector field $x_l(g) := g.x := 0_g \cdot x$. *The Lie bracket of two left invariant vector fields is left invariant and we thus obtain on $T_{\mathbf{1}}(G)$ a locally convex Lie algebra structure. We call $\mathbf{L}(G) := \mathbf{L}(G, D, m_G, \mathbf{1}) := (T_{\mathbf{1}}(G), [\cdot,\cdot])$ the Lie algebra of the local group G. For more details on local Lie groups, we refer to [GN06].* □

Remark II.1.11. If G is a Lie group and $U = U^{-1} \subseteq G$ an open identity neighborhood, then U carries a natural local Lie group structure with $D := \{(x,y) \in U \times U \colon xy \in U\}$ and $m_U := m_G|_D$. Clearly U and G have the same Lie algebras.

Local groups of this type are called *enlargeable*. As we shall see in Example VI.1.7 below, not all local Lie groups are enlargeable because not all Banach–Lie algebras are integrable (Example VI.1.16). □

II.2. From local data to global Lie groups

We now give the precise formulation of an elementary but extremely useful tool which helps to construct Lie group structures on groups containing a local Lie group. This theorem directly carries over from the finite-dimensional case, which can be found in [Ch46, §14, Prop. 2] or [Ti83, p.14]. In [GN06], it is our main method to endow groups with Lie group structures.

Theorem II.2.1. *Let G be a group and $U = U^{-1}$ a symmetric subset. We further assume that U is a smooth manifold such that*

(L1) there exists an open symmetric $\mathbf{1}$-neighborhood $V \subseteq U$ with $V \cdot V \subseteq U$ such that the group multiplication $m_V: V \times V \to U$ is smooth,
(L2) the inversion map $\eta_U: U \to U, u \mapsto u^{-1}$ is smooth, and
(L3) for each $g \in G$ there exists an open symmetric $\mathbf{1}$-neighborhood $U_g \subseteq U$ with $c_g(U_g) \subseteq U$, and such that the conjugation map $c_g: U_g \to U, x \mapsto gxg^{-1}$ is smooth.

Then there exists a unique Lie group structure on G for which there exists an open $\mathbf{1}$-neighborhood $U_0 \subseteq U$ such that the inclusion map $U_0 \to G$ induces a diffeomorphism onto an open subset of G.

If the group G is generated by V, then condition (L3) can be omitted. □

If V is as above, then $D := \{(x,y) \in V \times V : xy \in V\}$ defines on V the structure of a local Lie group, and the preceding theorem implies that the smooth structure of this local Lie group, together with the group structure of G, determines the global Lie group structure of G. The subtlety of condition (L3) is that it mixes local and global objects because it requires that each element of G induces an isomorphism of the corresponding germ of local groups. The following corollary is a converse of Remark II.1.11 (*cf.* [Swi65]). It is the central tool to pass from local to global subgroups of Lie groups.

Corollary II.2.2. *Let $(U, D, m_U, \mathbf{1})$ be a local Lie group, G a group, and $\eta: U \to G$ an injective morphism of local groups. Then the subgroup $H := \langle \eta(U) \rangle \subseteq G$ generated by $\eta(U)$ carries a unique Lie group structure for which η is a diffeomorphism onto an open subset of H.* □

The preceding corollary shows in particular that if, in addition to the assumptions of Theorem II.2.1, the group multiplication of G restricts to a smooth map on the domain $D_U := \{(x,y) \in U \times U : xy \in U\}$, then the inclusion $U \hookrightarrow G$ is a diffeomorphism onto an open subset of G, endowed with the Lie group structure determined by U.

Corollary II.2.3. *Let G be a group and $N \trianglelefteq G$ a normal subgroup that carries a Lie group structure. Then there exists a Lie group structure on G for which N is an open subgroup if and only if for each $g \in G$, the restriction $c_g|_N$ is a smooth automorphism of N.* □

The preceding corollary is of particular interest for abelian groups. In this case, it leads for each Lie group structure on any subgroup $N \subseteq G$ to a Lie group structure on G for which N is an open subgroup.

The following corollary implies in particular that quotients of Lie groups by discrete normal subgroups are Lie groups.

Corollary II.2.4. *Let $\varphi\colon G \to H$ be a covering of topological groups. If G or H is a Lie group, then the other group has a unique Lie group structure for which φ is a morphism of Lie groups which is a local diffeomorphism.* □

Remark II.2.5. (a) (Lie subgroups) If G is a Lie group with Lie algebra $\mathfrak{g}$ and $H \subseteq G$ is a submanifold which is a group, then H inherits a Lie group structure from G. Moreover, there exists a closed subspace $\mathfrak{h} \subseteq \mathfrak{g} \cong T_{\mathbf{1}}(G)$ and a chart (φ, U) of G with $\mathbf{1} \in U = U^{-1}$, $\varphi(\mathbf{1}) = 0$ and

$$\varphi(U \cap H) = \varphi(U) \cap \mathfrak{h}.$$

The local multiplication $x * y := \varphi(\varphi^{-1}(x)\varphi^{-1}(y))$ on

$$D := \{(x,y) \in \varphi(U) \times \varphi(U)\colon \varphi^{-1}(x)\varphi^{-1}(y) \in U\}$$

then satisfies

$$x * y \in \mathfrak{h} \quad \text{for} \quad (x,y) \in D \cap (\mathfrak{h} \times \mathfrak{h}) \tag{2.2.1}$$

and

$$x^{-1} \in \mathfrak{h} \quad \text{for} \quad x \in \mathfrak{h} \cap \varphi(U). \tag{2.2.2}$$

In view of Remark II.1.8, this implies that $\mathfrak{h}$ is a closed Lie subalgebra of $\mathfrak{g}$.

If, conversely, $\mathfrak{h} \subseteq \mathfrak{g}$ is a closed Lie subalgebra for which there is a chart (φ, U) as above, satisfying (2.2.1/2), then $\varphi(U) \cap \mathfrak{h}$ carries a local Lie group structure and we can apply Corollary II.2.2 to the embedding $\varphi^{-1}\colon \varphi(U) \cap \mathfrak{h} \to G$, which leads to a Lie group structure on the subgroup $H := \langle \varphi^{-1}(\varphi(U) \cap \mathfrak{h}) \rangle$ of G. We know already from the finite dimensional theory that, in general, this does not lead to a submanifold of G.

(b) A weaker concept of a "Lie subgroup" is obtained by requiring only that $H \subseteq G$ a subgroup, for which there exists an identity neighborhood U^H whose smooth arc-component U_0^H of $\mathbf{1}$ is a submanifold of G (*cf.* [KYMO85, p.45]). Then we can use Theorem II.2.1 to obtain a Lie group stucture on H for which some identity neighborhood is diffeomorphic to an identity neighborhood in U_0^H. □

Remark II.2.6. Since it also makes sense to consider manifolds without assuming that they are Hausdorff (*cf.* [Pa57], [La99]), it is worthwhile to observe that this does not lead to a larger class of Lie groups.

In fact, let G be a Lie group which is not necessarily Hausdorff. Then G is in particular a topological group which possesses an identity neighborhood U homeomorphic to an open subset of a locally convex space. As U is Hausdorff, and since the subgroup $\overline{\{\mathbf{1}\}}$ of G coincides with the intersection of all $\mathbf{1}$-neighborhoods, the closedness of $\{\mathbf{1}\}$ in U implies that $\{\mathbf{1}\}$ is a closed subgroup of G and hence that G is a Hausdorff topological group. □

To see how Theorem II.2.1 can be applied, we now take a closer look at groups of differentiable maps. First we introduce a natural topology on these groups.

Definition II.2.7. (Groups of differentiable maps as topological groups) *Let M be a smooth manifold (possibly infinite-dimensional), K a Lie group with Lie algebra $\mathfrak{k}$ and $r \in \mathbb{N}_0 \cup \{\infty\}$. We endow the group $G := C^r(M,K)$ with the compact open C^r-topology (Definition I.5.1).*

We know already that the tangent bundle TK of K is a Lie group (Remark II.1.6). Iterating this procedure, we obtain a Lie group structure on all higher tangent bundles T^nK. For each $n \in \mathbb{N}_0$, we thus obtain topological groups $C(T^nM,T^nK)_c$ by using the topology of uniform convergence on compact subsets of T^nM, which coincides with the compact open topology (Definition I.5.1). We also observe that for two smooth maps $f_1, f_2 \colon M \to K$, the functoriality of T yields

$$T(f_1 \cdot f_2) = T(m_G \circ (f_1 \times f_2)) = T(m_G) \circ (Tf_1 \times Tf_2) = Tf_1 \cdot Tf_2.$$

Therefore the inclusion map

$$C^r(M,K) \hookrightarrow \prod_{n=0}^{r} C(T^nM,T^nK)_c, \quad f \mapsto (T^nf)_{0 \le n \le r}$$

is a group homomorphism, so that the inverse image of the product topology from the right hand side is a group topology on $C^r(M,K)$. Hence the compact open C^r-topology turns $C^r(M,K)$ into a topological group, even if M and K are infinite-dimensional. □

We define the *support* of a Lie group-valued map $f \colon M \to G$ by

$$\mathrm{supp}(f) := \overline{\{x \in M \colon f(x) \neq \mathbf{1}\}},$$

for a closed subset $X \subseteq M$ we put

$$C^r_X(M,K) := \{f \in C^r(M,K) \colon \mathrm{supp}(f) \subseteq X\},$$

and write $C^r_c(M,K)$ for the subgroup of compactly supported C^r-maps.

Theorem II.2.8. *Let K be a Lie group with Lie algebra $\mathfrak{k}$, M a finite-dimensional manifold (possibly with boundary), and $r \in \mathbb{N}_0 \cup \{\infty\}$.*

(a) If M is compact, then $C^r(M,K)$ carries a Lie group structure compatible with the compact open C^r-topology, and its Lie algebra is $C^r(M,\mathfrak{k})$, endowed with the pointwise bracket.

(b) If M is σ-compact, then $C^r_c(M,\mathfrak{k})$, endowed with the locally convex direct limit topology of the spaces $C^r_X(M,\mathfrak{k})$, $X \subseteq M$ compact, is a topological Lie algebra and $C^r_c(M,K)$ carries a natural Lie group structure with Lie algebra $C^r_c(M,\mathfrak{k})$.

Proof. (Sketch) (a) Let $G := C^r(M,K)$ and $\mathfrak{g} := C^r(M,\mathfrak{k})$. The Lie group structure on G can be constructed with Theorem II.2.1 as follows. Let $\varphi_K : U_K \to \mathfrak{k}$ be a chart of K. Then the set $U_G := \{f \in G : f(M) \subseteq U_K\}$ is an open subset of G. Assume, in addition, that $\mathbf{1} \in U_K$ and $\varphi_K(\mathbf{1}) = 0$. Then the map

$$\varphi_G : U_G \to \mathfrak{g}, \quad f \mapsto \varphi_K \circ f$$

defines a chart (φ_G, U_G) of G. To apply Theorem II.2.1, one has to verify that in this chart the inversion is a smooth map, that the multiplication map

$$D_G := \{(f,g) \in U_G \times U_G : fg \in U_G\} \to U_G$$

is smooth and that conjugation maps are smooth in some $\mathbf{1}$-neighborhood of U_G. For details, we refer to [Gl02c], resp., [GN06].

To calculate the Lie algebra of this group, we observe that for $m \in M$, we have for the multiplication in local coordinates

$$\begin{aligned}(f *_G g)(m) &:= \varphi_G\Big(\varphi_G^{-1}(f)\varphi_G^{-1}(g)\Big)(m) = \varphi_K\big(\varphi_K^{-1}(f(m))\varphi_K^{-1}(g(m))\big)\\ &= f(m) *_K g(m) = f(m) + g(m) + b_{\mathfrak{k}}(f(m), g(m)) + \cdots .\end{aligned}$$

In view of Remark II.1.8, this implies that $b_{\mathfrak{g}}(f,g)(m) = b_{\mathfrak{k}}(f(m),g(m))$, and hence that

$$[f,g](m) = b_{\mathfrak{g}}(f,g)(m) - b_{\mathfrak{g}}(g,f)(m) = b_{\mathfrak{k}}(f(m),g(m)) - b_{\mathfrak{k}}(g(m),f(m)) = [f(m),g(m)].$$

Therefore $\mathbf{L}(C^r(M,K)) = C^r(M,\mathfrak{k})$, endowed with the pointwise defined Lie bracket.

(b) is proved along the same lines. Note that it is not obvious that the Lie bracket on $C^r_c(M,\mathfrak{k})$ is continuous because it is a bilinear map. □

If K is finite-dimensional, then the preceding Lie group construction can be found in Michor's book [Mi80], and also in [AHMTT93] (which basically deals with the topological level). In [BCR81], one finds interesting variants of groups of smooth maps on open subsets $U \subseteq \mathbb{R}^n$ which are rapidly decreasing at the boundary with respect to certain weight functions. In particular, there is a Lie group $\mathscr{S}(\mathbb{R}^n, K)$ whose Lie algebra is the space $\mathscr{S}(\mathbb{R}^n, \mathfrak{k})$ of $\mathfrak{k}$-valued Schwartz functions on $\mathbb{R}^n$.

Remark II.2.9. (a) If M is a non-compact finite-dimensional manifold, then one cannot expect the topological groups $C^r(M,K)$ to be Lie groups. A typical example arises for $M = \mathbb{N}$ (a 0-dimensional manifold) and $K = \mathbb{T} := \mathbb{R}/\mathbb{Z}$. Then $C^r(M,K) \cong \mathbb{T}^{\mathbb{N}}$ is a compact topological group for which no $\mathbf{1}$-neighborhood is contractible, so that it carries no smooth manifold structure.

(b) Non-linear maps on spaces of compactly supported functions such as $E := C^\infty_c(\mathbb{R},\mathbb{R})$ (Examples I.1.3) require extreme caution. E.g. the map

$$f\colon C^\infty_c(\mathbb{R},\mathbb{R}) \to C^\infty_c(\mathbb{R},\mathbb{R}), \quad \gamma \mapsto \gamma \circ \gamma - \gamma(0)$$

is smooth on each closed Fréchet subspace $E_n := C^\infty_{[-n,n]}(\mathbb{R},\mathbb{R})$, but it is discontinuous in 0 ([Gl06a]). Therefore the LF space $E = \varinjlim E_n$ is a direct limit in the category of locally convex spaces, but not in the category of topological spaces. □

Remark II.2.10. (a) Let A be a commutative unital locally convex algebra with a smooth exponential function

$$\exp_A\colon A \to A^\times,$$

i.e., $\exp_A\colon (A,+) \to (A^\times,\cdot)$ is a group homomorphism with $T_0(\exp_A) = \mathrm{id}_A$.

Then $\Gamma_A := \ker(\exp_A)$ is a closed subgroup of A not containing any line. Suppose that Γ_A is discrete. Then $N := A/\Gamma_A$ carries a natural Lie group structure (Corollary II.2.4) and the exponential function factors through an injection $N \hookrightarrow A^\times$. We may therefore use Corollary II.2.3 to define a Lie group structure on the group $A^\times$ for which the identity component is $\exp_A(A) \cong N$.

(b) If M is a σ-compact finite-dimensional manifold, then $A := C^\infty(M,\mathbb{C})$ is a complex locally convex algebra with respect to the compact open C^∞ topology, and

$$\exp_A\colon A \to A^\times = C^\infty(M,\mathbb{C}^\times), \quad f \mapsto e^f$$

is a smooth exponential function.

If M is non-compact, then $A^\times$ is not open because for each unbounded function $f\colon X \to \mathbb{C}$ the element $\mathbf{1} + \lambda f$ is not invertible for $\lambda \in \mathbb{C}$ arbitrarily close to 0. It follows that A is a CIA if and only if M is compact.

The closed subgroup $\Gamma_A = \ker(\exp_A) = C^\infty(M, 2\pi i\mathbb{Z}) \cong C^\infty(M,\mathbb{Z})$ is discrete if and only if M has only finitely many connected components. In this case, (a) implies that $A^\times$ carries a Lie group structure for which $\exp_A$ is a local diffeomorphism.

A typical example is $M = \mathbb{R}$ and $A = C^\infty(\mathbb{R},\mathbb{C})$ with

$$A^\times = C^\infty(\mathbb{R},\mathbb{C}^\times) \cong C^\infty_*(\mathbb{R},\mathbb{C}^\times) \times \mathbb{C}^\times \cong C^\infty_*(\mathbb{R},\mathbb{C}) \times \mathbb{C}^\times$$

as topological groups, where C^∞_* denotes functions mapping 0 to 1, resp., to 0. For $M = \mathbb{N}$, we have $A \cong \mathbb{C}^{\mathbb{N}}$, and $\Gamma_A \cong \mathbb{Z}^{\mathbb{N}}$ is not discrete.

If M is connected, it is not hard to see that the map

$$\delta\colon A^\times \cong C^\infty(M,\mathbb{C}^\times) \to Z^1_{\mathrm{dR}}(M,\mathbb{C}), \quad f \mapsto \frac{df}{f}$$

induces a topological isomorphism of $A^\times/\mathbb{C}^\times$ onto the group $Z^1_{\mathrm{dR}}(M,\mathbb{Z})$ of closed 1-forms whose periods are contained in $2\pi i\mathbb{Z}$, and the arc-component $A^\times_a$ of the identity is mapped onto the set of exact 1-forms. We conclude that, as topological groups,

$$\pi_0(A^\times) \cong A^\times/A^\times_a \cong H^1_{\mathrm{dR}}(M,\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(M),\mathbb{Z}) \cong \mathrm{Hom}(H_1(M),\mathbb{Z}),$$

and this group is discrete if and only if $H^1_{\mathrm{sing}}(M,\mathbb{Z}) \cong \mathrm{Hom}(H_1(M),\mathbb{Z})$ is finitely generated (*cf.* [NeWa06b]). This shows that the arc-component of the identity in $A^\times$ is open if and only if $H^1_{\mathrm{sing}}(M,\mathbb{Z})$ is finitely generated.

For $M := \mathbb{C} \setminus \mathbb{N}$, the group $H_1(M) \cong \mathbb{Z}^{(\mathbb{N})}$ is of infinite rank, $H^1_{\mathrm{sing}}(M,\mathbb{Z}) \cong \mathbb{Z}^{\mathbb{N}}$ is not discrete, but M is connected, so that $A^\times$ carries a Lie group structure whose underlying topology is finer than the original topology of $A^\times$ induced from A. □

II.3. Smoothness of maps into diffeomorphism groups

Although the notion of a smooth manifold provides us with a natural notion of a smooth map between such manifolds, it turns out to be convenient to have a notion of a smooth map of a manifold into spaces of smooth maps which do not carry a natural manifold structure. In this subsection, we discuss this notion of smoothness with an emphasis on maps with values in groups of diffeomorphisms of locally convex manifolds.

Definition II.3.1. *Let M be a smooth locally convex manifold and* $\mathrm{Diff}(M)$ *the group of diffeomorphisms of M. Further let N be a smooth manifold. Although, in general,* $\mathrm{Diff}(M)$ *has no natural Lie group structure, we call a map $\varphi\colon N \to$* $\mathrm{Diff}(M)$ smooth *if the map*

$$\widehat{\varphi}\colon N \times M \to M \times M, \quad (n,x) \mapsto (\varphi(n)(x), \varphi(n)^{-1}(x))$$

is smooth. If N is an interval in $\mathbb{R}$, we obtain in particular a notion of a smooth curve. □

To discuss derivatives of such smooth map, we take a closer look at the "tangent bundle" of $\mathrm{Diff}(M)$, which can be done without a Lie group structure on $\mathrm{Diff}(M)$ (which does not exist in a satisfactory fashion for non-compact M; *cf.* Theorem VI.2.6). We think of the set

$$T(\mathrm{Diff}(M)) := \{X \in C^\infty(M,TM) \colon \pi_{TM} \circ X \in \mathrm{Diff}(M)\}$$

as the *tangent bundle of* $\mathrm{Diff}(M)$, with the map

$$\pi \colon T(\mathrm{Diff}(M)) \to \mathrm{Diff}(M), \quad X \mapsto \pi_{TM} \circ X$$

as the bundle projection, and $T_\varphi(\mathrm{Diff}(M)) := \pi^{-1}(\varphi)$ is considered as the *tangent space* in $\varphi \in \mathrm{Diff}(M)$. We have natural left and right actions of $\mathrm{Diff}(M)$ on $T(\mathrm{Diff}(M))$ by

$$\varphi.X = T(\varphi) \circ X \quad \text{and} \quad X.\varphi := X \circ \varphi.$$

The action

$$\mathrm{Ad} \colon \mathrm{Diff}(M) \times \mathcal{V}(M) \to \mathcal{V}(M), \quad (\varphi, X) \mapsto \varphi_* X := \mathrm{Ad}(\varphi).X := T(\varphi) \circ X \circ \varphi^{-1}$$

is called the *adjoint action of* $\mathrm{Diff}(M)$ *on* $\mathcal{V}(M)$.

Smooth curves $\varphi \colon J \subseteq \mathbb{R} \to \mathrm{Diff}(M)$ have (left) logarithmic derivatives

$$\delta(\varphi) \colon J \to \mathcal{V}(M), \quad \delta(\varphi)_t := \varphi(t)^{-1}.\varphi'(t)$$

which are smooth curves in the Lie algebra $\mathcal{V}(M)$ of smooth vector fields on M, i.e., time-dependent vector fields. For general N, the logarithmic derivatives are $\mathcal{V}(M)$-valued 1-forms on N, defined as follows:

If $\varphi \colon N \to \mathrm{Diff}(M)$ is smooth and $\widehat{\varphi}_1 \colon N \times M \to M, (n,x) \mapsto \varphi(n)(x)$, then we have a smooth tangent map

$$T(\widehat{\varphi}_1) \colon T(N \times M) \cong T(N) \times T(M) \to T(M),$$

and for each $v \in T_p(N)$ the partial map

$$T_p(\varphi)v \colon M \to T(M), \quad m \mapsto T_{(p,m)}(\widehat{\varphi}_1)(v,0)$$

is an element of $T_{\varphi(p)}(\mathrm{Diff}(M))$. We thus obtain a *tangent map*

$$T(\varphi) \colon T(N) \to T(\mathrm{Diff}(M)), \quad v \in T_p(N) \mapsto T_p(\varphi)v.$$

Definition II.3.2. *We define the* (left) logarithmic derivative of φ in p *by*

$$\delta(\varphi)_p \colon T_p(N) \to \mathcal{V}(M), \quad v \mapsto \varphi(p)^{-1}.T_p(\varphi)(v) = T(\varphi(p)^{-1}) \circ T_p(\varphi)(v).$$

It can be shown that $\delta(\varphi)$ is a smooth $\mathcal{V}(M)$-valued 1-form on N (see [GN06] for details), but recall that $\mathcal{V}(M)$ need not be a topological Lie algebra if M is not finite-dimensional (Remark I.5.3). □

For calculations, it is convenient to observe the Product- and Quotient Rule, both easy consequences of the Chain Rule:

Lemma II.3.3. *For two smooth maps* $f,g\colon N \to \mathrm{Diff}(M)$*, define* $(fg)(n) := f(n)\circ g(n)$ *and* $(fg^{-1})(n) := f(n)\circ g(n)^{-1}$*. Then we have the*
(1) Product Rule: $\delta(fg) = \delta(g) + \mathrm{Ad}(g^{-1}).\delta(f)$*, and the*
(2) Quotient Rule: $\delta(fg^{-1}) = \mathrm{Ad}(g).(\delta(f) - \delta(g))$*,*
where we write $(\mathrm{Ad}(f).\alpha)_n := \mathrm{Ad}(f(n)).\alpha_n$ *for a* $\mathcal{V}(M)$*-valued* 1*-form* α *on* N*.* □

Remark II.3.4. Although we shall only use the left logarithmic derivative, we note that one can also define the *right logarithmic derivative* of a smooth map $\varphi\colon N \to \mathrm{Diff}(M)$ by

$$\delta^r(\varphi)_p(v) = \big(T_p(\varphi)v\big)\circ\varphi(p)^{-1},$$

which also defines an element of $\Omega^1(N,\mathcal{V}(M))$, satisfying $\delta^r(\varphi) = \mathrm{Ad}(\varphi).\delta(\varphi) = -\delta(\varphi^{-1})$.

We then have for two smooth maps $f,g\colon N \to \mathrm{Diff}(M)$ the
(1) Product Rule: $\delta^r(fg) = \delta^r(f) + \mathrm{Ad}(f).\delta^r(g)$, and the
(2) Quotient Rule: $\delta^r(fg^{-1}) = \delta^r(f) - \mathrm{Ad}(fg).\delta^r(g)$. □

The following lemma generalizes Lemma 7.4 in [Mil84] which deals with Lie group-valued curves.

Lemma II.3.5. (Uniqueness Lemma) *Suppose that* N *is connected. For two smooth maps* $f,g\colon N \to \mathrm{Diff}(M)$*, the relation* $\delta(f) = \delta(g)$ *is equivalent to the existence of* $\varphi \in \mathrm{Diff}(M)$ *with* $g(p) = \varphi\circ f(p)$ *for all* $p \in N$*. In particular,* $g(p_0) = f(p_0)$ *for some* $p_0 \in N$ *implies* $f = g$*.*

Proof. If $g(p) = \varphi\circ f(p)$ for each $p \in N$, then $T_p(g) = \varphi(p).T_p(f)$, and therefore $\delta(g) = \delta(f)$.

If, conversely, $\delta(g) = \delta(f)$ and $\gamma := gf^{-1}$, then the Quotient Rule implies $\delta(\gamma) = \delta(gf^{-1}) = 0$, which in turn implies that for each $x \in M$ the map $p \mapsto \gamma(p)(x)$ has vanishing derivative, hence is locally constant (Proposition I.3.4). Since N is connected, γ is constant. We conclude that $g = \varphi\circ f$ for some $\varphi \in \mathrm{Diff}(M)$. □

The Uniqueness Lemma is a key tool which implies in particular that solutions to certain initial value problems are unique whenever they exist (which need not be the case). In this generality, this is quite remarkable because there are ordinary linear differential equations with constant coefficients on Fréchet spaces E for which solutions are not unique (*cf.* Example II.3.11 below). Nevertheless, the Uniqueness Lemma implies that solutions of the corresponding operator-valued initial value problems on the group $\mathrm{GL}(E) \subseteq \mathrm{Diff}(E)$ are unique whenever they exist.

Remark II.3.6. Smooth maps with values in $\mathrm{Diff}(M)$ can be specialized in several ways:

(a) Let E be a locally convex space and $\mathrm{GL}(E)$ the group of linear topological automorphisms of E. Then $\mathrm{GL}(E)$ consists of all diffeomorphisms of E commuting with the scalar multiplications $\mu_t(v) = tv$, $t \in \mathbb{K}^\times$, and $\mathfrak{gl}(E) = (\mathscr{L}(E), [\cdot,\cdot])$ can be identified with the Lie subalgebra of $\mathscr{V}(E)$ consisting of linear vector fields which can be characterized in a similar way. This observation implies that the logarithmic derivative of a smooth map $\varphi \colon N \to \mathrm{GL}(E)$ is a $\mathfrak{gl}(E)$-valued 1-form on N and that the Uniqueness Lemma applies to $\mathrm{GL}(E)$-valued smooth maps.

(b) If K is a Lie group with Lie algebra $\mathfrak{k}$, then we consider the group $C^\infty(M,K)$ of smooth maps, endowed with the pointwise bracket, as a subgroup of $\mathrm{Diff}(M\times K)$, by letting $f \in C^\infty(M,K)$ act on $M \times K$ by $\widetilde{f}(m,k) := (m, f(m)k)$. The corresponding Lie algebra of vector fields on $M \times K$ is $C^\infty(M,\mathfrak{k})$, where $\xi \in C^\infty(M,\mathfrak{k})$ corresponds to the vector field given by

$$\widetilde{\xi}(m,k) = T_{\mathbf{1}}(\rho_k)\xi(m) \in T_k(K) \subseteq T_{(m,k)}(M \times K).$$

A map $\varphi \colon N \to C^\infty(M,K)$ is smooth as a map into $\mathrm{Diff}(M\times K)$ if and only if the map

$$N \times M \times K \to (M \times K)^2, \quad (n,m,k) \mapsto \big((m, \varphi(n)(m)k), (m, \varphi(n)(m)^{-1}k)\big)$$

is smooth, which in turn means that the map $\widehat{\varphi} \colon N \times M \to K, (n,m) \mapsto \varphi(n)(m)$ is smooth. Hence the Uniqueness Lemma also applies to functions $\varphi \colon N \to C^\infty(M,K)$ which are smooth in the sense that $\widehat{\varphi}$ is smooth. Their logarithmic derivatives $\delta(\varphi)$ can be viewed as $C^\infty(M,\mathfrak{k})$-valued 1-forms on N.

(c) If G is a Lie group, then G itself can be identified with the subgroup $\{\lambda_g \colon g \in G\}$ of $\mathrm{Diff}(G)$, consisting of all left translations. On the Lie algebra level, this corresponds to the embedding $\mathbf{L}(G) \hookrightarrow \mathscr{V}(G)$ as the right invariant vector fields. Then a map $\varphi \colon N \to G \subseteq \mathrm{Diff}(G)$ is smooth if and only if it is smooth as a G-valued map, and we thus obtain a Uniqueness Lemma for G-valued smooth maps and $\mathbf{L}(G)$-valued 1-forms. □

Remark II.3.7. (a) The Uniqueness Lemma implies in particular that a smooth left action of a connected Lie group G on a smooth manifold M, given by a homomorphism $\sigma \colon G \to \mathrm{Diff}(M)$, is uniquely determined by the corresponding homomorphism of Lie algebras

$$\dot{\sigma} := -\delta(\sigma)_{\mathbf{1}} \colon \mathbf{L}(G) \to \mathscr{V}(M)$$

because $\delta(\sigma)$ is a left invariant $\mathscr{V}(M)$-valued 1-form on G, hence determined by its value in $\mathbf{1}$.

(b) It likewise follows that any smooth representation $\pi\colon G \to \mathrm{GL}(E)$ of a connected Lie group G on some locally convex space E is uniquely determined by its derived representation

$$\mathbf{L}(\pi) := \delta(\pi)_{\mathbf{1}}\colon \mathbf{L}(G) \to \mathfrak{gl}(E) \subseteq \mathcal{V}(E). \qquad \square$$

Remark II.3.8. (Complete vector fields) (a) Another consequence of the Uniqueness Lemma is that we may define a *complete vector field X on M* as a vector field for which there exists a smooth one-parameter group $\gamma_X\colon \mathbb{R} \to \mathrm{Diff}(M)$ with $\gamma_X'(0) = X$. In this sense, we consider the complete vector fields as the domain of the exponential function $\exp(X) := \gamma_X(1)$ of $\mathrm{Diff}(M)$.

(b) Likewise, the domain of the exponential function of $\mathrm{GL}(E)$, E a locally convex space, is the set of all continuous linear operators D on E for which the corresponding linear vector field $X_D(v) = Dv$ is complete, i.e., there exists a smooth representation $\alpha\colon \mathbb{R} \to \mathrm{GL}(E)$ with $\alpha'(0) = D$. We call these operators *D integrable*.

(c) We may further define for each Lie group G the domain of the exponential function of G as those elements $x \in \mathbf{L}(G)$ for which the corresponding left invariant vector field x_l is complete. $\square$

Example II.3.9. (The adjoint representation of a Lie group) The adjoint action encodes a good deal of structural information of a Lie group G. It provides a linearized picture of the non-commutativity of G.

For each $g \in G$, the map $c_g\colon G \to G, x \mapsto gxg^{-1}$, is a smooth automorphism of G, hence induces a continuous linear automorphism

$$\mathrm{Ad}(g) := \mathbf{L}(c_g)\colon \mathbf{L}(G) \to \mathbf{L}(G).$$

We thus obtain a smooth action $G \times \mathbf{L}(G) \to \mathbf{L}(G), (g,x) \mapsto \mathrm{Ad}(g).x$, called the *adjoint action* of G on $\mathbf{L}(G)$. By considering the Taylor expansion of the map $(g,h) \mapsto ghg^{-1}$, one shows that the derived representation of $\mathbf{L}(G)$ on $\mathbf{L}(G)$ satisfies

$$\text{(2.3.1)} \qquad \mathbf{L}(\mathrm{Ad}) = \mathrm{ad}, \quad \text{i.e.,} \quad \mathbf{L}(\mathrm{Ad})(x)(y) = [x,y] \quad \text{for} \quad x,y \in \mathbf{L}(G).$$

If $\mathbf{L}(G)' := \mathcal{L}(\mathbf{L}(G), \mathbb{K})$ denotes the *topological dual of* $\mathbf{L}(G)$, then we also obtain a representation of G on $\mathbf{L}(G)'$ by $\mathrm{Ad}^*(g).f := f \circ \mathrm{Ad}(g)^{-1}$, called the *coadjoint action*. Since we do not endow $\mathbf{L}(G)'$ with a topology, we will not specify any smoothness or continuity properties of this action. $\square$

The following lemma shows that, whenever there is a smooth curve $\gamma\colon J \to \mathrm{Diff}(M)$ satisfying the initial value problem

$$\text{(2.3.2)} \qquad \gamma(0) = \mathrm{id}_M \quad \text{and} \quad \gamma'(t) = X_t \circ \gamma(t)$$

for a time-dependent vector field $X\colon J \to \mathcal{V}(M)$, then all integral curves of X on M are of the form

$$\eta(t) = \gamma(t)(m), \tag{2.3.3}$$

hence unique. It follows in particular, that the existence of multiple integral curves of X implies that (2.3.2) has no solution. Below we shall see examples where this situation arises, even for linear differential equations.

Lemma II.3.10. *Let $J \subseteq \mathbb{R}$ be an interval containing 0 and $\gamma\colon J \to \mathrm{Diff}(M)$ be a smooth curve with $\gamma(0) = \mathrm{id}_M$. Let $X_t := \delta^r(\gamma)_t$ be the corresponding time-dependent vector field on M with $X_t \circ \gamma(t) = \gamma'(t)$, $m_0 \in M$, and assume that $\eta : J \to M$ is a solution of the initial value problem:*

$$\eta(0) = m_0 \quad \text{and} \quad \eta'(t) = X_t(\eta(t)) \quad \text{for} \quad t \in J.$$

Then $\eta(t) = \gamma(t)(m_0)$ holds for all $t \in J$.

Proof. The smooth curve $\alpha\colon J \to M, t \mapsto \gamma(t)^{-1}(\eta(t))$ satisfies $\alpha(0) = m_0$ and

$$\begin{aligned}\alpha'(t) &= (\gamma^{-1})'(\eta(t)) + T(\gamma(t)^{-1})(\eta'(t)) = T(\gamma(t)^{-1})\big(\delta(\gamma^{-1})_t(\eta(t)) + \eta'(t)\big)\\ &= T(\gamma(t)^{-1})\big(-\delta^r(\gamma)_t(\eta(t)) + \eta'(t)\big) = T(\gamma(t)^{-1})\big(-X_t(\eta(t)) + \eta'(t)\big) = 0.\end{aligned}$$

Hence α is constant m_0, and the assertion follows. □

In [OMYK82], one finds the particular version of the preceding lemma dealing with solutions of the initial value problem

$$\eta'(t) = [\eta(t), \xi(t)] + \eta(t), \quad \eta(0) = x$$

in the Lie algebra of a regular Lie group (see also [KYMO85, 2.5/2/6]).

Example II.3.11. (A linear ODE with multiple solutions) (*cf.* [Ham82, 5.6.1], [Mil84]) We give an example of a linear ODE for which solutions to initial value problems exist, but are not unique. We consider the Fréchet space $E := C^\infty([0,1],\mathbb{R})$ of smooth functions on the closed unit interval, and the continuous linear operator $Df := f'$ on E. We are asking for solutions of the initial value problem

$$\dot\gamma(t) = D\gamma(t), \quad \gamma(0) = v_0, \quad \gamma\colon I \subseteq \mathbb{R} \to E. \tag{2.3.4}$$

As a consequence of E. Borel's Theorem that each power series is the Taylor series of a smooth function, each $v_0 \in E$ has an extension to a smooth function on $\mathbb{R}$. Let h be such a function and consider the curve

$$\gamma\colon \mathbb{R} \to E, \quad \gamma(t)(x) := h(t+x).$$

Then $\gamma(0) = h|_{[0,1]} = v_0$ and $\dot\gamma(t)(x) = h'(t+x) = \gamma(t)'(x) = (D\gamma(t))(x)$. It is clear that these solutions of (2.3.4) depend on the choice of the extension h of v_0.

Lemma II.3.10 and the discussions preceding it now imply that D is not integrable. In fact, for any smooth homomorphism $\alpha\colon \mathbb{R} \to \mathrm{GL}(E)$ with $\alpha'(0) = D$, we would have $\delta^r(\alpha) = D$, so that any solution of (2.3.4) is of the form $\gamma(t) = \alpha(t).v_0$, contradicting the existence of multiple solutions. □

Example II.3.12. (A linear ODE without solutions; [Mil84]) We identify $E := C^\infty(\mathbb{S}^1, \mathbb{C})$ with the space of 2π-periodic smooth functions on the real line. We consider the linear operator $Df := -f''$ and the equation (2.3.4), which in this case is the heat equation with reversed time. If γ is a solution of (2.3.4) and $\gamma(t)(x) = \sum_{n\in\mathbb{Z}} a_n(t)e^{inx}$ its Fourier expansion, then $a_n'(t) = n^2 a_n(t)$ for each $n \in \mathbb{Z}$ leads to $a_n(t) = a_n(0)e^{tn^2}$. If the Fourier coefficients $a_n(0)$ of γ_0 do not satisfy $\sum_n |a_n(0)|e^{\varepsilon n^2} < \infty$ for any $\varepsilon > 0$ (which need not be the case for a smooth function γ_0), then (2.3.4) does not have a solution on $[0,\varepsilon]$.

As a consequence, the operator $\exp(tD)$ is not defined in $\mathrm{GL}(E)$ for any $t > 0$. Nevertheless, we may use the Fourier series expansion to see that $\beta(t) := (1+it^2)\mathbf{1} + tD$ defines a smooth curve $\beta\colon \mathbb{R} \to \mathrm{GL}(E)$. We further have $\beta'(0) = D$, so that D arises as the tangent vector of a smooth curve in $\mathrm{GL}(E)$, but not of any smooth one-parameter group. □

The following example is of some interest for the integrability of Lie algebras of formal vector fields (Example VI.2.8).

Example II.3.13. We consider the space $E := \mathbb{R}[[x]]$ of formal power series $\sum_{n=0}^\infty a_n x^n$ in one variable. We endow it with the Fréchet topology for which the map $\mathbb{R}^{\mathbb{N}_0} \to \mathbb{R}[[x]], (a_n) \mapsto \sum_n a_n x^n$ is a topological isomorphism. Then $Df := f'$ with $f'(x) := \sum_{n=1}^\infty a_n n x^{n-1} = \sum_{n=0}^\infty a_{n+1}(n+1)x^n$ for $f(x) = \sum_{n=0}^\infty a_n x^n$ defines a continuous linear operator on E. We claim that this operator is not integrable.

We argue by contradiction, and assume that $\alpha\colon \mathbb{R} \to \mathrm{GL}(E)$ is a smooth $\mathbb{R}$-action of E with $\alpha'(0) = D$. For each $n \in \mathbb{N}$, the curve $\gamma\colon \mathbb{R} \to E, \gamma(t) := (x+t)^n$, satisfies $\dot\gamma(t) = n(x+t)^{n-1} = D\gamma(t)$, so that Lemma II.3.10 implies that $\alpha(t)x^n = (x+t)^n$ for all $t \in \mathbb{R}$. Then we obtain $\alpha(1)x^n = 1 + nx + \cdots$. In view of $\lim_{n\to\infty} x^n \to 0$ in E, this contradicts the continuity of the operator $\alpha(1)$. Therefore D is not integrable. □

Example II.3.14. Let M be a compact manifold and $\mathfrak{g} = \mathcal{V}(M)$, the Lie algebra of smooth vector fields on M. We now sketch how the group $G := \mathrm{Diff}(M)$ can be turned into a Lie group, modeled on $\mathcal{V}(M)$, endowed with its natural Fréchet topology (Definition I.5.2) ([Les67]).

If $\mathrm{Fl}^X\colon \mathbb{R} \times M \to M, (t,m) \mapsto \mathrm{Fl}_t^X(m)$ denotes the flow of the vector field X, then the exponential function of the group $\mathrm{Diff}(M)$ should be given by the time-1-map of the flow of a vector field:

$$\exp_{\mathrm{Diff}(M)}\colon \mathcal{V}(M) \to \mathrm{Diff}(M), \quad X \mapsto \mathrm{Fl}_1^X.$$

For the Lie group structure described below, this is indeed the case. Unfortunately, it is not a local diffeomorphism of a 0-neighborhood in $\mathcal{V}(M)$ onto any identity neighborhood in $\mathrm{Diff}(M)$. Therefore we cannot use it to define a chart around $\mathbf{1} = \mathrm{id}_M$ (*cf.* [Grab88], [Pali68/74], and also [Fre68], which deals with local smooth diffeomorphisms in two dimensions).

Fortunately, there is an easy way around this problem. Let g be a Riemannian metric on M and $\mathrm{Exp}\colon TM \to M$ be its exponential function, which assigns to $v \in T_m(M)$ the point $\gamma_v(1)$, where $\gamma_v\colon [0,1] \to M$ is the geodesic segment with $\gamma_v(0) = m$ and $\gamma_v'(0) = v$. We then obtain a smooth map

$$\Phi\colon TM \to M \times M, \quad v \mapsto (m, \mathrm{Exp}\, v), \quad v \in T_m(M).$$

There exists an open neighborhood $U \subseteq TM$ of the zero section such that Φ maps U diffeomorphically onto an open neighborhood of the diagonal in $M \times M$. Now

$$U_{\mathfrak{g}} := \{X \in \mathcal{V}(M)\colon X(M) \subseteq U\}$$

is an open subset of the Fréchet space $\mathcal{V}(M)$, and we define a map

$$\varphi\colon U_{\mathfrak{g}} \to C^\infty(M,M), \quad \varphi(X)(m) := \mathrm{Exp}(X(m)).$$

It is clear that $\varphi(0) = \mathrm{id}_M$. One can show that after shrinking $U_{\mathfrak{g}}$ to a sufficiently small 0-neighborhood in the compact open C^1-topology on $\mathcal{V}(M)$, we achieve that $\varphi(U_{\mathfrak{g}}) \subseteq \mathrm{Diff}(M)$. To see that $\mathrm{Diff}(M)$ carries a Lie group structure for which φ is a chart, one has to verify that the group operations are smooth in a 0-neighborhood when transferred to $U_{\mathfrak{g}}$ via φ, so that Theorem II.2.1 applies. We thus obtain a Lie group structure on $\mathrm{Diff}(M)$ (*cf.* [Omo70], [GN06]).

From the smoothness of the map $U_{\mathfrak{g}} \times M \to M, (X,m) \mapsto \varphi(X)(m) = \mathrm{Exp}(X(m))$ it follows that the canonical left action $\sigma\colon \mathrm{Diff}(M) \times M \to M, (\varphi, m) \mapsto \varphi(m)$ is smooth in an identity neighborhood of $\mathrm{Diff}(M)$, and hence smooth, because it is an action by smooth maps. The corresponding homomorphism of Lie algebras $\dot{\sigma}\colon \mathbf{L}(\mathrm{Diff}(M)) \to \mathcal{V}(M)$ (Remark II.3.7(a)) is given by

$$\dot{\sigma}(X)(m) = -T\sigma(X, 0_m) = -X(m),$$

i.e., $\dot{\sigma} = -\mathrm{id}_{\mathcal{V}(M)}$, which leads to

$$\mathbf{L}(\mathrm{Diff}(M)) = (\mathcal{V}(M), [\cdot,\cdot])^{\mathrm{op}},$$

where $\mathfrak{g}^{\mathrm{op}}$ is the *opposite* of the Lie algebra $\mathfrak{g}$ with the bracket $[x,y]_{\mathrm{op}} := [y,x]$.

This "wrong" sign is caused by the fact that we consider $\mathrm{Diff}(M)$ as a group acting on M from the left. If we consider it as a group acting on the right, we obtain the opposite multiplication $\varphi * \psi := \psi \circ \varphi$ and $\mathbf{L}(\mathrm{Diff}(M)^{\mathrm{op}}) \cong (\mathcal{V}(M), [\cdot,\cdot])$. Here we write G^{op} for the *opposite group* with the order of multiplication reversed. □

II.4. Applications to Lie group-valued smooth maps

In this subsection, we describe some applications of the Uniqueness Lemma to Lie group-valued smooth maps (*cf.* Remark II.3.6(c)).

Let G be a Lie group with Lie algebra $\mathfrak{g} = \mathbf{L}(G)$. The *Maurer–Cartan form* $\kappa_G \in \Omega^1(G,\mathfrak{g})$ is the unique left invariant 1-form on G with $\kappa_{G,\mathbf{1}} = \mathrm{id}_{\mathfrak{g}}$, i.e., $\kappa_G(v) = g^{-1}.v$ for $v \in T_g(G)$. In various disguises, this form plays a central role in the approach to (local) (Banach–)Lie groups via partial differential equations ([Mau88], [CaE01], [Lie95], [Bir38], [MicA48], [Lau56]).

Identifying G with the subgroup of left translations in $\mathrm{Diff}(G)$, the concepts of the preceding subsection apply to any smooth map $f\colon M \to G$ (Remark II.3.6(c)). The logarithmic derivative of f can be described as a pull-back of the Maurer–Cartan form:

$$\delta(f) = f^*\kappa_G \in \Omega^1(M,\mathfrak{g}).$$

Proposition II.4.1. *Let G and H be Lie groups.*

(1) If $\varphi\colon G \to H$ is a morphism of Lie groups, then $\delta(\varphi) = \mathbf{L}(\varphi)\circ\kappa_G$.
(2) If G is connected and $\varphi_1,\varphi_2 : G \to H$ are morphisms of Lie groups with $\mathbf{L}(\varphi_1) = \mathbf{L}(\varphi_2)$, then $\varphi_1 = \varphi_2$.
(3) For a smooth function $f\colon G \to H$ with $f(\mathbf{1}) = \mathbf{1}$, the following are equivalent:
   (a) $\delta(f)$ is a left invariant 1-form.
   (b) f is a group homomorphism.

Proof. (1) is a simple computation, and (2) follows with (1) and the Uniqueness Lemma (*cf.* Remark II.3.6(c)).

The proof of (3) follows a similar pattern, applying the Uniqueness Lemma to the relations $\lambda_g^*\delta(f) = \delta(f)$. □

Applying (2) to the conjugation automorphisms $c_g \in \mathrm{Aut}(G)$, we obtain:

Corollary II.4.2. *If G is a connected Lie group, then* $\ker\mathrm{Ad} = Z(G)$. □

It follows in particular, that the adjoint action of a connected Lie group G is trivial if and only if G is abelian. In view of Remark II.3.6(b), this is equivalent to the triviality of the corresponding derived action, which is the adjoint action of $\mathbf{L}(G)$ (Example II.3.9). We thus obtain the following affirmative answer to a question of J. Milnor ([Mil84]):

Proposition II.4.3. *A connected Lie group is abelian if and only if its Lie algebra is abelian.* □

This argument can be refined by investigating the structure of logarithmic derivatives of iterated commutators of smooth curves in a Lie group G. A systematic use of the Uniqueness Lemma then leads to the following result (see [GN06]):

Theorem II.4.4. *A connected Lie group G is nilpotent, resp., solvable, if and only if its Lie algebra $\mathbf{L}(G)$ is nilpotent, resp., solvable.* □

II.5. The exponential function and regularity

In the Lie theory of finite-dimensional and Banach–Lie groups, the exponential function is a central tool used to pass information from the group to the Lie algebra and vice versa. Unfortunately, the exponential function is less powerful in the context of locally convex Lie groups. Here we take a closer look at its basic properties, and in Section IV below we study the class of locally exponential Lie groups for which the exponential function behaves well in the sense that it is a local diffeomorphism in 0.

Definition II.5.1. *For a Lie group G with Lie algebra $\mathfrak{g} = \mathbf{L}(G)$, we call a smooth function* $\exp_G: \mathfrak{g} \to G$ *an* exponential function for G *if for each $x \in \mathfrak{g}$ the curve $\gamma_x(t) := \exp_G(tx)$ is a one-parameter group with $\gamma_x'(0) = x$.* □

It is easy to see that any such curve is a solution of the initial value problem (IVP)

$$\gamma(0) = \mathbf{1}, \quad \delta(\gamma) = x,$$

so that the Uniqueness Lemma implies that solutions are unique whenever they exist. Hence a Lie group G has at most one exponential function.

The question for the existence of an exponential function leads to the more general question when for a smooth curve $\xi \in C^\infty(I, \mathfrak{g})$ ($I = [0,1]$), the initial value problem (IVP)

$$\gamma(0) = \mathbf{1}, \quad \delta(\gamma) = \xi, \tag{2.5.1}$$

has a solution. If this is the case for constant functions $\xi(t) = x$, the corresponding solutions are the curves γ_x required to obtain an exponential function. The solutions of (2.5.1) are unique by the Uniqueness Lemma (Remark II.3.6(c)).

Definition II.5.2. *A Lie group G is called* regular *if for each $\xi \in C^\infty(I, \mathfrak{g})$, the initial value problem (2.5.1) has a solution $\gamma_\xi \in C^\infty(I, G)$, and the* evolution map

$$\mathrm{evol}_G: C^\infty(I, \mathfrak{g}) \to G, \quad \xi \mapsto \gamma_\xi(1)$$

is smooth. □

Remark II.5.3. (a) If G is regular, then G has a smooth exponential function, given by

$$\exp_G(x) := \mathrm{evol}_G(\xi_x),$$

where $\xi_x(t) = x$ for $t \in I$.

(b) For any Lie group G, the logarithmic derivative

$$\delta\colon C^\infty_*([0,1],G) \to C^\infty(I,\mathbf{L}(G)) \cong \Omega^1(I,\mathbf{L}(G))$$

is a smooth map with $T_{\mathbf{1}}(\delta)\xi = \xi'$. If G is regular, this fact can be used to show that $T_{\mathbf{1}}(\delta)$ is surjective, hence that $\mathbf{L}(G)$ is Mackey complete (*cf.* [GN06]). □

Remark II.5.4. As a direct consequence of the existence of solutions to ordinary differential equations on open domains of Banach spaces and their smooth dependence on parameters (*cf.* [La99]), every Banach–Lie group is regular. □

All Lie groups known to the author which are modeled on Mackey complete spaces are regular. In concrete situations, it is sometimes hard to verify regularity, and in some case it is not known if the Lie groups under consideration are regular. We shall take a closer look at criteria for regularity in Section III below. In particular, we shall see that essentially all groups belonging to the major classes discussed in the introduction are in fact regular.

Example II.5.5. If the model space is no longer assumed to be Mackey complete, one can construct non-regular Lie groups as follows (*cf.* [Gl02b, Sect. 7]): Let $A \subseteq C([0,1],\mathbb{R})$ denote the unital subalgebra of all rational functions, i.e., of all quotients $p(x)/q(x)$, where $q(x)$ is a polynomial without zeros in $[0,1]$. We endow A with the induced norm $\|f\| := \sup_{0\leq t\leq 1}|f(t)|$. If an element $f \in A$ is invertible in $C([0,1],\mathbb{R})$, then it has no zero in $[0,1]$, which implies that it is also invertible in A, i.e.,

$$A^\times = C([0,1],\mathbb{R})^\times \cap A.$$

This shows that $A^\times$ is open in A, and since the Banach algebra $C([0,1],\mathbb{R})$ is a CIA, the continuity of the inversion is inherited by A, so that A is a CIA. In particular, $A^\times$ is a Lie group (Example II.1.4).

Let $f \in A$ and assume that there exists a smooth homomorphism $\gamma_f\colon \mathbb{R} \to A^\times$ with $\gamma'_f(0) = f$. Then Proposition II.4.1, applied to γ_f as a map $\mathbb{R} \to C([0,1],\mathbb{R})^\times$, leads to $\gamma_f(t) = e^{tf}$ for each $t \in \mathbb{R}$. Since e^f is not rational if f is not constant, we conclude that f is constant. Therefore the Lie group $A^\times$ does not have an exponential function and in particular it is not regular. □

The following proposition illustrates the relation between regularity and Mackey completeness.

Proposition II.5.6. *The additive Lie group $(E,+)$ of a locally convex space E is regular if and only if E is Mackey complete.*

Proof. For a smooth curve $\xi\colon I \to E$, any solution $\gamma_\xi\colon I \to E$ of (2.5.1) satisfies $\gamma'_\xi = \xi$ and vice versa. Therefore regularity implies that E is Mackey complete (Definition I.1.4). Conversely, Mackey completeness of E implies that $\mathrm{evol}_G(\xi) := \int_0^1 \xi(s)\,ds$ defines a continuous linear map $\mathrm{evol}_G\colon C^\infty(I,E) \to E$, so that it is in particular smooth. □

Proposition II.5.7. *Suppose that the Lie group G has a smooth exponential function* $\exp_G\colon \mathfrak{g} \to G$*. Then its logarithmic derivative is given by*

$$\delta(\exp_G)(x) = \int_0^1 \mathrm{Ad}(\exp_G(-tx))\,dt, \tag{2.5.2}$$

where the operator-valued integral is defined pointwise, i.e.,

$$\delta(\exp_G)(x)y = \int_0^1 \mathrm{Ad}(\exp_G(-tx))y\,dt \quad \textit{for each } y \in \mathfrak{g}.$$

Proof. ([Grab93]) For $t,s \in \mathbb{R}$, we consider the three smooth functions $f, f_t, f_s\colon \mathfrak{g} \to G$, given by

$$f(x) := \exp_G((t+s)x), \quad f_t(x) := \exp_G(tx) \quad \text{and} \quad f_s(x) := \exp_G(sx),$$

satisfying $f = f_t f_s$ pointwise on $\mathfrak{g}$. The Product Rule (Lemma II.3.3) implies that

$$\delta(f) = \delta(f_s) + \mathrm{Ad}(f_s)^{-1}\delta(f_t).$$

For the smooth curve $\psi\colon \mathbb{R} \to \mathfrak{g}$, $\psi(t) := \delta(\exp_G)_{tx}(ty)$, we therefore obtain

$$\psi(t+s) = \delta(f)_x(y) = \delta(f_s)_x(y) + \mathrm{Ad}(f_s)^{-1}.\delta(f_t)_x(y) = \psi(s) + \mathrm{Ad}(\exp_G(-sx)).\psi(t). \tag{2.5.3}$$

We have $\psi(0) = 0$ and $\psi'(0) = \lim_{t\to 0} \delta(\exp_G)_{tx}(y) = \delta(\exp_G)_0(y) = y$, so that taking derivatives with respect to t in 0, (2.5.3) leads to $\psi'(s) = \mathrm{Ad}(\exp_G(-sx)).y$. Now the assertion follows by integration from $\delta(\exp_G)_x(y) = \psi(1) = \int_0^1 \psi'(s)\,ds$. □

If $\mathfrak{g}$ is integrable to a group with exponential function, then the one-parameter groups $\mathrm{Ad}(\exp_G(tx))$ have the infinitesimal generator $\mathrm{ad}\,x$ (Remark II.3.7(b), Example II.3.9), so that we may also write

$$\mathrm{Ad}(\exp_G(tx)) = e^{t\,\mathrm{ad}\,x}. \tag{2.5.4}$$

If, in addition, $\mathfrak{g}$ is Mackey complete, then the operator-valued integral

$$\kappa_{\mathfrak{g}}(x) := \int_0^1 e^{-t\,\mathrm{ad}\,x}\,dt \tag{2.5.5}$$

exists pointwise because the curves $t \mapsto e^{-t\,\mathrm{ad}\,x}y$ are smooth, and the preceding theorem states that for each $x \in \mathfrak{g}$:

$$\delta(\exp_G)_x = \kappa_{\mathfrak{g}}(x). \tag{2.5.6}$$

The advantage of $\kappa_{\mathfrak{g}}(x)$ is that it is expressed completely in Lie algebraic terms.

Remark II.5.8. If $\mathfrak{g}$ is a Banach–Lie algebra, then $\kappa_{\mathfrak{g}}(x)$ can be represented by a convergent power series

$$\kappa_{\mathfrak{g}}(x) = \int_0^1 e^{-t\,\mathrm{ad}\,x}\,dt = \frac{\mathbf{1} - e^{-\mathrm{ad}\,x}}{\mathrm{ad}\,x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!}(\mathrm{ad}\,x)^k.$$

This means that $\kappa_{\mathfrak{g}}(x) = f(\mathrm{ad}\,x)$ holds for the entire function

$$f(z) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} z^k = \frac{1-e^{-z}}{z}.$$

As $f^{-1}(0) = 2\pi i\mathbb{Z} \setminus \{0\}$, and $\mathrm{Spec}(\kappa_{\mathfrak{g}}(x)) = f(\mathrm{Spec}(\mathrm{ad}\,x))$ by the Spectral Mapping Theorem, we see that $\kappa_{\mathfrak{g}}(x)$ is invertible if and only if $\mathrm{Spec}(\mathrm{ad}\,x) \cap 2\pi i\mathbb{Z} \subseteq \{0\}$.

Part of this observation can be saved in the general case. If $\mathfrak{g}$ is Mackey complete one can show that $\kappa_{\mathfrak{g}}(x)$ is not injective if and only if there exists some $n \in \mathbb{N}$ with

$$\ker((\mathrm{ad}\,x)^2 + 4\pi^2 n^2 \mathbf{1}) \neq \{0\}.$$

If $\mathfrak{g}$ is a complex Lie algebra, this means that some $2\pi i n \in 2\pi i\mathbb{Z} \setminus \{0\}$ is an eigenvalue of $\mathrm{ad}\,x$ (*cf.* [GN06] for details). □

Examples II.5.9. Let $\alpha \colon \mathbb{R} \to \mathrm{GL}(E)$ be a smooth representation of $\mathbb{R}$ on the Mackey complete locally convex space E with the infinitesimal generator $D = \alpha'(0)$. Then the semi-direct product group

$$G := E \rtimes_{\alpha} \mathbb{R}, \quad (v,t)(v',t') = (v + \alpha(t)v', t + t')$$

is a Lie group with Lie algebra $\mathfrak{g} = E \rtimes_D \mathbb{R}$ and exponential function

$$\exp_G(v,t) = \big(\beta(t)v, t\big) \quad \text{with} \quad \beta(t) = \int_0^1 \alpha(st)\,ds = \begin{cases} \mathrm{id}_E & \text{for } t = 0 \\ \frac{1}{t}\int_0^t \alpha(s)\,ds & \text{for } t \neq 0. \end{cases}$$

From this formula it is clear that $(w,t) \in \mathrm{im}(\exp_G)$ is equivalent to $w \in \mathrm{im}(\beta(t))$. We conclude that $\exp_G$ is injective on some 0-neighborhood if and only if $\beta(t)$ is injective for t close to 0, and it is surjective onto some **1**-neighborhood in G if and only if $\beta(t)$ is surjective for t close to 0 (*cf.* Problem IV.4 below).

Note that the eigenvector equation $Dv = \lambda v$ for $t\lambda \neq 0$ implies that

$$\beta(t)v = \int_0^1 e^{st\lambda} v\,ds = \frac{e^{t\lambda} - 1}{t\lambda} v,$$

so that $\beta(t)v = 0$ is equivalent to $t\lambda \in 2\pi i\mathbb{Z} \setminus \{0\}$.

(a) For the Fréchet space $E = \mathbb{C}^{\mathbb{N}}$ and the diagonal operator D given by $D(z_n) = (2\pi i n z_n)$, we see that $\beta(\frac{1}{n})e_n = 0$ holds for $e_n = (\delta_{mn})_{m \in \mathbb{N}}$, and $e_n \notin$

$\operatorname{im}(\beta(\frac{1}{n}))$. We conclude that $(e_n, \frac{1}{n})$ is not contained in the image of $\exp_G$, and since $(e_n, \frac{1}{n}) \to (0,0)$, the identity of G, $\operatorname{im}(\exp_G)$ does not contain any identity neighborhood of G. Hence the exponential function of the Fréchet–Lie group $G = E \rtimes_\alpha \mathbb{R}$ is neither locally injective nor locally surjective in 0.

(b) For the Fréchet space $E = \mathbb{R}^{\mathbb{N}}$ and the diagonal operator D given by $D(z_n) = (nz_n)$, it is easy to see that all operators $\beta(t)$ are invertible and that $\beta \colon \mathbb{R} \to \operatorname{GL}(E)$ is a smooth map. This implies that $\exp_G \colon \mathfrak{g} \to G$ is a diffeomorphism. □

Remark II.5.10. If G is the unit group $A^\times$ of a Mackey complete CIA, then we identify $T(G) \subseteq T(A) \cong A \times A$ with $A^\times \times A$ and note that $\operatorname{ad} x = \lambda_x - \rho_x$ and $e^{\operatorname{ad} x} y = e^x y e^{-x}$. Therefore (2.5.2) can be written as

$$T_x(\exp_G)y = e^x \int_0^1 e^{-tx} y e^{tx}\, dt = \int_0^1 e^{(1-t)x} y e^{tx}\, dt$$

(*cf.* [MicA45] for the case of Banach algebras). □

A closer investigation of (2.5.5) leads to the following results on the behavior of the exponential function (*cf.* [GN06]; and [LaTi66] for the finite-dimensional case):

Proposition II.5.11. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and a smooth exponential function. Then the following assertions hold for $x, y \in \mathfrak{g}$:*

(1) If $\kappa_{\mathfrak{g}}(x)y = 0$, then

$$\exp_G(e^{t \operatorname{ad} y}.x) = \exp_G(x) \quad \textit{for all} \quad t \in \mathbb{R}.$$

(2) If $\kappa_{\mathfrak{g}}(x)$ is not injective and $\mathfrak{g}$ is Mackey complete, then $\exp_G$ is not injective in any neighborhood of x.

(3) If $\kappa_{\mathfrak{g}}(x)$ is injective, then
   - *(a) $\exp_G(y) = \exp_G(x)$ implies $[x,y] = 0$ and $\exp_G(x-y) = \mathbf{1}$.*
   - *(b) $\exp_G(x) \in Z(G)$ implies $x \in \mathfrak{z}(\mathfrak{g})$ and equivalence holds if G is connected.*
   - *(c) $\exp_G(x) = \mathbf{1}$ implies $x \in \mathfrak{z}(\mathfrak{g})$.*

(4) Suppose that 0 is isolated in $\exp_G^{-1}(\mathbf{1})$. Then x is isolated in $\exp_G^{-1}(\exp_G(x))$ if and only if $\kappa_{\mathfrak{g}}(x)$ is injective.

(5) If $\mathfrak{a} \subseteq \mathfrak{g}$ is an abelian subalgebra, then $\exp_{\mathfrak{a}} := \exp_G|_{\mathfrak{a}} \colon \mathfrak{a} \to G$ is a morphism of Lie groups. Its kernel $\Gamma_{\mathfrak{a}} := \ker(\exp_{\mathfrak{a}})$ is a closed subgroup of $\mathfrak{a}$ in which all C^1-curves are constant. It intersects each finite-dimensional subspace of $\mathfrak{a}$ in a discrete subgroup. □

Remark II.5.12. (a) Let $U \subseteq \mathfrak{g}$ be a 0-neighborhood with the property that $\kappa_{\mathfrak{g}}(z)$ is injective for each $z \in U - U$. Then the preceding proposition implies for $x, y \in$

U with $\exp_G x = \exp_G y$ that $\exp_G(x-y) = \mathbf{1}$, $[x,y]=0$, and since $x-y \in U$, it further follows that $x-y \in \mathfrak{z}(\mathfrak{g})$. If we assume, in addition, that the closed subgroup $\Gamma_{\mathfrak{z}(\mathfrak{g})} := \ker(\exp_{\mathfrak{z}(\mathfrak{g})})$ intersects $U-U$ only in $\{0\}$, $\exp_G$ is injective on U.

(b) If G is a Banach–Lie group and $\mathfrak{g} = \mathbf{L}(G)$ carries a norm with $\|[x,y]\| \leq \|x\| \cdot \|y\|$, then $\|\operatorname{ad} x\| \leq \|x\|$. Therefore $\|x\| < 2\pi$ implies that $\kappa_{\mathfrak{g}}(x)$ is invertible (Remark II.5.8). If $\exp_G|_{\mathfrak{z}(\mathfrak{g})}$ is injective, i.e., $Z(G)$ is simply connected, the preceding remark implies that $\exp_G$ is injective on the open ball $B_\pi := \{x \in \mathfrak{g} \colon \|x\| < \pi\}$ (*cf.* [LaTi66]). In general, we may put

$$\delta_G := \inf\{\|x\| \colon 0 \neq x \in \Gamma_{\mathfrak{z}(\mathfrak{g})}\}$$

to see that $\exp_G$ is injective on the ball of radius $r := \min\{\pi, \frac{\delta_G}{2}\}$ (*cf.* [GN03], [Bel04, Rem. 2.3]). □

Example II.5.13. In [Omo70], [Ham82] and [Mil84], it is shown that for the group $G := \mathrm{Diff}(\mathbb{S}^1)$ of diffeomorphisms of the circle, the image of the exponential function is not a neighborhood of $\mathbf{1}$ (*cf.* also [KM97, Ex. 43.2], [PS86, p.28]). Small perturbations of rigid rotations of order n lead to a sequence of diffeomorphisms converging to $\mathrm{id}_{\mathbb{S}^1}$ which do not lie on any one-parameter group.

More generally, for any compact manifold M, the image of the exponential function of $\mathrm{Diff}(M)$ does not contain any identity neighborhood (*cf.* [Grab88], [Pali68/74], and [Fre68] for some 2-dimensional cases).

Identifying the Lie algebra $\mathfrak{g} := \mathcal{V}(\mathbb{S}^1)$ of $\mathrm{Diff}(\mathbb{S}^1)$ with smooth 2π-periodic functions on $\mathbb{R}$, the Lie bracket corresponds to

$$[f,g] = fg' - f'g.$$

For the constant function $f_0 = 1$ and $c_n(t) := \cos(nt)$ and $s_n(t) = \sin(nt)$, this leads to

$$[f_0, s_n] = nc_n \quad \text{and} \quad [f_0, c_n] = -ns_n,$$

so that $\mathrm{span}\{f_0, s_n, c_n\} \subseteq \mathcal{V}(\mathbb{S}^1)$ is a Lie subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. It further follows that $((\operatorname{ad} f_0)^2 + n^2\mathbf{1})s_n = 0$, so that $\kappa_{\mathfrak{g}}(\frac{2\pi}{n} f_0)s_n = 0$ implies that $\exp_G$ is not injective in any neighborhood of $\frac{2\pi}{n} f_0$ (Proposition II.5.11(1)) (*cf.* [Mil82, Ex. 6.6]). Therefore $\exp_G$ is neither locally surjective nor injective. □

Remark II.5.14. (Surjectivity of $\exp_G$) The global behavior of the exponential function and in particular the question of its surjectivity is a quite complicated issue, depending very much on specific properties of the groups under consideration (*cf.* [Wü03/05]).

(a) For finite-dimensional Lie groups, the most basic general result is that if G is a connected Lie group with compact Lie algebra $\mathfrak{g}$, then $\exp_G$ is surjective. Since the compactness of $\mathfrak{g}$ is equivalent to the existence of an $\mathrm{Ad}(G)$-invariant scalar product, which in turn leads to a biinvariant Riemannian metric on G, the surjectivity of $\exp_G$ can be derived from the Hopf–Rinow Theorem in Riemannian geometry.

(b) A natural generalization of the notion of a compact Lie algebra to the Banach context is to say that a real Banach–Lie algebra $(\mathfrak{g}, \|\cdot\|)$ is *elliptic* if the norm on $\mathfrak{g}$ is invariant under the group $\mathrm{Inn}(\mathfrak{g}) := \langle e^{\mathrm{ad}\,\mathfrak{g}} \rangle \subseteq \mathrm{Aut}(\mathfrak{g})$ of *inner automorphisms* (*cf.* [Ne02c, Def. IV.3]). A finite-dimensional Lie algebra $\mathfrak{g}$ is elliptic with respect to some norm if and only if it is compact. In this case, the requirement of an invariant scalar product leads to the same class of Lie algebras, but in the infinite-dimensional context this is different. Here the existence of an invariant scalar product turning $\mathfrak{g}$ into a real Hilbert space leads to the structure of an L^*-algebra. Simple L^*-algebras can be classified, and each L^*-algebra is a Hilbert space direct sum of simple ideals and its center (*cf.* [Sc60/61], [dlH72], [CGM90], [Neh93], [St99]). In particular, the classification shows that every L^*-algebra can be realized as a closed subalgebra of the L^*-algebra $B_2(H)$ of Hilbert–Schmidt operators on a complex Hilbert space H. Therefore the requirement of an invariant scalar product on $\mathfrak{g}$ leads to the embeddability into the Lie algebra $\mathfrak{u}_2(H)$ of skew-hermitian Hilbert–Schmidt operators on a Hilbert space H.

The class of elliptic Lie algebras is much bigger. It contains the algebra $\mathfrak{u}(A)$ of skew-hermitian elements of any C^*-algebra A and in particular the Lie algebra $\mathfrak{u}(H)$ of the full unitary group $\mathrm{U}(H)$ of a Hilbert space H.

Although finite-dimensional connected Lie groups with compact Lie algebra have a surjective exponential function, this is no longer true for connected Banach–Lie groups with elliptic Lie algebra. This is a quite remarkable phenomenon discovered by Putnam and Winter in [PW52]: the orthogonal group $\mathrm{O}(H)$ of a real infinite-dimensional Hilbert space is a connected Banach–Lie group with elliptic Lie algebra, but its exponential function is *not* surjective. This contrasts the fact that the exponential function of the unitary group $\mathrm{U}(H)$ of a complex Hilbert space is always surjective, as follows from the spectral theory of unitary operators. □

II.6. Initial Lie subgroups

It is one of the fundamental problems of Lie theory (FP5) to understand to which extent subgroups of Lie groups carry natural Lie group structures. In this subsection, we briefly discuss the rather weak concept of an initial Lie subgroup. As a consequence of the universal property built into its definition, such a structure is unique whenever it exists. As the discussion in Remark II.6.5 and further results in Section IV below show, it is hard to prove that a subgroup does not carry any initial Lie group structure (*cf.* Problem II.6).

Definition II.6.1. *An injective morphism $\iota\colon H \to G$ of Lie groups is called an* initial Lie subgroup *if* $\mathbf{L}(\iota)\colon \mathbf{L}(H) \to \mathbf{L}(G)$ *is injective, and for each C^k-map $f\colon M \to G$ ($k \in \mathbb{N} \cup \{\infty\}$) from a C^k-manifold M to G with* $\mathrm{im}(f) \subseteq H$, *the corresponding map $\iota^{-1} \circ f\colon M \to H$ is C^k.* □

The following lemma shows that the existence of an initial Lie group structure only depends on the subgroup H, considered as a subset of G.

Lemma II.6.2. *Any subgroup H of a Lie group G carries at most one structure of an initial Lie subgroup.*

Proof. If $\iota'\colon H' \hookrightarrow G$ is another initial Lie subgroup with the same range as $\iota\colon H \to G$, then $\iota^{-1} \circ \iota'\colon H' \to H$ and $\iota'^{-1} \circ \iota\colon H \to H'$ are smooth morphisms of Lie groups, so that H and H' are isomorphic. □

A priori, any subgroup H of a Lie group G can be an initial Lie subgroup. A first step to a better understanding of initial subgroups is to find a natural candidate for the Lie algebra of such a subgroup. In the following, we write $C^1_*(I,G)$ for the set of all C^1-curves $\gamma\colon I = [0,1] \to G$ with $\gamma(0) = \mathbf{1}$. Then the following definition works well for all subgroups (*cf.* [Lau56]; see also [vN29; pp.18/19]):

Proposition II.6.3. *Let $H \subseteq G$ be a subgroup of the Lie group G. Then the differential tangent set*

$$\mathbf{L}^d(H) := \{\alpha'(0) \in \mathbf{L}(G) = T_{\mathbf{1}}(G)\colon \alpha \in C^1_*([0,1],G),\ \mathrm{im}(\alpha) \subseteq H\}$$

is a Lie subalgebra of $\mathbf{L}(G)$. *If, in addition, H carries the structure $\iota_H\colon H \to G$ of an initial Lie subgroup, then* $\mathbf{L}^d(H) = \mathrm{im}(\mathbf{L}(\iota_H))$.

Proof. If $\alpha, \beta \in C^1_*(I,G)$, then $(\alpha\beta)'(0) = \alpha'(0) + \beta'(0)$, $(\alpha^{-1})'(0) = -\alpha(0)$, and for $0 \leq \lambda \leq 1$ the curve $\alpha_\lambda(t) := \alpha(\lambda t)$ satisfies $\alpha'_\lambda(0) = \lambda\alpha'(0)$. This implies that $\mathbf{L}^d(H)$ is a real linear subspace of $\mathbf{L}(G)$.

Next we recall that $[x,y]$ is the lowest order term in the Taylor expansion of the commutator map $(x,y) \mapsto xyx^{-1}y^{-1}$ in any local chart around $\mathbf{1}$ (Remark II.1.8). This implies that the curve

$$\gamma(t) := \alpha(\sqrt{t})\beta(\sqrt{t})\alpha(\sqrt{t})^{-1}\beta(\sqrt{t})^{-1}$$

with $\gamma(0) = \mathbf{1}$ is C^1 with $\gamma'(0) = [\alpha'(0), \beta'(0)]$.[1] We conclude that $\mathbf{L}^d(H)$ is a Lie subalgebra of $\mathbf{L}(G)$.

If, in addition, H is initial, then $C^1_*([0,1],H) = \{\alpha \in C^1_*([0,1],G)\colon \mathrm{im}(\alpha) \subseteq H\}$ implies that $\mathbf{L}(\iota_H)(\mathbf{L}(H)) = \mathbf{L}^d(H)$. □

[1] Note that in general this curve is not twice differentiable.

We put the superscript d (for differentiable) to distinguish $\mathbf{L}^d(H)$ from the Lie algebra of a Lie group. Later, we shall encounter another approach to the Lie algebra of a subgroup which works well for closed subgroups of locally exponential Lie groups.

Remark II.6.4. Let $\alpha \in C^1_*([0,1],G)$ and $H \subseteq G$ be a subgroup. If $\mathrm{im}(\alpha) \subseteq H$, then the image of the continuous curve $\delta(\alpha) \in C([0,1],\mathbf{L}(G))$ is contained in $\mathbf{L}^d(H)$. If, conversely, $\mathrm{im}(\delta(\alpha)) \subseteq \mathbf{L}^d(H)$, then it is not clear why this should imply that $\mathrm{im}(\alpha) \subseteq H$. We shall see below that the concept of regularity helps to deal with this problem. □

Remark II.6.5. The following facts demonstrate that it is not easy to find subgroups with no initial Lie subgroup structure ([Ne05, Lemma I.7]):

(a) Let $H \subseteq G$ be a subgroup such that all C^1-arcs in H are constant. Then the discrete topology defines on H an initial Lie subgroup structure.

(b) If $\dim G < \infty$, then any subgroup $H \subseteq G$ carries an initial Lie group structure: According to Yamabe's Theorem ([Go69]), the arc-component H_a of G is of the form $\langle \exp \mathfrak{h} \rangle$ for some Lie subalgebra $\mathfrak{h} \subseteq \mathbf{L}(G)$, which can be identified with $\mathbf{L}^d(H)$. To obtain the initial Lie group structure on H, we endow H_a with its intrinsic Lie group structure and extend it with Corollary II.2.3 to all of H.

(c) If the connected Lie group G has a smooth exponential function, the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g} = \mathbf{L}(G)$ is Mackey complete, and the subgroup $\Gamma_{\mathfrak{z}(\mathfrak{g})} := \exp_G^{-1}(\mathbf{1}) \cap \mathfrak{z}(\mathfrak{g})$ is discrete, then $Z(G)$ carries an initial Lie group structure with Lie algebra $\mathfrak{z}(\mathfrak{g})$.

We endow $\exp_G(\mathfrak{z}(\mathfrak{g})) \cong \mathfrak{z}(\mathfrak{g})/\Gamma_{\mathfrak{z}(\mathfrak{g})}$ with the quotient Lie group structure (Corollary II.2.4) and use Corollary II.2.3 to extend it to all of $Z(G)$. □

Remark II.6.6. If $H \subseteq G$ is a Lie subgroup in the sense of Remark II.2.5(b), then some identity neighborhood of H is a submanifold of G and its intrinsic Lie group structure turns H into an initial Lie subgroup of G. □

Open Problems for Section II

Problem II.1. Show that every Lie group G modeled on a Mackey complete locally convex space has a smooth exponential function, or find a counterexample (*cf.* Example II.5.5). □

The following assertion is even stronger:

Problem II.2. ([Mil84]) Show that every Lie group G modeled on a Mackey complete locally convex space is regular, or find a counterexample.

The assumption of Mackey completeness of $\mathbf{L}(G)$ is necessary because for any regular Lie group the differential of the evolution map $\mathrm{evol}_G\colon C^\infty([0,1],\mathfrak{g}) \to G$ is given by

$$T_0(\mathrm{evol}_G)\xi = \int_0^1 \xi(t)\,dt.$$

Therefore the regularity of G implies the Mackey completeness of $\mathbf{L}(G)$ (*cf.* Proposition II.5.6). □

Problem II.3. ([Mil84]) Show that two 1-connected Lie groups G with isomorphic Lie algebras are isomorphic. For groups with Mackey complete Lie algebras, this would follow from Theorem III.1.5 and a positive solution to Problem II.2. □

Problem II.4. Prove or disprove the following claims for all Lie groups G with a smooth exponential function $\exp_G\colon \mathfrak{g} = \mathbf{L}(G) \to G$:

(1) 0 is isolated in $\exp_G^{-1}(\mathbf{1})$.
(2) 0 is isolated in $\Gamma_{\mathfrak{z}(\mathfrak{g})} := \exp_G^{-1}(\mathbf{1}) \cap \mathfrak{z}(\mathfrak{g})$, where $\mathfrak{z}(\mathfrak{g})$ denotes the center of $\mathfrak{g}$.

In view of Remark II.6.5(c), a solution of (2) would be of particular interest to classify classes of extensions of Lie groups by non-abelian Lie groups (*cf.* Theorem V.1.5). Note that (2) is equivalent to the discreteness of the group $\Gamma_{\mathfrak{z}(\mathfrak{g})}$. We know that all C^1-curves in this closed subgroup of $\mathfrak{z}(\mathfrak{g})$ are constant and that all intersections with finite-dimensional subspaces are discrete (Proposition II.5.11(5)). □

Problem II.5. (Small Torsion Subgroup Problem; (FP8)) Show that for any Lie group G there exists an identity neighborhood U such that $\mathbf{1}$ is the only element of finite order generating a subgroup lying in U.

If the answer to Problem II.4(1) is negative for some Lie group G, then each identity neighborhood contains the range of a homomorphism $\mathbb{T} \cong \mathbb{R}/\mathbb{Z} \to G$ obtained by $\exp_G(\mathbb{R}x)$ for $x \in \exp_G^{-1}(\mathbf{1})$ sufficiently close to 0. This implies in particular that each identity neighborhood contains non-trivial torsion subgroups.

It is a classical result that Banach–Lie groups do not contain small subgroups, i.e., there exists a $\mathbf{1}$-neighborhood U for which $\{\mathbf{1}\}$ is the only subgroup contained in U. This is no longer true for locally convex vector groups, such as $G = \mathbb{R}^{\mathbb{N}}$, with the product topology. Then each 0-neighborhood contains nonzero vector subspaces, so that G has small subgroups. However, G is torsion free.

For a locally convex space E, the non-existence of small subgroups is equivalent to the existence of a continuous norm on E. Every locally exponential Lie group G for which $\mathbf{L}(G)$ has a continuous norm has no small subgroups (*cf.* Section IV). Since any real vector space is torsion free, this implies that no locally exponential Lie group contains small torsion subgroups. For strong ILB–Lie groups, it is also known that they do not contain small subgroups (*cf.* Theorem III.2.3), and this implies in particular that for each compact manifold M the group $\mathrm{Diff}(M)$ does not contain small subgroups. We also know that direct limits of finite-dimensional Lie groups do not contain small subgroups (Theorem VII.1.3). □

Problem II.6. (Initial Subgroup Problem) Give an example of a subgroup H of some infinite-dimensional Lie group which does not possess any initial Lie subgroup structure.

We think that such examples exist, but in view of Remark II.6.5(b), there is no such example in any finite-dimensional Lie group. Moreover, Theorem IV.4.17 below implies that all closed subgroups of Banach–Lie groups carry initial Lie subgroup structures. Therefore the most natural candidates of groups to consider are (non-closed) subgroups of Banach spaces which are connected by smooth arcs. For $E := C([0,1],\mathbb{R})$, the subgroup $H \subseteq E$ generated by the smooth curve $\gamma\colon [0,1] \to E, \gamma(t)(x) := e^{tx} - 1$ is a natural candidate. Since the values $\gamma(t)$ for $t > 0$ are linearly independent, $H = \sum_{t\in]0,1]} \mathbb{Z}\gamma(t)$ is a free abelian group. □

Problem II.7. (Canonical factorization for Lie groups) Let $\varphi\colon G \to H$ be a morphism of Lie groups. Does the quotient group $G/\ker\varphi \cong \varphi(G) \subseteq H$ carry a natural Lie group structure for which the induced map $G/\ker(\varphi) \to H$ is smooth and each other morphism $\psi\colon G \to H'$ with $\ker\varphi \subseteq \ker\psi$ factors through $G/\ker(\varphi)$? Does $\varphi(G)$ carry the structure of an initial Lie subgroup of H? Maybe it helps to assume that G is a regular Lie group (*cf.* Section III below).□

Problem II.8. (Locally Compact Subgroup Problem; (FP9)) Show that any locally compact subgroup of a Lie group G is a (finite-dimensional) Lie group. Since locally compact subgroups are Lie groups if and only if they have no small subgroups, this is closely related to Problem II.5. We shall see below that this problem has a positive solution for most classes of concrete groups (*cf.* Theorem IV.3.15 for locally exponential Lie groups; [MZ55, Th. 5.2.2, p.208] for the Lie group $\mathrm{Diff}(M)$ of diffeomorphisms of a compact manifold, and Theorem VII.1.3 for direct limits of finite-dimensional Lie groups). □

Problem II.9. (Completeness of Lie groups) Suppose that the Lie algebra $\mathbf{L}(G)$ of the Lie group G is a complete locally convex space. Does this imply that the group G is complete with respect to the left, resp., right uniform structure? □

Problem II.10. (Large tori in Lie groups) Suppose that G is a Lie group with a smooth exponential function and that $\mathfrak{a} \subseteq \mathbf{L}(G)$ is a closed abelian subalgebra for which the closed subgroup $\Gamma_{\mathfrak{a}} := \exp_G^{-1}(\mathbf{1}) \cap \mathfrak{a}$ spans a dense subspace of

$\mathfrak{a}$. Then the exponential function $\mathrm{Exp}_{\mathfrak{a}} := \exp_G|_{\mathfrak{a}} : \mathfrak{a} \to G$ factors through a continuous map $\mathfrak{a}/\Gamma_{\mathfrak{a}} \to G$. Characterize the groups $A := \mathfrak{a}/\Gamma_{\mathfrak{a}}$ for which this may happen.

If $\mathfrak{a}$ is finite-dimensional, then A is a torus (Proposition II.5.11(5)), so that we may think of these groups A as *generalized tori*. If $\Gamma_{\mathfrak{a}}$ is discrete, then A is a Lie group. If, in addition, $\mathfrak{a}$ is separable, then $\Gamma_{\mathfrak{a}}$ is a free group ([Ne02a, Rem. 9.5(c)]). If Problem II.4 has a positive solution, then $\Gamma_{\mathfrak{a}}$ is always discrete.

An interesting example in this context is $E = \mathbb{R}^{\mathbb{N}}$ with the closed subgroup $\Gamma_E := \mathbb{Z}^{\mathbb{N}}$. In this case, the quotient $E/\Gamma_E \cong \mathbb{T}^{\mathbb{N}}$ is the compact torus which is not a Lie group because it is not locally contractible. Do pairs $(\mathfrak{a}, \Gamma_{\mathfrak{a}}) \cong (\mathbb{R}^{\mathbb{N}}, \mathbb{Z}^{\mathbb{N}})$ occur? For the free vector space $E = \mathbb{R}^{(\mathbb{N})}$ over $\mathbb{N}$, the subgroup $\Gamma_E := \mathbb{Z}^{(\mathbb{N})}$ is discrete and E/Γ_E is a Lie group, a direct limit of finite-dimensional tori (*cf.* Theorem VII.1.1). □

Problem II.11. Does the adjoint group $\mathrm{Ad}(G) \subseteq \mathrm{Aut}(\mathbf{L}(G))$ of a Lie group G always carry a natural Lie group structure for which the adjoint representation $\mathrm{Ad} : G \to \mathrm{Ad}(G)$ is a quotient morphism of topological groups? Since, in general, the group $\mathrm{Aut}(\mathbf{L}(G))$ is not a Lie group if $\mathbf{L}(G)$ is not Banach, this does not follow from a positive solution of Problem II.7. Closely related is the question if $\exp_G x \in Z(G)$ for x small enough implies $x \in \mathfrak{z}(\mathfrak{g})$. □

Problem II.12. Let G be a connected Lie group with a smooth exponential function and $\mathfrak{a} \subseteq \mathbf{L}(G)$ a Mackey complete abelian subalgebra for which the group $\Gamma_{\mathfrak{a}} := \exp_G^{-1}(\mathbf{1}) \cap \mathfrak{a}$ is discrete. Then $\exp_G|_{\mathfrak{a}}$ factors through an injective smooth map $A := \mathfrak{a}/\Gamma_{\mathfrak{a}} \hookrightarrow G$ and A carries a natural Lie group structure (Corollary II.2.4). Is this Lie group always initial? According to Remark II.6.5(c), this is the case for $\mathfrak{a} = \mathfrak{z}(\mathfrak{g})$. □

III. Regularity

In this section, we discuss regularity of Lie groups in some more detail. In particular, we shall see how regularity of a Lie group can be used to obtain a Fundamental Theorem of Calculus for Lie group-valued smooth functions. This implies solutions to many integrability questions. For example, for each homomorphism $\psi : \mathbf{L}(G) \to \mathbf{L}(H)$ from the Lie algebra of a 1-connected Lie group G into the Lie algebra of a regular Lie group H, there exists a unique morphism of Lie groups φ with $\mathbf{L}(\varphi) = \psi$. In Section III.2, we turn to the concepts of strong ILB–Lie groups and μ-regularity and their relation to our context. In the remaining two subsections III.3 and III.4, we discuss some applications to groups of diffeomorphisms and groups of smooth maps, resp., gauge groups.

III.1. The Fundamental Theorem for Lie group-valued functions

Definition III.1.1. *Let G be a Lie group with Lie algebra $\mathfrak{g} = \mathbf{L}(G)$. We call a $\mathfrak{g}$-valued 1-form $\alpha \in \Omega^1(M,\mathfrak{g})$* integrable *if there exists a smooth function $f\colon M \to G$ with $\delta(f) = \alpha$. The 1-form α is said to be* locally integrable *if each point $m \in M$ has an open neighborhood U such that $\alpha|_U$ is integrable.* □

We recall from Definition I.4.1(b) the brackets $\Omega^p(M,\mathfrak{g}) \times \Omega^q(M,\mathfrak{g}) \to \Omega^{p+q}(M,\mathfrak{g})$. If f is a solution of the equation $\delta(f) = f^*\kappa_G = \alpha \in \Omega^1(M,\mathfrak{g})$, then the fact that κ_G satisfies the Maurer–Cartan equation $d\kappa_G + \frac{1}{2}[\kappa_G,\kappa_G] = 0$ implies that so does α:

$$(MC) \qquad d\alpha + \frac{1}{2}[\alpha,\alpha] = 0.$$

The following theorem is a version of the Fundamental Theorem of Calculus for functions with values in regular Lie groups ([GN06]).

Theorem III.1.2. (Fundamental Theorem for Lie group-valued functions) *Let M be a smooth manifold, G a Lie group and $\alpha \in \Omega^1(M,\mathbf{L}(G))$. Then the following assertions hold:*

(1) If G is regular and α satisfies the Maurer–Cartan equation, then α is locally integrable.
(2) If M is 1-connected and α is locally integrable, then it is integrable.
(3) If M is connected, $m_0 \in M$, and α is locally integrable, then there exists a homomorphism
$$\mathrm{per}_\alpha\colon \pi_1(M,m_0) \to G$$
that vanishes if and only if α is integrable. For a piecewise smooth representative $\sigma\colon [0,1] \to M$ of a loop in M, the element $\mathrm{per}_\alpha([\sigma])$ is given by $\gamma(1)$ for $\gamma\colon [0,1] \to G$ satisfying $\delta(\gamma) = \sigma^*\alpha$. □

Remark III.1.3. If M is one-dimensional, then each $\mathfrak{g}$-valued 2-form on M vanishes, so that $[\alpha,\beta] = 0 = d\alpha$ for $\alpha,\beta \in \Omega^1(M,\mathfrak{g})$. Therefore all 1-forms trivially satisfy the Maurer–Cartan equation. □

This remark applies in particular to the manifold with boundary $M = I = [0,1]$. The requirement that for each smooth curve $\xi \in C^\infty(I,\mathfrak{g}) \cong \Omega^1(I,\mathfrak{g})$, the IVP

$$\gamma(0) = \mathbf{1}, \quad \gamma'(t) = \gamma(t).\xi(t) \quad \text{for} \quad t \in I,$$

has a solution depending smoothly on ξ leads to the concept of a regular Lie group.

Remark III.1.4. (a) If M is a complex manifold, G is a complex Lie group and $\alpha \in \Omega^1(M,\mathfrak{g})$ is a holomorphic 1-form, then for any smooth function $f\colon M \to G$

with $\delta(f) = \alpha$, the differential of f is complex linear in each point, so that f is holomorphic. Conversely, the logarithmic derivative of any holomorphic function f is a holomorphic 1-form.

If, in addition, M is a one-dimensional complex manifold, then for each holomorphic 1-form $\alpha \in \Omega^1(M, \mathfrak{g})$ the 2-forms $d\alpha$ and $[\alpha, \alpha]$ are holomorphic, which implies that they vanish. Therefore the Maurer–Cartan equation is automatically satisfied by all holomorphic 1-forms. □

The following theorem is one of the main motivations for introducing the notion of regularity. It was proved in [OMYK82] under the stronger assumption of μ-regularity (*cf.* Subsection III.2 below) and by Milnor (who attributed it to Thurston) in the following form ([Mil82/84]):

Theorem III.1.5. *If H is a regular Lie group, G is a* 1*-connected Lie group, and $\varphi\colon \mathbf{L}(G) \to \mathbf{L}(H)$ is a continuous homomorphism of Lie algebras, then there exists a unique Lie group homomorphism $f\colon G \to H$ with $\mathbf{L}(f) = \varphi$.*

Proof. This is Theorem 8.1 in [Mil84] (see also [KM97, Th. 40.3]). The uniqueness assertion follows from Proposition II.4.1 and does not require the regularity of H.

On G, we consider the smooth $\mathbf{L}(H)$-valued 1-form $\alpha := \varphi \circ \kappa_G$ and it is easily verified that α satisfies the MC equation. Therefore the Fundamental Theorem implies the existence of a unique smooth function $f\colon G \to H$ with $\delta(f) = \alpha$ and $f(\mathbf{1}_G) = \mathbf{1}_H$. In view of Proposition II.4.1(3), the function f is a homomorphism of Lie groups with $\mathbf{L}(f) = \alpha_{\mathbf{1}} = \varphi$. □

Corollary III.1.6. *If G_1 and G_2 are regular* 1*-connected Lie groups with isomorphic Lie algebras, then G_1 and G_2 are isomorphic.* □

Corollary III.1.7. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{n} \trianglelefteq \mathfrak{g}$ a closed ideal which is not* $\mathrm{Ad}(G)$*-invariant. Then the quotient Lie algebra $\mathfrak{g}/\mathfrak{n}$ is not integrable to a regular Lie group.*

Proof. If Q is a regular Lie group with Lie algebra $\mathfrak{q} := \mathfrak{g}/\mathfrak{n}$, then the quotient map $q\colon \mathfrak{g} \to \mathfrak{n}$ integrates to a morphism of Lie groups $\varphi\colon \widetilde{G} \to Q$ with $\mathbf{L}(\varphi) = q$ (Theorem III.1.5), so that $\mathfrak{n} = \ker(\mathbf{L}(\varphi))$, contradicting its non-invariance under $\mathrm{Ad}(\widetilde{G}) = \mathrm{Ad}(G)$. □

Remark III.1.8. Let G be a regular Lie group and $\mathfrak{h} \subseteq \mathbf{L}(G)$ a closed Lie subalgebra. Let $\iota\colon H \to G$ be a regular connected initial Lie subgroup of G with $\mathbf{L}^d(H) = \mathfrak{h}$. Then for each smooth curve $\gamma\colon I \to H$ the curve $\delta(\gamma)$ has values in $\mathfrak{h}$, and, conversely, for any smooth curve $\xi\colon I \to \mathfrak{h}$, the regularity of H and the Uniqueness Lemma imply that the corresponding curve γ_ξ has values in H. Hence H coincides with the set of endpoints of all curves γ_ξ, $\xi \in C^\infty(I, \mathfrak{h})$. In particular, H is uniquely determined by the Lie algebra $\mathfrak{h}$. □

For the groups of smooth maps on a compact manifold, it is quite easy to find charts of the corresponding mapping groups, such as $C^\infty(M,K)$, by composing with charts of K (Theorem II.2.8). This does no longer work for non-compact manifolds, as the discussion in Remark II.2.9(a) shows. The Fundamental Theorem implies that for any regular Lie group K with Lie algebra $\mathfrak{k}$, any 1-connected manifold M, $m_0 \in M$ and

$$C^\infty_*(M,K) := \{f \in C^\infty(M,K) \colon f(m_0) = \mathbf{1}\},$$

the map

$$\delta \colon C^\infty_*(M,K) \to \{\alpha \in \Omega^1(M,\mathfrak{k}) \colon d\alpha + \tfrac{1}{2}[\alpha,\alpha] = 0\}$$

is a bijection, which can be shown to be a homeomorphism. If the solution set of the MC equation carries a natural manifold structure, we thus obtain a manifold structure on the group $C^\infty_*(M,K)$ and hence on $C^\infty(M,K)$. This is the case if K is abelian, M is one-dimensional (all 2-forms vanish), and for holomorphic 1-forms on complex one-dimensional manifolds (*cf.* Remark III.1.4). Following this strategy and using Glöckner's Implicit Function Theorem to take care of the period conditions if M is not simply connected, we get the following result ([NeWa06b]). To formulate the real and complex case in one statement, let $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, K be a $\mathbb{K}$-Lie group, and $C^\infty_{\mathbb{K}}(M,K)$ be the group of $\mathbb{K}$-smooth K-valued maps. For $\mathbb{K} = \mathbb{C}$, these are the holomorphic maps, and in this case the smooth C^∞-topology on $C^\infty_{\mathbb{C}}(M,K) = \mathscr{O}(M,K)$ coincides with the compact open topology.

Theorem III.1.9. *Let K be a regular $\mathbb{K}$-Lie group and M a finite-dimensional connected σ-compact $\mathbb{K}$-manifold. We endow the group $C^\infty_{\mathbb{K}}(M,K)$ with the compact open C^∞-topology, turning it into a topological group. This topology is compatible with a Lie group structure if*

(1) $\dim_{\mathbb{K}} M = 1$, *$\pi_1(M)$ is finitely generated and K is a Banach–Lie group.*
(2) $H^1_{\rm sing}(M,\mathbb{Z})$ *is finitely generated and K is abelian.*
(3) $H^1_{\rm sing}(M,\mathbb{Z})$ *is finitely generated and K is finite-dimensional and solvable.*
(4) K is diffeomorphic to a locally convex space. □

III.2. Strong ILB–Lie groups and μ-regularity

An important criterion for regularity of a Lie group rests on the concept of a (strong) ILB–Lie group, a concept developed by Omori by abstracting a common feature from groups of smooth maps and diffeomorphism groups ([Omo 74]). The bridge from ILB–Lie groups to Milnor's regularity concept was built in [OMYK82], where even a stronger regularity concept, called μ-regularity below, is used. In this subsection, we explain some of the key results concerning μ-regularity and how they apply to diffeomorphism groups. In his book

[Omo97], Omori works with a slight variant of the axiomatics of μ-regularity, as defined below, but since it is quite close to the original concept, we shall not go into details on this point.

Definition III.2.1. (ILB–Lie groups; [Omo74, p.2])

(a) An ILB chain *is a sequence* $(E_n)_{n\geq d}$*,* $d \in \mathbb{N}$*, of Banach spaces with continuous dense inclusions* $\eta_n\colon E_n \hookrightarrow E_{n+1}$*. The projective limit* $E := \varprojlim E_n$ *of this system is a Fréchet space.*

Realizing E *as* $\{(x_n)_{n\geq d} \in \prod_{n\geq d} E_n\colon (\forall n)\ \eta_n(x_n) = x_{n+1}\}$*, we see that for each* $k \geq d$ *the projection map*

$$q_k\colon E \to E_k, \quad (x_n) \mapsto x_k$$

is injective. We may therefore think of E *and all spaces* E_n *as subspaces of* E_d*, which leads to the identification of* E *with the intersection* $\bigcap_{n\geq d} E_n$*.*

(b) A topological group G *is called an* ILB–Lie group modeled on the ILB chain $(E_n)_{n\geq d}$ *if there exists a sequence of topological groups* G_n*,* $n \geq d$*, satisfying the conditions (G1)–(G7) below. If, in addition, (G8) holds, then* G *is called a* strong ILB–Lie group.

- *(G1)* G_n *is a smooth Banach manifold modeled on* E_n*.*
- *(G2)* G_{n+1} *is a dense subgroup of* G_n *and the inclusion map* $G_{n+1} \hookrightarrow G_n$ *is smooth.*
- *(G3)* $G = \varprojlim G_n$ *as topological groups, so that we may identify* G *with* $\bigcap_{n\geq d} G_n \subseteq G_d$*.*
- *(G4) The group multiplication of* G *extends to a* C^ℓ*-map* $\mu_G^{n,\ell}\colon G_{n+\ell} \times G_n \to G_n$*.*
- *(G5) The inversion map of* G *extends to a* C^ℓ*-map* $G_{n+\ell} \to G_n$*.*
- *(G6) The right translations in the groups* G_n *are smooth.*
- *(G7) The tangent map* $T(\mu_G^{n,\ell})$ *induces a* C^ℓ*-map* $T_{\mathbf{1}}(G_{n+\ell}) \times G_n \to T(G_n)$*.*
- *(G8) There exists a chart* (φ_d, U_d) *of* G_d *with* $\mathbf{1} \in U_d$ *and* $\varphi_d(\mathbf{1}) = 0$ *such that* $U_n := U_d \cap G_n$ *and* $\varphi_n := \varphi_d|_{U_n}$ *define an* E_n*-chart* (φ_n, U_n) *of* G_n*.*

If all spaces E_n *are Hilbert, we call* $(E_n)_{n\geq d}$ *an* ILH chain *and* G *an* ILH–Lie group. □

Remark III.2.2. (Omori) Every strong ILB–Lie group G carries a natural Fréchet–Lie group structure with $\mathbf{L}(G) \cong \bigcap_{n\geq d} E_n = E$. A chart (φ, U) in the identity is obtained by $U := G \cap U_d$ and $\varphi := \varphi_d|_U$ (notation as in (G8)). □

A complete solution to (FP8) for a large class of Lie groups is provided by:

Theorem III.2.3. ([Omo74, Th.1.4.2]) *Strong ILB-Lie groups have no small subgroups, i.e., there exists an identity neighborhood containing no non-trivial subgroups.* □

We now give the slightly involved definition of the regularity concept introduced in [OMYK82/83a].

Definition III.2.4. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\Delta := \{t_0, \ldots, t_m\}$ a division of the real interval $J := [a,b]$ with $a = t_0$ and $b = t_m$. We write*

$$|\Delta| := \max\{t_{j+1} - t_j : j = 0, \ldots, m-1\}.$$

For $|\Delta| \leq \varepsilon$, a pair (h, Δ) is called a step function *on $[0,\varepsilon] \times J$ if $h\colon [0,\varepsilon] \times J \to G$ is a map satisfying*

(1) $h(0,t) = \mathbf{1}$ for all $t \in J$ and all maps $h^t(s) := h(s,t)$ are C^1.
(2) $h(s,t) = h(s,t_j)$ for $t_j \leq t < t_{j+1}$.

For a step function (h, Δ), we define the product integral *$\prod_a^t (h, \Delta) \in G$ by*

$$\prod_a^t (h,\Delta) := h(t - t_k, t_k) h(t_k - t_{k-1}, t_{k-1}) \cdots h(t_1 - t_0, t_0) \quad \textit{for} \quad t_k \leq t < t_{k+1}.$$

Now let (h_n, Δ_n) be a sequence of step functions with $|\Delta_n| \to 0$ for which the sequence $(h_n, \frac{\partial h_n}{\partial s})$ converges uniformly to a pair $(h, \frac{\partial h}{\partial s})$ for a function h: $[0,\varepsilon] \times J \to G$. Then the limit function h is a C^1-hair in $\mathbf{1}$*, i.e., it is continuous, differentiable with respect to s, and $\frac{\partial h}{\partial s}$ is continuous on $[0,\varepsilon] \times J$.*

The Lie group G is called μ-regular[2] *(called "regular" in [OMYK82/83a]) if the product integrals $\prod_a^t (h_n, \Delta_n)$ converges uniformly on J for each sequence (h_n, Δ_n) converging in the sense explained above to some C^1-hair in $\mathbf{1}$. Then the limit is denoted $\prod_a^t (h, d\tau)$ and called the* product integral of *h.* □

Remark III.2.5. The First Fundamental Theorem in [OMYK82] asserts that the product integral $\prod_a^t (h, d\tau)$ is C^1 with respect to t and satisfies

$$\frac{d}{dt} \prod_a^t (h, d\tau) = u(t) \cdot \prod_a^t (h, d\tau) \quad \text{for} \quad u(t) = \frac{\partial h}{\partial s}(0,t),$$

where $u \in C(J, \mathfrak{g})$ is a continuous curve. Hence the product integral is the unique C^1-curve $\gamma_u\colon J \to G$ with $\gamma_u(a) = \mathbf{1}$ and $\delta^r(\gamma_u) = u$. The Second Fundamental Theorem in [OMYK82] is that the right logarithmic derivative

$$\delta^r \colon C^1_*(J,G) \to C(J,\mathfrak{g})$$

is a C^∞-diffeomorphism, where $C^1_*(J,G)$ is the group of C^1-paths $\gamma\colon J \to G$ with $\gamma(a) = \mathbf{1}$, endowed with the compact open C^1-topology (*cf.* Theorem II.2.8).

Since the inclusion map $C^\infty([0,1],\mathfrak{g}) \to C^0([0,1],\mathfrak{g})$ is continuous and the evaluation map $\mathrm{ev}_1 \colon C^1_*([0,1],G) \to G, \gamma \mapsto \gamma(1)$ is smooth, it follows in particular that each μ-regular Lie group is regular. □

[2] μ stands for "multiplicative."

Theorem III.2.6. ([OMYK82, Th. 6.9]) *Strong ILB-Lie groups are μ-regular, hence in particular regular.* □

Lemma III.2.7. ([OMYK83a, Lemma 1.1]) *In each μ-regular Fréchet–Lie group G, we have for each C^1-curve $\gamma\colon [0,1] \to G$ with $\gamma(0) = \mathbf{1}$ the relation*

$$\lim_{n\to\infty} \gamma\big(\tfrac{t}{n}\big)^n = \exp_G(t\gamma'(0)) \quad \textit{for} \quad 0 \leq t \leq 1. \qquad \square$$

Theorem III.2.8. ([OMYK83a, Th. 4.2]) *Let G be a μ-regular Fréchet–Lie group. For each closed finite-codimensional subalgebra $\mathfrak{h} \subseteq \mathbf{L}(G)$, there exists a connected Lie group H with $\mathbf{L}(H) = \mathfrak{h}$ and an injective morphism of Lie groups $\eta_{\mathfrak{h}}\colon H \to G$ for which $\mathbf{L}(\eta_{\mathfrak{h}})\colon \mathbf{L}(H) \to \mathbf{L}(G)$ is the inclusion of $\mathfrak{h}$.* □

Theorem III.2.9. ([OMYK83a, Prop. 6.6]) *Let M be a compact manifold, G a locally exponential μ-regular Fréchet–Lie group, $r \in \mathbb{N}_0 \cup \{\infty\}$, and $q\colon \mathbb{G} \to M$ a smooth fiber bundle whose fibers are groups isomorphic to G, for which the transition functions are group automorphisms. Then the group $C^r(M,\mathbb{G})$ of C^r-sections of this bundle is a group with respect to pointwise multiplication, and it carries a natural Lie group structure, turning it into a μ-regular Fréchet–Lie group.* □

A slightly weaker version of the preceding theorem can already be found in [Les68]. Note that it applies in particular to gauge groups of G-bundles over M. We have added the assumption that G is locally exponential because this is needed for the standard constructions of charts of the group $C^r(M,\mathbb{G})$ (*cf.* Theorem IV.1.12 below for gauge groups).

Theorem III.2.10. ([OMYK83a, Prop. 2.4]) *Let G be a μ-regular Fréchet–Lie group and $H \subseteq G$ a subgroup for which there exists an identity neighborhood U^H whose smooth arc-component of $\mathbf{1}$ is a submanifold of G. Then H carries the structure of an initial Lie subgroup* (Remark II.6.6) *which is μ-regular.* □

III.3. Groups of diffeomorphisms

As an important consequence of Theorem III.2.6, several classes of groups of diffeomorphisms are regular:

Theorem III.3.1. *Let M be a compact smooth manifold. Then the following groups carry natural structures of strong ILH–Lie groups, and hence are μ-regular:*

(1) $\mathrm{Diff}(M)$.
(2) $\mathrm{Diff}(M,\omega) := \{\varphi \in \mathrm{Diff}(M)\colon \varphi^*\omega = \omega\}$*, where ω is a symplectic 2-form on M.*

(3) $\mathrm{Diff}(M,\mu)$*, where* μ *is a volume form on* M*.*
(4) $\mathrm{Diff}(M,\alpha)$*, where* α *is a contact form on* M*.*

The corresponding Lie algebras are $\mathcal{V}(M)$*,* $\mathcal{V}(M,\omega) := \{X \in \mathcal{V}(M)\colon \mathcal{L}_X\omega = 0\}$*,* $\mathcal{V}(M,\mu)$*, resp.,* $\mathcal{V}(M,\alpha)$*.* □

The preceding results on the Lie group structure of groups of diffeomorphisms have a long history. The Lie group structure on $\mathrm{Diff}(M)$ for a compact manifold M has first been constructed by J. Leslie in [Les67], and Omori proved in [Omo70] that $\mathrm{Diff}(M)$ can be given the structure of a strong ILH–Lie group (*cf.* also [Eb68] for the ILH structure). Ebin and Marsden extended the ILH results to compact manifolds with boundary ([EM69/70]). Later expositions of this result can be found in [Gu77], [Mi80] and [Ham82]. The regularity of $\mathrm{Diff}(M)$ is proved in [Mil84/82] with direct arguments, not using ILB techniques.

In [Arn66], Arnold studies the group $\mathrm{Diff}(M,\mu)$, where μ is a volume form on the compact manifold M, as the configuration space of a perfect fluid. Arguing by analogy with finite-dimensional groups, he showed that, for a suitable right invariant Riemannian metric on this group, the Euler equation of a perfect fluid corresponds to the geodesic equation for a left invariant Riemannian metric on $\mathrm{Diff}(M,\mu)$. This was made rigorous by Marden and Abraham in [MA70]. For a more recent survey on this circle of ideas, we refer to [EMi99].

Using Hodge theory, Ebin and Marsden show in [EM70] that if ω either is a volume form or a symplectic form on a compact manifold M, then $\mathrm{Diff}(M,\omega)$ carries the structure of an ILH–Lie group (see also [Wei69]). They further show that the group $\mathrm{Diff}_+(M)$ of orientation preserving diffeomorphisms of M is diffeomorphic to the direct product $\mathrm{Diff}(M,\mu)\times \mathrm{Vol}_1(M)$, where $\mathrm{Vol}_1(M)$ denotes the convex set of volume forms of total mass 1 on M. This refines a result of Omori on the topological level (*cf.* [KM97, Th. 43.7]). For the symplectic case, a more direct proof of the regularity assertion can be found in [KM97, Th. 43.12], where it is also shown that $\mathrm{Diff}(M,\omega)$ is a submanifold of $\mathrm{Diff}(M)$.

In [EM70], one also finds that the following groups are ILH–Lie groups:

(1) $\mathrm{Diff}(M,N) := \{\varphi \in \mathrm{Diff}(M)\colon \varphi(N) = N\}$ and $\mathrm{Diff}_N(M) := \{\varphi \in \mathrm{Diff}(M)\colon \varphi|_N = \mathrm{id}_N\}$, where $N \subseteq M$ is a closed submanifold and M compact without boundary.
(2) $\mathrm{Diff}_{\partial M}(M) := \{\varphi \in \mathrm{Diff}(M)\colon (\forall x \in \partial M)\ \varphi(x) = x\}$, if M has a boundary.
(3) $\mathrm{Diff}(M,\mu)$ and $\mathrm{Diff}_{\partial M}(M,\mu) := \mathrm{Diff}(M,\mu)\cap \mathrm{Diff}_{\partial M}(M)$ for any volume form μ on M.
(4) If, in addition, $\omega = d\theta$ is an exact symplectic 2-form on M, then $\mathrm{Diff}_{\partial M}(M,\omega)$ and

$$\mathrm{Ham}(M,\omega) := \{\varphi \in \mathrm{Diff}_{\partial M}(M)\colon \varphi^*\theta - \theta \in B^1_{\mathrm{dR}}(M,\mathbb{R})\}$$

are ILH–Lie groups (see also Remark V.2.14(c)).

The following result of Michor ([Mi91]) concerns the Lie group structure of a gauge group in a setting where the gauge group is a Lie subgroup of a diffeomorphism group of a compact manifold.

Theorem III.3.2. *If $q\colon B \to M$ is a locally trivial fiber bundle over the compact manifold M with compact fiber F, then the gauge group* $\mathrm{Gau}(B)$ *is a split submanifold of the regular Fréchet–Lie group* $\mathrm{Diff}(B)$. □

III.4. Groups of compactly supported smooth maps and diffeomorphisms

In the preceding subsection, we discussed diffeomorphisms of compact manifolds. We now briefly take a look at the corresponding picture for compactly supported maps on σ-compact manifolds.

It is interesting that if M is a σ-compact finite-dimensional manifold, then for each locally convex space E, the space $C^\infty_c(M,E)$ has two natural topologies. The first one is the locally convex direct limit structure

$$C^\infty_c(M,E) = \varinjlim C^\infty_{M_n}(M,E),$$

where $(M_n)_{n\in\mathbb{N}}$ is an exhaustion of M, which for the case that E is Fréchet, defines an LF space structure on $C^\infty_c(M,E)$ (*cf.* Examples I.1.3 and Theorem II.2.8). The other locally convex topology is obtained by endowing for each $r \in \mathbb{N}$ the space $C^r_c(M,E)$ with the direct limit structure $\varinjlim C^r_{M_n}(M,E)$ and then topologize $C^\infty_c(M,E)$ as the projective limit $\varprojlim C^r_c(M,E)$ of these spaces. These two topologies do not coincide (*cf.* [Gl02b], see also [Gl06a]).

Similar phenomena occur for the space $C^\infty_c(M,\mathbb{E})$ of smooth compactly supported sections of a vector bundle $\mathbb{E} \to M$ whose fibers are locally convex spaces. In the context of Lie algebras, this problem affects the model spaces $C^\infty_c(M,\mathfrak{k})$ of the Lie groups $C^\infty_c(M,K)$ and the space $\mathcal{V}_c(M)$ of compactly supported vector fields on M. For the natural LF space structure on $\mathcal{V}_c(M)$, the corresponding Lie group structure on $\mathrm{Diff}_c(M)$ has been constructed by Michor in [Mi80, pp.39, 197], where he even endows $\mathrm{Diff}(M)$ with the Lie group structure for which $\mathrm{Diff}_c(M)$ is an open subgroup (Corollary II.2.3).

The following theorem complements Theorem II.2.8 in a natural way (*cf.* [GN06], based on [Gl02d]).

Theorem III.4.1. *Let M be a σ-compact finite-dimensional smooth manifold and K a regular Lie group. Both natural topologies turn $C^\infty_c(M,\mathbf{L}(K))$ into a topological Lie algebra. Accordingly, the group $C^\infty_c(M,K)$ carries two regular Lie group structures for which the Lie algebra is $C^\infty_c(M,\mathbf{L}(K))$, endowed with these two topologies. If M is non-compact, these two regular Lie groups are* not *isomorphic.* □

The corresponding result for diffeomorphism groups is proved by Glöckner in [Gl02b] (for corresponding statements without proof see also [Mil82]).

Theorem III.4.2. *Let M be a σ-compact finite-dimensional manifold. Both natural topologies turn $\mathcal{V}_c(M)$ into a topological Lie algebra, and the group* $\mathrm{Diff}_c(M)^{\mathrm{op}}$ *carries two corresponding regular Lie group structure turning $\mathcal{V}_c(M)$ into its Lie algebra. For M non-compact, these two regular Lie groups are not isomorphic.* □

Open Problems for Section III

Problem III.1. Show that every abelian Lie group G modeled on a Mackey complete locally convex space $\mathfrak{g}$ is regular.

We may w.l.o.g. assume that G is 1-connected (*cf.* Theorem V.1.8 below). Then the regularity of the additive group of $\mathfrak{g} = \mathbf{L}(G)$ (Proposition II.5.6) implies that $\mathrm{id}_{\mathfrak{g}}$ integrates to a smooth homomorphism $\mathrm{Log}_G \colon G \to \mathfrak{g}$ (Theorem III.1.5), so that the assumption implies the existence of a logarithm function, but it is not clear how to get an exponential function (*cf.* [Mil82, p.36]). One would like to show that Log_G is an isomorphism of Lie groups, but also weaker information would be of interest: Is Log_G surjective or injective?

If $H := \mathrm{im}(\mathrm{Log}_G)$ were a proper subgroup of $\mathfrak{g}$, it would be a strange object: Since $\mathbf{L}(\mathrm{Log}_G) = \mathrm{id}_{\mathfrak{g}}$, we have $\mathbf{L}^d(H) = \mathfrak{g}$ (Remark II.6.4). For any $\alpha \in C^1_*([0,1],H)$, the relation $\alpha'(0) = \lim_{n\to\infty} n\alpha(\frac{1}{n})$ implies that H is dense in $\mathfrak{g}$. Is H a vector space? Let

$$P := \{\xi \in C^\infty([0,1],\mathfrak{g}) \colon (\exists \gamma \in C^\infty([0,1],G)\, \delta(\gamma) = \xi\}.$$

For $\gamma(0) = \mathbf{1}$ and $\delta(\gamma) = \xi$, we then have $\mathrm{Log}_G(\gamma(1)) = \int_0^1 \xi(t)\,dt$. Therefore H is the image of the additive group P under the integration map. Is P a vector subspace of $C^\infty([0,1],\mathfrak{g})$? □

Problem III.2. Let G be a regular Lie group (not necessarily Fréchet or μ-regular). Show that for each closed finite-codimensional subalgebra $\mathfrak{h} \subseteq \mathbf{L}(G)$ there exists a connected Lie group H with $\mathbf{L}(H) = \mathfrak{h}$ and an injective morphism of Lie groups $\eta_{\mathfrak{h}} \colon H \to G$ for which $\mathbf{L}(\eta_{\mathfrak{h}}) \colon \mathbf{L}(H) \to \mathbf{L}(G)$ is the inclusion of $\mathfrak{h}$ (*cf.* Theorem III.2.8). □

Problem III.3. Let $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, M be a σ-compact finite-dimensiomal $\mathbb{K}$-manifold, and K a (finite-dimensional) $\mathbb{K}$-Lie group. We endow the group $C^\infty_{\mathbb{K}}(M,K)$ of K-valued $\mathbb{K}$-smooth functions $M \to K$ with the compact open C^∞-topology, turning it into a topological group. For $\mathbb{K} = \mathbb{C}$, this is the group of holomorphic functions and the compact open C^∞-topology coincides with the compact open topology (*cf.* [NeWa06b]). When is this topology on the group $C^\infty_{\mathbb{K}}(M,K)$ compatible with a Lie group structure? See Theorem III.1.9, for partial results in this direction. □

Problem III.4. Consider a topological group $G = \varprojlim G_j$ which is a projective limit of the Banach–Lie groups G_j (or more general locally exponential groups).

(1) Characterize the situations where G is locally exponential in the sense that it carries a (locally exponential) Lie group structure (*cf.* Remark IV.1.22). For the case where all G_j are finite-dimensional, this is done in [HoNe06] (*cf.* Theorem X.1.9).
(2) Can we say more in the special case $G = \prod_{j\in J} G_j$?
(3) Suppose that G carries a compatible Fréchet–Lie group structure. Does this imply that G is regular? (*cf.* [Ga97] for some results in this direction). □

Problem III.5. Let $\iota_j\colon H_j \to G$, $j = 1,2$, be two initial Lie subgroups of the Lie group G with $\mathbf{L}^d(H_1) = \mathbf{L}^d(H_2)$. Which additional assumptions are necessary to conclude that $H_1 = H_2$ as subgroups of G, hence that H_1 and H_2 are isomorphic as Lie groups? Note that Remark III.1.8 implies that this is the case if H_1 and H_2 are μ-regular or at least if the maps $\delta\colon C^1_*([0,1],H_j) \to C^0([0,1],\mathbf{L}^d(H_j))$ are surjective. □

IV. Locally exponential Lie groups

In this section, we turn to Lie groups with an exponential function $\exp_G\colon \mathbf{L}(G) \to G$ which is well-behaved in the sense that it maps a 0-neighborhood in $\mathbf{L}(G)$ diffeomorphically onto a $\mathbf{1}$-neighborhood in G. We call such Lie groups *locally exponential*.

This class of Lie groups has been introduced by Milnor in [Mil84][3], where one finds some of the basic results explained below. In [GN06], we devote a long chapter to this important class of infinite-dimensional Lie groups, which properly contains the class of BCH–Lie groups as those for which the BCH–series defines an analytic local multiplication on a 0-neighborhood in $\mathbf{L}(G)$ (*cf.* [Gl02c] for basic results in the BCH context). In particular, it contains all Banach–Lie groups, but also many other interesting types of groups such as unit groups of Mackey complete CIAs, groups of the form $C^\infty_c(M,K)$, where M is σ-compact and K is locally exponential, and moreover, all projective limits of nilpotent Lie groups. It therefore includes many classes of “formal” Lie groups. The appeal of this class is due to its large scope and the strength of the general

[3] In [Rob96/97], these groups are called “of the first kind.”

Lie theoretic results that can be obtained for these groups. Up to certain refinements of assumptions, a substantial part of the theory of Banach–Lie groups carries over to locally exponential groups.

One of the most important structural consequences of local exponentiality is that it provides canonical local coordinates given by the exponential function. This in turn permits us to develop a good theory of subgroups and there even is a characterization of those subgroups for which we may form Lie group quotients. Moreover, we shall see in Section VI below that integrability of a locally exponential Lie algebra (to be defined below) can be characterized similarly as for Banach algebras.

Not all regular Lie groups are locally exponential. The simplest examples can be found among groups of the form $G = E \rtimes_\alpha \mathbb{R}$ for a smooth $\mathbb{R}$-action on E (Example II.5.9). Another prominent example of a regular Lie group which is not locally exponential is the group $\mathrm{Diff}(\mathbb{S}^1)$ of diffeomorphisms of the circle (Example II.5.13).

IV.1. Locally exponential Lie groups and BCH–Lie groups

Definition IV.1.1. *We call a Lie group G* locally exponential *if it has a smooth exponential function* $\exp_G\colon \mathbf{L}(G) \to G$ *which is a local diffeomorphism in* 0, *i.e., there exists an open* 0*-neighborhood* $U \subseteq \mathbf{L}(G)$ *mapped diffeomorphically onto an open* $\mathbf{1}$*-neighborhood of* G*.*

A Lie group is called exponential *if, in addition,* $\exp_G$ *is a global diffeomorphism.* □

If $\exp_G\colon \mathbf{L}(G) \to G$ is an exponential function, then $T_0(\exp_G) = \mathrm{id}_{\mathbf{L}(G)}$ by definition. This observation is particularly useful in the finite-dimensional or Banach context, where it follows from the Inverse Function Theorem that $\exp_G$ is a local diffeomorphism in 0, so that we can use the exponential function to obtain charts around $\mathbf{1}$:

Proposition IV.1.2. *Banach–Lie groups are locally exponential.* □

We shall see below that a similar conclusion does not work for general Fréchet–Lie groups, because in this context there is no general Inverse Function Theorem. From that it follows that to integrate a Lie algebra homomorphism $\varphi\colon \mathbf{L}(G) \to \mathbf{L}(H)$ to a group homomorphisms, it is in general not enough to start with the prescription $\exp_G x \mapsto \exp_H \varphi(x)$ to obtain a local homomorphism, because $\exp_G(\mathbf{L}(G))$ need not be a $\mathbf{1}$-neighborhood in G (*cf.* Example II.5.9).

For Banach–Lie groups, the existence of "canonical" coordinates provided by the exponential map leads to a description of the local multiplication in a canonical form, given by the BCH series:

Definition IV.1.3. *For two elements x,y in a Lie algebra $\mathfrak{g}$, we define*

$$H_1(x,y) := x+y, \quad H_2(x,y) := \frac{1}{2}[x,y],$$

and for $n \geq 3$:

$$H_n(x,y) := \sum_{\substack{k,m\geq 0\\ p_i+q_i>0}} \frac{(-1)^k}{(k+1)(q_1+\cdots+q_k+1)} \frac{(\mathrm{ad}\,x)^{p_1}(\mathrm{ad}\,y)^{q_1}\cdots(\mathrm{ad}\,x)^{p_k}(\mathrm{ad}\,y)^{q_k}(\mathrm{ad}\,x)^m}{p_1!q_1!\cdots p_k!q_k!m!}y,$$

where the sum is extended over all summands with $p_1+q_1+\cdots+p_k+q_k+m+1=n$. The formal series $\sum_{n=1}^\infty H_n(x,y)$ is called the Baker–Campbell–Hausdorff series. □

There are many different looking ways to write the polynomials $H_n(x,y)$. We have chosen the one obtained from the integral formula

$$x*y = x+\int_0^1 \psi(e^{\mathrm{ad}\,x}e^{t\,\mathrm{ad}\,y})y\,dt, \tag{4.1.1}$$

where ψ denotes the analytic function $\psi(z) := \frac{z}{z-1}\log z$, defined in a neighborhood of 1. Formula (4.1.1) is valid for sufficiently small elements x and y in a Banach–Lie algebra, because we may use functional calculus in Banach algebras to make sense of $\psi(e^{\mathrm{ad}\,x}e^{\mathrm{ad}\,y})$ for x,y close to 0. Then the explicit expansion of the BCH series is obtained from the series expansion of ψ and the exponential series of $e^{\mathrm{ad}\,x}$ and $e^{\mathrm{ad}\,ty}$.

Remark IV.1.4. (History of the BCH series) In [SchF90a], F. Schur derived recursion formulas for the summands of the series describing the multiplication of a Lie group in canonical coordinates (i.e., in an exponential chart). He also proved the local convergence of the series given by this recursion relations, which can in turn be used to obtain the associativity of the BCH multiplication (*cf.* [BCR81, p.93], [Va84, Sect. 2.15]). His approach is quite close to our treatment of locally exponential Lie algebras in the sense that he derived the series from the Maurer–Cartan form by integration of a partial differential equation of the form $f^*\kappa_{\mathfrak{g}} = \kappa_{\mathfrak{g}}$ with $f(0) = x$, whose unique solution is the left multiplication $f = \lambda_x$ in the local group.

The BCH series was made more explicit by Campbell in [Cam97/98], and in [Hau06] Hausdorff approached the BCH series on a formal level, showing that the formal expansion of $\log(e^xe^y)$ can be expressed in terms of Lie polynomials. Part of his results had been obtained earlier by Baker ([Bak01/05]). See [Ei68] for a more recent short argument that all terms in the BCH series are Lie brackets. □

Definition IV.1.5. *A topological Lie algebra* $\mathfrak{g}$ *is called* BCH–Lie algebra *if there exists an open* 0*-neighborhood* $U \subseteq \mathfrak{g}$ *such that for* $x, y \in U$ *the BCH series*

$$\sum_{n=1}^{\infty} H_n(x,y)$$

converges and defines an analytic function $U \times U \to \mathfrak{g}, (x,y) \mapsto x * y$ *(cf. Definition I.2.1). In view of [Gl02a, 2.9], the analyticity of the product* $x * y$ *is automatic if* $\mathfrak{g}$ *is a Fréchet space.* □

Example IV.1.6. (a) If $\mathfrak{g}$ is a nilpotent locally convex Lie algebra of nilpotency class m, then the BCH series defines a polynomial multiplication

$$x * y = x + y + \frac{1}{2}[x,y] + \sum_{n \leq m} H_n(x,y)$$

on $\mathfrak{g}$. From the structure of the series it follows immediately that for $t, s \in \mathbb{R}$ and $x \in \mathfrak{g}$ we have

$$tx * sx = (t+s)x,$$

so that $(\mathfrak{g}, *)$ is an exponential nilpotent Lie group.

(b) If $\mathfrak{g}$ is a Banach–Lie algebra whose norm is submultiplicative in the sense that $\|[x,y]\| \leq \|x\| \cdot \|y\|$ for $x, y \in \mathfrak{g}$, then the BCH series $x * y = \sum_{n=1}^{\infty} H_n(x,y)$ converges for $\|x\|, \|y\| < \frac{1}{3}\log(\frac{3}{2})$ ([Bir38]). □

The following result is quite useful to show that certain Lie algebras are not BCH:

Theorem IV.1.7. (Robart's Criterion; [Rob04]) *If* $\mathfrak{g}$ *is a sequentially complete BCH–Lie algebra, then there exists a* 0*-neighborhood* $U \subseteq \mathfrak{g}$ *such that* $f(x,y) := \sum_{n=0}^{\infty} (\operatorname{ad} x)^n y$ *converges and defines an analytic function on* $U \times \mathfrak{g}$. □

On the global level we have the following result whose proof requires the uniqueness assertion from Theorem IV.2.8 below:

Theorem IV.1.8. *For a Lie group G the following are equivalent:*

(1) G is analytic with an analytic exponential function which is a local analytic diffeomorphism in 0*.*
(2) G is locally exponential and $\mathbf{L}(G)$ *is BCH.* □

In Examples IV.1.14(b) and IV.1.16 below, we describe an analytic Lie group with an analytic exponential function which is a smooth diffeomorphism, but such that $\mathbf{L}(G)$ is not BCH. This is a negative answer to a question raised in [Mil84, p.31].

Definition IV.1.9. *A group satisfying the equivalent conditions of the preceding theorem is called a* BCH–Lie group. □

Our introductory discussion now can be stated as:

Corollary IV.1.10. *Each Banach–Lie group is BCH.* □

The Lie group concept used in [BCR81] is stronger than our concept of a BCH–Lie group because additional properties of the Lie algebra are required, namely that it is a so-called AE–Lie algebra, a property which encodes the existence of certain seminorms, compatible with the Lie bracket.

The following two theorems show that many interesting classes of Lie groups are in fact BCH.

Theorem IV.1.11. *If A is a Mackey complete CIA, then its unit group $A^\times$ is BCH. If, in addition, A is sequentially complete, then $A^\times$ is regular.*

Proof. (Sketch) If A is a Mackey complete complex CIA, then the fact that $A^\times$ is open implies that for each $a \in A$ the spectrum $\mathrm{Spec}(a)$ is a compact subset of $\mathbb{C}$, and the holomorphic functional calculus works as for Banach algebras (*cf.* [Wae54a/b][4], [Al65], [Gl02b]). This provides an analytic exponential function, and on the open star-like subset

$$U := \{a \in A\colon\ \mathrm{Spec}(a) \cap]-\infty, 0] = \emptyset\} \subseteq A^\times$$

we likewise obtain an analytic logarithm function $\log\colon U \to A$. From that, local exponentiality for complex CIAs follows easily.

That the multiplication on $A^\times$ is analytic follows from its bilinearity on A, and the analyticity of the inversion is obtained from functional calculus, which in turn leads to the expansion by the Neumann series $(\mathbf{1} - x)^{-1} = \sum_{n=0}^\infty x^n$. We conclude that $A^\times$ is a BCH–Lie group.

The real case can be reduced to the complex case, because for each real CIA A its complexification $A_\mathbb{C}$ is a complex CIA ([Gl02b]).

Now assume that A is sequentially complete. For $u \in C([0,1], A)$ we want to solve the linear initial value problem

$$\gamma(0) = \mathbf{1}, \quad \gamma'(t) = \gamma(t)u(t). \tag{4.1.2}$$

According to an idea of T. Robart ([Rob04]), the BCH property of $A^\times$ implies that this can be done by Picard iteration:

$$\gamma_0(t) := \mathbf{1}, \quad \gamma_{n+1}(t) := \mathbf{1} + \int_0^t \gamma_n(\tau)u(\tau)\,d\tau,$$

which leads to

$$\gamma_n(t) = \mathbf{1} + \sum_{k=1}^n \int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_2} u(\tau_1)u(\tau_2)\cdots u(\tau_n)\ d\tau_1\,d\tau_2 \cdots d\tau_n.$$

[4] Waelbroeck even introduces a functional calculus in several variables for tuples in complete locally convex algebras which are not necessarily CIAs, but where spectra and resolvents satisfy certain regularity conditions.

Now one argues that the analyticity of the function $(\mathbf{1}-x)^{-1} = \sum_{n=0}^{\infty} x^n$ implies that all sums of the form $\sum_{n=0}^{\infty} x_{n1} \cdots x_{nn}$ converge for x_{ij} in some sufficiently small 0-neighborhood. A closer inspection of the limiting process implies that the limit curve $\gamma := \lim_{n\to\infty} \gamma_n$ is C^1, solves the initial value problem (4.1.2), and depends analytically on u. This implies the regularity of $A^\times$. □

Theorem IV.1.12. *If K is a locally exponential Lie group and $q\colon P \to M$ a smooth K-principal bundle over the σ-compact finite-dimensional manifold M, then the group $\mathrm{Gau}_c(P)$ of compactly supported gauge transformations is a locally exponential Lie group. In particular, the Lie group $C_c^\infty(M,K)$ is locally exponential.*

If, in addition, K is regular, then $\mathrm{Gau}_c(P)$ is regular and if K is BCH, then so is $\mathrm{Gau}_c(P)$.

Proof. (Sketch; *cf.* [GN06] and Theorem II.2.8) Let $\exp_K\colon \mathbf{L}(K) \to K$ be the exponential function of K and realize $\mathrm{Gau}(P)$ as the subgroup $C^\infty(P,K)^K$ of K-fixed points in $C^\infty(P,K)$ with respect to the K-action given by $(k.f)(p) := kf(p.k)k^{-1}$. Then we put

$$\begin{aligned}\mathfrak{gau}(P) &:= C^\infty(P,\mathbf{L}(K))^K \\ &= \{\xi \in C^\infty(M,\mathbf{L}(K))\colon (\forall p \in P)(\forall k \in K)\ \mathrm{Ad}(k).\xi(p.k) = \xi(p)\},\end{aligned}$$

and observe that for the group $G := \mathrm{Gau}_c(P)$ the map

$$\exp_G\colon \mathfrak{g} := \mathfrak{gau}_c(P) \to G, \quad \xi \mapsto \exp_K \circ \xi$$

is a local homeomorphism in 0. Using Theorem II.2.1, this can be used to define a Lie group structure on G. Then $\exp_G$ is an exponential function of G, and, by construction, it is a local diffeomorphism in 0. □

Various special cases of the preceding theorem can be found in the literature: [OMYK82], [Sch04] (for M compact, K finite-dimensional), [Mil84] (without proofs), [KM97, 42.21] (in the convenient setting), and [Gl02c], [Wo05a]). That $\mathrm{Gau}(P)$ is μ-regular if K is μ-regular follows from Theorem III.2.9.

Example IV.1.13. (Pro-nilpotent Lie groups) If $\mathfrak{g} = \varprojlim \mathfrak{g}_j$ is a projective limit of a family of nilpotent Lie algebras $(\mathfrak{g}_j)_{j\in J}$ (a so-called *pro-nilpotent Lie algebra*), then the corresponding connecting homomorphisms of Lie algebras are also morphisms for the corresponding group structures (Example IV.1.6(a)), so that $(\mathfrak{g},*) := \varprojlim (\mathfrak{g}_j,*)$ defines on $\mathfrak{g}$ a Lie group structure with $\mathbf{L}(\mathfrak{g},*) = \mathfrak{g}$. We thus obtain an exponential Lie group $G := (\mathfrak{g},*)$ with $\exp_G = \mathrm{id}_{\mathfrak{g}}$. This group is *pro-nilpotent* in the sense that it is a projective limit of nilpotent Lie groups.

Example IV.1.14. (Formal diffeomorphisms) (a) Important examples of pro-nilpotent Lie groups arise as certain groups of formal diffeomorphisms. We write $\mathrm{Gf}_n(\mathbb{K})$ for the group of formal diffeomorphisms of $\mathbb{K}^n$ fixing 0, where

$\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The elements of this group are represented by formal power series of the form

$$\varphi(x) = gx + \sum_{|\mathbf{m}|>1} c_{\mathbf{m}} x^{\mathbf{m}},$$

where $g \in \mathrm{GL}_n(\mathbb{K})$,

$$\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}_0^n, \quad |\mathbf{m}| := m_1 + \cdots + m_n, \quad x^{\mathbf{m}} := x_1^{m_1} \cdots x_n^{m_n}, \quad c_{\mathbf{m}} \in \mathbb{K}^n,$$

and the group operation is given by composition of power series. We call φ *pro-unipotent* if $g = \mathbf{1}$. It is easy to see that the pro-unipotent formal diffeomorphisms form a pro-nilpotent Lie group $\mathrm{Gf}_n(\mathbb{K})_1 = \varprojlim G_k$, where G_k is the finite-dimensional nilpotent group obtained by composing polynomials of the form

$$\varphi(x) = x + \sum_{1<|\mathbf{m}|\leq k} c_{\mathbf{m}} x^{\mathbf{m}}$$

modulo terms of order $> k$. The group $\mathrm{Gf}_n(\mathbb{K})$ of all formal diffeomorphisms of $\mathbb{K}^n$ fixing 0 is a semidirect product

$$\mathrm{Gf}_n(\mathbb{K}) \cong \mathrm{Gf}_n(\mathbb{K})_1 \rtimes \mathrm{GL}_n(\mathbb{K}), \tag{4.1.3}$$

where the group $\mathrm{GL}_n(\mathbb{K})$ of linear automorphisms acts by conjugation. As this action is smooth, $\mathrm{Gf}_n(\mathbb{K})$ is a Fréchet–Lie group.

These groups are μ-regular Lie groups (*cf.* [Omo80]): In view of the semidirect decomposition and the fact that μ-regularity is an extension property (Theorem V.1.8), it suffices to observe that pro-nilpotent Lie groups are μ-regular, which follows by an easy projective limit argument.

The group $\mathrm{Gf}_n(\mathbb{K})$ has been studied by Sternberg in [St61], where he shows in particular that for $\mathbb{K} = \mathbb{C}$ and $n = 1$ the elements

$$\varphi_m(x) = e^{\frac{2\pi i}{m}} x + p x^{m+1}, \quad m \in \mathbb{N} \setminus \{1\}, p \in \mathbb{C}^\times,$$

are not contained in the image of the exponential function. This is of particular interest because $\varphi_m \to \mathbf{1}$ in the Lie group $\mathrm{Gf}_n(\mathbb{C})$, so that the image of the exponential function in this group is not an identity neighborhood. A detailed analysis of the exponential function of this group can also be found in Lewis's paper [Lew39].

To see that φ_m is not in the image of the exponential function of $\mathrm{Gf}_n(\mathbb{C})$, it suffices to verify this in the finite-dimensional solvable quotient group $G_{m+1} \rtimes \mathbb{C}^\times$, i.e., modulo terms of order $m+2$. The subgroup $\mathbb{C}x^{m+1} \rtimes \mathbb{C}^\times$ is isomorphic to $\mathbb{C} \rtimes \mathbb{C}^\times$ with the multiplication

$$(z,w)(z',w') = (z + w^m z', ww')$$

and the exponential function

$$\exp(z,w) = \Big(\frac{e^{wm}-1}{wm}z, e^w\Big) = \Big(\frac{(e^w)^m-1}{wm}z, e^w\Big),$$

showing that φ_m is not contained in the exponential image of this subgroup. However, one can use Proposition II.5.11(3) to see that any element ξ with $\exp\xi = \varphi_m$ must be contained in the plane $\mathrm{span}\{x, x^{m+1}\}$. This completes the proof.

(b) For $\mathbb{K} = \mathbb{R}$ the identity component $\mathrm{Gf}_1(\mathbb{R})_0$ is exponential and analytic, but not BCH. For $n \geq 2$ the group $\mathrm{Gf}_n(\mathbb{R})$ is analytic, but not locally exponential. If a subgroup $H \subseteq \mathrm{GL}_n(\mathbb{R})$ consists of matrices with real eigenvalues, then the subgroup $\mathrm{Gf}_n(\mathbb{R})_1 \rtimes H \subseteq \mathrm{Gf}_n(\mathbb{R})$ is locally exponential ([Rob02, Ths. 6/7]). □

Example IV.1.15. Let $F = \mathbb{K}[x_1,\ldots,x_n]$ be the free associative algebra in n generators $S := \{x_1,\ldots,x_n\}$. Then F has a natural filtration

$$F_k := \mathrm{span}\{s_1\cdots s_m \colon s_i \in S, m \geq k\}.$$

Each quotient F/F_m is a finite-dimensional unital algebra, hence a CIA. Therefore the algebra $\widehat{F} := \varprojlim F/F_n$, which can be identified with the algebra of non-commutative formal power series in the generators $x_1,\ldots,x_n$, is a complete CIA ([GN06]).

We conclude that the unit group $\widehat{F}^\times$ is a BCH–Lie group (Theorem IV.1.11). Let $\varepsilon\colon \widehat{F} \to \mathbb{K}$ denote the homomorphism sending each x_i to 0. Then the normal subgroup $U := \mathbf{1} + \ker\varepsilon$ is a pro-nilpotent Lie group and $\widehat{F}^\times \cong U \rtimes \mathbb{K}^\times$. In particular, the exponential function

$$\mathrm{Exp}\colon \ker\varepsilon \to U = \mathbf{1} + \ker\varepsilon, \quad x \mapsto \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

is an analytic diffeomorphism with the analytic inverse $\mathrm{Log}(x) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x-\mathbf{1})^k$. We thus obtain on $\ker\varepsilon$ a global analytic multiplication

$$x * y := \mathrm{Log}(\mathrm{Exp}\,x\,\mathrm{Exp}\,y)$$

given by the BCH series, so that its values lie in the completion $\widehat{L}$ of the free Lie algebra L generated by $x_1,\ldots,x_n$, which is a closed Lie subalgebra of $\widehat{F}$. □

In Section XI below, we shall discuss more aspects of projective limits of finite- and infinite-dimensional Lie groups.

Example IV.1.16. We recall the group $G = E \rtimes_\alpha \mathbb{R}$ from Example II.5.9(b), where $E = \mathbb{R}^{\mathbb{N}}$ and $\alpha(t) = e^{tD}$ with the diagonal operator $D(z_n) = (nz_n)$. Then the Lie group structure on G is analytic and the explicit formula shows that

$$\exp_G\colon \mathbf{L}(G) \to G, \quad (v,t) \mapsto (\beta(t)v, t)$$

is analytic. Further, it is a smooth diffeomorphism whose inverse

$$\log_G\colon G \to \mathbf{L}(G), \quad (v,t) \mapsto (\beta(t)^{-1}v, t)$$

is smooth but not analytic. In fact, $\beta(t)^{-1}e_n = \frac{tn}{e^{nt}-1}e_n$, and for n fixed, the radius of convergence of the Taylor series of this function in 0 is $\frac{2\pi}{n}$. A similar argument shows that the corresponding global multiplication $x * y := \log_G(\exp_G(x)\exp_G(y))$ on $\mathbf{L}(G)$ is smooth but not analytic. With Robart's Criterion (Theorem IV.1.7), this follows from the fact that the power series

$$\sum_{k=0}^{\infty} t^k D^k e_n = (1-tn)^{-1}e_n$$

is not convergent for $|t| > \frac{1}{n}$. □

For details concerning the following results, we refer to [GN06] (see also [Mil82, 4.3] for some of the statements). Minor modifications of the corresponding argument for finite-dimensional, resp., Banach–Lie groups lead to the following lemma, which in turn is the key to the following theorem:

Lemma IV.1.17. *Let G be a locally exponential Lie group. For $x, y \in \mathbf{L}(G)$, we have the Trotter Product Formula*

$$\exp_G(x+y) = \lim_{n\to\infty}\Big(\exp_G\big(\frac{x}{n}\big)\exp_G\big(\frac{y}{n}\big)\Big)^n$$

and the Commutator Formula

$$\exp_G([x,y]) = \lim_{n\to\infty}\Big(\exp_G\big(\frac{x}{n}\big)\exp_G\big(\frac{y}{n}\big)\exp_G\big(-\frac{x}{n}\big)\exp_G\big(-\frac{y}{n}\big)\Big)^{n^2}. \quad □$$

Theorem IV.1.18. (Automatic Smoothness Theorem) *Each continuous homomorphism $\varphi\colon G \to H$ of locally exponential (BCH) Lie groups is smooth (analytic).* □

For (local) Banach–Lie groups, Theorem IV.1.18 can already be found in [Bir38], and for BCH–Lie groups in [Gl02c] (see also [Mil84] for the statement without proof). The special case of one-parameter groups $\mathbb{R} \to A^\times$, where A is a Banach algebra is due to Nagumo ([Nag36]) and Nathan ([Nat35]). Mostly such "automatic smoothness" theorems concern continuous homomorphisms $\varphi\colon G \to H$ of Lie groups, where H is a Lie group with an exponential function for which each continuous one-parameter group is of the form $\gamma_x(t) = \exp_H(tx)$ and G is locally exponential. We then obtain a map $\mathbf{L}(\varphi)\colon \mathbf{L}(G) \to \mathbf{L}(H)$ by $\varphi(\exp_G(tx)) = \exp_H(t\,\mathbf{L}(\varphi)x)$ for $t \in \mathbb{R}$ and $x \in \mathbf{L}(G)$, and then it remains to show that $\mathbf{L}(\varphi)$ is continuous and linear.

The following theorem is due to Maissen for Banach–Lie groups ([Mais62, Satz 10.3]):

Theorem IV.1.19. *Let G and H be Lie groups and $\psi\colon \mathbf{L}(G) \to \mathbf{L}(H)$ a continuous homomorphism of Lie algebras. Assume that G is locally exponential and 1-connected and that H has a smooth exponential function. Then there exists a unique morphism of Lie groups $\varphi\colon G \to H$ with $\mathbf{L}(\varphi) = \psi$.*

Proof. (Idea) Let $U_{\mathfrak{g}} \subseteq \mathfrak{g} = \mathbf{L}(G)$ be a convex balanced 0-neighborhood mapped diffeomorphically by the exponential function to an open subset U_G of G.

First one observes that $\psi^*\kappa_{\mathbf{L}(H)} = \psi \circ \kappa_{\mathbf{L}(G)}$ (*cf.* (2.5.5)). For the map

$$f\colon U_G \to H, \quad \exp_G(x) \mapsto \exp_H(\psi(x)),$$

this leads to $\delta(f) = f^*\kappa_H = \psi \circ \kappa_G$, showing that the $\mathbf{L}(H)$-valued 1-form $\psi \circ \kappa_G$ is locally integrable. Since this form on G is left invariant and G is 1-connected, it is globally integrable to a function $\varphi\colon G \to H$ with $\varphi(\mathbf{1}) = \mathbf{1}$ and $\delta(\varphi) = \psi \circ \kappa_G$ (Theorem III.1.2). Now Proposition II.4.1 implies that φ is a group homomorphism, and by construction $\mathbf{L}(\varphi) = \alpha_{\mathbf{1}} = \psi$. □

Since we do not know if all Lie groups with an exponential function are regular, the preceding theorem is not a consequence of Theorem III.1.5.

Corollary IV.1.20. *If G_1 and G_2 are locally exponential 1-connected Lie groups with isomorphic Lie algebras, then G_1 and G_2 are isomorphic.* □

Remark IV.1.21. It is instructive to compare Corollary IV.1.20 with the corresponding statement for regular Lie groups (Corollary III.1.6). They imply that there exists for each locally convex Lie algebra $\mathfrak{g}$ at most one 1-connected Lie group G which is regular and at most one 1-connected locally exponential Lie group H with $\mathbf{L}(G) = \mathbf{L}(H) = \mathfrak{g}$. The regularity of G implies that $\mathrm{id}_{\mathfrak{g}}$ integrates to a smooth homomorphism $\varphi\colon H \to G$, but we do not know if there is a morphism $\psi\colon G \to H$ with $\mathbf{L}(\psi) = \mathrm{id}_{\mathfrak{g}}$ (*cf.* Problem III.1).

Presently we do not know if all locally exponential Lie groups (modeled on Mackey complete spaces) are regular, therefore it is still conceivable that there might be locally exponential Lie algebras which are the Lie algebra of a 1-connected regular Lie group and a non-isomorphic 1-connected locally exponential Lie group which is not regular. □

Remark IV.1.22. Theorem IV.1.18 implies in particular that being a locally exponential Lie group is a *topological property*: Any topological group G carries at most one structure of a locally exponential Lie group. We thus adjust our terminology in the sense that we call a topological group *locally exponential* if it carries a locally exponential Lie group structure compatible with the topology.

Forgetting the differentiable structure on G, it becomes an interesting issue how to recover it. In view of Theorem IV.1.18, we recover the Lie algebra $\mathbf{L}(G)$, as a set, by identifying $x \in \mathbf{L}(G) \cong T_{\mathbf{1}}(G)$ with the corresponding one-parameter group $\gamma_x(t) = \exp_G(tx)$. Starting from G, as a topological group, we may then put $\mathfrak{L}(G) := \mathrm{Hom}_c(\mathbb{R}, G)$, the set of continuous homomorphisms $\mathbb{R} \to G$. The scalar multiplication of $\mathfrak{L}(G)$ can be written as

$$(4.1.4) \qquad (\lambda\alpha)(t) := \alpha(\lambda t), \quad \lambda \in \mathbb{R}, \alpha \in \mathrm{Hom}_c(\mathbb{R}, G),$$

and, in view of Lemma IV.1.17, addition and Lie bracket may be written on the level of one-parameter groups by

$$(4.1.5) \qquad (\alpha + \beta)(t) := \lim_{n\to\infty} \Big(\alpha(\frac{t}{n})\beta(\frac{t}{n})\Big)^n$$

and

$$(4.1.6) \qquad [\alpha, \beta](t^2) := \lim_{n\to\infty} \Big(\alpha(\frac{t}{n})\beta(\frac{t}{n})\alpha(-\frac{t}{n})\beta(-\frac{t}{n})\Big)^{n^2}.$$

We can also recover the topology on $\mathbf{L}(G)$ as the compact open topology on $\mathfrak{L}(G)$, and the exponential function as the evaluation map

$$(4.1.7) \qquad \exp_G\colon \mathrm{Hom}_c(\mathbb{R}, G) \to G, \quad \gamma \mapsto \gamma(1).$$

□

In [HoMo05/06], Hofmann and Morris use (4.1.4-7) as the starting point in the investigation of a remarkable class of topological groups:

Definition IV.1.23. *Let G be a topological group and* $\mathfrak{L}(G) := \mathrm{Hom}_c(\mathbb{R}, G)$ *the set of one-parameter groups, endowed with the compact open topology. Then G is said to be a* topological group with Lie algebra *if the limits in (4.1.5/6) exist for* $\alpha, \beta \in \mathrm{Hom}_c(\mathbb{R}, G)$ *and define elements of* $\mathfrak{L}(G)$*, addition and bracket are continuous maps* $\mathfrak{L}(G) \times \mathfrak{L}(G) \to \mathfrak{L}(G)$*, and* $\mathfrak{L}(G)$ *is a real Lie algebra with respect to the scalar multiplication (4.1.4), the addition (4.1.5), and the bracket (4.1.6). This implies that* $\mathfrak{L}(G)$ *is a topological Lie algebra. The exponential function of G is defined by (4.1.7).* □

In [BCR81], Boseck, Czichowski and Rudolph define smooth functions on a topological group in terms of restrictions to one-parameter groups, which leads them to (4.1.4-7), together with the assumption that $\mathfrak{L}(G)$ can be identified with the set of derivations of the algebra of germs of smooth functions in **1** ([BCR81, Sect. 1.5]).

We have just seen that any locally exponential Lie group is a topological group with Lie algebra. Since $\mathbb{R}$ is connected, a topological group G has a Lie algebra if and only if its identity component G_0 does. In [HoMo05, Th. 2.3], it is also observed that any abelian topological group is a group with Lie algebra, where the addition on $\mathfrak{L}(G)$ is pointwise multiplication and the bracket is trivial (*cf.* Problem IV.7).

Theorem IV.1.24. *Each* 2*-step nilpotent topological group has a Lie algebra.*

Proof. (Sketch) The commutator map $c\colon G \times G \to Z(G)$ is an alternating bihomomorphism. Then direct calculations lead to the formulas

$$(\alpha+\beta)(t) = \alpha(t)\beta(t)c(\alpha(t),\beta(-\tfrac{t}{2})) \quad \text{and} \quad [\alpha,\beta](t) = c(\alpha(1),\beta(t)),$$

which can be used to verify all requirements. □

We shall return to topological groups with Lie algebras in our discussion of projective limits in Section X.

Remark IV.1.25. We have seen in Corollary IV.1.20 that a 1-connected locally exponential Lie group G is completely determined up to isomorphism (as a topological group) by its Lie algebra.

If G is connected but not simply connected, then we have a universal covering morphism $q_G\colon \widetilde{G} \to G$ and $\ker q_G \cong \pi_1(G)$ is a discrete central subgroup of $\widetilde{G}$ with $G \cong \widetilde{G}/\ker q_G$. It is easy to see that two discrete central subgroups $\Gamma_1, \Gamma_2 \subseteq Z(\widetilde{G})$ lead to isomorphic quotient groups $\widetilde{G}/\Gamma_1$ and $\widetilde{G}/\Gamma_2$ if and only if there exists an automorphism $\varphi \in \mathrm{Aut}(\widetilde{G}) \cong \mathrm{Aut}(\mathbf{L}(G))$ with $\varphi(\Gamma_1) = \Gamma_2$. Therefore the isomorphism classes of connected Lie groups G with a given Lie algebra $\mathfrak{g} \cong \mathbf{L}(G)$ are parametrized by the orbits of $\mathrm{Aut}(\mathfrak{g}) \cong \mathrm{Aut}(\widetilde{G})$ in the set of discrete central subgroup of $\widetilde{G}$.

If G_0 and a discrete group Γ are given, then the determination of all Lie groups G with identity component G_0 and component group $\pi_0(G) \cong \Gamma$ corresponds to the classification of all Lie group extensions

$$\mathbf{1} \to G_0 \hookrightarrow G \twoheadrightarrow \Gamma \to \mathbf{1},$$

i.e., to a description of the set $\mathrm{Ext}(\Gamma, G_0)$. Extension problems of this type are discussed in Section V.1 below. □

IV.2. Locally exponential Lie algebras

We now turn to the Lie algebras which are candidates for Lie algebras of locally exponential Lie groups. We call these Lie algebras "locally exponential". They are defined by the requirement that some 0-neighborhood carries a local group structure in "canonical" coordinates, i.e., the additive one-parameter groups $t \mapsto tx$, should also be one-parameter groups for the local group structure (*cf.* [Bir38], [Lau55]).

Definition IV.2.1. *A locally convex Lie algebra* $\mathfrak{g}$ *is called* locally exponential *if there exists a circular convex open* 0*-neighborhood* $U \subseteq \mathfrak{g}$ *and an open subset* $D \subseteq U \times U$ *on which we have a smooth map*

$$m_U\colon D \to U, \quad (x,y) \mapsto x * y$$

such that $(U, D, m_U, 0)$ *is a local Lie group (Definition II.1.10) satisfying:*

(E1) *For $x \in U$ and $|t|, |s|, |t+s| \leq 1$, we have $(tx, sx) \in D$ with $tx * sx = (t+s)x$.*
(E2) *The second order term in the Taylor expansion of m_U is $b(x,y) = \frac{1}{2}[x,y]$.*

The Lie algebra $\mathfrak{g}$ is called exponential *if $U = \mathfrak{g}$ and $D = \mathfrak{g} \times \mathfrak{g}$.* □

Since any local Lie group on an open subset of a locally convex space V leads to a Lie algebra structure on V (Definition II.1.10), condition (E2) only ensures that $\mathfrak{g}$ is the Lie algebra of the local group (*cf.* Remark II.1.8).

Using exponential coordinates, we directly get:

Lemma IV.2.2. *The Lie algebra $\mathbf{L}(G)$ of a locally exponential Lie group G is locally exponential.* □

Definition IV.2.3. *We call a locally exponential Lie algebra $\mathfrak{g}$* enlargeable *if it is integrable to a locally exponential Lie group G. As we shall see in Remark IV.2.5 below, this is equivalent to the enlargeability of some associated local group in $\mathfrak{g}$.* □

Examples IV.2.4. (a) All BCH–Lie algebras, hence in particular all Banach–Lie algebras and therefore all finite-dimensional Lie algebras are locally exponential (Example IV.1.6).

A different existence proof for the local multiplication on a Banach–Lie algebra $\mathfrak{g}$ is given by Laugwitz ([Lau56]): As a first step, we observe that $\kappa_{\mathfrak{g}}(x) := \frac{\mathbf{1}-e^{-\operatorname{ad} x}}{\operatorname{ad} x}$ defines a smooth map $\kappa_{\mathfrak{g}} \colon \mathfrak{g} \to \mathscr{L}(\mathfrak{g})$ with $\kappa_{\mathfrak{g}}(0) = \mathrm{id}_{\mathfrak{g}}$, so that its values are invertible on some 0-neighborhood. We consider $\kappa_{\mathfrak{g}}$ as a $\mathfrak{g}$-valued 1-form on $\mathfrak{g}$. Then one verifies that $\kappa_{\mathfrak{g}}$ satisfies the Maurer–Cartan equation, which implies the existence of an open 0-neighborhood U such that for each $x \in U$ the (partial differential) equation

$$f^*\kappa_{\mathfrak{g}} = \kappa_{\mathfrak{g}}, \quad f(0) = x$$

has a unique solution f_x on U. For x, y close to 0, the composition $f_x \circ f_y$ is then defined on some 0-neighborhood and satisfies $f_x \circ f_y(0) = f_x(y) = f_{f_x(y)}(0)$ as well as $(f_x \circ f_y)^*\kappa_{\mathfrak{g}} = f_y^* f_x^* \kappa_{\mathfrak{g}} = \kappa_{\mathfrak{g}}$, which implies $f_x \circ f_y = f_{f_x(y)}$ on some 0-neighborhood. For $x * y := f_x(y)$, this leads to the associativity condition

$$x * (y * z) = (x * y) * z$$

on some 0-neighborhood in $\mathfrak{g}$, hence to a local group structure. As $\kappa_{\mathfrak{g}}$ satisfies $\kappa_{\mathfrak{g}}(x)x = x$ for each $x \in \mathfrak{g}$, the curves $t \mapsto tx$ are local one-parameter groups. This corresponds to condition (E1).

(b) If $\mathfrak{g}$ is locally exponential and M a compact manifold, then $C^\infty(M, \mathfrak{g})$ is also locally exponential with respect to $(x * y)(m) := x(m) * y(m)$ for all $m \in M$ and x, y close to 0. □

Remark IV.2.5. A similar reasoning as in the proof of Theorem IV.1.19 implies that any morphism $f\colon \mathfrak{g} \to \mathfrak{h}$ of locally exponential Lie algebras satisfies $f(x*y) = f(x)*f(y)$ for x,y close to 0. Applying this to $f = \mathrm{id}_{\mathfrak{g}}$ shows in particular that the Lie algebra $\mathfrak{g}$ determines the germ of the local multiplication $x*y$ (*cf.* [Lau56] for the Banach case). We know that this multiplication need not be analytic, not even if it is defined on all of $\mathfrak{g} \times \mathfrak{g}$ (Example IV.1.16). $\square$

Suppose that $\mathfrak{g}$ is an exponential Lie algebra for which the group $(\mathfrak{g},*)$ is regular. Then $(\mathfrak{g},*)$ is the unique 1-connected regular Lie group with Lie algebra $\mathfrak{g}$. If G is any 1-connected Lie group (regular or not) with $\mathbf{L}(G) = \mathfrak{g}$, and G has an exponential function $\exp_G\colon \mathbf{L}(G) = \mathfrak{g} \to G$, then $\exp_G$ is a group homomorphism $(\mathfrak{g},*) \to G$ (*cf.* Propositions II.4.1 and II.5.7). The regularity of $(\mathfrak{g},*)$ implies the existence of a unique homomorphism $\mathrm{Log}_G\colon G \to (\mathfrak{g},*)$ with $\mathbf{L}(\mathrm{Log}_G) = \mathrm{id}_{\mathfrak{g}}$, and the uniqueness assertion of Proposition II.4.1 yields $\mathrm{Log}_G \circ \exp_G = \mathrm{id}_{\mathfrak{g}}$ and $\exp_G \circ \mathrm{Log}_G = \mathrm{id}_G$. Since on any Mackey complete nilpotent Lie algebra $\mathfrak{g}$, the BCH multiplication defines a regular Lie group structure ([GN06]), these arguments lead to the following theorem:

Theorem IV.2.6. *If G is a connected nilpotent Lie group with a smooth exponential function and $\mathbf{L}(G)$ is Mackey complete, then the exponential function*

$$\exp_G\colon (\mathbf{L}(G),*) \to G$$

is a covering morphism of Lie groups. In particular, $G \cong (\mathbf{L}(G),*)/\Gamma$ for a discrete subgroup $\Gamma \subseteq \mathfrak{z}(\mathfrak{g})$, isomorphic to $\pi_1(G)$. Moreover, G is regular and locally exponential. $\square$

This generalizes a result of Michor and Teichmann who showed in [MT99] that any connected regular abelian Lie group G is of the form $\mathbf{L}(G)/\Gamma$ for a discrete subgroup $\Gamma \cong \pi_1(G)$ of $\mathbf{L}(G)$. Related results can be found in [Ga96], where locally exponential abelian Fréchet–Lie groups are studied as projective limits of Banach–Lie groups.

Without any completeness assumption we obtain the following very natural intrinsic characterization of the BCH series as the only Lie series which leads on nilpotent Lie algebras to a group multiplication satisfying (E1).

Proposition IV.2.7. *If G is a 1-connected exponential nilpotent Lie group, then $G \cong (\mathbf{L}(G),*)$, where $*$ denotes the (polynomial) BCH multiplication on $\mathbf{L}(G)$.* $\square$

We have already seen that the Lie bracket on a locally exponential Lie algebra $\mathfrak{g}$ determines the germ of the corresponding local multiplication (Remark IV.2.5), hence in particular its Taylor series in $(0,0)$. The preceding proposition is the key step to the following theorem, identifying this series as the BCH series. In the Banach context, the corresponding result is due to Birkhoff ([Bir38]). Its statement can be found in [Mil82] as Lemma 4.4, with the hint that it can be proved with the methods used in [HS68] in the finite-dimensional case, which is based on formula (4.1.1). Since the spectra of the operators $\mathrm{ad}\,x$ and $\mathrm{ad}\,y$ are possibly unbounded, formula (4.1.1) makes no sense for general locally exponential Lie algebras. The situation is much better if $\mathfrak{g}$ is nilpotent. In this case, the operators $e^{\mathrm{ad}\,x}$ are unipotent, so that $\psi(e^{\mathrm{ad}\,x}e^{\mathrm{ad}\,y})$ is a polynomial in x and y. The reduction to this case is a key point in the proof of the following theorem.

Theorem IV.2.8. (Universality Theorem) *If $\mathfrak{g}$ is locally exponential, then the Taylor series of the local multiplication $x * y$ in $(0,0)$ is the BCH series.*

Proof. (Sketch) A central idea is the following. For each Lie algebra we obtain by extension of scalars from $\mathbb{R}$ to the two-dimensional algebra $\mathbb{R}[\varepsilon]$ of dual numbers ($\varepsilon^2 = 0$), the Lie algebra $T(\mathfrak{g}) := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{R}[\varepsilon]$. One can show that $T(\mathfrak{g})$ is also locally exponential. The local multiplication $m_{T(\mathfrak{g})}$ is the tangent map of the local multiplication $m_{\mathfrak{g}}$ of $\mathfrak{g}$ and $U_{T(\mathfrak{g})} = T(U_{\mathfrak{g}}) = U_{\mathfrak{g}} \times \mathfrak{g}$ is the tangent bundle of $U_{\mathfrak{g}} \subseteq \mathfrak{g}$.

Iterating this procedure, we obtain a sequence of locally exponential Lie algebras

$$T^n(\mathfrak{g}) := \mathfrak{g} \otimes \mathbb{R}[\varepsilon_1, \ldots, \varepsilon_n] \quad \text{with} \quad \varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i,\ \varepsilon_i^2 = 0,$$

whose local multiplication $T^n(m_{\mathfrak{g}})$ induces a global multiplication on the nilpotent ideal $J \trianglelefteq T^n(\mathfrak{g})$ which is the kernel of the augmentation map $T^n(\mathfrak{g}) \to \mathfrak{g}$. Applying Proposition IV.2.7 to $(J, T^n(m_{\mathfrak{g}}))$ now shows that the n-th order Taylor polynomial of $m_{\mathfrak{g}}$ in $(0,0)$ is given by the BCH series. □

For the discussion of quotients of locally exponential groups below, the following theorem is crucial:

Theorem IV.2.9. (Quotient Theorem for locally exponential Lie algebras) *Let $\mathfrak{g}$ be a locally exponential Mackey complete Lie algebra and $\mathfrak{n} \trianglelefteq \mathfrak{g}$ a closed ideal. Then $\mathfrak{g}/\mathfrak{n}$ is locally exponential if and only if*

(1) $\mathfrak{n}$ is stable, i.e., $e^{\mathrm{ad}\,x}(\mathfrak{n}) = \mathfrak{n}$ for each $x \in \mathfrak{g}$, and
(2) $\kappa_{\mathfrak{g}}(x)\mathfrak{n} = \mathfrak{n}$ for each x in some 0-neighborhood of $\mathfrak{g}$.

If $\mathfrak{n}$ is the kernel of a morphism $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ of locally exponential Lie algebras, then $\mathfrak{n}$ is locally exponential and both conditions are satisfied, so that φ factors through the quotient map $q\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$ to an injective morphism $\overline{\varphi}\colon \mathfrak{g}/\mathfrak{n} \to \mathfrak{h}$ of locally exponential Lie algebras. □

The preceding result is nicely complemented by the following observation on extensions:

Theorem IV.2.10. ([GN07]) *Let $\mathfrak{g}$ be a locally exponential Lie algebra and $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ be a central extension, i.e., a quotient morphism with central kernel $\mathfrak{z}$. If $\mathfrak{z}$ is Mackey complete, then $\widehat{\mathfrak{g}}$ is locally exponential.* □

Remark IV.2.11. In [Hof72/75], K. H. Hofmann advocates an approach to Banach–Lie groups by defining a Banach–Lie group as a topological group possessing an identity neighborhood isomorphic (as a topological local group) to the local group defined by the BCH multiplication in a 0-neighborhood of a Banach–Lie algebra. A key point of this perspective is that Banach–Lie groups form a full sub-category of the category of topological groups (Theorem IV.1.18, Remark IV.1.22). Due to the analyticity of the BCH multiplication, this approach works quite well for Banach–Lie groups, and also for the larger class of BCH Lie groups which behave in almost all respects like Banach–Lie groups. Although one may think that one can adopt a similar point of view for locally exponential Lie groups, a closer analysis of the arguments used in this theory to pass from infinitesimal to local information shows that the behavior of locally exponential groups is far from being controlled by topology. Actually the arguments we use are much closer to the original approach to Lie theory via differential equations (*cf.* Examples IV.2.4).

Giving up the analyticity requirement of the local multiplication in an identity neighborhood implies that we have to work in a smooth category to prove uniqueness assertions. The Maurer–Cartan form and the Uniqueness Lemma are the fundamental tools. In the analytic context, one can often argue quite directly by analytic continuation. □

IV.3. Locally exponential Lie subgroups

It is a well-known result in finite-dimensional Lie theory that each closed subgroup H of a Lie group G carries a natural Lie group structure turning it into a submanifold of G (see [vN29] for closed subgroups of $\mathrm{GL}_n(\mathbb{R})$). This becomes already false for closed subgroups of infinite-dimensional Hilbert spaces, which contain contractible subgroups not containing any smooth arc. Therefore additional assumptions on closed subgroups are needed to make them accessible by Lie theoretic methods. Since we already know that each topological group carries at most one locally exponential Lie group structure, it is clear that a closed subgroup deserves to be called a *Lie subgroup* if it is a locally exponential Lie group with respect to the induced topology. For Banach–Lie groups, this is precisely Hofmann's approach, and for several of the results described below, Banach versions can be found in [Hof75].

Lie subgroups and factor groups

Lemma IV.3.1. *For every closed subgroup H of the locally exponential Lie group G, we have*

$$\mathbf{L}^d(H) = \mathbf{L}^e(H) := \{x \in \mathbf{L}(G) \colon \exp_G(\mathbb{R}x) \subseteq H\},$$

and this is a closed Lie subalgebra of $\mathbf{L}(G)$*.*

Proof. The equality $\mathbf{L}^e(H) = \mathbf{L}^d(H)$ follows from $\lim_{n\to\infty} \gamma\big(\frac{t}{n}\big)^n = \exp_G(t\gamma'(0))$ for each curve $\gamma\colon [0,1] \to H$ with $\gamma(0) = \mathbf{1}$ which is differentiable in 0 because we can write it on some interval $[0,\varepsilon]$ as $\gamma = \exp_G \circ \eta$ with some C^1-curve η in $\mathbf{L}(G)$ with $\eta(0) = 0$. The closedness follows from the obvious closedness of $\mathbf{L}^e(H)$. ☐

In the following we shall keep the notation $\mathbf{L}^e(H) = \{x \in \mathbf{L}(G)\colon \exp_G(\mathbb{R}x) \subseteq H\}$ for a subgroup H of a Lie group G with an exponential function, because if H is not closed or G is not locally exponential, it is not clear that this set coincides with $\mathbf{L}^d(H)$

Definition IV.3.2. *A closed subgroup H of a locally exponential Lie group G is called a* locally exponential Lie subgroup*, or simply a* Lie subgroup*, if H is a locally exponential Lie group with respect to the induced topology (cf. Remark IV.1.22).* ☐

A Banach version of the following theorem is Proposition 3.4 in [Hof75].

Theorem IV.3.3. *For a closed subgroup H of the locally exponential Lie group G the following are equivalent:*

(1) H is a locally exponential Lie group.
(2) There exists an open 0*-neighborhood* $V \subseteq \mathbf{L}(G)$ *such that* $\exp_G|_V$ *is a diffeomorphism onto an open* $\mathbf{1}$*-neighborhood in G and* $\exp_G(V \cap \mathbf{L}^e(H)) = \exp_G(V) \cap H$*.*

In particular, each locally exponential Lie subgroup is a submanifold of G. ☐

Proposition IV.3.4. *If* $\varphi\colon G' \to G$ *is a morphism of locally exponential Lie groups and* $H \subseteq G$ *is a locally exponential Lie subgroup, then* $H' := \varphi^{-1}(H)$ *is a locally exponential Lie subgroup. In particular,* $\ker\varphi$ *is a locally exponential Lie subgroup of* G'*.* ☐

The preceding proposition implies in particular that if a quotient G/N by a closed normal subgroup N is locally exponential, then N is a locally exponential Lie subgroup. But the converse is more subtle:

Theorem IV.3.5. (Quotient Theorem for locally exponential groups) *For a closed normal subgroup* $N \trianglelefteq G$ *the following are equivalent:*

(1) G/N *is a locally exponential Lie group.*
(2) N is a locally exponential Lie subgroup and $\mathbf{L}(G)/\mathbf{L}^e(N)$ *is a locally exponential Lie algebra.*
(3) N is a locally exponential Lie subgroup and $\kappa_{\mathbf{L}(G)}(x)(\mathbf{L}^e(N)) = \mathbf{L}^e(N)$ *for* $x \in \mathbf{L}(G)$ *sufficiently close to* 0*.*

If N is the kernel of a morphism $\varphi\colon G \to H$ of locally exponential Lie groups, then $G/\ker\varphi$ is a Lie group, so that φ factors through a quotient map $G \to G/\ker\varphi$ and an injective morphism $\overline{\varphi}\colon G/\ker\varphi$ of locally exponential Lie groups. □

Since quotients of BCH–Lie algebras are BCH–Lie algebras, no matter whether they are complete or not ([Gl02c, Th. 2.20]), we get the following corollary, whose Banach version is also contained in [Hof75, Prop. 3.6] and [GN03].

Corollary IV.3.6. (Quotient Theorem for BCH–Lie groups) *A closed normal subgroup N of a BCH–Lie group G is a BCH–Lie group if and only if the quotient G/N is a BCH–Lie group.* □

If $\varphi\colon G \to H$ is an injective morphism of locally exponential Lie groups, then the preceding theorem provides no additional information. In Section IV.4, we shall encounter this situation for integral subgroups of G. The following example provides a bijective morphism φ for which $\mathbf{L}(\varphi)$ is not surjective. The only way to avoid this pathology is to assume that $\mathbf{L}(G)$ is separable (*cf.* Theorems IV.4.14/15 below). That not all surjective morphisms of locally exponential Lie groups are quotient morphisms can already be seen for surjective continuous linear maps between non-Fréchet spaces.

Example IV.3.7. We give an example of a proper closed subalgebra $\mathfrak{h}$ of the Lie algebra $\mathbf{L}(G)$ of some Banach–Lie group G for which $\langle \exp\mathfrak{h}\rangle = G$ ([HoMo98, p.157]).

We consider the abelian Lie group $\mathfrak{g} := \ell^1(\mathbb{R},\mathbb{R}) \times \mathbb{R}$, where the group structure is given by addition. We write $(e_r)_{r\in\mathbb{R}}$ for the canonical topological basis elements of $\ell^1(\mathbb{R},\mathbb{R})$. Then the subgroup D generated by the pairs $(e_r,-r)$, $r \in \mathbb{R}$, is closed and discrete, so that $G := \mathfrak{g}/D$ is an abelian Lie group. Now we consider the closed subalgebra $\mathfrak{h} := \ell^1(\mathbb{R},\mathbb{R})$ of $\mathfrak{g}$. As $\mathfrak{h} + D = \mathfrak{g}$, we have $H := \exp_G \mathfrak{h} = G$, and therefore $(0,1) \in \mathbf{L}^e(H) \setminus \mathfrak{h}$.

The map $\varphi := \exp_G|_{\mathfrak{h}}\colon (\mathfrak{h},+) \to G$ is a surjective morphism of Lie groups for which $\mathbf{L}(\varphi)$ is the inclusion of the proper subalgebra $\mathfrak{h}$. □

That for connected Banach–Lie groups G the center $Z(G) = \ker \mathrm{Ad}$ is a locally exponential Lie subgroup follows immediately from Proposition IV.3.4 (*cf.* [Lau55]). For non-Banach–Lie algebras $\mathfrak{g}$, $\mathrm{Aut}(\mathfrak{g})$ carries no natural Lie group structure, so that Proposition IV.3.4 does not apply. This makes the following theorem quite remarkable. The crucial point in its proof is to show that for the exponential function

$$\mathrm{Exp}\colon \mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \to \mathrm{Aut}(\mathfrak{g}), \quad x \mapsto e^{\mathrm{ad}\,x}, \tag{4.3.1}$$

the point 0 is isolated in $\mathrm{Exp}^{-1}(\mathrm{id}_{\mathfrak{g}})$ (*cf.* Problems II.4 and IX.1).

Theorem IV.3.8. *Let $\mathfrak{g}$ be a locally exponential Lie algebra. Then the adjoint group $G_{\mathrm{ad}} := \langle e^{\mathrm{ad}\,\mathfrak{g}} \rangle \subseteq \mathrm{Aut}(\mathfrak{g})$ carries the structure of a locally exponential Lie group whose Lie algebra is the quotient $\mathfrak{g}_{\mathrm{ad}} := \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ and (4.3.1) its exponential function.* □

Combining the preceding theorem with Proposition IV.3.4, we get:

Corollary IV.3.9. *If G is a connected locally exponential Lie group, then its center $Z(G) = \ker \mathrm{Ad}$ is a locally exponential Lie subgroup.* □

Algebraic subgroups

The concept of an algebraic subgroup of a Banach–Lie algebra, introduced by Harris and Kaup ([HK77]), provides very convenient criteria which in many concrete cases can be used to verify that a closed subgroup H of a Banach–Lie group is a Banach–Lie subgroup.

Definition IV.3.10. *Let A be a unital Banach algebra. A subgroup $G \subseteq A^\times$ is called* algebraic *if there exists a $d \in \mathbb{N}_0$ and a set $\mathscr{F}$ of Banach space-valued polynomial functions on $A \times A$ of degree $\leq d$ such that*

$$G = \{g \in A^\times : (\forall f \in \mathscr{F})\ f(g, g^{-1}) = 0\}.$$ □

Theorem IV.3.11. ([HK77], [Ne04b, Prop. IV.14]) *Every algebraic subgroup $G \subseteq A^\times$ of the unit group $A^\times$ of a Banach algebra A is a Banach–Lie subgroup.* □

Corollary IV.3.12. *Let E be a Banach space and $F \subseteq E$ a closed subspace. Then*

$$\mathrm{GL}(E,F) := \{g \in \mathrm{GL}(E) : g(F) = F\}$$

is a Banach–Lie subgroup of $\mathrm{GL}(E)$. □

Corollary IV.3.13. *Let E be a Banach space and $v \in E$. Then*

$$\mathrm{GL}(E)_v := \{g \in \mathrm{GL}(E) : g(v) = v\}$$

is a Banach–Lie subgroup of $\mathrm{GL}(E)$. □

Corollary IV.3.14. *For each continuous bilinear map $\beta : E \times E \to E$ on a Banach space E, the group*

$$\mathrm{Aut}(E,\beta) := \{g \in \mathrm{GL}(E) : \beta \circ (g \times g) = g \circ \beta\}$$

is a Banach–Lie subgroup of $\mathrm{GL}(E)$ *with Lie algebra*

$$\mathrm{der}(E,\beta) := \{D \in \mathfrak{gl}(E) : (\forall v, w \in E)\ D.\beta(v,w) = \beta(D.v,w) + \beta(v,D.w)\}.$$ □

Corollary IV.3.15. *For each bilinear map $\beta\colon E \times E \to \mathbb{K}$, the group*

$$\mathrm{O}(E,\beta) := \{g \in \mathrm{GL}(E)\colon \beta \circ (g \times g) = \beta\}$$

is a Banach–Lie subgroup of $\mathrm{GL}(E)$ *with Lie algebra*

$$\mathfrak{o}(E,\beta) := \{D \in \mathfrak{gl}(E)\colon (\forall v,w \in E)\ \beta(D.v,w) + \beta(v,D.w) = 0\}. \qquad \square$$

Closed subgroups versus Lie subgroups

For finite-dimensional Lie groups, closed subgroups are Lie subgroups (*cf.* [vN29]), but for Banach–Lie groups this is no longer true. What remains true is that locally compact subgroups (which are closed in particular) are Lie subgroups. For subgroups of Banach algebras the following theorem is due to Yosida ([Yo36]) and for general Banach–Lie groups to Laugwitz ([Lau55], [Les66]). Although the arguments in the Banach case do not immediately carry over because unit spheres for seminorms are no longer bounded, one can use Glöckner's Implicit Function Theorem ([Gl03a]) to get:

Theorem IV.3.16. ([GN06]) *Each locally compact subgroup of a locally exponential Lie group is a finite-dimensional Lie subgroup.* $\square$

Remark IV.3.17. How bad closed subgroups may behave is illustrated by the following example ([Hof75, Ex. 3.3(i)]): We consider the real Hilbert space $G := L^2([0,1],\mathbb{R})$ as a Banach–Lie group. Then the subgroup $H := L^2([0,1],\mathbb{Z})$ of all those functions which almost everywhere take values in $\mathbb{Z}$ is a closed subgroup. Since the one-parameter subgroups of G are of the form $\mathbb{R}f$, $f \in G$, we have $\mathbf{L}^e(H) = \{0\}$. On the other hand, the group H is arc-wise connected. It is contractible, because the map $F\colon [0,1] \times H \to H$ given by

$$F(t,f)(x) := \begin{cases} f(x) & \text{for } 0 \leq x \leq t \\ 0 & \text{for } t < x \leq 1 \end{cases}$$

is continuous with $F(1,f) = f$ and $F(0,f) = 0$. $\square$

IV.4. Integral subgroups

It is a well-known result in finite-dimensional Lie theory that for each subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g} = \mathbf{L}(G)$ of a finite-dimensional Lie group G, there exists a Lie group H with Lie algebra $\mathfrak{h}$ together with an injective morphism of Lie groups $\iota\colon H \to G$ for which $\mathbf{L}(\iota)\colon \mathfrak{h} \to \mathfrak{g}$ is the inclusion map. As a group, H coincides with $\langle \exp \mathfrak{h} \rangle$, the analytic subgroup corresponding to $\mathfrak{h}$, and $\mathfrak{h}$ can be recovered from this subgroup as the set $\mathbf{L}^e(H) = \{x \in \mathbf{L}(G)\colon \exp(\mathbb{R}x) \subseteq H\}$.

This nice and simple theory of analytic subgroups and integration of Lie algebra inclusions $\mathfrak{h} \hookrightarrow \mathbf{L}(G)$ becomes much more subtle for infinite-dimensional Lie groups. Even for Banach–Lie groups some pathologies arise. Here any inclusion $\mathfrak{h} \hookrightarrow \mathbf{L}(G)$ of Banach–Lie algebras integrates to an "integral" subgroup $H \hookrightarrow G$, but if the Banach–Lie algebra $\mathfrak{h}$ is not separable, then it may happen that $\mathfrak{h}$ cannot be recovered from the abstract subgroup H of G. In Example IV.3.7, it even occurs that $\mathfrak{h} \neq \mathbf{L}(G)$ and $H = G$.

Definition IV.4.1. *Let G be a Lie group with an exponential function, so that we obtain for each $x \in \mathbf{L}(G)$ an automorphism $e^{\operatorname{ad} x} := \operatorname{Ad}(\exp_G x) \in \operatorname{Aut}(\mathbf{L}(G))$. A subalgebra $\mathfrak{h} \subseteq \mathbf{L}(G)$ is called* stable *if*

$$e^{\operatorname{ad} x}(\mathfrak{h}) = \operatorname{Ad}(\exp_G x)(\mathfrak{h}) = \mathfrak{h} \quad \textit{for all } x \in \mathfrak{h}.$$

An ideal $\mathfrak{n} \trianglelefteq \mathbf{L}(G)$ is called a stable ideal *if $e^{\operatorname{ad} x}(\mathfrak{n}) = \mathfrak{n}$ for all $x \in \mathbf{L}(G)$.* □

The following lemma shows that stability of kernel and range is a necessary requirement for the integrability of a homomorphism of Lie algebras.

Lemma IV.4.2. *If $\varphi\colon G \to H$ is a morphism of Lie groups with an exponential function, then $\operatorname{im}(\mathbf{L}(\varphi))$ is a stable subalgebra of $\mathbf{L}(H)$, and $\ker(\mathbf{L}(\varphi))$ is a stable ideal of $\mathbf{L}(G)$.*

Proof. We have $\varphi \circ \exp_G = \exp_H \circ \mathbf{L}(\varphi)$, which leads to

$$\begin{aligned}\mathbf{L}(\varphi) \circ e^{\operatorname{ad} x} &= \mathbf{L}(\varphi) \circ \operatorname{Ad}(\exp_G x) = \mathbf{L}(\varphi \circ c_{\exp_G x}) = \mathbf{L}(c_{\varphi(\exp_G x)} \circ \varphi) \\ &= \operatorname{Ad}(\exp_H \mathbf{L}(\varphi)x) \circ \mathbf{L}(\varphi) = e^{\operatorname{ad} \mathbf{L}(\varphi)(x)} \circ \mathbf{L}(\varphi).\end{aligned}$$

We conclude in particular that $\operatorname{im}(\mathbf{L}(\varphi))$ is a stable subalgebra and that $\ker(\mathbf{L}(\varphi))$ is a stable ideal. □

Lemma IV.4.3. *(a) Each closed subalgebra which is finite-dimensional or finite-codimensional is stable.*

(b) Let $\mathfrak{g}$ be a BCH Lie algebra. Then each closed subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and each closed ideal $\mathfrak{n} \trianglelefteq \mathfrak{g}$ is stable.

(c) If $\mathfrak{h} \hookrightarrow \mathfrak{g}$ is a continuous inclusion of locally convex Lie algebras such that for each $x \in \mathfrak{h}$ the operators $\operatorname{ad}_{\mathfrak{h}} x$ and $\operatorname{ad}_{\mathfrak{g}} x$ are integrable on $\mathfrak{h}$, resp., $\mathfrak{g}$, then $\mathfrak{h}$ is a stable subalgebra of $\mathfrak{g}$.

Proof. (a) (*cf.* [Omo97, Lemma III.4.8]) If $\mathfrak{h}$ is finite-dimensional, then the Uniqueness Lemma implies for $x \in \mathfrak{h}$ the relation $e^{\operatorname{ad} x}|_{\mathfrak{h}} = e^{\operatorname{ad} x|\mathfrak{h}}$. If $\mathfrak{h}$ is finite-codimensional and $q\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ the projection map, then the curve $\gamma(t) := q(e^{t \operatorname{ad} x}y)$ satisfies the linear ODE $\gamma'(t) = \operatorname{ad}_{\mathfrak{g}/\mathfrak{h}}(x)\gamma(t)$, hence vanishes for $y \in \mathfrak{h}$.

(b) Since $\mathfrak{g}$ is BCH, the map $x \mapsto e^{\operatorname{ad} x}y = x * y * (-x)$ is analytic on some open 0-neighborhood, hence given by the power series $\sum_{n=0}^{\infty} \frac{1}{n!}(\operatorname{ad} x)^n y$. Therefore the closedness of $\mathfrak{h}$ implies that for x close to 0 we have $e^{\operatorname{ad} x}(\mathfrak{h}) \subseteq \mathfrak{h}$. This implies stability. A similar argument yields the stability of closed ideals.

(c) We apply Lemma II.3.10 to see that $e^{\mathrm{ad}_{\mathfrak{g}} x} y = e^{\mathrm{ad}_{\mathfrak{h}} x} y \in \mathfrak{h}$ holds for $x, y \in \mathfrak{h}$. □

In view of the preceding lemma, stability causes no problems for BCH–Lie algebras, but the condition becomes crucial in the non-analytic context.

Example IV.4.4. The first example of a closed Lie subalgebra $\mathfrak{h}$ of some $\mathbf{L}(G)$ which does not integrate to any group homomorphism is due to H. Omori (*cf.* [Mil84, 8.5]).

We consider the group $G := \mathrm{Diff}(\mathbb{T}^2)$ of diffeomorphisms of the 2-dimensional torus and use coordinates $(x,y) \in [0,1]^2$ corresponding to the identification $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$. Then

$$\mathfrak{h} := \Big\{ f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} \colon \frac{1}{2} \leq x \leq 1 \Rightarrow g(x,y) = 0 \Big\}$$

is easily seen to be a closed Lie subalgebra of $\mathfrak{g} = \mathscr{V}(\mathbb{T}^2)$. The vector field $X := \frac{\partial}{\partial x}$ generates the smooth action $\alpha \colon \mathbb{T} \to \mathrm{Diff}(\mathbb{T}^2)$ of $\mathbb{T}$ on $\mathbb{T}^2$ given by $[z].([x],[y]) = ([x+z],[y])$. This vector field is contained in $\mathfrak{h}$, but

$$e^{\frac{1}{2}\mathrm{ad} X}\mathfrak{h} = \mathrm{Ad}(\alpha(\tfrac{1}{2}))\mathfrak{h} = \Big\{ f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} \colon 0 \leq x \leq \frac{1}{2} \Rightarrow g(x,y) = 0 \Big\} \neq \mathfrak{h}.$$

This shows that $\mathfrak{h}$ is not stable and hence that it does not integrate to any subgroup of $\mathrm{Diff}(\mathbb{T}^2)$ with an exponential function. □

Example IV.4.5. Let $E := C^\infty(\mathbb{R},\mathbb{R})$ and consider the one-parameter group $\alpha \colon \mathbb{R} \to \mathrm{GL}(E)$ given by $\alpha(t)(f)(x) = f(x+t)$. Then $\mathbb{R}$ acts smoothly on E, so that we can form the corresponding semi-direct product group $G := E \rtimes_\alpha \mathbb{R}$. This is a Lie group with a smooth exponential function given by

$$\exp_G(v,t) = \Big(\int_0^1 \alpha(st).v\, ds, t \Big), \text{ where } \Big(\int_0^1 \alpha(st).v\, ds \Big)(x) = \int_0^1 v(x+st)\, ds.$$

The Lie algebra $\mathfrak{g} = \mathbf{L}(G)$ has the corresponding semi-direct product structure $\mathfrak{g} = E \rtimes_D \mathbb{R}$ with $Dv = v'$, i.e.,

$$[(f,t),(g,s)] = (tg' - sf', 0).$$

In $\mathfrak{g}$, we now consider the subalgebra $\mathfrak{h} := E_{[0,1]} \rtimes \mathbb{R}$, where

$$E_{[0,1]} := \{ f \in E \colon \mathrm{supp}(f) \subseteq [0,1] \}.$$

Then $\mathfrak{h}$ is a closed subalgebra of $\mathfrak{g}$. It is not stable because $\alpha(-t)E_{[0,1]} = E_{[t,t+1]}$. The subgroup of G generated by $\exp_G \mathfrak{h}$ contains $\{0\} \rtimes \mathbb{R}$, $E_{[0,1]}$, and hence all subspaces $E_{[t,t+1]}$, which implies that $\langle \exp_G \mathfrak{h} \rangle = C^\infty_c(\mathbb{R}) \rtimes \mathbb{R}$.

Lemma IV.4.2 implies that the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ does not integrate to a homomorphism $\varphi \colon H \to G$ for any Lie group H with an exponential function and $\mathbf{L}(H) = \mathfrak{h}$. □

Example IV.4.6. Let $E \subseteq C^\infty(\mathbb{R},\mathbb{C})$ be the closed subspace of 1-periodic functions, $\mu \in \mathbb{R}^\times$, and consider the homomorphism $\alpha\colon \mathbb{R} \to \mathrm{GL}(E)$ given by

$$(\alpha(t)f)(x) := e^{\mu t} f(x+t).$$

That the corresponding $\mathbb{R}$-action on E is smooth follows from the smoothness of the translation action and one can show that the group $G := E \rtimes_\alpha \mathbb{R}$ is exponential with Lie algebra $\mathfrak{g} = E \rtimes_D \mathbb{R}$ and $Df = \mu f + f'$ ([GN06]). In particular, the product $x * y := \exp_G^{-1}(\exp_G(x)\exp_G(y))$ is globally defined on $\mathfrak{g}$.

Let $M \subseteq [0,1]$ be an open subset which is not dense and put

$$E_M := \{f \in E\colon f|_M = 0\}.$$

Then E_M is a closed subspace of E with $DE_M \subseteq E_M$ but $\alpha(t)(E_M) = E_{M-t} \not\subseteq E_M$ for some $t \in \mathbb{R}$. Therefore $\mathfrak{h}_M := E_M \rtimes_D \mathbb{R} \subseteq \mathfrak{g} = E \rtimes_D \mathbb{R}$ is a closed subalgebra of the exponential Fréchet–Lie algebra $\mathfrak{g}$ which is not stable. Since $e^{\mathrm{ad}\, x} y = x * y * (-x)$ for all $x, y \in \mathfrak{g}$, this implies in particular that $\mathfrak{h}$ is not closed under the $*$-multiplication. ☐

Definition IV.4.7. *Let G be a Lie group. An* integral subgroup *is an injective morphism $\iota\colon H \to G$ of Lie groups such that H is connected and the differential $\mathbf{L}(\iota)\colon \mathbf{L}(H) \to \mathbf{L}(G)$ is injective.* ☐

Remark IV.4.8. Let $\iota\colon H \to G$ be an integral subgroup and assume that H and G have exponential functions. Then the relation

$$\exp_G \circ \mathbf{L}(\iota) = \iota \circ \exp_H \tag{4.4.1}$$

implies that $\ker(\mathbf{L}(\iota)) = \mathbf{L}(\ker \iota) = \{0\}$, so that $\mathbf{L}(\iota)\colon \mathbf{L}(H) \to \mathbf{L}(G)$ is an injective morphism of topological Lie algebras, which implies in particular that $\mathfrak{h} := \mathrm{im}(\mathbf{L}(\iota))$ is a stable subalgebra of $\mathbf{L}(G)$ (Lemma IV.4.2). Moreover, (4.4.1) shows that the subgroup $\iota(H)$ of G coincides, as a set, with the subgroup $\langle \exp_G \mathfrak{h}\rangle$ of G generated by $\exp_G \mathfrak{h}$. Therefore a locally exponential integral subgroup can be viewed as a locally exponential Lie group structure on the subgroup of G generated by $\exp_G \mathfrak{h}$. ☐

Theorem IV.4.9. (Integral Subgroup Theorem) *Let G be a Lie group with a smooth exponential function and $\alpha\colon \mathfrak{h} \to \mathbf{L}(G)$ an injective morphism of topological Lie algebras, where $\mathfrak{h}$ is locally exponential. We assume that the closed subgroup*

$$\Gamma := \{x \in \mathfrak{z}(\mathfrak{h})\colon \exp_G(\alpha(x)) = \mathbf{1}\}$$

is discrete. Then there exists a locally exponential integral subgroup $\iota\colon H \to G$ with $\mathbf{L}(H) = \mathfrak{h}$ and $\mathbf{L}(\iota) = \alpha$. In particular, $\mathfrak{h}$ is integrable to a locally exponential Lie group. ☐

The discreteness of Γ is automatic in the following two special cases (*cf.* Problem II.4).

Corollary IV.4.10. *Let G be a Lie group with a smooth exponential function and $\mathfrak{h}$ locally exponential. Then any injective morphism $\alpha\colon \mathfrak{h} \to \mathbf{L}(G)$ integrates to a locally exponential integral subgroup if one of the following two conditions is satisfied:*

(1) $\mathfrak{z}(\mathfrak{h})$ is finite-dimensional.
(2) G is locally exponential. □

Corollary IV.4.10(1) is a substantial generalization of the main result of [Pe95b] which assumes that G is regular and $\mathfrak{h}$ is Banach. Other special cases can be found in many places in the literature, such as [MR95, Th. 2]. The versions given in [RK97, Th. 2] and [Rob97, Cor. 2] contradict the existence of unstable closed subalgebras in locally exponential Lie algebras (Example IV.4.6). For Banach–Lie groups it is contained in [EK64], and for BCH–Lie groups in [Rob97].

Remark IV.4.11. (a) In [Rob97], Robart gives a criterion for the existence of integral subgroups of a locally exponential Lie group G for a prescribed injective morphism $\alpha\colon \mathfrak{h} \to \mathfrak{g} = \mathbf{L}(G)$: The Lie algebra morphism α can be integrated to an integral subgroup if and only if $\mathfrak{h}/\mathfrak{z}(\mathfrak{h})$ is the Lie algebra of a locally exponential Lie group isomorphic to $H_{\rm ad} := \langle e^{\operatorname{ad}\mathfrak{h}} \rangle \subseteq \operatorname{Aut}(\mathfrak{h})$ with exponential function as in (4.3.1). In view of Theorems IV.2.10 and IV.3.8, for Mackey complete Lie algebras, this condition is equivalent to $\mathfrak{h}$ being locally exponential. This argument shows in particular, that Robart's concept of a *Lie algebra of the first kind* coincides with our concept of a locally exponential Lie algebra. In the light of this remark, Theorem 5 in [Rob97] can be read as a version of our Corollary IV.4.10(2), whereas Corollary IV.4.10(1) corresponds to his Theorem 8. We do not understand the precise meaning of his remark concerning a generalization to the case where $\mathfrak{z}(\mathfrak{h})$ is infinite-dimensional by simply refining the topology.

(b) Even for a closed subalgebra $\mathfrak{h} \subseteq \mathfrak{g} := \mathbf{L}(G)$, the condition that it is locally exponential is quite subtle. It means that for x,y sufficiently close to 0 in $\mathfrak{h}$, we have $x * y \in \mathfrak{h}$. If $\mathfrak{g}$ is BCH and $\mathfrak{h}$ is closed, this is clearly satisfied, but if $\mathfrak{g}$ is not BCH, not every closed subalgebra satisfies this condition because it implies stability (Example IV.4.5). □

To verify this condition, one would like to show that the integral curve $\gamma(t) := x * ty$ of the left invariant vector field y_l through x does not leave the closed subspace $\mathfrak{h}$ of $\mathfrak{g}$. This leads to the necessary condition $T_0(\lambda_x)(\mathfrak{h}) \subseteq \mathfrak{h}$, which, under the assumption that $\mathfrak{h}$ is stable, means that the operator $\kappa_{\mathfrak{g}}(x) = \int_0^1 e^{-t\operatorname{ad}x}\,dt$ satisfies $\kappa_{\mathfrak{g}}(x)(\mathfrak{h}) = \mathfrak{h}$ for $x \in \mathfrak{h}$ sufficiently close to 0 (*cf.* Theorem IV.3.8 and Problem IV.5 below).

Example IV.4.12. (a) Applying Corollary IV.4.10 to the CIA $\widehat{F}$ obtained by completing the free associative algebra in n generators $x_1, \ldots, x_n$ (Example IV.1.15), it follows that the closed Lie subalgebra generated by $x_1, \ldots, x_n$, i.e., the completion of the free Lie algebra, integrates to a subgroup. As $\widehat{F}$ is topologically isomorphic to $\mathbb{R}^{\mathbb{N}}$, each closed subspace is complemented ([HoMo98, Th. 7.30(iv)]), so that the existence of the corresponding integral subgroup could also be obtained by the methods developed in [Les92, Sect. 4] which require complicated assumptions on groups and Lie algebras.

(b) If K is a Banach–Lie group with Lie algebra $\mathfrak{k}$ and M a compact manifold, then the group $C^\infty(M,K)$ is BCH (Theorem IV.1.12), so that the Integral Subgroup Theorem also applies to each closed subalgebra $\mathfrak{h} \subseteq C^\infty(M,\mathfrak{k})$ (*cf.* [Les92, Sect. 4]). □

Remark IV.4.13. (a) In [La99], S. Lang calls a subgroup H of a Banach–Lie group G a "Lie subgroup" if H carries a Banach–Lie group structure for which there exists an immersion $\eta \colon H \to G$. This requires the Lie algebra $\mathbf{L}(H)$ of H to be a closed subalgebra of $\mathbf{L}(G)$ which is complemented in the sense that there exists a closed vector space complement (*cf.* Remark I.2.7). From that, it follows that his Lie subgroups coincide with the integral subgroups with closed complemented Lie algebra (*cf.* Corollary IV.4.10).

The advantage of Lang's more restrictive concept is that for a closed complemented Lie subalgebra $\mathfrak{h} \subseteq \mathbf{L}(G)$ one obtains the existence of corresponding integral subgroups from the Frobenius Theorem for Banach manifolds ([La99, Th. VI.5.4]). But it excludes in particular closed non-complemented subspaces of Banach spaces.

(b) The most restrictive concept of a Lie subgroup is the one used in [Bou89, Ch. 3], where a "Lie subgroup" of a Banach–Lie group G is a Banach–Lie subgroup H with the additional property that $\mathbf{L}(H)$ is complemented, i.e., H is required to be a split submanifold of G. This concept has the advantage that it implies that the quotient space G/H carries a natural manifold structure for which the quotient map $q \colon G \to G/H$ is a submersion ([Bou89, Ch. 3, §1.6, Prop. 11]). However, the condition that $\mathbf{L}(H)$ is complemented is very hard to check in concrete situations and, as the Quotient Theorem and the Integral Subgroup Theorem show, not necessary.

(c) For closed subalgebras which are not necessarily complemented, the Integral Subgroup Theorem can already be found in [Mais62] who also shows that kernels are Banach–Lie subgroups and that G/N is a Lie group if N is a Banach–Lie subgroup with complemented Lie algebra as in (b). This case is also dealt with in [Hof68], [Hof75, Th. 4.1], and a local version can be found in [Lau56]. □

The following theorem generalizes [Hof75, Prop. 4.3] from Banach–Lie groups to locally exponential ones, which is quite straightforward ([GN06]). The necessity of the separability assumption follows from Example IV.3.7.

Theorem IV.4.14. (Initiality Theorem for integral subgroups) *Let G be a Lie group with a smooth exponential function* $\exp_G\colon \mathbf{L}(G) \to G$ *which is injective on some* 0*-neighborhood. Further let $\iota_H\colon H \hookrightarrow G$ be a locally exponential integral subgroup whose Lie algebra $\mathbf{L}(H)$ is separable. Then the subgroup $\iota_H(H)$ of G satisfies*

$$\mathbf{L}^e(\iota_H(H)) = \{x \in \mathbf{L}(G)\colon \exp_G(\mathbb{R}x) \subseteq \iota_H(H)\} = \mathrm{im}(\mathbf{L}(\iota_H)).$$

In particular, the surjectivity of ι_H implies the surjectivity of $\mathbf{L}(\iota_H)$.

If, in addition, G is locally exponential and $\mathbf{L}(H)$ is a closed subalgebra of $\mathbf{L}(G)$, then $\iota_H\colon H \to G$ is an initial Lie subgroup of G. □

Theorem IV.4.15. ([Hof75, Prop. 4.6]) *Let G be a separable Banach–Lie group and assume that $\iota_H\colon H \to G$ is an integral subgroup with closed range. Then ι_H is an embedding. In particular, H is a Banach–Lie subgroup of G.* □

We conclude this section with a discussion of initial Lie subgroup structures on closed subgroups of Banach–Lie groups and locally convex spaces.

Theorem IV.4.16. (Initiality Theorem for closed subgroups of Banach–Lie groups) *Let G be a Banach–Lie group and $H \subseteq G$ a closed subgroup. Then H carries the structure of an initial Lie subgroup with Lie algebra $\mathbf{L}^e(H) = \mathbf{L}^d(H)$. Its identity component is an integral Lie subgroup for the closed Lie subalgebra $\mathbf{L}^e(H)$ of $\mathbf{L}(G)$.*

Proof. (Idea) We know from Lemma IV.3.1 that $\mathbf{L}^d(H) = \mathbf{L}^e(H)$ is a closed Lie subalgebra of $\mathbf{L}(G)$. Let $\iota\colon H_0 \to G$ be the corresponding integral Lie subgroup. Since each smooth curve $\gamma\colon [0,1] \to \iota(H_0) \subseteq H$ satisfies $\delta(\gamma)(t) \in \mathbf{L}^d(H)$ for each t and H_0 is regular (Remark II.5.4), γ is smooth as a curve to H_0, and this further permits us to conclude that H_0 is initial and coincides with the smooth arc–component of H. Now one uses Corollary II.2.3 to extend the Lie group structure to all of H. □

Theorem IV.4.17. (Initiality Theorem for closed subgroups of locally convex spaces) *Let E be a locally convex space and $H \subseteq (E,+)$ a closed subgroup. Then H carries an initial Lie group structure, for which $H_0 = \mathbf{L}^d(H) = \mathbf{L}^e(H)$ is the largest vector subspace contained in H.*

Proof. (Idea) For each curve $\alpha \in C^1_*(I,E)$ with $\mathrm{im}(\alpha) \subseteq H$ we have $tx = \lim_{n\to\infty} n\alpha(\frac{t}{n}) \in H$, which leads to $\mathbf{L}^d(H) = \mathbf{L}^e(H)$, a closed subspace of E. For each C^1-curve $\gamma\colon [0,1] \to E$ with range in H, all tangent vectors lie in $\mathbf{L}^d(H)$. This implies that γ lies in a coset of $\mathbf{L}^d(H)$. Defining the Lie group structure in such a way that $\mathbf{L}^d(H)$ becomes an open subgroup of H, it follows easily that H is initial (Corollary II.2.3). □

Remark IV.4.18. (Stability and distributions) (a) For a subset $D \subseteq \mathcal{V}(M)$, we call the subset $\Delta_D \subseteq T(M)$ defined by $\Delta_D(m) := \operatorname{span}\{X(m)\colon X \in D\}$ the corresponding *smooth distribution*. Conversely, we associate to a (smooth) distribution $\Delta \subseteq T(M)$ the subspace $D_\Delta := \{X \in \mathcal{V}(M)\colon (\forall m \in M)\ X(m) \in \Delta\}$. The distribution Δ is said to be *involutive* if D_Δ is a Lie subalgebra of $\mathcal{V}(M)$. A smooth distribution Δ_D is called *D-invariant* if it is preserved by the local flows generated by elements of D, and *integrable* it possesses (maximal) integral submanifolds through each point of M.

Sussman's Theorem asserts that Δ_D is integrable if and only if it is D-invariant ([Sus73, Th. 4.2]). As a special case, where all subspaces $\Delta_D(m)$, $m \in M$, are of the same dimension, we obtain Frobenius' Theorem. The invariance condition on Δ_D implies that it is involutive, but the converse does not hold. E.g. consider on $M = \mathbb{R}^2$ the set D, consisting of two vector fields

$$\frac{\partial}{\partial x_1}, \quad f(x_1)\frac{\partial}{\partial x_2} \quad \text{with} \quad f^{-1}(0) =]-\infty, 0]$$

(see also Example IV.4.4).

If M is analytic and D consists of analytic vector fields, then Nagano shows in [Naga66] that the involutivity of the corresponding distribution is sufficient for the existence of integral submanifolds.

(b) The invariance condition for a distribution is quite analogous to the stability condition for a Lie subalgebra $\mathfrak{h} \subseteq \mathbf{L}(G)$. If, furthermore, G is analytic with an analytic exponential function, then each closed subalgebra is stable, which is analogous to Nagano's result (*cf.* Lemma IV.4.3(b)).

To relate this to stability of Lie algebras of vector fields, assume that M is compact. If $\mathfrak{h} \subseteq \mathcal{V}(M)$ is a stable subalgebra (not necessarily closed), then the corresponding distribution $\Delta_{\mathfrak{h}}$ is stable, and Sussman's Theorem implies that its maximal integral submanifolds are the orbits of the subgroup $H := \langle \exp_{\mathrm{Diff}(M)}(\mathfrak{h}) \rangle$ of $\mathrm{Diff}(M)$ generated by the flows of elements of $\mathfrak{h}$ ([KYMO85, Sect. 3.1]).

If, conversely, we start with a smooth distribution Δ, then the closed space $\mathfrak{h}_\Delta$ of all vector fields with values in Δ is a Lie subalgebra if and only if Δ is involutive. Furthermore, it is not hard to see that Sussman's Theorem implies that Δ is involutive if and only if $\mathfrak{h}_\Delta$ is stable. If this is the case, the corresponding subgroup H_Δ of $\mathrm{Diff}(M)$ satisfies $\mathfrak{h}_\Delta \subseteq \mathbf{L}^e(H_\Delta) \subseteq \mathbf{L}^d(H_\Delta) \subseteq \mathfrak{h}_\Delta$. Now it is a natural question whether H_Δ carries the structure of a Lie group. For more details and related examples, we refer to Section 3.1 in [KYMO85]. $\square$

Open Problems for Section IV

Problem IV.1. Show that for each subgroup H of a locally exponential Lie group G, the set $\mathbf{L}^e(H) = \{x \in \mathbf{L}(G)\colon \exp_G(\mathbb{R}x) \subseteq H\}$ is a Lie subalgebra of G or find a counterexample.

If G is finite-dimensional, then Yamabe's Theorem implies that the arc-component of H is an integral subgroup (Remark II.6.5) which proves the assertion in this case. If H is closed, then $\mathbf{L}^e(H)$ is a Lie subalgebra by Lemma IV.3.1.

We further observe that $\mathbf{L}^e(H)$ is invariant under all operators $\mathrm{Ad}(\exp_G(x)) = e^{\mathrm{ad}\,x}$ for $x \in \mathbf{L}^e(H)$. If $\mathbf{L}^e(H)$ is closed (which is the case for each closed subgroup) and closed under addition, then it is a closed vector subspace of $\mathbf{L}(G)$, and for $x, y \in \mathfrak{h}$ it contains the derivative of the curves $t \mapsto e^{t\,\mathrm{ad}\,x}y$ in 0. This implies that it is a Lie subalgebra. □

Problem IV.2. Show that for each closed subgroup H of a locally exponential Lie group G the closed Lie subalgebra $\mathbf{L}^e(H) \subseteq \mathbf{L}(G)$ is locally exponential. □

Problem IV.3. Find an example of a locally exponential normal Lie subgroup N of a locally exponential Lie group G for which $\mathbf{L}(G)/\mathbf{L}(N)$ is not locally exponential or prove that it always is. In view of Theorem IV.3.5, this would imply that G/N is locally exponential. □

Problem IV.4. (One-parameter groups and local exponentiality) Let $\alpha\colon \mathbb{R} \to \mathrm{GL}(E)$ define a smooth action of $\mathbb{R}$ on the Mackey complete locally convex space and $D := \alpha'(0)$ be its infinitesimal generator. We then obtain a 2-step solvable Lie group $G := E \rtimes_\alpha \mathbb{R}$ with the product

$$(v,t)(v',t') = (v + \alpha(t).v', t + t')$$

and the Lie algebra $\mathfrak{g} = E \rtimes_D \mathbb{R}$. Characterize local exponentiality of G in terms of the infinitesimal generator D.

Writing the exponential function as $\exp_G(v,t) = (\beta(t).v, t)$ with $\beta(t) = \int_0^1 \alpha(st)\,ds$, we obtain the curve $\beta\colon \mathbb{R} \to \mathscr{L}(E)$. We are looking for a characterization of those operators D for which there exists some $T > 0$ such that

(1) $\beta(]-T,T[) \subseteq \mathrm{GL}(E)$, and
(2) $\widetilde{\beta}\colon]-T,T[\times E \to E, (t,v) \mapsto \beta(t)^{-1}v$ is smooth.

Note that $(t,v) \mapsto \beta(t)v$ is always smooth. If E is a Banach space, then G is a Banach–Lie group, hence locally exponential. In this case, D is a bounded operator and we have for each $t \neq 0$:

$$\beta(t) = \frac{1}{t}\int_0^t e^{sD}\,ds = \frac{1}{t}\frac{e^{tD} - \mathbf{1}}{D} = \frac{e^{tD} - \mathbf{1}}{tD} = \sum_{k=0}^{\infty} \frac{t^k}{(k+1)!} D^k.$$

As $\beta\colon \mathbb{R} \to \mathscr{L}(E)$ is analytic w.r.t. to the operator norm on $\mathscr{L}(E)$, $\beta(0) = \mathbf{1}$, and $\mathrm{GL}(E)$ is open, conditions (1) and (2) follow immediately. Moreover, the Spectral Theorem implies that

$$\mathrm{Spec}(\beta(t)) = \left\{ \frac{e^{t\lambda} - 1}{t\lambda} : \lambda \in \mathrm{Spec}(D) \right\},$$

which means that $\beta(t)$ is invertible for $|t| < \frac{2\pi}{\sup\{\mathrm{Im}(\lambda)\colon \lambda \in \mathrm{Spec}(D)\}}$. □

Problem IV.5. (Invariant subspaces) Let $\alpha\colon \mathbb{R} \to \mathrm{GL}(E)$ be a smooth action of $\mathbb{R}$ on the Mackey complete locally convex space E and $\beta(t)$ as in Problem IV.4.

Suppose that $F \subseteq E$ is a closed invariant subspace. Then we also have $\beta(t)(F) \subseteq F$ for each $t \in \mathbb{R}$. Assume that for some $\varepsilon > 0$ the operator $\beta(t)$ is invertible for $|t| \leq \varepsilon$. Show that $\beta(t)^{-1}(F) \subseteq F$ for $|t| \leq \varepsilon$ or find a counterexample. Note that this is trivially the case if F is of finite dimension or codimension. □

Problem IV.6. Show that BCH–Lie groups are regular. In [Rob04], Robart has obtained substantial results in this direction, including that for each BCH–Lie group G with Lie algebra $\mathfrak{g}$ and each smooth path $\xi \in C^\infty(I, \mathfrak{g})$, the initial value problem

$$\eta(0) = \mathrm{id}_{\mathfrak{g}}, \quad \eta'(t) = [\eta(t), \xi(t)]$$

has a solution in $\mathscr{L}(\mathfrak{g})$. Unfortunately, it is not clear whether these solutions define curves in $\mathrm{GL}(\mathfrak{g})$. □

Problem IV.7. Show that each nilpotent topological group is a topological group with Lie algebra in the sense of Definition IV.1.23 (*cf.* Theorem IV.1.24). □

Problem IV.8. In Theorem IV.3.3, we have seen that a locally exponential Lie subgroup H of a locally exponential Lie group G is a submanifold, where the submanifold chart in the identity can be obtained from the exponential function of G.

It is an interesting question whether every Lie group H which is a submanifold of a locally exponential Lie group G is in fact a locally exponential Lie group. This is true if G is a Banach–Lie group because this property is inherited by every subgroup which is a submanifold.

This point also concerns the use of the term “Lie subgroup,” which would also be natural for subgroups which are submanifolds. □

Problem IV.9. Develop a theory of algebraic subgroups for CIAs in the context of locally exponential, resp., BCH–Lie groups. A typical question such a theory should answer is: For which linear actions of a locally exponential Lie group G on a locally convex space E are the stabilizers G_v, $v \in E$, locally exponential Lie subgroups? □

Problem IV.10. Show that Theorems IV.4.15/16 remain valid for locally exponential Lie groups. □

Problem IV.11. Let G be a regular Lie group and $H \subseteq G$ a closed subgroup. Then $\mathbf{L}^e(H)$ is a closed subset of $\mathbf{L}(G)$, stable under scalar multiplication. On the other hand, $\mathbf{L}^d(H)$ is a Lie subalgebra containing $\mathbf{L}^e(H)$. Do these two sets always coincide? If, in addition, G is μ-regular, this follows from Lemma III.2.7. □

Problem IV.12. (a) Let G be a Lie group with a smooth exponential function. Find examples where the Trotter Formula and/or the Commutator Formula (Lemma IV.1.17) do not hold. For which classes of groups (beyond the locally exponential ones) are these formulas, or the more general Lemma III.2.7, valid? What about the group $\mathrm{Diff}_c(M)$?

(b) What can be said about the sequence of power maps $p_n(x) = x^n$ in a local Lie group? In local coordinates with 0 as neutral element, it is interesting to consider for an element x the sequence $\left(\frac{x}{n}\right)^n$. Does it converge (to x)? □

Problem IV.13. Let Δ be an integrable distribution on the compact manifold M and H_Δ the corresponding subgroup of $\mathrm{Diff}(M)$ preserving the maximal integral submanifolds of Δ (Remark IV.4.18(b)). Show that H_Δ is a regular Lie group. □

Problem IV.14. (*cf.* [Rob97, p.837, Prop. 3]) Let G be a μ-regular Lie group and $\mathfrak{h} \subseteq \mathbf{L}(G)$ a closed stable subalgebra. Does $\mathfrak{h}$ integrate to an integral Lie subgroup? Since product integrals converge in G, for two smooth curves $\alpha, \beta \colon [0,1] \to \mathfrak{h}$, the curve

$$(\alpha * \beta)(t) := \alpha(t) + \mathrm{Ad}(\gamma_\alpha(t)).\beta(t)$$

has values in $\mathfrak{h}$, which leads to a Lie group structure on $C^\infty(I, \mathfrak{h})$ with Lie algebra $C^\infty([0,1], \mathfrak{h})$, where the bracket is given by

$$[\xi, \eta](t) := \left[\xi(t), \int_0^t \eta(\tau)\, d\tau\right] + \left[\int_0^t \xi(\tau)\, d\tau, \eta(t)\right]$$

([Rob97, Th. 9]). The map $E \colon \alpha \mapsto \gamma_\alpha(1)$ is a group homomorphism, so that the problem is to see that the quotient group $(C^\infty(I, \mathfrak{h}), *)/\ker E \cong \mathrm{im}(E)$ carries a natural Lie group structure. □

V. Extensions of Lie groups

In this section, we turn to some general results on extensions of infinite-dimensional Lie groups. In Section V.1, we explain how an extension of G by N is described in terms of data associated to G and N. This description is easily adapted from the abstract group theoretic setting (*cf.* [ML68]). In Section V.2, we describe the appropriate cohomological setup for Lie theory and explain criteria for the integrability of Lie algebra cocycles to group cocycles. This is applied in Section V.3 to integrate abelian extensions of Lie algebras to corresponding group extensions.

V.1. General extensions

Definition V.1.1. *An* extension of Lie groups *is a short exact sequence*

$$\mathbf{1} \to N \xrightarrow{\iota} \widehat{G} \xrightarrow{q} G \to \mathbf{1}$$

of Lie group morphisms, for which $\widehat{G}$ is a smooth (locally trivial) principal N-bundle over G with respect to the right action of N given by $(\widehat{g}, n) \mapsto \widehat{g}n$. In the following, we identify N with the subgroup $\iota(N) \trianglelefteq \widehat{G}$.

We call two extensions $N \hookrightarrow \widehat{G}_1 \twoheadrightarrow G$ and $N \hookrightarrow \widehat{G}_2 \twoheadrightarrow G$ of the Lie group G by the Lie group N equivalent *if there exists a Lie group morphism $\varphi\colon \widehat{G}_1 \to \widehat{G}_2$ such that the following diagram commutes:*

$$\begin{array}{ccccc} N & \hookrightarrow & \widehat{G}_1 & \twoheadrightarrow & G \\ \downarrow{\scriptstyle \mathrm{id}_N} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}_G} \\ N & \hookrightarrow & \widehat{G}_2 & \twoheadrightarrow & G. \end{array}$$

It is easy to see that any such φ is an isomorphism of Lie groups and that we actually obtain an equivalence relation. We write $\mathrm{Ext}(G,N)$ *for the set of equivalence classes of Lie group extensions of G by N.*[5]

We call an extension $q\colon \widehat{G} \to G$ with $\ker q = N$ split *if there exists a Lie group morphism $\sigma\colon G \to \widehat{G}$ with $q \circ \sigma = \mathrm{id}_G$. This implies that $\widehat{G} \cong N \rtimes_S G$ for $S(g)(n) := \sigma(g)n\sigma(g)^{-1}$.* □

Remark V.1.2. A Lie group extension $N \hookrightarrow \widehat{G} \twoheadrightarrow G$ can also be described in terms of data associated to G and N as follows: Let $q\colon \widehat{G} \to G$ be a Lie group extension of G by N. By assumption, the map q has a smooth local section. Hence there exists a global section $\sigma\colon G \to \widehat{G}$ smooth in an identity neighborhood and *normalized* by $\sigma(\mathbf{1}) = \mathbf{1}$. Then the map

$$\Phi\colon N \times G \to \widehat{G}, \quad (n,g) \mapsto n\sigma(g)$$

[5] From the description of Lie group extensions as in Theorem V.1.4 below, one obtains cardinality estimates showing that the equivalence classes actually form a set.

is a bijection which restricts to a local diffeomorphism on an identity neighborhood. In general, Φ is not continuous, but we may nevertheless use it to identify $\widehat{G}$ with the product set $N \times G$, endowed with the multiplication

$$(n,g)(n',g') = (nS(g)(n')\omega(g,g'), gg'), \tag{5.1.1}$$

where

$$S := C_N \circ \sigma \colon G \to \operatorname{Aut}(N) \quad \text{for} \quad C_N \colon \widehat{G} \to \operatorname{Aut}(N),\ C_N(g) = gng^{-1}, \tag{5.1.2}$$

and

$$\omega \colon G \times G \to N, \quad (g,g') \mapsto \sigma(g)\sigma(g')\sigma(gg')^{-1}. \tag{5.1.3}$$

Note that ω is smooth in an identity neighborhood and that the map $\widehat{S} \colon G \times N \to N, (g,n) \mapsto S(g)(n)$ is smooth in a set of the form $U_G \times N$, where U_G is an identity neighborhood of G. The maps S and ω satisfy the relations

(C1) $\sigma(g)\sigma(g') = \omega(g,g')\sigma(gg')$,
(C2) $S(g)S(g') = C_N(\omega(g,g'))S(gg')$,
(C3) $\omega(g,g')\omega(gg',g'') = S(g)\big(\omega(g',g'')\big)\omega(g,g'g'')$. □

Definition V.1.3. *Let G and N be Lie groups. A* smooth outer action of G on N *is a map $S \colon G \to \operatorname{Aut}(N)$ with $S(\mathbf{1}) = \mathrm{id}_N$ for which*

$$\widehat{S} \colon G \times N \to N, \quad (g,n) \mapsto S(g)(n)$$

is smooth on a set of the form $U_G \times N$, where $U_G \subseteq G$ is an open identity neighborhood, and for which there exists a map $\omega \colon G \times G \to N$ with $\omega(\mathbf{1},\mathbf{1}) = \mathbf{1}$, smooth in an identity neighborhood, such that (C2) holds. We call (S,ω) a locally smooth non-abelian 2-cocycle.

We define an equivalence relation on the set of all smooth outer actions of G on N by $S' \sim S$ if $S' = (C_N \circ \alpha) \cdot S$ for some map $\alpha \colon G \to N$ with $\alpha(\mathbf{1}) = \mathbf{1}$, smooth in an identity neighborhood. We write $[S]$ for the equivalence class of S. □

Remark V.1.2 implies that for each extension $q \colon \widehat{G} \to G$ of Lie groups and any section $\sigma \colon G \to \widehat{G}$ which is smooth in an identity neighborhood with $\sigma(\mathbf{1}) = \mathbf{1}$, (5.1.2) defines a smooth outer action of G on N. Different choices of such sections lead to equivalent outer actions.

Theorem V.1.4. *Let G be a connected Lie group, N a Lie group and (S,ω) a smooth outer action of G on N. Then (5.1.1) defines a group structure on $N \times G$ if and only if (C3) holds. If this is the case, then this group carries a unique Lie group structure, denoted $N \times_{(S,\omega)} G$, for which the identity $N \times G \to N \times_{(S,\omega)} G$ is smooth in an identity neighborhood and*

$$q \colon N \times_{(S,\omega)} G \to G, \quad (n,g) \mapsto g$$

defines a Lie group extension of G by N. □

All Lie group extensions of G by N arise in this way, so that we obtain a partition

$$\mathrm{Ext}(G,N) = \bigcup_{[S]} \mathrm{Ext}(G,N)_{[S]},$$

where $\mathrm{Ext}(G,N)_{[S]}$ denote the set of equivalence classes of extensions corresponding to the equivalence class $[S]$.

If N is abelian, then each class $[S]$ contains a unique representative S, which is a smooth action of G on N. Fixing S, the set $\mathrm{Ext}(G,N)_S$ carries a natural abelian group structure, where the addition is given by the *Baer sum*: For two extensions $q_1\colon \widehat{G}_1 \to G$, $q_2\colon \widehat{G}_2 \to G$ of G by N, the Baer sum is defined by

$$\widehat{G} := \{(\widehat{g}_1, \widehat{g}_2) \in \widehat{G}_1 \times \widehat{G}_2 \colon q_1(\widehat{g}_1) = q_2(\widehat{g}_2)\}/\Delta'_N, \quad \Delta'_N := \{(n,n^{-1})\colon n \in N\},$$

and the projection map $q(\widehat{g}_1, \widehat{g}_2) := q_1(\widehat{g}_1)$. This defines an abelian group structure on the set $\mathrm{Ext}(G,N)_S$ whose neutral element is the class of the split extension $\widehat{G} = N \rtimes_S G$ (*cf.* [ML63, Sect. IV.4]). In Theorem V.2.8 below, we shall recover this group structure in terms of group cohomology.

Theorem V.1.5. ([Ne05]) *Assume that $Z(N)$ carries an initial Lie subgroup structure (Remark II.6.5). Then each class $[S]$ determines a smooth G-action on $Z(N)$ by $g.z := S(g)(z)$ and the abelian group* $\mathrm{Ext}(G,Z(N))_S$ *acts simply transitively on* $\mathrm{Ext}(G,N)_{[S]}$ *by*

$$[H] * [\widehat{G}] := [(\alpha^*\widehat{G})/\Delta'_{Z(N)}],$$

where $\alpha\colon H \to G$ is a Lie group extension of G by the G-module $Z(N)$,

$$\alpha^*\widehat{G} = \{(\widehat{g}_1, \widehat{g}_2) \in H \times \widehat{G}\colon \alpha(\widehat{g}_1) = q(\widehat{g}_2)\} \quad \textit{and} \quad \Delta'_{Z(N)} := \{(n,n^{-1})\colon n \in Z(N)\}.$$

□

Examples V.1.6. Interesting classes of extensions of Lie groups arise as follows.

(a) Projective unitary representations: Let H be a complex Hilbert space, $\mathrm{U}(H)$ its unitary group with center $Z(\mathrm{U}(H)) = \mathbb{T}\mathbf{1}$, and $\mathrm{PU}(H) := \mathrm{U}(H)/\mathbb{T}\mathbf{1}$ the projective unitary group (all these groups are Banach–Lie groups). If H is a complex Hilbert space and $\pi\colon G \to \mathrm{PU}(H)$ a projective representation of the Lie group G with at least one smooth orbit in the projective space $\mathbb{P}(H)$, then the pull-back diagram

$$\begin{array}{ccccc} \mathbb{T} = \mathbb{R}/\mathbb{Z} & \hookrightarrow & \mathrm{U}(H) & \twoheadrightarrow & \mathrm{PU}(H) \\ \uparrow = & & \uparrow & & \uparrow \pi \\ \mathbb{T} & \hookrightarrow & \widehat{G} & \twoheadrightarrow & G \end{array}$$

defines a central Lie group extension of G by the circle group (*cf.* [Lar99]). This leads to a partition of the set of equivalence classes of projective unitary representations according to the set $\mathrm{Ext}(G,\mathbb{T})$ of central extensions of G by $\mathbb{T}$ (*cf.* [Ne02a]).

(b) Hamiltonian actions: Let G be a connected Lie group, M a locally convex manifold and $\omega \in \Omega^2(M,\mathbb{R})$ a closed 2-form which is the curvature of a pre-quantum line bundle $p\colon P \to M$ with connection 1-form $\theta \in \Omega^1(P,\mathbb{R})$. Assume further that $\sigma\colon G \to \mathrm{Diff}(M)$ defines a smooth action of G on M for which all associated vector fields $\dot\sigma(x) \in \mathcal{V}(M)$ are Hamiltonian, i.e., the closed 1-forms $i_{\dot\sigma(x)}\omega$ are exact. Then the range of the map $\sigma\colon G \to \mathrm{Diff}(M)$ lies in the group $\mathrm{Ham}(M,\omega)$ of Hamiltonian automorphisms of (M,ω), and the diagram

$$\begin{array}{ccccc} \mathbb{T} & \hookrightarrow & \mathrm{Aut}(P,\theta) & \twoheadrightarrow & \mathrm{Ham}(M,\omega) \\ \uparrow{=} & & \uparrow & & \sigma\uparrow \\ \mathbb{T} & \hookrightarrow & \widehat{G} & \twoheadrightarrow & G \end{array}$$

defines a central Lie group extension of G by $\mathbb{T}$ (*cf.* [Kos70], [RS81], [NV03]).

(c) (Extensions by gauge groups) Let $q\colon P \to M$ be a smooth K-principal bundle, where M is compact and K is locally exponential (*cf.* Theorem IV.1.12). Further, let $\sigma\colon G \to \mathrm{Diff}(M)$ define a smooth action of G on M, whose range lies in the subgroup $\mathrm{Diff}(M)_{[P]}$ of diffeomorphisms φ with $\varphi^*P \sim P$, i.e., fixing the equivalence class $[P]$. Then the diagram

$$\begin{array}{ccccc} \mathrm{Gau}(P) & \hookrightarrow & \mathrm{Aut}(P) = \mathrm{Diff}(P)^K & \twoheadrightarrow & \mathrm{Diff}(M)_{[P]} \\ \uparrow{=} & & \uparrow & & \sigma\uparrow \\ \mathrm{Gau}(P) & \hookrightarrow & \widehat{G} & \twoheadrightarrow & G \end{array}$$

defines an extension of G by the gauge group $\mathrm{Gau}(P)$ of this bundle.

(d) If $q\colon \mathbb{V} \to M$ is a finite-dimensional vector bundle and $K = \mathrm{GL}(V)$, the preceding remark applies to the corresponding frame bundle, and leads to the diagram

$$\begin{array}{ccccc} \mathrm{Gau}(\mathbb{V}) & \hookrightarrow & \mathrm{Aut}(\mathbb{V}) & \twoheadrightarrow & \mathrm{Diff}(M)_{[\mathbb{V}]} \\ \uparrow{=} & & \uparrow & & \sigma\uparrow \\ \mathrm{Gau}(\mathbb{V}) & \hookrightarrow & \widehat{G} & \twoheadrightarrow & G\,. \end{array}$$

In view of [KYMO85, p.89], the Lie group $\mathrm{Aut}(\mathbb{V})$ is μ-regular if M is compact.

(e) (Non-commutative generalizations; *cf.* [GrNe06], [KYMO85, Sect. 3.2]) Let A be a CIA and E a finitely generated projective right A-module. Then the group $\mathrm{GL}_A(E)$ of A-linear endomorphisms of E is a Lie group, which is a subgroup of the larger group

$$\Gamma\mathrm{L}(E) := \{\varphi \in \mathrm{GL}(E)\colon (\exists \varphi_A \in \mathrm{Aut}(A))(\forall s \in E)(\forall a \in A)\ \varphi(s.a) = \varphi(s).\varphi_A(a)\}$$

of semilinear automorphisms of E. These are the linear automorphisms $\varphi \in \mathrm{GL}(E)$ for which there exists an automorphism φ_A of A with $\varphi(s.a) = \varphi(s).\varphi_A(a)$

for $s \in E, a \in A$. Then for each homomorphism $\sigma\colon G \to \mathrm{Aut}(A)$, G a connected Lie group, whose range lies in the set $\mathrm{Aut}(A)_E$ of those automorphisms of A preserving E by pull-backs, the diagram

$$\begin{array}{ccccc} \mathrm{GL}_A(E) & \hookrightarrow & \Gamma\mathrm{L}(E) & \twoheadrightarrow & \mathrm{Aut}(A)_E \\ \uparrow{=} & & \uparrow & & \sigma\uparrow \\ \mathrm{GL}_A(E) & \hookrightarrow & \widehat{G} & \twoheadrightarrow & G \end{array}$$

defines an extension of G by the linear Lie group $\mathrm{GL}_A(E)$.

For the special case $A = C^\infty(M, \mathbb{R})$, finitely generated projective modules correspond to vector bundles over M (*cf.* [Ros94], [Swa62]), so that (e) specializes to (d). □

Remark V.1.7. Non-abelian extensions of Lie groups also play a crucial role in the structural analysis of the group of invertible Fourier integral operators of order zero on a compact manifold M ([OMYK81], [ARS86a/b]), which is an extension of a group of symplectomorphisms of the complement of the zero section in the cotangent bundle $T^*(M)$ whose Lie algebra corresponds to smooth functions homogeneous of degree 1 on the fibers. □

The following theorem is an important tool to verify that given Lie groups are regular (*cf.* [KM97], [OMYK83a, Th. 5.4] in the context of μ-regularity, and [Rob04]). A variant of this result for ILB–Lie groups is Theorem 3.4 in [ARS86b].

Theorem V.1.8. *Let $\widehat{G}$ be a Lie group extension of the Lie group G by N. Then $\widehat{G}$ is regular if and only if the groups G and N are regular.* □

Remark V.1.9. A typical class of examples illustrating the difference between abelian and central extensions of Lie groups arises from abelian principal bundles. If $q\colon P \to M$ is a smooth principal bundle with the abelian structure group Z over the compact connected manifold M, then the group $\mathrm{Aut}(P) = \mathrm{Diff}(P)^Z$ of all diffeomorphisms of P commuting with Z (the automorphism group of the bundle) is an extension of the open subgroup $\mathrm{Diff}(M)_{[P]}$ of $\mathrm{Diff}(M)$ by the gauge group $\mathrm{Gau}(P) \cong C^\infty(M, Z)$ of the bundle (Example V.1.6(c)). Here the conjugation action of $\mathrm{Diff}(M)$ on $\mathrm{Gau}(P)$ is given by composing functions with diffeomorphisms. Central extensions corresponding to the bundle $q\colon P \to M$ are obtained by choosing a principal connection 1-form $\theta \in \Omega^1(P, \mathfrak{z})$. Let $\omega \in \Omega^2(M, \mathfrak{z})$ denote the corresponding curvature form. Then the subgroup $\mathrm{Aut}(P, \theta)$ of those elements of $\mathrm{Aut}(P)$ preserving θ is a central extension of an open subgroup of $\mathrm{Diff}(M, \omega)$, which is substantially smaller that $\mathrm{Diff}(M)$. This example shows that the passage from central extensions to abelian extensions is similar to the passage from symplectomorphism groups to diffeomorphism groups (see also [Ne06a]).

As the examples of principal bundles over compact manifolds show, abelian extensions of Lie groups occur naturally in geometric contexts and in particular in symplectic geometry, where the pre-quantization problem is to find for a symplectic manifold (M,ω) a $\mathbb{T}$-principal bundle with curvature ω, which leads to an abelian extension of $\mathrm{Diff}(M)_0$ by the group $C^\infty(M,\mathbb{T})$. Conversely, every abelian extension $q\colon \widehat{G} \to G$ of a Lie group G by an abelian Lie group A is in particular an A-principal bundle over G. This leads to an interesting interplay between abelian extensions of Lie groups and abelian principal bundles over (finite-dimensional) manifolds.

A shift from central to abelian extensions occurs naturally as follows: Suppose that a connected Lie group G acts on a smooth manifold M which is endowed with a Z-principal bundle $q\colon P \to M$ (Z an abelian Lie group) and that each element of G lifts to an automorphism of the bundle. If all elements of G lift to elements of the group $\mathrm{Aut}(P,\theta)$ for some principal connection 1-form θ, then we obtain a central extension as in Example V.1.6(b). But if there is no such connection 1-form θ, then we are forced to consider the much larger abelian extension of G by the group $\mathrm{Gau}(P) \cong C^\infty(M,Z)$ or at least some subgroup containing non-constant functions. The case where M is a restricted Graßmannian of a polarized Hilbert space and the groups are restricted operator groups of Schatten class $p > 2$, resp., mapping groups $C^\infty(M,K)$, where K is finite-dimensional and M is a compact manifold of dimension ≥ 2, is discussed in detail in [Mick89] (see also [PS86] for a discussion of related topics). □

V.2. Cohomology of Lie groups and Lie algebras

Any good setting for a cohomology theory on Lie groups should be fine enough to take the smooth structure into account and flexible enough to parameterize equivalence classes of group extensions. All these criteria are met by the locally smooth cohomology we describe in this subsection (*cf.* [Ne02a], [Ne04a]). The traditional approach in finite dimensions uses globally smooth cochains ([Ho51]), which is too restrictive in infinite dimensions.

From Lie group cohomology to Lie algebra cohomology

Definition V.2.1. *(a) Let $\mathfrak{g}$ be a topological Lie algebra and E a locally convex space. We call E a* topological $\mathfrak{g}$-module *if E is a $\mathfrak{g}$-module for which the action map $\mathfrak{g} \times E \to E$ is continuous.*

(b) Let G be a Lie group and A an abelian Lie group. We call A a smooth G-module *if it is endowed with a G-module structure defined by a smooth action map $G \times A \to A$.* □

Definition V.2.2. *Let $\mathfrak{g}$ be a topological Lie algebra and E a topological $\mathfrak{g}$-module. For $p \in \mathbb{N}_0$, let $C^p_c(\mathfrak{g},E)$ denote the space of continuous alternating maps $\mathfrak{g}^p \to E$, i.e., the* Lie algebra p-cochains with values in the module E. *We identify $C^0_c(\mathfrak{g},E)$ with E and put $C^\bullet_c(\mathfrak{g},E) := \bigoplus_{p=0}^\infty C^p_c(\mathfrak{g},E)$. We then obtain a cochain complex with the* Lie algebra differential $d_\mathfrak{g}\colon C^p_c(\mathfrak{g},E) \to C^{p+1}_c(\mathfrak{g},E)$ *given on $f \in C^p_c(\mathfrak{g},E)$ by*

$$(d_\mathfrak{g} f)(x_0,\ldots,x_p) := \sum_{j=0}^p (-1)^j x_j.f(x_0,\ldots,\widehat{x}_j,\ldots,x_p) + \sum_{i<j} (-1)^{i+j} f([x_i,x_j],x_0,\ldots,\widehat{x}_i,\ldots,\widehat{x}_j,\ldots,x_p),$$

where $\widehat{x}_j$ indicates omission of x_j ([ChE48]). In view of $d_\mathfrak{g}^2 = 0$, the space $Z^p_c(\mathfrak{g},E) := \ker(d_\mathfrak{g}\mid_{C^p_c(\mathfrak{g},E)})$ of p-cocycles *contains the space $B^p_c(\mathfrak{g},E) := d_\mathfrak{g}(C^{p-1}_c(\mathfrak{g},E))$ of* p-coboundaries. *The quotient*

$$H^p_c(\mathfrak{g},E) := Z^p_c(\mathfrak{g},E)/B^p_c(\mathfrak{g},E)$$

is the p-th continuous cohomology space of $\mathfrak{g}$ with values in the $\mathfrak{g}$-module E. *We write $[f] := f + B^p_c(\mathfrak{g},E)$ for the cohomology class of the cocycle f.* □

Definition V.2.3. *Let G be a Lie group and E a* smooth locally convex G-module*, i.e., a smooth G-module which is a locally convex space. We write*

$$\rho_E\colon G \times E \to E, \quad (g,v) \mapsto \rho_E(g,v) =: \rho_E(g)(v) =: g.v$$

for the action map. We call a p-form $\alpha \in \Omega^p(G,E)$ equivariant *if we have for each $g \in G$ the relation*

$$\lambda_g^*\alpha = \rho_E(g) \circ \alpha.$$

If E is a trivial module, then an equivariant form is a left invariant E-valued form on G.

We write $\Omega^p(G,E)^G$ for the subspace of equivariant p-forms in $\Omega^p(G,E)$ and note that this is the space of G-fixed elements with respect to the action given by $g.\alpha := \rho_E(g) \circ (\lambda_{g^{-1}})^*\alpha$. The subcomplex $(\Omega^\bullet(G,E)^G, d)$ of equivariant differential forms in the E-valued de Rham complex on G has been introduced in the finite-dimensional setting by Chevalley and Eilenberg in [ChE48].

Let $\mathfrak{g} := \mathbf{L}(G) \cong T_\mathbf{1}(G)$. An equivariant p-form α is uniquely determined by the corresponding element $\alpha_\mathbf{1} \in C^p_c(\mathfrak{g},E)$:

$$\alpha_g(g.x_1,\ldots,g.x_p) = \rho_E(g) \circ \alpha_\mathbf{1}(x_1,\ldots,x_p) \quad \text{for} \quad g \in G, x_i \in \mathfrak{g}. \tag{5.2.1}$$

Conversely, (5.2.1) provides for each $\omega \in C^p_c(\mathfrak{g},E)$ a unique equivariant p-form ω^{eq} on G with $\omega^{\mathrm{eq}}_\mathbf{1} = \omega$. □

The following observation is due to Chevalley/Eilenberg ([ChE48, Th. 10.1]). For an adaptation to the infinite-dimensional setting, we refer to [Ne04a].

Proposition V.2.4. *The evaluation maps*

$$\mathrm{ev}_{\mathbf{1}}\colon \Omega^p(G,E)^G \to C^p_c(\mathfrak{g},E), \quad \omega \mapsto \omega_{\mathbf{1}}$$

define an isomorphism from the cochain complex $(\Omega^\bullet(G,E)^G, d)$ *of equivariant* E*-valued differential forms on* G *to the continuous* E*-valued Lie algebra complex* $(C^\bullet_c(\mathfrak{g},E), d_{\mathfrak{g}})$. □

Definition V.2.5. *Let* A *be a smooth* G*-module and* $C^n_s(G,A)$ *denote the space of all functions* $f\colon G^n \to A$ *which are smooth in an identity neighborhood and normalized in the sense that* $f(g_1,\ldots,g_n)$ *vanishes if* $g_j = \mathbf{1}$ *holds for some* j*. We call these functions* normalized locally smooth group cochains*. The differential* $d_G\colon C^n_s(G,A) \to C^{n+1}_s(G,A)$*, defined by*

$$(d_G f)(g_0,\ldots,g_n) := g_0.f(g_1,\ldots,g_n) + \sum_{j=1}^{n} (-1)^j f(g_0,\ldots,g_{j-1}g_j,\ldots,g_n) + (-1)^{n+1} f(g_0,\ldots,g_{n-1}).$$

turns $(C^\bullet_s(G,A), d_G)$ *into a differential complex. We write* $Z^n_s(G,A) := \ker(d_G \mid_{C^n_s(G,A)})$ *for the corresponding group of cocycles,* $B^n_s(G,A) := d_G(C^{n-1}_s(G,A))$ *for the subgroup of coboundaries, and*

$$H^n_s(G,A) := Z^n_s(G,A)/B^n_s(G,A)$$

for the n-th locally smooth cohomology group with values in the smooth module A. □

Let $M_1,\ldots,M_k$ be smooth manifolds, A an abelian Lie group and $f\colon M_1 \times \cdots \times M_k \to A$ a smooth function. For $v_k \in T_{m_k}(M_k)$ we obtain a smooth function

$$\partial_k(v_k)f\colon M_1 \times \cdots \times M_{k-1} \to \mathfrak{a} := \mathbf{L}(A),$$
$$(m_1,\ldots,m_{k-1}) \mapsto \delta(f)_{(m_1,\ldots,m_k)}(0,\ldots,0,v_k).$$

Iterating this process, we obtain for each tuple $(m_1,\ldots,m_k) \in M_1 \times \cdots \times M_k$ a continuous k-linear map

$$T_{m_1}(M_1) \times \cdots \times T_{m_k}(M_k) \to \mathfrak{a}, \quad (v_1,\ldots,v_k) \mapsto \big(\partial_1(v_1)\cdots\partial_k(v_k)f\big)(m_1,\ldots,m_k).$$

The following theorem describes the natural map from Lie group to Lie algebra cohomology ([Ne04a, App. B]; see also [EK64]):

Theorem V.2.6. *For* $f \in C^n_s(G,A)$*,* $n \geq 1$*, and* $x_1,\ldots,x_n \in \mathfrak{g} \cong T_{\mathbf{1}}(G)$ *we put*

$$(D_n f)(x_1,\ldots,x_n) := \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\big(\partial_1(x_{\sigma(1)})\cdots\partial_n(x_{\sigma(n)})f\big)(\mathbf{1},\ldots,\mathbf{1}).$$

Then $D_n(f) \in C^n_c(\mathfrak{g},\mathfrak{a})$, and these maps induce a morphism of cochain complexes

$$D\colon (C^n_s(G,A),d_G)_{n\geq 1} \to (C^n_c(\mathfrak{g},\mathfrak{a}),d_{\mathfrak{g}})_{n\geq 1}$$

and in particular homomorphisms $D_n\colon H^n_s(G,A) \to H^n_c(\mathfrak{g},\mathfrak{a})$ for $n \geq 2$.

For $A = \mathfrak{a}$ these assertions hold for all $n \in \mathbb{N}_0$, and if $A \cong \mathfrak{a}/\Gamma_A$ holds for a discrete subgroup Γ_A of $\mathfrak{a}$, then D_1 also induces a homomorphism $D_1\colon H^1_s(G,A) \to H^1_c(\mathfrak{g},\mathfrak{a}), [f] \mapsto [df(\mathbf{1})]$. □

Integrability of Lie algebra cocycles

We have seen above that for $n \geq 2$ there is a natural *derivation map*

$$D_n\colon H^n_s(G,A) \to H^n_c(\mathfrak{g},\mathfrak{a})$$

from locally smooth Lie group cohomology to continuous Lie algebra cohomology. Since the Lie algebra cohomology spaces $H^n_c(\mathfrak{g},\mathfrak{a})$ are much better accessible by algebraic means than those of G, it is important to understand the amount of information lost by the map D_n. Thus one is interested in kernel and cokernel of D_n. A determination of the cokernel can be considered as describing integrability conditions on cohomology classes $[\omega] \in H^n_c(\mathfrak{g},\mathfrak{a})$ which have to be satisfied to ensure the existence of $f \in Z^n_s(G,A)$ with $D_n f = \omega$.

Before we turn to the complete solution for $n = 2$ ([Ne04a, Sect. 7]), we take a closer look at the much simpler case $n = 1$.

Remark V.2.7. For $n = 1$ we consider the more general setting of a Lie group action on a non-abelian group: Let G and N be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{n}$ and $\sigma\colon G \times N \to N$ a smooth action of G on N by automorphisms. A *crossed homomorphism*, or a 1-*cocycle*, is a smooth map $f\colon G \to N$ with

$$f(gh) = f(g)\cdot g.f(h) \quad \text{for} \quad g,h \in G,$$

which is equivalent to $(f,\mathrm{id}_G)\colon G \to N \rtimes G$ being a group homomorphism. We note that for a 1-cocycle smoothness in an identity neighborhood implies smoothness and write $Z^1_s(G,N)$ for the set of smooth 1-cocycles $G \to N$.

It is easy to see that for each crossed homomorphism $f\colon G \to N$, the logarithmic derivative $\delta(f) \in \Omega^1(G,\mathfrak{n})$ is an equivariant 1-form with values in the smooth G-module $\mathfrak{n}$, hence uniquely determined by $D_1(f) := T_{\mathbf{1}}(f)\colon \mathfrak{g} \to \mathfrak{n}$. Conversely, an easy application of the Uniqueness Lemma shows that if G is connected, then a smooth function $f\colon G \to N$ is a crossed homomorphism if and only if $f(\mathbf{1}) = \mathbf{1}$ and $\delta(f)$ is an equivariant 1-form.

To see the infinitesimal picture, we call a continuous linear map $\alpha\colon \mathfrak{g} \to \mathfrak{n}$ a *crossed homomorphism*, or a 1-*cocycle*, if

$$\alpha([x,y]) = x.\alpha(y) - y.\alpha(x) + [\alpha(x),\alpha(y)], \tag{5.2.2}$$

which is equivalent to $(\alpha, \mathrm{id}_{\mathfrak{g}})\colon \mathfrak{g} \to \mathfrak{n} \rtimes \mathfrak{g}$ being a homomorphism of Lie algebras. We write $Z^1_c(\mathfrak{g},\mathfrak{n})$ for the set of continuous 1-cocycles $\mathfrak{g} \to \mathfrak{n}$. If G is connected, we obtain an injective map

$$D_1\colon Z^1_s(G,N) \to Z^1_c(\mathfrak{g},\mathfrak{n}).$$

The cocycle condition (5.2.2) for α holds if and only if $\alpha^{\mathrm{eq}} \in \Omega^1(G,\mathfrak{n})$ satisfies the Maurer–Cartan equation. Therefore the Fundamental Theorem (Theorem III.1.2) shows that if G is connected and N is regular, then a 1-cocycle $\alpha \in Z^1_c(\mathfrak{g},\mathfrak{n})$ is integrable to some group 1-cocycle if and only if the period homomorphism

$$\mathrm{per}_\alpha := \mathrm{per}_{\alpha^{\mathrm{eq}}}\colon \pi_1(G) \to N$$

vanishes. This can also be expressed by the exactness of the sequence

$$\mathbf{0} \to Z^1_s(G,N) \xrightarrow{D_1} Z^1_c(\mathfrak{g},\mathfrak{n}) \xrightarrow{\mathrm{per}} \mathrm{Hom}(\pi_1(G),N)$$

which already gives an idea of what kind of obstructions to expect for 2-cocycles. $\square$

The special importance of the group $H^2_s(G,A)$ stems from the following theorem, which can be derived easily from the construction in Section V.1.

Theorem V.2.8. *If G is a connected Lie group and $S\colon G \to \mathrm{Aut}(A)$ a smooth action of G on the abelian Lie group A, then we obtain an isomorphism of abelian groups*

$$H^2_s(G,A) \to \mathrm{Ext}(G,A)_S, \quad [\omega] \mapsto A \times_{(S,\omega)} G. \qquad \square$$

Remark V.2.9. (a) If the group G is not connected, then condition (L3) in Theorem II.2.1 requires an additional smoothness condition on cocycles, which is equivalent to smoothness of the functions

$$f_g\colon G \to A, \quad f_g(g') := f(g,g') - f(gg'g^{-1},g)$$

on an identity neighborhood for each $g \in G$. For $g \in G_0$ this is automatically the case for each $f \in Z^2_s(G,A)$. We write $Z^2_{ss}(G,A) \subseteq Z^2_s(G,A)$ for the set of all cocycles satisfying this additional condition. Then $B^2_s(G,A) \subseteq Z^2_{ss}(G,A)$, and we put $H^2_{ss}(G,A) := Z^2_{ss}(G,A)/B^2_s(G,A)$. Theorem V.2.8 remains valid for general G with $H^2_{ss}(G,A)$ instead of $H^2_s(G,A)$.

(b) The second cohomology groups do not only classify abelian extensions of G. In view of Theorem V.1.5, the sets $\mathrm{Ext}(G,N)_{[S]}$ are principal homogeneous spaces of the groups $\mathrm{Ext}(G,Z(N))_S \cong H^2_{ss}(G,Z(N))$, provided $Z(N)$ is an initial Lie subgroup of the Lie group N (Remark II.6.5(c)). Therefore the knowledge of the second cohomology groups is also crucial for an understanding of non-abelian extension classes. $\square$

On the Lie algebra level, we similarly have for topologically split extensions of Lie algebras (*cf.* Remark I.2.7):

Proposition V.2.10. *Let* $(\mathfrak{a}, S)$ *be a topological* $\mathfrak{g}$*-module, where* $S\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{a})$ *denotes the module structure, and write* $\mathrm{Ext}(\mathfrak{g}, \mathfrak{a})_S$ *for the set of all equivalence classes of topologically split* $\mathfrak{a}$*-extensions* $\widehat{\mathfrak{g}}$ *of* $\mathfrak{g}$ *for which the adjoint action of* $\widehat{\mathfrak{g}}$ *on* $\mathfrak{a}$ *induces the given* $\mathfrak{g}$*-module structure on* $\mathfrak{a}$*. Then the map*

$$Z_c^2(\mathfrak{g}, \mathfrak{a}) \to \mathrm{Ext}(\mathfrak{g}, \mathfrak{a})_S, \quad \omega \mapsto [\mathfrak{a} \oplus_\omega \mathfrak{g}],$$

where $\mathfrak{a} \oplus_\omega \mathfrak{g}$ *denotes* $\mathfrak{a} \times \mathfrak{g}$*, endowed with the bracket*

$$[(a,x),(a',x')] := (x.a' - x'.a + \omega(x,x'), [x,x']),$$

factors through a bijection $H_c^2(\mathfrak{g}, \mathfrak{a}) \to \mathrm{Ext}(\mathfrak{g}, \mathfrak{a})_S, [\omega] \mapsto [\mathfrak{a} \oplus_\omega \mathfrak{g}]$. □

We now turn to the description of the obstruction for the integrability of Lie algebra 2-cocycles.

Theorem V.2.11. (Approximation Theorem; [Ne02a; Th. A.3.7]) *Let* M *be a compact manifold. Then the inclusion map* $C^\infty(M,G) \hookrightarrow C(M,G)$ *is a morphism of Lie groups which is a weak homotopy equivalence, i.e., it induces isomorphisms of homotopy groups*

$$\pi_k(C^\infty(M,G)) \to \pi_k(C(M,G))$$

for each $k \in \mathbb{N}_0$*. In particular, we have*

$$[M,G] \cong \pi_0(C^\infty(M,G))$$

for the group $[M,G]$ *of homotopy classes of maps* $M \to G$*.* □

Below, $\mathfrak{a}$ denotes a smooth Mackey complete G-module.

Definition V.2.12. *(a) If* M *is a compact oriented manifold of dimension* k *and* $\Omega \in \Omega^k(G, \mathfrak{a})$ *a closed* $\mathfrak{a}$*-valued* k*-form, then the map*

$$\widetilde{\mathrm{per}}_\Omega\colon C^\infty(M,G) \to \mathfrak{a}, \quad \sigma \mapsto \int_\sigma \Omega := \int_M \sigma^*\Omega$$

is locally constant. If, in addition, Ω *is equivariant, then its values lie in the closed subspace* $\mathfrak{a}^\mathfrak{g}$ *of* $\mathfrak{g}$*-fixed elements in* $\mathfrak{a}$*, hence defines a period map* $[M,G] \to \mathfrak{a}^\mathfrak{g}$ *([Ne02a, Lemma 5.7]). If* $M = \mathbb{S}^k$ *is a sphere, so that* $\pi_k(G) \subseteq [\mathbb{S}^k, G]$ *corresponds to base point preserving maps, then restriction to* $\pi_k(G)$ *defines a group homomorphism*

$$\mathrm{per}_\Omega\colon \pi_k(G) \to \mathfrak{a}^\mathfrak{g},$$

called the period homomorphism *defined by* Ω*.*

(b) For $k = 2$ *and* $\omega \in Z_c^2(\mathfrak{g}, \mathfrak{a})$*, we obtain a Lie algebra* 1*-cocycle*

$$f_\omega\colon \mathfrak{g} \to C_c^1(\mathfrak{g}, \mathfrak{a})/d_\mathfrak{g}\mathfrak{a}, \quad x \mapsto [i_x\omega],$$

and it is shown in [Ne04a, Lemma 6.2] that this 1*-cocycle gives rise to a well-defined period homomorphism, called the* flux homomorphism*,*

$$F_\omega\colon \pi_1(G) \to H^1_c(\mathfrak{g},\mathfrak{a})$$

as follows. For each piecewise smooth loop $\gamma\colon \mathbb{S}^1 \to G$, we define a 1*-cocycle*

$$I_\gamma\colon \mathfrak{g} \to \mathfrak{a}, \quad I_\gamma(x) := \int_\gamma i_{x_r}\omega^{\mathrm{eq}},$$

where x_r is the right invariant vector field on x with $x_r(\mathbf{1}) = x$, and put $F_\omega([\gamma]) := [I_\gamma]$. □

Now we turn to the main result of this section ([Ne04a, Th. 7.2]):

Theorem V.2.13. *Let G be a connected Lie group, A a smooth G-module of the form $A \cong \mathfrak{a}/\Gamma_A$, where $\Gamma_A \subseteq \mathfrak{a}$ is a discrete subgroup of the Mackey complete space $\mathfrak{a}$ and $q_A\colon \mathfrak{a} \to A$ the quotient map. Then the map*

$$\widetilde{P}\colon Z^2_c(\mathfrak{g},\mathfrak{a}) \to \mathrm{Hom}\big(\pi_2(G),A\big) \times \mathrm{Hom}\big(\pi_1(G),H^1_c(\mathfrak{g},\mathfrak{a})\big),\ \widetilde{P}(\omega) = (q_A \circ \mathrm{per}_\omega, F_\omega)$$

factors through a homomorphism

$$P\colon H^2_c(\mathfrak{g},\mathfrak{a}) \to \mathrm{Hom}\big(\pi_2(G),A\big) \times \mathrm{Hom}\big(\pi_1(G),H^1_c(\mathfrak{g},\mathfrak{a})\big),\ P([\omega]) = (q_A \circ \mathrm{per}_\omega, F_\omega),$$

and the following sequence is exact:

$$\mathbf{0} \to H^1_s(G,A) \xrightarrow{I} H^1_s(\widetilde{G},A) \xrightarrow{R} H^1\big(\pi_1(G),A\big)^G \cong \mathrm{Hom}\big(\pi_1(G),A^G\big) \xrightarrow{\delta}$$
$$\xrightarrow{\delta} H^2_s(G,A) \xrightarrow{D_2} H^2_c(\mathfrak{g},\mathfrak{a}) \xrightarrow{P} \mathrm{Hom}\big(\pi_2(G),A\big) \times \mathrm{Hom}\big(\pi_1(G),H^1_c(\mathfrak{g},\mathfrak{a})\big).$$

Here the map δ assigns to a group homomorphism $\gamma\colon \pi_1(G) \to A^G$ the quotient of the semi-direct product $A \rtimes \widetilde{G}$ by the graph $\{(\gamma(d),d)\colon d \in \pi_1(G)\}$ of γ which is a discrete central subgroup, I denotes the inflation map and R the restriction map to the subgroup $\ker q_G \cong \pi_1(G)$ of $\widetilde{G}$.

If, in particular, $\pi_1(G)$ and $\pi_2(G)$ vanish, we obtain an isomorphism

$$D_2\colon H^2_s(G,A) \to H^2_c(\mathfrak{g},\mathfrak{a}).$$ □

Remark V.2.14. (a) If G is 1-connected, things become much simpler and the criterion for the integrability of a Lie algebra cocycle ω to a group cocycle is that $\mathrm{im}(\mathrm{per}_\omega) \subseteq \Gamma_A$. Similar conditions arise in the theory of abelian principal bundles on smoothly paracompact presymplectic manifolds (M,Ω) (Ω is a closed 2-form on M). Here the integrality of the cohomology class $[\Omega]$ is equivalent to the existence of a pre-quantum bundle, i.e., a $\mathbb{T}$-principal bundle $\mathbb{T} \hookrightarrow \widehat{M} \twoheadrightarrow M$ whose curvature 2-form is Ω (*cf.* [Bry93], [KYMO85]).

(b) For finite-dimensional Lie groups the integrability criteria simplify significantly because $\pi_2(G)$ vanishes ([CaE36]). This has been used by É. Cartan to construct central extensions and thus to prove that each finite-dimensional Lie algebra belongs to a global Lie group, which is known as Lie's third theorem (*cf.* [CaE30/52], [Est88]).

(c) Let (M,ω) denote a compact symplectic manifold and $\widetilde{\mathrm{Diff}}(M,\omega)$ the universal covering group of the identity component $\mathrm{Diff}(M,\omega)_0$ of the Fréchet–Lie group $\mathrm{Diff}(M,\omega)$ (Theorem III.3.1). Then the Lie algebra homomorphism

$$(5.2.3)\quad f_\omega\colon \mathscr{V}(M,\omega):=\{X\in\mathscr{V}(M)\colon \mathscr{L}_X\omega=0\}\to H^1_{\mathrm{dR}}(M,\mathbb{R}),\quad X\mapsto[i_X\omega],$$

where $H^1_{\mathrm{dR}}(M,\mathbb{R})$ is considered as an abelian Lie algebra, integrates to a Lie group homomorphism

$$\mathscr{F}\colon \widetilde{\mathrm{Diff}}(M,\omega)\to H^1_{\mathrm{dR}}(M,\mathbb{R}),$$

whose restriction per_{f_ω} to the discrete subgroup $\pi_1(\mathrm{Diff}(M,\omega))$ is called the *flux homomorphism*. Let $\mathfrak{ham}(M,\omega):=\ker f_\omega$ denote the Lie subalgebra of Hamiltonian vector fields. In [KYMO85, 2.2], it is shown that

$$\widetilde{\mathrm{Ham}}(M,\omega):=\ker\mathscr{F}$$

is a μ-regular Lie group.

Recently, there has been quite some activity concerning the flux homomorphism for symplectic manifolds and generalizations thereof ([Ban97, Ch. 3], [KKM05], [Ne06a]), including a proof of the flux conjecture ([Ono04]), formulated by Calabi ([Cal70]). It asserts that the image of the flux homomorphism per_{f_ω} is discrete for each compact symplectic manifold (*cf.* [MD05], [LMP98] for a survey).

(d) In [RS81], Ratiu and Schmid address the existence problem of ILH–Lie group structures for the following three classes of groups: Under the assumption that the image of the flux homomorphism is discrete, which is always the case ([Ono04]), they show that the group $\mathrm{Ham}(M,\omega)$ of Hamiltonian diffeomorphisms carries an ILH–Lie group structure. If $q\colon P\to M$ is a pre-quantum $\mathbb{T}$-bundle with curvature ω and connection 1-form θ, they further show that the quantomorphism group $\mathrm{Aut}(P,\theta)$, a central $\mathbb{T}$-extension of $\mathrm{Ham}(M,\omega)$ (*cf.* Example IV.1.6(b)), is an ILH–Lie group, and they obtain an ILH–Lie group G for the Lie algebra of real-valued smooth functions on $T^*(M)$ which are homogeneous of degree 1 with respect to the Poisson bracket. The latter group is of particular interest for the Lie group structure on the group of invertible Fourier–integral operators of order zero, which is a Lie group extension of G ([ARS84,86a/b]).

For a discussion of the relation between quantomorphisms and Hamiltonian diffeomorphisms, extending some of these structures, such as Kostant's Theorem ([Kos70]) to infinite dimensional manifolds, we refer to [NV03].

(e) The period and the flux homomorphism annihilate the torsion subgroups of $\pi_2(G)$, resp., $\pi_1(G)$. Hence they factor through the rational homotopy groups $\pi_2(G)\otimes\mathbb{Q}$, resp., $\pi_1(G)\otimes\mathbb{Q}$.

(f) If $\mathfrak{a}$ is a trivial module and $\omega\in Z^2_c(\mathfrak{g},\mathfrak{a})$, then $\widehat{\mathfrak{g}}:=\mathfrak{a}\oplus_\omega\mathfrak{g}$ is a central extension of $\mathfrak{g}$, and $x.(a,y):=(\omega(x,y),[x,y])$ turns $\widehat{\mathfrak{g}}$ into a topological $\mathfrak{g}$-module. A 1-cocycle $f\colon\mathfrak{g}\to\mathfrak{a}$ is the same as a Lie algebra homomorphism, and $B^1_c(\mathfrak{g},\mathfrak{a})=\{0\}$, so that $H^1_c(\mathfrak{g},\mathfrak{a})=\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g},\mathfrak{a})\cong\mathscr{L}(\mathfrak{g}/\overline{[\mathfrak{g},\mathfrak{g}]},\mathfrak{a})$. In this case, the flux homomorphism

$$F_\omega\colon\pi_1(G)\to\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g},\mathfrak{a})$$

vanishes if and only if the action of $\mathfrak{g}$ on $\widehat{\mathfrak{g}}$ integrates to a smooth action of the group G on $\widehat{\mathfrak{g}}$ ([Ne02a, Prop. 7.6]). □

We emphasize that Theorem V.2.13 holds for Lie groups which are not necessarily smoothly paracompact. On these groups de Rham's Theorem is not available, so that one has to get along without it and to use more direct methods. This is important because many interesting Banach–Lie groups are not smoothly paracompact, because their model spaces do not permit smooth bump functions (*cf.* Remark I.4.5).

Remark V.2.15. Let G be a connected smoothly paracompact Lie group and A a smooth G-module of the form $\mathfrak{a}/\Gamma_A$, where Γ_A is a discrete subgroup of $\mathfrak{a}$. Let $Z^2_{gs}(G,A)$ denote the group of smooth 2-cocycles $G\times G\to A$ and $B^2_{gs}(G,A)\subseteq Z^2_{gs}(G,A)$ the cocycles of the form d_Gh, where $h\in C^\infty(G,A)$ is a smooth function with $h(\mathbf{1})=0$. Then we have an injection

$$H^2_{gs}(G,A):=Z^2_{gs}(G,A)/B^2_{gs}(G,A)\hookrightarrow H^2_s(G,A),$$

and the space $H^2_{gs}(G,A)$ classifies those A-extensions of G with a smooth global section. We further have an exact sequence

$$\mathrm{Hom}(\pi_1(G),\mathfrak{a}^G)\xrightarrow{\delta}H^2_{gs}(G,A)\xrightarrow{D}H^2_c(\mathfrak{g},\mathfrak{a})\xrightarrow{P}$$
$$\xrightarrow{P}H^2_{\mathrm{dR}}(G,\mathfrak{a})\times\mathrm{Hom}\big(\pi_1(G),H^1_c(\mathfrak{g},\mathfrak{a})\big),$$

where $P([\omega])=([\omega^{\mathrm{eq}}],F_\omega)$ (*cf.* Section 8 in [Ne02a] and Remark 8.5 in [Ne 04a]). □

Remark V.2.16. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{a}:=C^\infty(G,\mathbb{R})$, endowed with the compact open C^∞-topology. Note that G acts on $\mathfrak{a}$ by $(g.f)(x):=f(xg)$, and that the corresponding action of $\mathfrak{g}$ corresponds to the embedding $\mathfrak{g}\to\mathscr{V}(G), x\mapsto x_l$. Using the left trivialization of $T(G)$, we see that $\mathbb{R}$-valued p-forms are in one-to-one correspondence with those smooth functions $G\times\mathfrak{g}^p\to\mathbb{R}$ which are p-linear and alternating in the last p arguments. This implies in particular, that each p-form $\omega \in \Omega^p(G,\mathbb{R})$ defines an element of $C^p_c(\mathfrak{g},\mathfrak{a})$, and it is easy to see that this leads to an injection of cochain complexes

$$\eta\colon (\Omega^\bullet(G,\mathbb{R}),d) \hookrightarrow (C^\bullet(\mathfrak{g},\mathfrak{a}),d_{\mathfrak{g}}).$$

If G is a Fréchet–Lie group, then the cartesian closedness argument from the convenient calculus (*cf.* [KM97, p.30]) implies that η is bijective, which leads to an isomorphism

$$H^p_{\mathrm{dR}}(G,\mathbb{R}) \cong H^p_c(\mathfrak{g},\mathfrak{a}).$$

If, in addition, G is smoothly paracompact, we thus obtain a description of real-valued singular cohomology of G in terms of Lie algebra cohomology (*cf.* [Mi87], [Ne04a, Ex. 7.6]).

In [Mi87], Michor applies this construction in particular to the group $\mathrm{Diff}(M)$ for a compact manifold M. For more detailed information on the de Rham cohomology of groups like $\mathrm{Diff}(M)$ or $C^\infty(M,K)$, where M is compact and K finite-dimensional, we refer to [Beg87]. □

We have seen above that period homomorphisms arise naturally in the integration theory of Lie algebra extensions to group extensions. Below we describe some interesting classes of Lie algebra 2-cocycles which have some independent topological interpretation.

Let $G = C^\infty_c(M,K)$, where K is a Lie group with Lie algebra $\mathfrak{k}$ and M is a σ-compact finite-dimensional manifold, so that $\mathfrak{g} = \mathbf{L}(G) \cong C^\infty_c(M,\mathfrak{k})$, endowed with the natural locally convex direct limit structure (Theorem II.2.8). For detailed proofs of the results below we refer to [MN03] for the compact case and to [Ne04c] for the non-compact case.

The Lie algebra cocycles we are interested in are those of *product type*, constructed as follows. Let E be a Mackey complete space and $\kappa\colon \mathfrak{k}\times\mathfrak{k} \to E$ an invariant continuous symmetric bilinear form. Then the quotient space $\mathfrak{z} := \Omega^1_c(M,E)/dC^\infty_c(M,E)$ carries a natural locally convex topology because the space of exact forms is closed with respect to the natural direct limit topology. We then obtain a continuous Lie algebra cocycle

$$\omega \in Z^2_c(\mathfrak{g},\mathfrak{z}) \quad \text{by} \quad \omega(\xi,\eta) := [\kappa(\xi,d\eta)].$$

Of particular interest is the case $E = V(\mathfrak{k})$, where $\kappa\colon \mathfrak{k}\times\mathfrak{k} \to V(\mathfrak{k})$ is the universal invariant symmetric bilinear form and the case $E=\mathbb{R}$, where κ is the Cartan–Killing form of a finite-dimensional Lie algebra. We write $\Pi^M_\kappa \subseteq \mathfrak{z}$ for the corresponding period group. Other types of cocycles, which are not of product type, are described in [NeWa06a]. If $\mathfrak{k}$ is a compact simple Lie algebra and $M = \mathbb{S}^1$, then $H^2_c(\mathfrak{g},\mathbb{R})$ is one-dimensional and generated by the cocycle defined by the Cartan–Killing form κ. The associated central extensions and their integrability to Lie groups are discussed in some detail in Section 4.2 in [PS86].

Theorem V.2.17. *The following assertions hold:*

(1) For $M = \mathbb{S}^1$ we have $\mathfrak{z} \cong E$, and the period group $\Pi_\kappa^{\mathbb{S}^1}$ is the image of a homomorphism

$$\mathrm{per}_\kappa \colon \pi_3(K) \to E,$$

obtained by identifying the subgroup $\pi_2(C_*(\mathbb{S}^1, K))$ of $\pi_2(G)$ with $\pi_3(K)$.

(2) Π_κ^M is contained in $H^1_{\mathrm{dR,c}}(M,E)$ and coincides with the set of all those cohomology classes $[\alpha]$ for which integration over circles and properly embedded copies of $\mathbb{R}$, we obtain elements of $\Pi_\kappa^{\mathbb{S}^1}$.

(3) Π_κ^M is discrete if and only if $\Pi_\kappa^{\mathbb{S}^1}$ is discrete.

(4) If $\dim K < \infty$ and κ is universal, then $\Pi_\kappa^{\mathbb{S}^1} \subseteq V(\mathfrak{k})$ is discrete.

(5) If K is compact and simple, then the Cartan–Killing form κ is universal, for a suitable normalization of κ we have $\Pi_\kappa^{\mathbb{S}^1} = \mathbb{Z}$, and $\Pi_\kappa^M \subseteq H^1_{\mathrm{dR,c}}(M,\mathbb{R})$ is the subgroup of all cohomology classes with integral periods in the sense of (2). □

Remark V.2.18. A particularly interesting class of corresponding central extensions has been studied by Etingof and Frenkel in [EF94]. They investigate the situation where M is a compact complex manifold, K is a simple complex Lie group, κ is the Cartan–Killing form, and by projecting onto the subspace of $H^1_{\mathrm{dR}}(M,\mathbb{C}) \subseteq \Omega^1(M,\mathbb{C})/dC^\infty(M,\mathbb{C})$ generated by the holomorphic 1-forms, they obtain a central extension of the complex Lie group $C^\infty(M,K)$ by a compact complex Lie group, which in some cases is an elliptic curve or an abelian variety. □

V.3. Abelian extensions of Lie groups

In this subsection, we use the results of the preceding subsection to integrate abelian extensions of Lie algebras to Lie group extensions.

If $S \colon G \to \mathrm{Aut}(A)$ defines on A the structure of a smooth G-module, G is connected and $A \cong \mathfrak{a}/\Gamma_A$ with $\Gamma_A \subseteq \mathfrak{a}$ discrete, then $H^2_s(G,A) \cong \mathrm{Ext}(G,A)_S$ (Theorem V.2.8), so that Theorem V.2.13 provides in particular necessary and sufficient conditions for a Lie algebra cocycle $\omega \in Z^2_c(\mathfrak{g},\mathfrak{a})$ to correspond to a global Lie group extension ([Ne04a, Th. 6.7]):

Theorem V.3.1. (Integrability Criterion) *Let G be a connected Lie group and A a smooth G-module with $A_0 \cong \mathfrak{a}/\Gamma_A$, where Γ_A is a discrete subgroup of the Mackey complete space $\mathfrak{a}$. For each $\omega \in Z^2_c(\mathfrak{g},\mathfrak{a})$, the abelian Lie algebra extension $\mathfrak{a} \hookrightarrow \widehat{\mathfrak{g}} := \mathfrak{a} \oplus_\omega \mathfrak{g} \twoheadrightarrow \mathfrak{g}$ integrates to a Lie group extension $A \hookrightarrow \widehat{G} \twoheadrightarrow G$ with a connected Lie group $\widehat{G}$ if and only if*

(1) $\Pi_\omega := \mathrm{im}(\mathrm{per}_\omega) \subseteq \Gamma_A$, and

(2) there exists a surjective homomorphism $\gamma\colon \pi_1(G) \to \pi_0(A)$ such that the flux homomorphism $F_\omega\colon \pi_1(G) \to H^1_c(\mathfrak{g},\mathfrak{a})$ is related to the characteristic homomorphism

$$\overline{\theta}_A\colon \pi_0(A) \to H^1_c(\mathfrak{g},\mathfrak{a}), \quad [a] \mapsto [D_1(d_G(a))] \quad \textit{by} \quad F_\omega = \overline{\theta}_A \circ \gamma.$$

If A is connected, then (2) is equivalent to $F_\omega = 0$. □

Corollary V.3.2. *Let G be a connected Lie group, $\mathfrak{a}$ a smooth Mackey complete G-module and $\omega \in Z^2_c(\mathfrak{g},\mathfrak{a})$. Then there exists a smooth G-module A with Lie algebra $\mathfrak{a}$ such that the abelian Lie algebra extension $\mathfrak{a} \hookrightarrow \widehat{\mathfrak{g}} := \mathfrak{a} \oplus_\omega \mathfrak{g} \twoheadrightarrow \mathfrak{g}$ integrates to a Lie group extension $A \hookrightarrow \widehat{G} \twoheadrightarrow G$ with a connected Lie group $\widehat{G}$ if and only if Π_ω is a discrete subgroup of $\mathfrak{a}^G$.*

Proof. The necessity is immediate from Theorem V.3.1. For the converse, we first use this theorem to find an extension $q_0\colon G^\sharp \to \widetilde{G}$ of the universal covering group $\widetilde{G}$ of G by the smooth G-module $A_0 := \mathfrak{a}/\Pi_\omega$. Then $A := q_0^{-1}(\pi_1(G)) \subseteq G^\sharp$ is a Lie group with identity component A_0, so that $G^\sharp$ is an A-extension of G. □

Note that it may happen that the group A constructed in the preceding proof is not abelian. Since A_0 and $\pi_1(G)$ are abelian, it is at most 2-step nilpotent.

Remark V.3.3. (a) Suppose that only (1) in Theorem V.3.1 is satisfied, and that A is connected. Consider the corresponding extension $q^\sharp\colon G^\sharp \to \widetilde{G}$ of $\widetilde{G}$ by $A \cong \mathfrak{a}/\Gamma_A$. Then $G \cong G^\sharp/\widehat{\pi}_1(G)$, where $\widehat{\pi}_1(G) := (q^\sharp)^{-1}(\pi_1(G))$ is a central A-extension of $\pi_1(G)$, hence 2-step nilpotent. This group is abelian if and only if the induced commutator map

$$C\colon \pi_1(G) \times \pi_1(G) \to A$$

vanishes. It is shown in [Ne04a, Rem. 6.8] that, up to sign, this map is given by

$$C([\gamma],[\eta]) = \int_{\gamma*\eta} \omega^{\mathrm{eq}}, \quad \text{where} \quad \gamma*\eta\colon \mathbb{T}^2 \to G, \quad (t,s) \mapsto \gamma(t)\eta(s).$$

(b) According to a result of H. Hopf ([Hop42]), we have for each arcwise connected topological space X an exact sequence

$$\mathbf{0} \to H^2(\pi_1(X),A) \to H^2_{\mathrm{sing}}(X,A) \cong \mathrm{Hom}(H_2(X),A) \to \mathrm{Hom}(\pi_2(X),A) \to \mathbf{0}$$

(*cf.* [ML78, p.5]). If G is smoothly paracompact, then the closed 2-form ω^{eq} defines a singular cohomology class in $H^2_{\mathrm{sing}}(G,\mathfrak{a}) \cong \mathrm{Hom}(H_2(M),\mathfrak{a})$ and after composition with the quotient map $q_A\colon \mathfrak{a} \to A$, a singular cohomology class $c_\omega \in H^2_{\mathrm{sing}}(G,A)$. The inclusion $\Pi_\omega \subseteq \Gamma_A$ means that this class vanishes on the spherical cycles, i.e., the image of $\pi_2(G)$ in $H_2(G)$. Hence it determines a central extension of $\pi_1(G)$ by A, and if A is divisible, this central extension is determined by the commutator map $C\colon \pi_1(G) \times \pi_1(G) \to A$. If this map vanishes, then $c_\omega = 0$, but Example V.3.5(b) below shows that this does not imply the existence of a corresponding global group cocycle. If G is 1-connected, then c_ω vanishes if and only if ω integrates to a group cocycle (*cf.* [EK64]), but in general this simple criterion fails.

(c) If $F'_\omega([\gamma]) \in H^1_{\rm dR}(G,\mathfrak{a})$ denotes the de Rham class obtained as in Proposition V.2.4, then we have for each piecewise smooth loop $\eta : \mathbb{S}^1 \to G$ the formula $\int_\eta F'_\omega(\gamma) = \int_{\gamma*\eta} \omega^{\rm eq}$. □

The following proposition displays another facet of Hopf's result mentioned under (b) above for the special case of topological groups (*cf.* [Ne04a, Prop. 6. 11]). In the context of rational homotopy theory, it can be extended to the Cartan–Serre Theorem, that the rational homology algebra of an arcwise connected topological group is generated by the homology classes defined by maps $\mathbb{S}^k \to G$, $k \in \mathbb{N}$ (*cf.* [BuGi02, Th. 3.17]).

Proposition V.3.4. *Let G be a topological group, $S_2(G) \subseteq H_2(G)$ the subgroup of spherical 2-cycles, i.e., the image of $\pi_2(G)$ under the Hurewicz homomorphism $\pi_2(G) \to H_2(G)$, and $\Lambda_2(G) := H_2(G)/S_2(G)$ the quotient group. Then $\Lambda_2(G)$ is generated by the images of cycles defined by maps of the form*

$$\alpha * \beta \colon \mathbb{T}^2 \to G, \quad (t,s) \mapsto \alpha(t)\beta(s),$$

where $\alpha, \beta \colon \mathbb{T} \to G$ are loops in G. □

Example V.3.5. (a) Let $G := \mathrm{Diff}(M)_0^{\rm op}$ be the opposite group of the identity component of $\mathrm{Diff}(M)$ for a connected compact manifold M. Recall that its Lie algebra is $\mathfrak{g} := \mathcal{V}(M)$ (Example II.3.14). For each Fréchet space E, the abelian Lie group $\mathfrak{a} = C^\infty(M,E)$ is a smooth G-module with respect to $\varphi.f := f \circ \varphi$. Each closed E-valued 2-form ω_M defines a continuous Lie algebra 2-cocycle by $\omega(X,Y) := \omega_M(X,Y)$. In this case, the period map and the flux cocycle can be described in geometrical terms. In [Ne04a, Sect. 9], it is shown that the period map

$$\mathrm{per}_\omega \colon \pi_2(\mathrm{Diff}(M)) \to \mathfrak{a}^{\mathfrak{g}} = C^\infty(M,E)^{\mathcal{V}(M)} = E$$

factors for each $m_0 \in M$ through the evaluation map $\mathrm{ev}_{m_0} \colon \mathrm{Diff}(M) \to M, \varphi \mapsto \varphi(m_0)$, to the map

$$\mathrm{per}_{\omega_M} \colon \pi_2(M,m_0) \to E, \quad [\sigma] \mapsto \int_\sigma \omega_M.$$

Likewise, the flux homomorphism can be interpreted as a map

$$F_\omega \colon \pi_1(\mathrm{Diff}(M)) \to H^1_{\rm dR}(M,E) \cong \mathrm{Hom}(\pi_1(M),E),$$

that vanishes if and only if all integrals of the 2-form ω_M over smooth cycles of the form $H \colon \mathbb{T}^2 \to M, (s,t) \mapsto \alpha(s).\beta(t)$ with loops α in $\mathrm{Diff}(M)$ and β in M vanish.

This easily leads to the sufficient condition for the integrability of ω that the period group Γ_E of the 2-form ω_M should be discrete in E. This in turn implies the existence of a Z-principal bundle for $Z := E/\Gamma_E$ with curvature ω_M over M, and the identity component of the group $\mathrm{Aut}(P) = \mathrm{Diff}(P)^Z$ is a Lie group extension of G by $\mathrm{Gau}(P) \cong C^\infty(M,Z)$, integrating ω (Example V.1.6(c)).

It would be very interesting to understand to which extent the discreteness of the periods of ω_M is necessary for the discreteness of the period group of ω (see also the discussion in [KYMO85, p.86] and Problem V.4).

(b) We consider the special case $M = \mathbb{T}^2$, realized as the unit torus in $\mathbb{C}^2$ and let ω_M be an invariant 2-form on M with $\int_M \omega_M = 1$.

Since $\pi_2(M,m_0)$ is trivial, per_ω vanishes. By $\alpha(z)(w_1,w_2) = (zw_1,w_2)$, we obtain a loop α in $\mathrm{Diff}(M)$, and the loop $\beta(z) := (1,z)$ in M satisfies $\alpha(z_1).\beta(z_2) = (z_1,z_2)$, so that

$$\int_{\alpha*\beta} \omega_M = 1.$$

We conclude that $F_\omega \neq 0$. Hence the Lie algebra cocycle ω on $\mathcal{V}(M)$ does not integrate to a group cocycle with values in the connected group $\mathfrak{a} = C^\infty(\mathbb{T}^2,\mathbb{R})$.

Since ω_M is integral, it is the curvature of a natural $\mathbb{T}$-bundle $q\colon P \to M$, which leads to an abelian extension

$$\mathbf{1} \to A := \mathrm{Gau}(P) \cong C^\infty(M,\mathbb{T}) \hookrightarrow \widehat{\mathrm{Diff}}(M)_0 \twoheadrightarrow \mathrm{Diff}(M)_0 \to \mathbf{1}$$

whose Lie algebra cocycle coincides with ω. Note that $\pi_0(A) \cong [\mathbb{T}^2,\mathbb{T}] \cong \mathbb{Z}^2$ is non-trivial.

(c) The same phenomenon occurs already for the subgroup $T := \mathbb{T}^2$, acting on itself by translations, and accordingly on $\mathfrak{a}$. By restriction, we obtain an abelian extension

$$\mathbf{1} \to A = C^\infty(\mathbb{T}^2,\mathbb{T}) \hookrightarrow \widehat{\mathbb{T}}^2 \twoheadrightarrow \mathbb{T}^2 \to \mathbf{1}$$

whose flux homomorphism $F_\omega\colon \pi_1(\mathbb{T}^2) \to H^1(\mathbb{R}^2,\mathfrak{a}) \cong H^1_{\mathrm{dR}}(\mathbb{T}^2,\mathbb{R}) \cong \mathbb{R}^2$ is injective. In this case, there is a reduction of the extension of T to an extension by the subgroup

$$B := \mathbb{T} \times \mathrm{Hom}(T,\mathbb{T}) \cong \mathbb{T} \times \mathbb{Z}^2 \subseteq A = C^\infty(\mathbb{T}^2,\mathbb{T}),$$

generated by the constant maps and the characters of T. The corresponding extension $\widetilde{T}$ of T by B is isomorphic to the Heisenberg group

$$H := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a,b,c \in \mathbb{R} \right\} = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{modulo} \quad \begin{pmatrix} 1 & 0 & \mathbb{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad \square$$

Example V.3.6. Let $G := \mathrm{SL}_2(\mathbb{R})$. From the natural action of G on $\mathbb{P}_1(\mathbb{R}) \cong \mathbb{S}^1$, we derive an action on the space $\mathfrak{a} := \Omega^1(\mathbb{S}^1, \mathbb{R})$. In Section 10 of [Ne04a], it is shown that there exists a non-trivial $\omega \in Z^2(\mathfrak{sl}_2(\mathbb{R}), \mathfrak{a})$ which integrates to an abelian extension

$$\widehat{\mathrm{SL}_2}(\mathbb{R}) = \Omega^1(\mathbb{S}^1, \mathbb{R}) \rtimes_f \mathrm{SL}_2(\mathbb{R}),$$

so that we obtain a non-trivial infinite-dimensional abelian extension of $\mathrm{SL}_2(\mathbb{R})$ which is a Fréchet–Lie group.

Since all finite-dimensional Lie group extensions of $\mathrm{SL}_2(\mathbb{R})$ by vector spaces split on the Lie algebra level, this example illustrates the difference between the finite- and infinite-dimensional theory. □

For more references dealing specifically with central extensions, we refer to [Ne02a]. In particular, [CVLL98] is a nice survey on central $\mathbb{T}$-extensions of Lie groups and their role in quantum physics (see also [Rog95]). It also contains a description of the universal central extension for finite-dimensional groups. For infinite-dimensional groups, universal central extensions are constructed in [Ne02d], and for root graded Lie algebras in [Ne03] (*cf.* Subsection VI.1).

Example V.3.7. (The Virasoro group) Let $G := \mathrm{Diff}_+(\mathbb{T})$ be the group of orientation preserving diffeomorphisms of the circle $\mathbb{T}$. Then the inclusion $\mathbb{T} \hookrightarrow G$ of the rigid rotations is a homotopy equivalence, so that $\pi_2(G)$ vanishes and $\pi_1(G) \cong \mathbb{Z}$ (*cf.* [Fu86, p.302]).

Furthermore, $H^2_c(\mathfrak{g}, \mathbb{R}) = \mathbb{R}[\omega]$ is one-dimensional ([PS86]), and the corresponding flux homomorphism F_ω vanishes ([Ne02a, Ex. 9.3]), so that Theorem V.3.1 implies the existence of a corresponding central $\mathbb{R}$-extension of G, called the *Bott–Virasoro group* Vir. Remark V.2.15 implies that this extension has a smooth global section, hence can be described by a smooth global cocycle. Such a cocycle, and other related ones, are described explicitly by Bott in [Bo77]. A more direct construction of this and related cocycles has been described recently by Billig ([BiY03]).

In [Se81], G. Segal studies projective unitary representations of $\mathrm{Diff}(\mathbb{S}^1)$ via representations of loop groups, which implicitly define unitary representations of the Bott–Virasoro group. In [GW84/85], Goodman and Wallach give an analytic construction of the unitary highest weight representations of Vir by directly integrating the corresponding Lie algebra representation on the representations of loop groups, using scales of Banach spaces.

The Bott–Virasoro group is also a very interesting geometric object. One aspect of its rich geometric structure is that, although it is only a smooth real Lie group which is not analytic (Remark VI.2.3 below), it carries the structure of a complex Fréchet manifold, which is obtained by identifying it with the complement of the zero section in the holomorphic line bundles over $\mathrm{Diff}_+(\mathbb{S}^1)/\mathbb{T}$ ([Lem95], [KY87]). □

Open Problems for Section V

Problem V.1. Generalize Theorem V.2.13 in an appropriate way to non-connected Lie groups G and A.

The generalization to non-connected Lie groups G means to derive accessible criteria for the extendibility of a 2-cocycle on the identity component G_0 to the whole group G. From the short exact sequence $G_0 \hookrightarrow G \twoheadrightarrow \pi_0(G)$, we obtain maps

$$H^2(\pi_0(G),A) \xrightarrow{I} H^2(G,A) \xrightarrow{R} H^2(G_0,A)^G,$$

but it is not clear how to describe the image of the restriction map R from G to G_0.

If A is a trivial module, one possible approach is to introduce additional structures on a central extension $\widehat{G}$ of G_0 by A, so that the map $q\colon \widehat{G} \to G$ describes a crossed module, which requires an extension of the natural G_0-action of G on $\widehat{G}$ to an action of G (*cf.* [Ne05]).

To deal with non-connected groups A seems to be tractable if we assume that $A_0 \cong \mathfrak{a}/\Gamma_A$ as in Theorem V.2.13. Under the assumption that G is connected, the crucial information is contained in an exact sequence

$$\mathbf{0} \to H^2_s(G,A_0) \to H^2_s(G,A) \xrightarrow{\gamma} \mathrm{Hom}(\pi_1(G),\pi_0(A)) \to H^3_s(G,A_0),$$

where γ assigns to an extension of G by A the corresponding connecting homomorphism $\pi_1(G) \to \pi_0(A)$ in the long exact homotopy sequence (*cf.* [Ne04a, App. E]). To determine $H^2_s(G,A)$ in terms of $H^s(G,A_0)$ and known data, one has to determine the image of $H^2_s(G,A)$ in $\mathrm{Hom}(\pi_1(G),\pi_0(A))$. □

Problem V.2. Do the spaces $Z^2_s(G,A)$ and $Z^2_{ss}(G,A)$ (Remark V.2.9) coincide for each non-connected Lie group G and each smooth G-module A?

This is true if G is connected ([Ne04a, Prop. 2.6]), but in general we do not know if $Z^2_{ss}(G,A)$ is a proper subgroup of $Z^2_s(G,A)$, which is equivalent to $H^2_{ss}(G,A)$ being a proper subgroup of $H^2_s(G,A)$. In terms of abelian extensions, this means that there exists an abelian extension $\widehat{G}$ of G by the G-module A for which the restriction $\widehat{G}_0$ to the identity component G_0 is a Lie group extension, but $\widehat{G}$ cannot be turned into a Lie group because for certain elements $\widehat{g} \in \widehat{G}$ the conjugation action on $\widehat{G}_0$ is not an action by smooth group automorphisms (*cf.* condition (L3) in Theorem II.2.1). □

Problem V.3. Give an explicit description of kernel and cokernel of the derivation maps

$$D_n\colon H^n_s(G,A) \to H^n_c(\mathfrak{g},\mathfrak{a}) \quad \text{for} \quad n \geq 3.$$

For $A \cong \mathfrak{a}/\Gamma_A$ for some discrete subgroup $\Gamma_A \subseteq \mathfrak{a}$, the first necessary condition for $[\omega] \in H^n_c(\mathfrak{g}, \mathfrak{a})$ to lie in the image of D, one obtains quite easily is that the range of the period homomorphism

$$\mathrm{per}_\omega \colon \pi_n(G) \to \mathfrak{a}$$

must be contained in $\Gamma_A \cong \pi_1(A)$ (*cf.* [GN07]). □

Problem V.4. An interesting special case of the preceding problem arises for $G = \mathrm{Diff}(M)_0^{\mathrm{op}}$, M a compact manifold, $\mathfrak{a} = C^\infty(M,\mathbb{R})$, where G acts by $(\varphi.f)(m) := f(\varphi(m))$, and $\omega \in \Omega^2(M,\mathbb{R})$ is a closed 2-form. Then ω defines a Lie algebra cocycle in $Z^2_c(\mathcal{V}(M), \mathfrak{a})$, and it is an interesting question when this cocycle integrates to a group cocycle on G. We know that this is the case if the period group $\langle [\omega], H_2(M) \rangle \subseteq \mathbb{R}$ is discrete, but this is not necessary (*cf.* [KYMO85, p.86]). The approach described in Example V.3.5 may be useful to analyze this problem. The crucial point is to understand the range of the homomorphism $\pi_2(\mathrm{Diff}(M)) \to \pi_2(M, m_0)$ and of the natural map $\pi_1(\mathrm{Diff}(M)) \times \pi_1(M, m_0) \to [\mathbb{T}^2, M] \to H_2(M)$ (Example V.3.5) (see [Ban97, Ch. 3] for more details on such maps). □

Problem V.5. Give a characterization of those principal K-bundles $q \colon P \to M$ for which the extension $\mathrm{Aut}(P)$ of the subgroup $\mathrm{Diff}(M)_{[P]}$ by the gauge group $\mathrm{Gau}(P)$ splits on the group level (*cf.* Example V.1.6). On the Lie algebra level, such conditions are given by Lecomte in [Lec85]. Note that this is obviously the case if the bundle is trivial, which implies $\mathrm{Aut}(P) \cong C^\infty(M,K) \rtimes \mathrm{Diff}(M)$. It is also the case for natural bundles to which the action of $\mathrm{Diff}(M)$ lifts, such as the frame bundle and other natural bundles. □

VI. Integrability of locally convex Lie algebras

In this section, we take a systematic look at the integrability problem for locally convex Lie algebras with an emphasis on locally exponential ones, because they permit a quite satisfying general theory. For Lie algebras which are not locally exponential only isolated results are available.

VI.1. Enlargeability of locally exponential Lie algebras

Definition VI.1.1. *A locally convex Lie algebra* $\mathfrak{g}$ *is said to be* integrable *if there exists a Lie group* G *with* $\mathbf{L}(G) \cong \mathfrak{g}$. *It is called* locally integrable *if there exists a local Lie group* $(G, D, m_G, \mathbf{1})$ *with Lie algebra* $\mathbf{L}(G) \cong \mathfrak{g}$. *A locally exponential Lie algebra is called* enlargeable *if it is integrable to a locally exponential Lie group, i.e., if some of the corresponding local groups are enlargeable (cf. Definition IV.2.3).* □

Although every finite-dimensional Lie algebra is integrable, integrability of infinite-dimensional Lie algebras turns out to be a very subtle property.

Examples VI.1.2. (a) If $\mathfrak{g}$ is a finite-dimensional Lie algebra, endowed with its unique locally convex topology, then $\mathfrak{g}$ is integrable. This is Lie's Third Theorem. One possibility to prove this is first to use Ado's Theorem to find an embedding $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n(\mathbb{R})$ and then to endow the integral subgroup $G := \langle \exp \mathfrak{g} \rangle \subseteq \mathrm{GL}_n(\mathbb{R})$ with a Lie group structure such that $\mathbf{L}(G) = \mathfrak{g}$ (*cf.* Corollary IV.4.10).

(b) If $\mathfrak{g}$ is locally exponential, then it is locally integrable by definition. In particular, every Banach–Lie algebra is locally integrable (Examples IV.2.4). □

Enlargeability and generalized central extensions

The criteria described in Section V.3 provide good tools to understand the difference between the group and Lie algebra picture for abelian extensions. However, not all quotient maps $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ of Lie algebras are topologically split in the sense that there is a continuous linear section, therefore they are not extensions of the type just discussed. An important example is the map $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$, where $\mathfrak{z}(\mathfrak{g})$ is the center. The fact that for each locally exponential Lie algebra $\mathfrak{g}$, the Lie algebra $\mathfrak{g}_{\mathrm{ad}} := \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is always integrable (Theorem IV.3.8) shows that the question of the integrability of central extensions has to be addressed even for those which are not topologically split. Fortunately, there is a method to circumvent the problems caused by this topological difficulty by reducing all assertions to topologically split central extensions. The key concept is that of a generalized central extension (*cf.* [Ne03], [GN07]).

Definition VI.1.3. *A morphism $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ of locally convex Lie algebras is called a* generalized central extension *if it has dense range and there exists a continuous bilinear map $b\colon \mathfrak{g} \times \mathfrak{g} \to \widehat{\mathfrak{g}}$ for which $b \circ (q \times q)$ is the Lie bracket on $\widehat{\mathfrak{g}}$. It is called a* central extension *if, in addition, q is a quotient map.* □

The subtlety of generalized central extensions is that q need not be surjective and if it is surjective, it need not be a quotient map. Fortunately, these difficulties are compensated by the following nice fact. Let us call a locally convex Lie algebra $\mathfrak{g}$ *topologically perfect* if its commutator algebra is dense. We call a generalized central extension $q_{\mathfrak{g}}\colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ *universal* if for any generalized central extension $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ there exists a unique morphism of locally convex Lie algebras $\alpha\colon \widetilde{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ with $q \circ \alpha = q_{\mathfrak{g}}$. Then one can show that each topologically perfect locally convex Lie algebra $\mathfrak{g}$ has a universal generalized central extension (unique up to isomorphism). For the basic results on generalized central extensions we refer to [Ne03, Sect. III], where one also finds descriptions of the universal generalized central extensions of several classes of Lie algebras.

Remark VI.1.4. If $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ is a central extension, then $q \times q\colon \widehat{\mathfrak{g}} \times \widehat{\mathfrak{g}} \to \mathfrak{g} \times \mathfrak{g}$ also is a quotient map. Therefore the Lie bracket of $\widehat{\mathfrak{g}}$ factors through a continuous bilinear map $b\colon \mathfrak{g} \times \mathfrak{g} \to \widehat{\mathfrak{g}}$ with $b(q(x),q(y)) = [x,y]$ for $x,y \in \widehat{\mathfrak{g}}$, showing that q is a generalized central extension of $\mathfrak{g}$. □

Proposition VI.1.5. ([Ne03, Lemma III.4])[6] *For a generalized central extension $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ the following assertions hold:*

(1) The corresponding map b is a Lie algebra cocycle in $Z^2_c(\mathfrak{g}, |\widehat{\mathfrak{g}}|)$, where $|\widehat{\mathfrak{g}}|$ denotes $\widehat{\mathfrak{g}}$, considered as a trivial $\mathfrak{g}$-module.
(2) If $|\mathfrak{g}|$ denotes the space $\mathfrak{g}$, endowed with the trivial Lie bracket, then the maps

$$\psi\colon \widehat{\mathfrak{g}} \to |\widehat{\mathfrak{g}}| \oplus_b \mathfrak{g}, \quad x \mapsto (x, q(x)) \quad \textit{and} \quad \eta\colon |\widehat{\mathfrak{g}}| \oplus_b \mathfrak{g} \to |\mathfrak{g}|, \quad (x,y) \mapsto y - q(x)$$

are homomorphisms of Lie algebras, ψ is a topological embedding, η is a quotient map, and the sequence

$$\mathbf{0} \to \widehat{\mathfrak{g}} \xrightarrow{\psi} |\widehat{\mathfrak{g}}| \oplus_b \mathfrak{g} \xrightarrow{\eta} |\mathfrak{g}| \to \mathbf{0}$$

is exact. □

For the following theorem from [GN07], we recall that central extensions of locally exponential Lie algebras by Mackey complete spaces are locally exponential (Theorem IV.2.10).

Theorem VI.1.6. (Enlargeability criterion for generalized central extensions) *Let G be a connected locally exponential Lie group with Lie algebra $\mathfrak{g}$ and $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ a generalized central extension for which $\widehat{\mathfrak{g}}$ is Mackey complete. Let $\omega \in Z^2_c(\mathfrak{g}, |\widehat{\mathfrak{g}}|)$ be the associated Lie algebra cocycle and* $\mathrm{per}_\omega\colon \pi_2(G) \to |\widehat{\mathfrak{g}}|$ *the corresponding period homomorphism. Then the following assertions hold:*

(1) $\Pi_\omega := \mathrm{im}(\mathrm{per}_\omega)$ *is contained in $\mathfrak{z}$.*
(2) $\widehat{\mathfrak{g}}$ is enlargeable if Π_ω is discrete.
(3) If q is a central extension, then $\widehat{\mathfrak{g}}$ is enlargeable if and only if Π_ω is discrete.

Proof. (1) follows from the fact that the cocycle $q \circ \omega = -d_{\mathfrak{g}} \mathrm{id}_{\mathfrak{g}}$ is trivial. It is the Lie bracket of $\mathfrak{g}$.

(2) Corollary V.3.2 implies that $\widetilde{\mathfrak{g}} := |\widehat{\mathfrak{g}}| \oplus_b \mathfrak{g}$ is enlargeable if and only if Π_ω is discrete. If this is the case, then the closed ideal $\widehat{\mathfrak{g}}$ of $\widetilde{\mathfrak{g}}$ is also enlargeable because $\widehat{\mathfrak{g}} \cong \ker \eta$ implies that it is locally exponential (Theorem IV.2.9), so that Corollary IV.4.10 applies.

(3) Suppose that q is a quotient map, i.e., a central extension, and that $\widehat{\mathfrak{g}}$ is enlargeable. Since the cocycle $\widetilde{b} := q^*b$ coincides with the Lie bracket on $\widehat{\mathfrak{g}}$, the

[6] For the case of central extensions of Banach–Lie algebras, part of the assertions below can be found in a footnote in [ES73, p.58].

corresponding central extension $\widehat{\mathfrak{g}}^\sharp := |\widehat{\mathfrak{g}}| \oplus_{\widetilde{b}} \widehat{\mathfrak{g}}$ is split by the section $\sigma(x) := (x,x)$, hence is enlargeable. Furthermore,

$$\widetilde{\mathfrak{g}} = |\widehat{\mathfrak{g}}| \oplus_b \mathfrak{g} \cong \widehat{\mathfrak{g}}^\sharp/(\{0\} \times \mathfrak{z})$$

is locally exponential by Theorem IV.2.9, which applies in particular to all quotients by central ideals. In view of Theorem VI.1.10 below, it now suffices to show that the integral subgroup Z generated by $\mathfrak{z}$ is a locally exponential Lie subgroup. But this follows from the fact that the projection onto $|\widehat{\mathfrak{g}}| \times \{0\}$ along $\mathrm{im}(\sigma)$ restricts to a homeomorphism on $\mathfrak{z}$. Hence the corresponding subgroup is a locally exponential Lie subgroup, and this completes the proof. □

The preceding theorem applies in particular to central extensions $\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g} = \mathbf{L}(G)$ of Banach–Lie algebras, for which it characterizes integrability in terms of the discreteness of Π_b. In this case, a similar criterion is given by van Est and Korthagen in [EK64]. On the surface, their criterion has the same formulation, but their period homomorphism arises as an element of $H^2_{\mathrm{sing}}(G,\mathfrak{z}) \cong \mathrm{Hom}(H_2(G),\mathfrak{z})$ obtained from the enlargeability theory of local groups ([Est62]). Under their assumption that G is 1-connected, the Hurewicz homomorphism $\pi_2(G) \to H_2(G)$ is an isomorphism, so that their period homomorphism also is a homomorphism $\pi_2(G) \to \mathfrak{z}$, and one can even show that both coincide up to sign. We think that the definition of the period homomorphism in terms of integration of differential forms makes it much more accessible than the implicit construction in [EK64].

Definition VI.1.7. *Let $\mathfrak{g}$ be a locally exponential Lie algebra and consider the central extension*

$$\mathbf{0} \to \mathfrak{z}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}_{\mathrm{ad}} := \mathfrak{g}/\mathfrak{z} \to \mathbf{0}.$$

Let $G_{\mathrm{ad}} \subseteq \mathrm{Aut}(\mathfrak{g})$ be endowed with its locally exponential group structure with Lie algebra $\mathfrak{g}_{\mathrm{ad}}$ (Theorem IV.3.8) and

$$\mathrm{per}_{\mathfrak{g}} \colon \pi_2(G_{\mathrm{ad}}) \to \mathfrak{z}(\mathfrak{g})$$

the corresponding period homomorphism (Theorem VI.1.6(1)). We write $\Pi(\mathfrak{g}) := \mathrm{im}(\mathrm{per}_{\mathfrak{g}})$ for its image and call it the period group of $\mathfrak{g}$. □

The following theorem generalizes the enlargeability criterion of [EK64] for Banach algebras. It follows immediately from Theorem IV.3.8 on the integrability of $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ and Theorem VI.1.6.

Theorem VI.1.8. (Enlargeability Criterion for locally exponential Lie algebras) *A Mackey complete locally exponential Lie algebra $\mathfrak{g}$ is enlargeable if and only if its period group $\Pi(\mathfrak{g})$ is discrete.* □

Proposition VI.1.9. *If $\mathfrak{g}$ is a separable locally exponential Lie algebra, then $\Pi(\mathfrak{g})$ is countable. If, in addition, $\mathfrak{g}$ is Fréchet, then $\Pi(\mathfrak{g})$ is closed if and only if it is discrete.*

Proof. If $\mathfrak{g}$ is separable, then the same holds for the connected group $G_{\rm ad}$ and hence for the identity component $C_*(\mathbb{S}^1, G_{\rm ad})_0$ of the loop group. Its universal covering group is also separable, so that its fundamental group, which is isomorphic to $\pi_2(G_{\rm ad})$, is countable. This implies that $\Pi(\mathfrak{g})$ is countable.

If $\Pi(\mathfrak{g})$ is closed and $\mathfrak{g}$ is Fréchet, it is a countable complete metric space, hence discrete. □

For the second part of the preceding proposition, the Fréchet assumption on $\mathfrak{g}$ is crucial: the space $\mathbb{R}^{\mathbb{R}}$ contains a non-discrete closed subgroup isomorphic to $\mathbb{Z}^{(\mathbb{N})}$ ([HMP04, Cor. 3.2(i)]).

Combining the fact that kernels of morphisms are locally exponential Lie subgroups (Proposition IV.3.4) and Theorem IV.1.19 on the integration of morphisms of Lie algebras, one obtains the equivalence of (1) and (2) in the following integrability criterion for quotient algebras ([GN06]):

Theorem VI.1.10. (Enlargeability Criterion for quotients) *Let G be a 1-connected locally exponential Lie group and $\mathfrak{n} \trianglelefteq \mathfrak{g}$ a closed ideal for which the quotient Lie algebra $\mathfrak{q} := \mathfrak{g}/\mathfrak{n}$ is locally exponential. Let*

$$Z(G,\mathfrak{n}) := \{g \in G\colon (\mathrm{Ad}(g) - \mathbf{1})(\mathfrak{g}) \subseteq \mathfrak{n}\}.$$

Then $Z(G,\mathfrak{n}) \trianglelefteq G$ is a normal locally exponential Lie subgroup with Lie algebra

$$\mathfrak{z}(\mathfrak{g},\mathfrak{n}) := \{x \in \mathfrak{g}\colon [x,\mathfrak{g}] \subseteq \mathfrak{n}\},$$

and the Lie algebra homomorphism $q\colon \mathfrak{z}(\mathfrak{g},\mathfrak{n}) \to \mathfrak{z}(\mathfrak{q})$ defines a period homomorphism

$$\mathrm{per}_q\colon \pi_1(Z(G,\mathfrak{n})) \to \mathfrak{z}(\mathfrak{q}), \quad \mathrm{per}_q([\gamma]) = \int_0^1 q(\delta(\gamma)_t)\,dt,$$

where $\gamma\colon [0,1] \to Z(G,\mathfrak{n})$ is a piecewise smooth loop. The following assertions are equivalent:

(1) The locally exponential Lie algebra $\mathfrak{q} = \mathfrak{g}/\mathfrak{n}$ is enlargeable.
(2) The normal integral subgroup $N := \langle \exp_G \mathfrak{n} \rangle \trianglelefteq G$ is a locally exponential Lie subgroup.
(3) The image of per_q is a discrete subgroup of $\mathfrak{z}(\mathfrak{q})$. □

Remark VI.1.11. In addition to the assumptions of the preceding theorem, suppose that G is a Fréchet–Lie group. We then have $Q_{\rm ad} \cong G/Z(G,\mathfrak{n})$, and since

Michael's Selection Theorem ([MicE59]) applies to the quotient map $\mathfrak{g} \to \mathfrak{q}$, this leads to a surjective homomorphism

$$\delta\colon \pi_2(Q_{\mathrm{ad}}) \to \pi_1(Z(G,\mathfrak{n})).$$

The surjectivity of δ follows from the 1-connectedness of G and the exactness of the long exact homotopy sequence of the bundle $G \to Q_{\mathrm{ad}}$. Then it is not hard to see that

$$\mathrm{per}_q \circ \delta = \mathrm{per}_{\mathfrak{q}}\colon \pi_2(Q_{\mathrm{ad}}) \to \mathfrak{z}(\mathfrak{q}),$$

which shows that (3) in Theorem VI.1.10 is equivalent to the discreteness of the $\Pi(\mathfrak{q})$ (Theorem VI.1.8). □

Proposition VI.1.12. (Functoriality of the period group) *Let $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ be a morphism of Mackey complete locally exponential Lie algebras with $\varphi(\mathfrak{z}(\mathfrak{g})) \subseteq \mathfrak{z}(\mathfrak{h})$ and $\varphi_{\mathrm{ad}}\colon \mathfrak{g}_{\mathrm{ad}} \to \mathfrak{h}_{\mathrm{ad}}$ the induced homomorphism. Then φ_{ad} integrates to a group homomorphism $\widetilde{\varphi}_{\mathrm{ad}}\colon \widetilde{G}_{\mathrm{ad}} \to \widetilde{H}_{\mathrm{ad}}$, $\varphi(\Pi(\mathfrak{g})) \subseteq \Pi(\mathfrak{h})$, and the following diagram commutes*

$$\begin{array}{ccc} \pi_2(G_{\mathrm{ad}}) & \xrightarrow{\pi_2(\widetilde{\varphi}_{\mathrm{ad}})} & \pi_2(H_{\mathrm{ad}}) \\ \downarrow \mathrm{per}_{\mathfrak{g}} & & \downarrow \mathrm{per}_{\mathfrak{h}}. \\ \mathfrak{z}(\mathfrak{g}) & \xrightarrow{\varphi} & \mathfrak{z}(\mathfrak{h}) \end{array}$$

□

Corollary VI.1.13. *If $\mathfrak{g}_1, \mathfrak{g}_2$ are Mackey complete locally exponential Lie algebras, then*

$$\Pi(\mathfrak{g}_1 \times \mathfrak{g}_2) = \Pi(\mathfrak{g}_1) \times \Pi(\mathfrak{g}_2).$$ □

Remark VI.1.14. (Constructing non-enlargeable Lie algebras) Suppose that $\mathfrak{g}$ is a locally exponential Lie algebra with $\Pi(\mathfrak{g}) \cong \mathbb{Z}$. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Then $\mathfrak{z} := \{(x, \theta x)\colon x \in \mathfrak{z}(\mathfrak{g})\}$ is a central ideal of $\mathfrak{g} \times \mathfrak{g}$, so that $\mathfrak{h} := (\mathfrak{g} \times \mathfrak{g})/\mathfrak{z}$ is locally exponential. Corollary VI.1.13 and Proposition VI.1.12 imply that, writing $\Pi(\mathfrak{g}) = \mathbb{Z}d$, we get

$$\Pi(\mathfrak{h}) \cong \mathbb{Z}[(d,0)] + \mathbb{Z}[(0,d)] = (\mathbb{Z} + \mathbb{Z}\theta)[(d,0)],$$

which is not discrete. Hence $\mathfrak{h}$ is not integrable. □

Using the construction of the group G via Theorem VI.1.6 and the long exact homotopy sequence, one can identify the period group of enlargeable Fréchet–Lie algebras in terms of the center:

Proposition VI.1.15. *If G is a locally exponential 1-connected Fréchet–Lie group and $\mathfrak{g} = \mathbf{L}(G)$ its Lie algebra, then*

$$\Pi(\mathfrak{g}) = \ker(\exp_G|_{\mathfrak{z}(\mathfrak{g})}) \cong \pi_1(Z(G)).$$ □

Example VI.1.16. The first example of a non-enlargeable Banach–Lie algebra was given by van Est and Korthagen with the method described in Remark VI.1.14 ([EK64]). It is the central extension $\mathfrak{g}$ of the Banach–Lie algebra $C^1(\mathbb{S}^1, \mathfrak{su}_2(\mathbb{C}))$ by $\mathbb{R}$, defined by the cocycle

$$\omega(f,g) := \int_0^1 \operatorname{tr}(f(t)g'(t))\,dt,$$

where we identify functions on $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ with 1-periodic functions on $\mathbb{R}$. Then $\mathfrak{g}_{\mathrm{ad}} \cong C^1(\mathbb{S}^1, \mathfrak{su}_2(\mathbb{C}))$ and $G_{\mathrm{ad}} \cong C^1(\mathbb{S}^1, \mathrm{SU}_2(\mathbb{C}))$ leads to $\pi_2(G_{\mathrm{ad}}) \cong \pi_3(\mathrm{SU}_2(\mathbb{C})) \cong \pi_3(\mathbb{S}^3) \cong \mathbb{Z}$. Now one shows that $\mathrm{per}_{\mathfrak{g}} = \mathrm{per}_\omega$ is non-trivial to verify that $\Pi(\mathfrak{g}) \cong \mathbb{Z}$.

Using Kuiper's Theorem ([Ku65]), Douady and Lazard gave a simpler example ([DL66]): by observing that the 1-connectedness of the unitary group $\mathrm{U}(H)$ of an infinite-dimensional complex Hilbert space H implies that its Lie algebra $\mathfrak{u}(H) := \{X \in \mathscr{L}(H) \colon X^* = -X\}$ satisfies

$$\Pi(\mathfrak{u}(H)) \cong \pi_1(Z(\mathrm{U}(H))) = \pi_1(\mathbb{T}) \cong \mathbb{Z}$$

(Proposition VI.1.15).

Based on the fact that $\mathrm{U}(H)$ is 1-connected, one can also give the following direct argument. For any irrational $\theta \in \mathbb{R} \setminus \mathbb{Q}$ the line $\mathfrak{n} := \mathbb{R}i(\mathbf{1}, \theta\mathbf{1})$ generates a dense subgroup of the center $Z(\mathrm{U}(H) \times \mathrm{U}(H)) \cong \mathbb{T}^2$ of the 1-connected group $\mathrm{U}(H) \times \mathrm{U}(H)$, so that Theorem VI.1.10 implies that the quotient Lie algebra $(\mathfrak{u}(H) \times \mathfrak{u}(H))/\mathfrak{n}$ is not enlargeable. □

Enlargeability of quotients

One may take Theorem VI.1.10 as a starting point of a theory of certain topological groups which are more general than Lie groups, namely quotients of Lie groups. This leads to the concept of a *scheme of Lie groups*, or *S-Lie group* (*cf.* [Ser65], [DL66] and [Est84]). The strength of this concept for Banach–Lie algebras and, more generally, locally exponential Lie algebras, follows from the fact that each such Lie algebra is a quotient of an enlargeable one:

Theorem VI.1.17. ([Swi71] for the Banach case) *For each locally exponential Fréchet–Lie algebra* $\mathfrak{g}$*, the Lie algebra*

$$\Lambda(\mathfrak{g}) := C_*([0,1], \mathfrak{g}) := \{\gamma \in C([0,1], \mathfrak{g}) \colon \gamma(0) = 0\}$$

is enlargeable.

Proof. Clearly, $\mathfrak{z}(\Lambda(\mathfrak{g})) = \Lambda(\mathfrak{z}(\mathfrak{g}))$, so that $\Lambda(\mathfrak{g})_{\mathrm{ad}} \cong \Lambda(\mathfrak{g}_{\mathrm{ad}})$ follows from Michael's Theorem ([MicE59]). The corresponding group $C_*([0,1], G_{\mathrm{ad}})$ is contractible, and this leads to $\Pi(\Lambda(\mathfrak{g})) = \{0\}$, which implies enlargeability. □

A central point of the preceding theorem is that it implies that each locally exponential Fréchet–Lie algebra $\mathfrak{g}$ is a quotient of an enlargeable Fréchet–Lie algebra (*cf.* [Rob02, Th. 5]): the evaluation map $\mathrm{ev}_1 \colon \Lambda(\mathfrak{g}) \to \mathfrak{g}, \gamma \mapsto \gamma(1)$ is a quotient map. Now one can address the enlargeability problem along the lines of Theorem VI.1.10.

Remark VI.1.18. (a) In [Woj06], Wojtyński describes a variant of this approach for Banach–Lie algebras. Instead of considering the Banach–Lie algebra $\Lambda(\mathfrak{g})$, he considers analytic paths $\gamma(t) := \sum_{n=1}^{\infty} a_n t^n$, for which $\|\gamma\|_1 := \sum_{n=1}^{\infty} \|a_n\|$ is finite. Identifying these curves with their coefficient sequences, we denote this space by $\ell^1(\mathfrak{g}) := \ell^1(\mathbb{N}, \mathfrak{g})$. The Lie bracket on this sequence space is given by

$$[(a_n),(b_n)] = (c_n) \quad \text{with} \quad c_n = \sum_{j=1}^{n-1} [a_j, b_{n-j}]. \tag{6.1.1}$$

With the same Lie bracket, we also turn the full sequence space $\mathfrak{g}^{\mathbb{N}}$ into a pro-nilpotent Fréchet–Lie algebra, which is exponential for trivial reasons. Since the Banach–Lie algebra $\ell^1(\mathfrak{g})$ injects into the exponential Lie algebra $\mathfrak{g}^{\mathbb{N}}$, it is enlargeable by Corollary IV.4.10. Again, we have an evaluation map

$$q \colon \ell^1(\mathfrak{g}) \to \mathfrak{g}, \quad (a_n) \mapsto \sum_{n=1}^{\infty} a_n,$$

which is a quotient morphism of Lie algebras and since the subgroup $\langle \exp \ell^1(\mathfrak{g}) \rangle$ is contractible (*cf.* [Woj06]), one may proceed with Theorem VI.1.10 as for $\Lambda(\mathfrak{g})$.

(b) In [Pe93a/95a], Pestov shows that if E is a Banach space of $\dim E > 1$, then the free Banach–Lie algebra over E has trivial center. As a consequence, every Banach–Lie algebra $\mathfrak{g}$ of dimension > 1 is a quotient of a centerless Banach–Lie algebra $F(\mathfrak{g})$, the free Banach–Lie algebra over the Banach space $\mathfrak{g}$, which is enlargeable because its center is trivial (Theorem IV.3.8). Again, we can proceed with Theorem VI.1.10 to obtain enlargeability criteria. □

The following enlargeability criterion of Swierczkowski for extensions by not necessarily abelian ideals is a powerful tool. It would be very interesting to see if it can be extended to the locally exponential setting. It applies in particular to all situations where $\mathfrak{q}$ is finite-dimensional or abelian (*cf.* [Swi65]; Remark V.2.14(b)).

Theorem VI.1.19. ([Swi67, Th., Sect. 12]) *Suppose that $\mathfrak{g}$ is a Banach–Lie algebra and $\mathfrak{n} \trianglelefteq \mathfrak{g}$ a closed enlargeable ideal for which $\mathfrak{q} := \mathfrak{g}/\mathfrak{n}$ is enlargeable to some group Q with vanishing $\pi_2(Q)$, then $\mathfrak{g}$ is enlargeable.* □

Definition VI.1.20. *A Banach–Lie algebra is said to be* lower solvable *if there exists an ordinal number* α *and an ascending chain of closed subalgebras*

$$\{0\} = \mathfrak{g}_0 \subseteq \mathfrak{g}_1 \subseteq \mathfrak{g}_2 \subseteq \cdots \subseteq \mathfrak{g}_\beta \subseteq \mathfrak{g}_{\beta+1} \subseteq \cdots \subseteq \mathfrak{g}_\alpha = \mathfrak{g}$$

such that

(a) If $\beta \leq \alpha$ *is not a limit ordinal, then* $X_{\beta-1}$ *is an ideal of* X_β *containing all commutators.*
(b) If $\beta \leq \alpha$ *is a limit ordinal, then* X_β *is the closure of* $\bigcup_{\gamma<\beta} X_\gamma$*.* □

The following theorem is an immediate consequence of Theorem VI.1.19, applied to the situation where $\mathfrak{q}$ is abelian:

Theorem VI.1.21. ([Swi65, Th. 2]) *Each lower solvable Banach–Lie algebra is enlargeable.* □

Some of the methods used above for Banach–Lie groups have some potential to work in greater generality. Here are some ideas:

Remark VI.1.22. If G is a Lie group with Lie algebra $\mathfrak{g}$, then

$$P(G) := C^\infty_*(I,G) := \{\gamma \in C^\infty(I,G)\colon\ \gamma(0) = \mathbf{1}\}$$

also is a Lie group with Lie algebra $P(\mathfrak{g}) := C^\infty_*(I,\mathfrak{g})$, endowed with the pointwise bracket. The logarithmic derivative $\delta\colon P(G) \to C^\infty(I,\mathfrak{g})$ is a smooth map satisfying $\delta(\alpha\beta) = \delta(\beta) + \mathrm{Ad}(\beta)^{-1}.\delta(\alpha)$ and $T_{\mathbf{1}}(\delta)(\xi) = \xi'$. (Lemma II.3.3). As $[\xi,\eta]' = [\xi',\eta] + [\xi,\eta']$, it follows that $T_1(\delta)\colon P(\mathfrak{g}) \to C^\infty(I,\mathfrak{g})$ becomes a topological isomorphism of Lie algebras if $C^\infty(I,\mathfrak{g})$ is endowed with the bracket

$$[\xi,\eta](t) := \Big[\xi(t), \int_0^t \eta(\tau)\,d\tau\Big] + \Big[\int_0^t \xi(\tau)\,d\tau, \eta(t)\Big]. \tag{6.1.2}$$

The evaluation map $\mathrm{ev}_1\colon P(\mathfrak{g}) \to \mathfrak{g}$ corresponds to the quotient map

$$C^\infty(I,\mathfrak{g}) \to \mathfrak{g}, \quad \xi \mapsto \int_0^1 \xi(\tau)\,d\tau.$$

If, in addition, G is regular, then δ is a diffeomorphism, and it follows that $C^\infty(I,\mathfrak{g})$, endowed with the bracket (6.1.2), is integrable. Since this property is clearly necessary for the regular integrability of $\mathfrak{g}$, Lie algebras with this property are called *pre-integrable* in [RK97] (see also [Les93]). □

If G is a real BCH–Lie group, then a morphism $\kappa_G\colon G \to G_{\mathbb{C}}$ to a complex BCH–Lie group $G_{\mathbb{C}}$ is called a *universal complexification* if for each other morphism $\alpha\colon G \to H$ to a complex BCH–Lie group H, there exists a unique morphism $\beta\colon G_{\mathbb{C}} \to H$ with $\alpha = \beta \circ \kappa_G$. It is well known that if G is finite-dimensional, then a universal complexification always exists (*cf.* [Ho65] , [Ne99, Th. XIII.5.6]), but it need not be locally injective, so that it may occur that

$\dim_{\mathbb{C}} G_{\mathbb{C}} < \dim_{\mathbb{R}} G$. The following theorem shows that, due to the existence of non-enlargeable Lie algebras, the situation becomes more complicated in infinite dimensions.

Theorem VI.1.23. (Existence of universal complexifications; [GN03], [Gl02c]) *Given a real BCH–Lie group G, let N_G be the intersection of all kernels of smooth homomorphisms from G to complex BCH–Lie groups. Then G has a universal complexification if and only if N_G is a BCH–Lie subgroup of G and the complexification of $\mathbf{L}(G)/\mathbf{L}^e(N_G)$ is enlargeable.* □

Note that Theorem VI.1.10 implies that if G is 1-connected, the existence of a universal complexification is equivalent to the enlargeability of $\mathbf{L}(G)/\mathbf{L}^e(N_G)$. In [GN03], one finds an example of a Banach-Lie group for which N_G fails to be a Lie subgroup ([GN03, Sect. V]) and also examples where $N_G = \{\mathbf{1}\}$ but $\mathbf{L}(G)_{\mathbb{C}}$ is not enlargeable. The setting of BCH–Lie groups is the natural one for complexifications because if $\mathfrak{g}$ is a locally exponential Lie algebra for which $\mathfrak{g}_{\mathbb{C}}$ is locally exponential as a complex Lie algebra, then the local multiplication is complex analytic. This implies that $\mathfrak{g}_{\mathbb{C}}$ is BCH which in turns entails that $\mathfrak{g}$ is BCH.

Localizing enlargeability

We call a norm $\|\cdot\|$ on a Lie algebra $\mathfrak{g}$ *submultiplicative* if $\|[x,y]\| \leq \|x\|\|y\|$ for all $x, y \in \mathfrak{g}$. A Banach–Lie algebra $(\mathfrak{g}, \|\cdot\|)$ is called *contractive* if its norm is submultiplicative. For any contractive Lie algebra, we define

$$\delta_{\mathfrak{g}} := \inf\{\|x\| : 0 \neq x \in \Pi(\mathfrak{g})\} \in [0, \infty]$$

and note that $\mathfrak{g}$ is enlargeable if and only if $\delta_{\mathfrak{g}} > 0$, which is equivalent to the discreteness of the period group $\Pi(\mathfrak{g})$ (Theorem VI.1.8). The following theorem is originally due to Pestov who proved it with non-standard methods. A "standard" proof has been given in [Bel04] by Beltita.

Theorem VI.1.24. (Pestov's Local Theorem on Enlargeability) *A contractive Banach–Lie algebra $\mathfrak{g}$ is enlargeable if and only if there exists a directed family $\mathscr{H}$ of closed subalgebras of $\mathfrak{g}$ for which $\bigcup \mathscr{H}$ is dense in $\mathfrak{g}$ and* $\inf\{\delta_{\mathfrak{h}} : \mathfrak{h} \in \mathscr{H}\} > 0$. □

Since for each finite-dimensional Lie algebra $\mathfrak{g}$ the period group is trivial, we have $\delta_{\mathfrak{g}} = \infty$, and the preceding theorem, applied to the directed family of finite-dimensional subalgebras of $\mathfrak{g}$ leads to:

Corollary VI.1.25. ([Pe92], [Bel04]) *If $\mathfrak{g}$ is a Banach–Lie algebra containing a locally finite-dimensional dense subalgebra, then $\mathfrak{g}$ is enlargeable.* □

Corollary VI.1.26. ([Pe93b, Th. 7]) *A Banach–Lie algebra $\mathfrak{g}$ is enlargeable if and only if all its separable closed subalgebras are.* □

Period groups for continuous inverse algebras

Another interesting class of cocycles arises for complete CIAs A ([Ne06c]). A continuous alternating bilinear map $\alpha\colon A \times A \to E$, E a locally convex space, is said to be a *cyclic* 1-*cocycle* if

$$\alpha(ab,c) + \alpha(bc,a) + \alpha(ca,b) = 0 \quad \text{for} \quad a,b,c \in A.$$

We write $ZC^1(A,E)$ for the set of all cyclic 1-cocycles with values in E. Let $A_L = \mathfrak{gl}_1(A)$ denote the Lie algebra $(A,[\cdot,\cdot])$ obtained by endowing A with the commutator bracket. Then each cyclic cocycle defines a Lie algebra cocycle $\alpha \in Z^2_c(A_L,E)$ with respect to the trivial module structure on E. To describe the universal cyclic cocycle, we endow $A \otimes A$ with the projective tensor topology and define $\langle A,A\rangle$ as the completion of the quotient space

$$(A \otimes A)/\,\overline{\operatorname{span}\{a \otimes a, ab \otimes c + bc \otimes a + ca \otimes b; a,b,c \in A\}}.$$

We write $\alpha_u(a,b) := \langle a,b\rangle$ for the image of $a \otimes b$ in $\langle A,A\rangle$. Then the universal property of the projective tensor product implies that

$$\mathscr{L}(\langle A,A\rangle,E) \to ZC^1(A,E), \quad f \mapsto f \circ \alpha_u$$

is a bijection for each complete locally convex space E, so that α_u is a universal cyclic 1-cocycle. Of particular interest is the map $b_A\colon \langle A,A\rangle \to A, \langle a,b\rangle \to [a,b]$ defined by the commutator bracket. Its kernel

$$HC_1(A) := \ker b_A \subseteq \langle A,A\rangle$$

is the *first cyclic homology space of* A (*cf.* [Lo98]). We write ω_u for the universal cyclic 1-cocycle, interpreted as a Lie algebra 2-cocycle. Then the corresponding period map

$$\operatorname{per}_{\omega_u}\colon \pi_2(A^\times) \to \langle A,A\rangle$$

actually has values in the subspace $HC_1(A)$, which leads to a homomorphism

$$\operatorname{per}_{\omega_u}\colon \pi_2(A^\times) \to HC_1(A).$$

It is a remarkable fact that this structure behaves nicely if we replace A by a matrix algebra $M_n(A)$. Let $\eta_n\colon A \to M_n(A), a \mapsto aE_{11}$ denote the natural inclusion map and observe that it induces maps $\langle A,A\rangle \to \langle M_n(A),M_n(A)\rangle$, taking $HC_1(A)$ into $HC_1(M_n(A))$. In the other direction, we have maps

$$\operatorname{tr}^{(2)}\colon \langle M_n(A),M_n(A)\rangle \to \langle A,A\rangle, \quad \langle (a_{ij}),(b_{ij})\rangle \mapsto \sum_{i,j=1}^n \langle a_{ij},b_{ji}\rangle,$$

and the topological version of the Morita invariance of cyclic homology ([Lo98, Th. 2.2.9]) asserts that these maps restrict to isomorphisms $HC_1(M_n(A)) \to$

$HC_1(A)$. This leads to extensions of the universal cocycle to a cocycle $\omega_u^n \in Z_c^2(\mathfrak{gl}_n(A), \langle A,A\rangle)$ with $\eta_n^*\omega_u^n = \omega_u$ for each $n \in \mathbb{N}$. In terms of the tensor product structure $\mathfrak{gl}_n(A) \cong A \otimes \mathfrak{gl}_n(\mathbb{K})$, it is given by

$$\omega_u^n(a\otimes x, b\otimes y) = \mathrm{tr}(xy)\langle a,b\rangle.$$

To explain the corresponding compatibility on the level of period homomorphisms, we define the *topological K-groups* of A by

$$K_{i+1}(A) := \varinjlim \pi_i(\mathrm{GL}_n(A)) \quad \text{for} \quad i \in \mathbb{N}_0,$$

where the direct limit on the right hand side corresponds to the embeddings

$$\mathrm{GL}_n(A) \to \mathrm{GL}_{n+1}(A), \quad a \mapsto \begin{pmatrix} a & 0 \\ 0 & \mathbf{1} \end{pmatrix},$$

induced by the corresponding embeddings $M_n(A) \hookrightarrow M_{n+1}(A)$. The group $K_0(A)$ is defined as the Grothendieck group of the abelian monoid $\varinjlim \pi_0(\mathrm{Idem}(M_n(A)))$, endowed with the addition $[e]+[f] := \left[\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}\right]$ (*cf.* [Bl98]).

The naturality of the universal cocycles now implies that the period maps

$$\mathrm{per}_{\omega_u^n} \colon \pi_2(\mathrm{GL}_n(A)) \to HC_1(A)$$

combine to a group homomorphism

$$\mathrm{per}_A^1 \colon K_3(A) = \varinjlim \pi_2(\mathrm{GL}_n(A)) \to HC_1(A),$$

which is a natural transformation from the functor K_3 with values in abelian groups to the functor HC_1 with values in complete locally convex spaces.

It is of some interest to know whether the group

$$\Pi_A^1 := \mathrm{im}(\mathrm{per}_A^1) \subseteq HC_1(A)$$

is discrete. If this is the case, then each period homomorphism $\mathrm{per}_{\omega_u^n}$ has discrete image, which implies that the corresponding central extension $\widehat{\mathfrak{gl}}_n(A)$ of the Lie algebra $\mathfrak{gl}_n(A)$ by $\langle M_n(A), M_n(A)\rangle$ is enlargeable.

This central extension is of particular interest when restricted to the subalgebra $\mathfrak{sl}_n(A) := \overline{[\mathfrak{gl}_n(A), \mathfrak{gl}_n(A)]}$. We define a Lie bracket on $\langle M_n(A), M_n(A)\rangle$ by

$$[\langle a,b\rangle, \langle a',b'\rangle] := \langle [a,b], [a',b']\rangle,$$

turning it into a locally convex Lie algebra. Now the bracket map of $M_n(A)$ induces a generalized central extension

$$q \colon \widehat{\mathfrak{sl}}_n(A) := \langle M_n(A), M_n(A)\rangle \to \mathfrak{sl}_n(A), \quad \langle a,b\rangle \mapsto [a,b]$$

with $\ker q = HC_1(M_n(A)) \cong HC_1(A)$, which is a universal generalized central extension, called the *topological Steinberg–Lie algebra* ([Ne03, Ex. 4.10]). The enlargeability criterion in Theorem VI.1.6 immediately leads to:

Theorem VI.1.27. *If the subgroup Π_A^1 of $HC_1(A)$ is discrete, then all Steinberg–Lie algebras $\widehat{\mathfrak{sl}}_n(A)$ are enlargeable.* □

Remark VI.1.28. Let $\omega \in ZC^1(A,E)$ be a cyclic cocycle, considered as a Lie algebra cocycle on $(A,[\cdot,\cdot])$. Then the adjoint action of A on the Lie algebra $\widehat{A}_L := E \oplus_\omega A_L$ integrates to an action of $A^\times$ by

$$g.(z,a) = (z - \omega(ag^{-1},g), gag^{-1}).$$

In view of Remark V.2.14(f), this implies the triviality of the flux homomorphism

$$F_\omega \colon \pi_1(A^\times) \to H_c^1(A_L,E) \subseteq \mathscr{L}(A,E). \qquad \square$$

According to [Bos90], we have for each complex complete CIA A natural isomorphisms

$$\beta_A^i \colon K_i(A) \to K_{i+2}(A), \quad i \in \mathbb{N}_0.$$

This is an abstract version of Bott periodicity. In particular, the range of

$$P_A := \mathrm{per}_A^1 \circ \beta_A^1 \colon K_1(A) \to HC_1(A)$$

coincides with Π_1^A. The main advantage of this picture is that natural transformations from K_1 to HC_1 are unique, which leads to the explicit formula

$$P_A([g]) = \sum_{i,j} \langle (g^{-1})_{ij}, g_{ji} \rangle \quad \text{for} \quad [g] \in K_1(A), g \in \mathrm{GL}_n(A)$$

(*cf.* [Ne06c]). If, in addition, A is commutative, then $HC_1(A)$ is the completion of the quotient $\Omega^1(A)/d_A(A)$, where $\Omega^1(A)$ is the (topological) universal differential module of A. In these terms, we then have

$$P_A(g) = \langle \det(g)^{-1}, \det(g) \rangle = [\det(g)^{-1} d_A(\det(g))],$$

which leads to

$$\mathrm{im}(P_A) = P_A([A^\times]) = \{[a^{-1}d_A(a)] \colon a \in A^\times\}.$$

Examples VI.1.29. (1) For $A = C_c^\infty(M,\mathbb{C})$, M a σ-compact finite-dimensional manifold, we have

$$HC_1(A) \cong \Omega_c^1(M,\mathbb{C})/dC_c^\infty(M,\mathbb{C})$$

([Co94], [Mai02]). Moreover, $M_n(A) \cong C_c^\infty(M, M_n(\mathbb{C}))$ and

$$\omega_u^n(f,g) = [\mathrm{tr}(f \cdot dg)]$$

is a cocycle of product type, which implies that its period group coincides with the group

$$\mathrm{im}(P_A) = \delta(C_c^\infty(M,\mathbb{C}^\times))/dC_c^\infty(M,\mathbb{C})$$

of integral cohomology classes in $H^1_{\mathrm{dR,c}}(M,\mathbb{C})$, which is discrete (Theorem V. 2. 17).

(2) For $A = C(X,\mathbb{C})$, where X is a compact space, Johnson's Theorem entails that $\Omega^1(A)$, and hence $HC_1(A) \cong \Omega^1(A)/d_A(A)$, vanish ([BD73, Th. VI.12]). This further implies that for each C^*-Algebra A the homomorphism P_A vanishes. □

A particularly interesting class of Fréchet CIAs are the *d-dimensional smooth quantum tori*. These algebras are parametrized by skew-symmetric matrices $\Theta \in \mathrm{Skew}_d(\mathbb{R})$, as follows. They are topologically generated by d invertible elements $u_1,\ldots,u_d$, together with their inverses, satisfying the commutation relations

$$u_p u_q = e^{2\pi i \Theta_{pq}} u_q u_p \quad \text{for} \quad 1 \leq p,q \leq d.$$

Moreover,

$$A_\Theta = \Big\{ \sum_{I\in\mathbb{Z}^d} \alpha_I u^I : (\forall k \in \mathbb{N}) \sum_I |I|^k |\alpha_I| < \infty \Big\},$$

where $|I| = i_1 + \cdots + i_d$ and $u^I := u_1^{i_1} \cdots u_d^{i_d}$, so that, as a Fréchet space, A_Θ is isomorphic to the space of smooth functions on the d-dimensional torus. In particular, we have the commutative case $A_0 \cong C^\infty(\mathbb{T}^d,\mathbb{C})$. The following theorem characterizes those for which the image of P_A is discrete ([Ne06c]):

Theorem VI.1.30. *For the d-dimensional smooth quantum torus A_Θ, the group* $\mathrm{im}(P_{A_\Theta})$ *is discrete if and only if $d \leq 2$ or the matrix Θ has rational entries.* □

An interesting consequence of the preceding theorem is that there exists a CIA A for which $\mathrm{im}(P_A)$ is not discrete. The smallest examples are of the form $A := C^\infty(\mathbb{T}, A_\Theta)$, where $\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$, $\theta \in \mathbb{R}\setminus\mathbb{Q}$, so that A_Θ is a so-called *irrational rotation algebra*.

VI.2. Integrability of non-locally exponential Lie algebras

After the discussion of the enlargeability of locally exponential and Banach–Lie algebras in the preceding subsection, we now turn to more general classes of Lie algebras. Unfortunately, there is no general theory beyond the locally exponential class, so that all positive and negative results are quite particular.

We start with a discussion of some obstructions to the integrability to an analytic Lie group, then turn to complexifications of Lie algebras of vector fields, and finally to Lie algebras of formal vector fields, resp., Lie algebras of germs.

Proposition VI.2.1. ([Mil84, Lemma 9.1]) *Let G be a connected analytic Lie group. Then each closed ideal $\mathfrak{n} \trianglelefteq \mathbf{L}(G)$ is invariant under $\mathrm{Ad}(G)$.* □

Corollary VI.2.2. *If $\mathfrak{g}$ is a Lie algebra containing a closed ideal which is not stable, then $\mathfrak{g}$ is not integrable to an analytic Lie group with an analytic exponential function.* □

Remark VI.2.3. Proposition VI.2.1 implies that the Lie group $\mathrm{Diff}(M)$ of all diffeomorphisms of a compact manifold M does not possess an analytic Lie group structure for which its Lie algebra is $\mathcal{V}(M)$. Indeed, for each non-dense open subset $K \subseteq M$, the subspace

$$\mathcal{V}(M)_K := \{X \in \mathcal{V}(M) \colon X|_K = 0\}$$

is a closed ideal of $\mathcal{V}(M)$ not invariant under $\mathrm{Ad}(\mathrm{Diff}(M))$ because $\mathrm{Ad}(\varphi).\mathcal{V}(M)_K = \mathcal{V}(M)_{\varphi(K)}$ for $\varphi \in \mathrm{Diff}(M)$. □

The situation improves if we restrict our attention to analytic diffeomorphisms:

Theorem VI.2.4. ([Les82/83]) *Let M be a compact analytic manifold and $\mathcal{V}^\omega(M)$ the Lie algebra of analytic vector fields on M. Then $\mathcal{V}^\omega(M)$ carries a natural Silva space structure, turning it into a topological Lie algebra, and the group $\mathrm{Diff}^\omega(M)$ of analytic diffeomorphisms carries a smooth Lie group structure for which $\mathcal{V}^\omega(M)^{\mathrm{op}}$ is its Lie algebra.* □

It is shown by Tognoli in [Ta68] that the group $\mathrm{Diff}^\omega(M)$, M a compact analytic manifold, carries no analytic Lie group structure (*cf.* [Mil82, Ex. 6.12]). That there is no analytic Lie group with an analytic exponential function and Lie algebra $\mathcal{V}^\omega(M)$ can be seen by verifying that the map $(X,Y) \mapsto \mathrm{Ad}(\mathrm{Fl}^X_1).Y$ is not analytic on a 0-neighborhood in $\mathcal{V}^\omega(M) \times \mathcal{V}^\omega(M)$ (*cf.* [Mil82, Ex. 6.17]).

The following non-integrability result is quite strong because it does not assume the existence of an exponential function. Its outcome is that complexifications of Lie algebras of vector fields are rarely integrable. For complexifications of Lie algebras of ILB–Lie groups, similar results are described by Omori in [Omo97, Cor. 4.4].

Theorem VI.2.5. ([Lem97]) *Let M be a compact manifold of positive dimension. Then the complexifications $\mathfrak{g}_{\mathbb{C}}$ of the following Lie algebras $\mathfrak{g}$ are not integrable:*

(1) The Lie algebra $\mathcal{V}(M)$ of smooth vector fields on M.
(2) If M is analytic, the Lie algebra $\mathcal{V}^{\omega}(M)$ of analytic vector fields on M.
(3) If Ω is a symplectic 2-form on M, the Lie algebra $\mathcal{V}(M,\Omega) := \{X \in \mathcal{V}(M) : \mathcal{L}_X\Omega = 0\}$ of symplectic vector fields on M.
(4) If M is analytic and Ω is an analytic symplectic 2-form on M, the Lie algebra $\mathcal{V}^{\omega}(M,\Omega)$ of analytic symplectic vector fields on M.

Proof. (Idea) Lempert's proof is based on the following result, which is obtained by PDE methods: If $\xi : \mathbb{R} \to \mathfrak{g}_{\mathbb{C}}$ is a smooth curve such that for each $x \in \mathfrak{g}_{\mathbb{C}}$ the IVP

$$\gamma(0) = x, \quad \dot{\gamma}(t) = [\xi(t), \gamma(t)]$$

has a smooth solution, then $\xi(0) \in \mathfrak{g}$.

For (1) he gives another argument, based on the fact that

$$\mathrm{Aut}(\mathcal{V}(M)_{\mathbb{C}}) \cong \mathrm{Aut}(\mathcal{V}(M)) \rtimes \{\mathbf{1}, \sigma\} \cong \mathrm{Diff}(M) \rtimes \{\mathbf{1}, \sigma\}, \tag{6.2.1}$$

where σ denotes the complex conjugation on $\mathcal{V}(M)_{\mathbb{C}}$. The first isomorphism is obtained in [Lem97], using [Ame75], and the second is an older result of Pursell and Shanks ([PuSh54]; *cf.* Theorem IX.2.1).

Clearly (6.2.1) implies that for any connected Lie group G with Lie algebra $\mathbf{L}(G) = \mathcal{V}(M)_{\mathbb{C}}$, the group $\mathrm{Ad}(G) \subseteq \mathrm{Aut}(\mathcal{V}(M)_{\mathbb{C}})$ preserves the real subspace $\mathcal{V}(M)$. Taking derivatives of orbit maps, this leads to the contradiction $[\mathcal{V}(M)_{\mathbb{C}}, \mathcal{V}(M)] \subseteq \mathcal{V}(M)$. □

Theorem VI.2.6. ([Omo81]) *For any non-compact σ-compact smooth manifold M of positive dimension, the Lie algebra $\mathcal{V}(M)$ is not integrable to any Lie group with an exponential function.*

Proof. (Sketch) If G is a Lie group with Lie algebra $\mathbf{L}(G) = \mathcal{V}(M)$ and an exponential function, then for each $X \in \mathcal{V}(M)$ we obtain a smooth 1-parameter group $t \mapsto \mathrm{Ad}(\exp_G(tX))$ of automorphisms of $\mathcal{V}(M)$ with generator $\mathrm{ad}\,X$. By [Ame75, Thm. 2], $\mathrm{Aut}(\mathcal{V}(M)) \cong \mathrm{Diff}(M)$, so that we obtain a one-parameter group γ_X of $\mathrm{Diff}(M)$ which then is shown to coincide with the flow generated by X (*cf.* Lemma II.3.10; [KYMO85, Sect. 3.4]). This contradicts the existence of non-complete vector fields on M. □

Since the BCH series can be used to defined a Lie group structure on any nilpotent locally convex Lie algebra, all these Lie algebras are integrable. The following theorem shows that the integrability problem for solvable locally convex Lie algebras contains the integrability problem for continuous linear operators on locally convex spaces, which is highly non-trivial (Problem VI.1). E.g.

if M is a finite-dimensional σ-compact manifold and $X \in \mathcal{V}(M)$ a vector field, then the corresponding derivation of the Fréchet algebra $C^\infty(M,\mathbb{R})$ is integrable if and only if the vector field X is complete.

Theorem VI.2.7. *Let E be a locally convex space and $D \in \mathfrak{gl}(E)$. Then the solvable Lie algebra $\mathfrak{g} := E \rtimes_D \mathbb{R}$ with the bracket $[(v,t),(v',t')] := (tDv' - t'Dv, 0)$ is integrable if and only if D is integrable to a smooth $\mathbb{R}$-action on E.*

Proof. If D is integrable to a smooth representation $\alpha\colon \mathbb{R} \to \mathrm{GL}(E)$ with $\alpha'(0) = D$, then the semi-direct product $G := E \rtimes_\alpha \mathbb{R}$ is a Lie group with the Lie algebra $\mathfrak{g}$.

Suppose, conversely, that G is a connected Lie group with Lie algebra $\mathfrak{g}$. Replacing G by its universal covering group, we may assume that G is 1-connected. Then the regularity of the additive group $(\mathbb{R},+)$ implies the existence of a smooth homomorphism $\chi\colon G \to \mathbb{R}$ with $\mathbf{L}(\chi) = q$, where $q(v,t) = t$ is the projection $\mathfrak{g} = E \rtimes_D \mathbb{R} \to \mathbb{R}$ (Theorem III.1.5).

Using Glöckner's Implicit Function Theorem ([Gl03a]), it follows that $\ker\chi$ is a submanifold of G and there exists a smooth curve $\gamma\colon \mathbb{R} \to G$ with $\gamma(0) = \mathbf{1}$ and $\chi \circ \gamma = \mathrm{id}_\mathbb{R}$.

Next we observe that $[\mathfrak{g},\mathfrak{g}] \subseteq E$ implies that E is $\mathrm{Ad}(G)$-invariant, so that $\mathrm{Ad}_E(g) := \mathrm{Ad}(g)|_E$ defines a smooth action of G on E whose derived representation is given by $\mathrm{ad}_E(x,t) = tD$. We now put $\alpha(t) := \mathrm{Ad}_E(\gamma(t))$ and observe that

$$\delta(\alpha)(t) = \mathrm{ad}_E(\delta(\gamma)(t)) = q(\delta(\gamma)(t)) \cdot D = \delta(\chi \circ \gamma)(t) \cdot D = D.$$

Hence D is integrable. □

Example VI.2.8. Let $\mathfrak{gf}_n(\mathbb{R})_{-1} := \mathbb{R}^n[[x_1,\ldots,x_n]]$ denote the space of all $\mathbb{R}^n$-valued formal power series in n variables, considered as the Lie algebra of formal vector fields, endowed with the bracket

$$[f,g](x) := dg(x)f(x) - df(x)g(x),$$

which makes sense on the formal level because if f is homogeneous of degree p and g is homogeneous of degree q, then $[f,g]$ is of degree $p+q-1$.

We have already seen in Example IV.1.14 that the subalgebra $\mathfrak{gf}_n(\mathbb{R})$ of all elements with vanishing constant term is the Lie algebra of the Fréchet–Lie group $\mathrm{Gf}_n(\mathbb{R})$ of formal diffeomorphisms of $\mathbb{R}^n$ fixing 0. We obviously have the split short exact sequence

$$\mathbf{0} \to \mathfrak{gf}_n(\mathbb{R}) \hookrightarrow \mathfrak{gf}_n(\mathbb{R})_{-1} \twoheadrightarrow \mathbb{R}^n \to \mathbf{0},$$

where $\mathbb{R}^n$ is considered as an abelian Lie algebra, corresponding to the constant vector fields.

We claim that the Lie algebra $\mathfrak{gf}_n(\mathbb{R})_{-1}$ is not integrable to any Lie group with an exponential function. This strengthens a statement in [KYMO85, p.80], that it is not integrable to a μ-regular Fréchet–Lie group. Let us assume that G is a Lie group with Lie algebra $\mathfrak{gf}_n(\mathbb{R})_{-1}$. With a similar argument as in the proof of Theorem VI.2.7, one can show that for each constant function x the operator $\operatorname{ad} x$ on $\mathfrak{gf}_n(\mathbb{R})_{-1}$ is integrable. We consider the constant function e_1. Then $[e_1, g] = \frac{\partial g}{\partial x_1}$, and we can now justify as in Example II.3.13 that $\operatorname{ad} e_1$ is not integrable, hence that $\mathfrak{gf}_n(\mathbb{R})_{-1}$ cannot be integrable to any Lie group with an exponential function. □

The preceding example shows that the constant terms create problems in integrating Lie algebras of formal vector fields, which is very natural because the formal completion distinguishes the point $0 \in \mathbb{R}^n$. A similar phenomenon arises in the context of groups of germs of local diffeomorphisms. For germs of functions in 0, the non-integrability of vector fields with non-zero constant term follows from the fact that all automorphisms preserve the unique maximal ideal of functions vanishing in 0 (*cf.* [GN06]).

Let $\mathfrak{gs}_n(\mathbb{R})_{-1}$ denote the space of germs of smooth maps $\mathbb{R}^n \to \mathbb{R}^n$ in 0, identified with germs of vector fields in 0. According to [RK97, Sect. 5.2], this space carries a natural Silva structure, turning it into a locally convex Lie algebra. Let $\mathfrak{gs}_n(\mathbb{R})$ denote the subspace of all germs vanishing in 0 and $\mathfrak{gs}_n(\mathbb{R})_1$ the set of germs vanishing of second order in 0.

Theorem VI.2.9. ([RK97, Th. 3]) *The group* $\mathrm{Gs}_n(\mathbb{R})$ *of germs of diffeomorphism of* $\mathbb{R}^n$ *in* 0 *fixing* 0 *carries a Lie group structure for which the Lie algebra is the space* $\mathfrak{gs}_n(\mathbb{R})$ *of germs of vector fields vanishing in* 0.

We have a semidirect product decomposition $\mathrm{Gs}_n(\mathbb{R}) \cong \mathrm{Gs}_n(\mathbb{R})_1 \rtimes \mathrm{GL}_n(\mathbb{R})$, *where* $\mathrm{Gs}_n(\mathbb{R})_1$ *is the normal subgroup of those germs* $[\varphi]$ *for which* $\varphi - \mathrm{id}_{\mathbb{R}^n}$ *vanishes of order* 2. *The map*

$$\Phi\colon \mathfrak{gs}_n(\mathbb{R})_1 \to \mathrm{Gs}_n(\mathbb{R})_1, \quad \xi \mapsto \mathrm{id} + \xi$$

is a global diffeomorphism. □

In view of the preceding theorem, it is a natural problem to integrate Lie algebras of germs of vector fields vanishing in the base point to Lie groups of germs of diffeomorphisms. This program is carried out by Kamran and Robart in several papers (*cf.* [RK97], [KaRo01/04], [Rob02]). It results in several interesting classes of Silva–Lie groups of germs of smooth and also analytic local diffeomorphisms, where the corresponding Silva–Lie algebras depend on certain parameters which are used to obtain a good topology.

Example VI.2.10. The formal analog of the Lie algebra $\mathfrak{gs}_1(\mathbb{R})_1$ is the Lie algebra $\mathfrak{gf}_1(\mathbb{R})_1$ which is pro-nilpotent, hence in particular BCH. In contrast to this fact, Robart observed that $\mathfrak{gs}_1(\mathbb{R})_1$ is not BCH. In fact, for the elements $\xi(x) = ax^2$, $\lambda \in \mathbb{R}$ and $\eta(x) = x^3$, we have

$$\sum_{n=0}^{\infty} \big((\mathrm{ad}\,\xi)^n \eta\big)(x) = x^3 \sum_{n=0}^{\infty} a^n n! x^n,$$

which converges for no $x \neq 0$ if $a \neq 0$. With Floret's results from [Fl71, p.155], it follows that this series does not converge in the Silva space $\mathfrak{gs}_1(\mathbb{R})_1$, so that Theorem IV.1.7 shows that $\mathfrak{gs}_1(\mathbb{R})_1$ is not BCH. □

The following proposition is a variant of E. Borel's theorem on the Taylor series of smooth functions. It provides an interesting connection between the smooth global and the formal perspective on diffeomorphism groups.

Proposition VI.2.11. (Glöckner) *Let M be a smooth finite-dimensional manifold, $m_0 \in M$ and $\mathrm{Diff}_c(M)_{m_0}$ the stabilizer of m_0. For each $\varphi \in \mathrm{Diff}_c(M)_{m_0}$, let $T^\infty_{m_0}(\varphi) \in \mathrm{Gf}_n(\mathbb{R})$ denote the Taylor series of φ in m_0 with respect to some local chart. Then the map*

$$T^\infty_{m_0} \colon \mathrm{Diff}_c(M)_{m_0,0} \to \mathrm{Gf}_n(\mathbb{R})_0$$

is a surjective homomorphism of Lie groups, where $\mathrm{Gf}_n(\mathbb{R})_0$ is the subgroup of index 2, consisting of those formal diffeomorphisms ψ with $\det(T_0(\psi)) > 0$. □

Example VI.2.12. Let $\mathfrak{gh}_n(\mathbb{C})$ denote the space of germs of holomorphic maps $f \colon \mathbb{C}^n \to \mathbb{C}^n$ in 0 satisfying $f(0) = 0$. We endow this space with the locally convex direct limit topology of the Banach spaces E_k of holomorphic functions on the closed unit disc of radius $\frac{1}{k}$ in $\mathbb{C}^n$ (with respect to any norm). Thinking of the elements of $\mathfrak{gh}_n(\mathbb{C})$ as germs of vector fields in 0 leads to the Lie bracket

$$[f,g](z) := dg(z)f(z) - df(z)g(z),$$

which turns $\mathfrak{gh}_n(\mathbb{C})$ into a topological Lie algebra.

The set $\mathrm{Gh}_n(\mathbb{C})$ of all germs $[f]$ with $\det(f'(0)) \neq 0$ is an open subset of $\mathfrak{gh}_n(\mathbb{C})$ which is a group with respect to composition $[f][g] := [f \circ g]$. In [Pis77], Pisanelli shows that composition and inversion in $\mathrm{Gh}_n(\mathbb{C})$ are holomorphic, so that $\mathrm{Gh}_n(\mathbb{C})$ is a complex Lie group with respect to the manifold structure it inherits as an open subset of $\mathfrak{gh}_n(\mathbb{C})$. This Lie group has a holomorphic exponential function which is not locally surjective, where the latter fact can be obtained by adapting Sternberg's example $f(z) = e^{\frac{2\pi i}{m}} z + pz^{m+1}$ (Example IV.1.14) ([Pis76]).

Note that $\mathrm{Gh}_n(\mathbb{C}) \cong \mathrm{Gh}_n(\mathbb{C})_1 \rtimes \mathrm{GL}_n(\mathbb{C})$, where $\mathrm{Gh}_n(\mathbb{C})_1$ is the subgroup of all diffeomorphisms with linear term $\mathrm{id}_{\mathbb{C}^n}$. □

Remark VI.2.13. Let $\mathfrak{g}(A)$ be a symmetrizable Kac–Moody Lie algebra. In [Rod89], Rodriguez-Carrington describes certain Fréchet completions of $\mathfrak{g}(A)$, including smooth $\mathfrak{g}^\infty(A)$ and analytic versions $\mathfrak{g}^\omega(A)$, which are BCH–Lie algebras ([Rod89, Prop. 1]). Corresponding groups are constructed for the unitary real forms by unitary highest weight modules of $\mathfrak{g}(A)$, as subgroups of the unitary groups of a Hilbert space (Corollary IV.4.10). In [Su88], Suto obtains closely related results, but no Lie group structures.

In a different direction, Leslie describes in [Les90] a certain completion $\overline{\mathfrak{g}}(A)$ of $\mathfrak{g}(A)$ which leads to a Lie group structure on the space $C^\infty([0,1],\overline{\mathfrak{g}}(A))$, corresponding to the natural Lie algebra structure on this space. One thus obtains an integrable Lie algebra extension of $\overline{\mathfrak{g}}(A)$ in the spirit of pre-integrable Lie algebras (Remark IV.1.22). For an approach to Kac–Moody groups in the context of diffeological groups, we refer to [Les03] (*cf.* [So84]). □

Open Problems for Section VI

Problem VI.1. (Generators of smooth one-parameter groups) Let E be a locally convex space and $D\colon E \to E$ a continuous linear endomorphism. Characterize those linear operators D for which there exists a homomorphism $\alpha\colon \mathbb{R} \to \mathrm{GL}(E)$ defining a smooth action of $\mathbb{R}$ on E. In view of Theorem VI.2.7, this is equivalent to the integrability of the 2-step solvable Lie algebra $\mathfrak{g} := E \rtimes_D \mathbb{R}$.

If E is a Banach space, then each D integrates to a homomorphism α which is continuous with respect to the norm topology on $\mathrm{GL}(E)$ and given by the convergent exponential series $\alpha(t) := \sum_{k=0}^\infty \frac{t^k}{k!} D^k$.

Since for each smooth linear $\mathbb{R}$-action on E, given by some α as above, the infinitesimal generator $\alpha'(0)$ is everywhere defined, this problem is not a problem about operators which are unbounded in the sense that they are only defined on dense subspaces. In some sense, the passage from Banach spaces to locally convex spaces takes care of this problem. If, e.g. $\alpha\colon \mathbb{R} \to \mathrm{GL}(E)$ is a strongly continuous one-parameter group on a Banach space E, then the subspace $E^\infty \subseteq E$ of smooth vectors carries a natural Fréchet topology inherited from the embedding

$$E^\infty \hookrightarrow C^\infty(\mathbb{R},E), \quad v \mapsto \alpha^v, \quad \alpha^v(t) = \alpha(t)v,$$

and the induced one-parameter group $\alpha^\infty\colon \mathbb{R} \to \mathrm{GL}(E^\infty)$ defines a smooth action. In this sense, each generator of a strongly continuous one-parameter group also generates a smooth one-parameter group on a suitable Fréchet space. □

Problem VI.2. (Integrability of 2-step solvable Lie algebras) Theorem VI.2.7 gives an integrability criterion for solvable Lie algebras of the type $\mathfrak{g} = E \rtimes_D \mathbb{R}$.

Since abelian Lie algebras are integrable for trivial reasons, it is natural to address the integrability problem for solvable Lie algebras by first restricting to algebras of *solvable class* 2, i.e., $D^1(\mathfrak{g}) := \overline{[\mathfrak{g},\mathfrak{g}]}$ is an abelian ideal of $\mathfrak{g}$. Clearly, the adjoint action defines a natural topological module structure for the abelian Lie algebra $W := \mathfrak{g}/D^1(\mathfrak{g})$ on $E := D^1(\mathfrak{g})$. Here are some problems concerning this situation:

(1) Does the integrability of $\mathfrak{g}$ imply that the Lie algebra module structure of W on E integrates to a smooth action of the Lie group $(W,+)$ on E? If E is finite-dimensional, this can be proved by an argument similar to the proof of Theorem VI.2.7.
(2) Assume that the Lie algebra module structure of W on E integrates to a smooth action of $(W,+)$. Does this imply that $\mathfrak{g}$ is integrable?

If $\mathfrak{g} \cong V \rtimes W$ is a semidirect product, the latter is obvious, but if $\mathfrak{g}$ is a non-trivial extension of W by V, the situation is more complicated. Note that all solvable Banach–Lie algebras are integrable by Theorem VI.1.21. □

Problem VI.3. Is the group $\mathrm{Gs}_n(\mathbb{R})_1$ of germs of diffeomorphisms φ of $\mathbb{R}^n$ fixing 0, for which the linear term of $\varphi - \mathrm{id}_{\mathbb{R}^n}$ vanishes, exponential? □

Problem VI.4. Let G be a regular Lie group. Is every finite codimensional closed subalgebra $\mathfrak{h} \subseteq \mathbf{L}(G)$ integrable to an integral subgroup? For μ-regular groups this follows from Theorem III.2.8. □

Problem VI.5. Is the group $\mathrm{Gh}_n(\mathbb{C})$ defined in Example VI.2.12 a regular Lie group? Is the subgroup $\mathrm{Gh}_n(\mathbb{C})_1$ an exponential Lie group? (*cf.* Problem VI.3) □

Problem VI.6. Does Pestov's Theorem VI.1.24 generalize to locally exponential Lie algebras? □

Problem VI.7. For quotient maps $q\colon E \to Q$ of Fréchet spaces, we may use [MicE59] to find a continuous linear cross section $\sigma\colon Q \to E$, which implies in particular that q defines a topologically trivial fiber bundle. For more general locally convex spaces, cross sections might not exist, but it would still be interesting if quotient maps of locally convex spaces are Serre fibrations, i.e., have the homotopy lifting property for cubes (*cf.* [Bre93]). If this is the case, the long exact homotopy sequence would also be available for quotient maps of locally exponential Lie groups, which would be an important tool to calculate homotopy groups of such Lie groups. □

Problem VI.8. Prove an appropriate version of Theorem VI.1.23 on the existence of a universal complexification for locally exponential Lie algebras.

Note that this already becomes an interesting issue on the level of Lie algebras because the complexification of a locally exponential Lie algebra need not be locally exponential. In fact, in Example IV.4.6 we have seen an exponential Lie algebra $\mathfrak{g}$ containing an unstable closed subalgebra $\mathfrak{h}$. If $\mathfrak{g}_{\mathbb{C}}$ is locally exponential, as a complex Lie algebra, then the local multiplication in $\mathfrak{g}_{\mathbb{C}}$ is holomorphic, so that $\mathfrak{g}$ is BCH, contradicting the existence of unstable closed subalgebras. □

VII. Direct limits of Lie groups

The systematic study of Lie group structures on direct limit Lie groups $G = \varinjlim G_n$ was started in the 1990s by J. Wolf and his coauthors ([NRW91/93]). They used certain conditions on the groups G_n and the maps $G_n \to G_{n+1}$ to ensure that the direct limit groups are locally exponential. Since not all direct limit groups are locally exponential (Example VII.1.4(c)), their approach does not cover all cases. The picture for countable direct limits of finite-dimensional Lie groups was nicely completed by Glöckner who showed that arbitrary countable limits of finite-dimensional Lie groups exist ([Gl03b/05a]). The key to these results are general construction principles for direct limits of finite-dimensional manifolds. These results are discussed in Section VII.1. In Section VII.2, we briefly turn to other types of direct limit constructions where the groups G_n are infinite-dimensional Lie groups.

VII.1. Direct limits of finite-dimensional Lie groups

Theorem VII.1.1. ([Gl05]) (a) *For every sequence $(G_n)_{n\in\mathbb{N}}$ of finite-dimensional Lie groups G_n with morphisms $\varphi_n\colon G_n \to G_{n+1}$, the direct limit group $G := \varinjlim G_n$ carries a regular Lie group structure. The model space $\mathbf{L}(G) \cong \varinjlim \mathbf{L}(G_n)$ is countably dimensional and carries the finest locally convex topology, and G has the universal property of a direct limit in the category of Lie groups.*

(b) *Every countably dimensional locally finite Lie algebra $\mathfrak{g}$, endowed with the finest locally convex topology, is integrable to a regular Lie group G.*

(c) *Every connected regular Lie group G whose Lie algebra is countably dimensional, locally finite and carries the finest locally convex topology is a direct limit of finite-dimensional Lie groups.* □

In the following, we shall call the class of Lie groups described by the preceding theorem *locally finite-dimensional* (regular) Lie groups.

Remark VII.1.2. (a) Beyond countable directed systems, several serious obstacles arise. First of all, for countably dimensional vector spaces, the finest locally convex topology coincides with the finest topology for which all inclusions of finite-dimensional subspaces are continuous. This is crucial for many arguments in this context. If E is not of countable dimension, the addition on E is not continuous for the latter topology. Similar problems occur for uncountable direct limits of topological groups: in many cases the direct limit topology does not lead to a continuous multiplication (*cf.* [Gl03b] for more details).

(b) Any countably dimensional space E, endowed with the finest locally convex topology can be considered as a direct limit space of finite-dimensional subspaces E_n of $\dim E_n = n$. Since each E_n is a closed subspace which is Banach, and all inclusions $E_n \to E_{n+1}$ are compact operators, E is an LF space and a Silva space at the same time. □

Theorem VII.1.3. ([Gl05/06d]) *Let G be a locally finite-dimensional Lie group. Then the following assertions hold:*

(1) Every subalgebra $\mathfrak{h} \subseteq \mathbf{L}(G)$ integrates to an integral subgroup.
(2) Every closed subgroup H is a split submanifold, so that H is a locally finite-dimensional Lie group, and the quotient space G/H carries a natural manifold structure.
(3) Every locally compact subgroup $H \subseteq G$ is a finite-dimensional Lie group.
(4) G does not contain small subgroups. □

Example VII.1.4. (a) One of the most famous examples of a direct limit Lie group is the group

$$\mathrm{GL}_\infty(\mathbb{R}) := \varinjlim \mathrm{GL}_n(\mathbb{R})$$

with the connecting maps

$$\varphi_n\colon \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_{n+1}(\mathbb{R}), \quad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

Its Lie algebra is the Lie algebra $\mathfrak{gl}_\infty(\mathbb{R})$ of all $(\mathbb{N} \times \mathbb{N})$-matrices with only finitely many non-zero entries (*cf.* [NRW91], [Gl03b]).

In [KM97, Thm. 47.9], it is shown that every subalgebra $\mathfrak{h}$ of $\mathfrak{gl}_\infty(\mathbb{R})$ is integrable to an integral subgroup, which is a special case of Theorem VII.1.3. Here $\mathfrak{h}$ is even BCH.

(b) In the context of C^*-algebras, direct limits of finite-dimensional ones are particularly interesting objects. On the level of unit groups one encounters in particular groups of the form

$$G := \varinjlim \mathrm{GL}_{2^n}(\mathbb{C}), \quad \varphi_n(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

(c) Let $E := \mathbb{C}^{(\mathbb{N})}$ be the free vector space with basis $(e_n)_{n\in\mathbb{N}}$ and $D \in \mathscr{L}(E)$ be defined by $D(e_n) = 2\pi i n e_n$ (*cf.* Example II.5.9(a)). Then the Lie algebra $\mathfrak{g} := E \rtimes_D \mathbb{R}$ is locally finite and we obtain a corresponding locally finite-dimensional Lie group $G = E \rtimes_\alpha \mathbb{R}$, where $\alpha(t) = e^{tD}$. Since the sequence $(0, \frac{1}{n})_{n\in\mathbb{N}}$ consists of singular points for the exponential function, the Lie algebra $\mathfrak{g}$ is not locally exponential (*cf.* Remark II.5.8). ☐

Theorem VII.1.5. *Every continuous homomorphism between locally finite-dimensional Lie groups is smooth.* ☐

As the corresponding result for locally exponential Lie groups (Theorem IV. 1.18) did, the preceding theorem implies that locally finite-dimensional Lie groups form a full sub-category of topological groups. We even have the following stronger version of the preceding theorem:

Theorem VII.1.6. *Let $G = \varinjlim G_n$ be a locally finite-dimensional Lie group and H a Lie group.*

(a) A group homomorphism $\varphi\colon G \to H$ is smooth if and only if the corresponding homomorphisms $\varphi_n\colon G_n \to H$ are smooth.
(b) If H has a smooth exponential map, then each continuous homomorphism $\varphi\colon G \to H$ is smooth.

Proof. (a) is contained in [Gl05]. In view of (a), part (b) follows from the finite-dimensional case, which in turn follows from the existence of local coordinates of the second kind: $(t_1, \ldots, t_n) \mapsto \prod_{i=1}^n \exp_G(t_i x_i)$. ☐

VII.2. Direct limits of infinite-dimensional Lie groups

Direct limit constructions also play an important role when applied to sequences of infinite-dimensional Lie groups. On the level of Banach-, resp., Fréchet spaces, different types of directed systems lead to the important classes of LF spaces and Silva spaces (*cf.* Definition I.1.2).

If M is a σ-compact finite-dimensional manifold and K a Lie group, then the groups $C^\infty_c(M,K)$ of compactly supported smooth maps $M \to K$ are direct limits of the subgroups $C^\infty_X(M,K) := \{f \in C^\infty(M,K)\colon \operatorname{supp}(f) \subseteq X\}$, which, for Banach–Lie groups K, are Fréchet–Lie groups. On $C^\infty_c(M,K)$ this leads to the structure of an LF–Lie group if K is Fréchet, but the construction of a Lie group structure works for general K (Theorem II.2.8). For $\dim K < \infty$, these groups are also discussed in [NRW94] as direct limit Lie groups which are BCH.

Many interesting direct limits of mapping groups and other interesting classes embed naturally into certain direct sums, also called restricted direct products, often given by a nice atlas of a manifold. Therefore the following theorem turns out to be quite useful because it provides realizations as subgroups of a

Lie group, and it usually is easier to verify that subgroups of Lie groups are Lie groups, than to construct the Lie group structures directly.

Theorem VII.2.1. ([Gl03c]) *If $(G_i)_{i\in I}$ is a family of locally exponential Lie groups, then their direct sum*

$$G := \bigoplus_{i\in I} G_i := \Big\{ (g_i)_{i\in I} \in \prod_{i\in I} G_i \colon |\{i\colon g_i \neq \mathbf{1}\}| < \infty \Big\}$$

carries a natural Lie group structure, where $\mathbf{L}(G) \cong \bigoplus_{i\in I} \mathbf{L}(G_i)$ carries the locally convex direct sum topology. □

Theorem VII.2.2. ([Gl06c]) *For a σ-compact, non-compact manifold M of positive dimension, the Lie group $\mathrm{Diff}_c(M)$ of compactly supported diffeomorphisms, endowed with the Lie group structure modeled on the LF space $\mathcal{V}_c(M)$ is not a direct limit of the subgroups $\mathrm{Diff}_{M_n}(M)$, $(M_n)_{n\in\mathbb{N}}$ an exhaustion of M, in the category of smooth manifolds, but a homomorphism $\mathrm{Diff}_c(M) \to H$ to a Lie group H is smooth if and only if it is smooth on each subgroup $\mathrm{Diff}_{M_n}(M)$.* □

A crucial tool for the proof of the preceding theorem is the following lemma:

Lemma VII.2.3. (Fragmentation Lemma) *Let M be a σ-compact finite-dimensional manifold. Then there exists a locally finite cover $(K_n)_{n\in\mathbb{N}}$ of M by compact sets, an open identity neighborhood $U \subseteq \mathrm{Diff}_c(M)$ and a smooth mapping $\Phi\colon U \to \bigoplus_{n\in\mathbb{N}} \mathrm{Diff}_{K_n}(M)$ which satisfies $\gamma = \Phi(\gamma)_1 \circ \cdots \circ \Phi(\gamma)_n$ for each $\gamma \in U$.* □

Theorem VII.2.4. ([Gl06c]) *For a σ-compact, non-compact manifold M of positive dimension and a finite-dimensional Lie group K of positive dimension, the Lie group $C^\infty_c(M,K)$ of compactly supported K-valued smooth functions, endowed with the Lie group structure modeled on the direct limit space $C^\infty_c(M,K) = \varinjlim C^\infty_{M_n}(M,K)$, $(M_n)_{n\in\mathbb{N}}$ an exhaustion of M, is not a direct limit of the subgroups $C^\infty_{M_n}(M,K)$ in the category of smooth manifolds, but a homomorphism $C^\infty_c(M,K) \to H$ to a Lie group H is smooth if and only if it is smooth on each subgroup $C^\infty_{M_n}(M,K)$.* □

Open Problems for Section VII

Problem VII.1. Is every Lie group G whose Lie algebra $\mathbf{L}(G)$ is countably dimensional, locally finite, and endowed with the finest locally convex topology regular? (*cf.* Problem II.2). □

Problem VII.2. Are locally finite-dimensional Lie groups topological groups with Lie algebra? It is not clear that the compact open topology on $\mathfrak{L}(G) \cong \mathrm{Hom}(\mathbb{R}, G)$ coincides with the given one on $\mathbf{L}(G)$ if the group is not locally exponential. □

Problem VII.3. Does every subgroup H of a locally finite-dimensional Lie group G carry an initial Lie subgroup structure? (*cf.* (FP5)) □

Problem VII.4. Let M be a locally convex manifold and $\mathfrak{g} \subseteq \mathcal{V}(M)$ a countably dimensional locally finite-dimensional subalgebra consisting of complete vector fields. Does the inclusion $\mathfrak{g} \to \mathcal{V}(M)$ integrate to a smooth action of a corresponding Lie group G with $\mathbf{L}(G) = \mathfrak{g}$? (*cf.* (FP7))

The first step should be to prove this for finite-dimensional Lie algebras $\mathfrak{g}$, using local coordinates of the second kind and then to use that locally finite-dimensional Lie groups are direct limits in the category of smooth manifolds ([Gl05]). □

Problem VII.5. The methods developed in [Gl03b] for the analysis of direct limit Lie groups seem to have potential to apply to more general classes of Lie groups G which are direct limits of finite-dimensional manifolds M_n, $n \in \mathbb{N}$, with the property that for $n, m \in \mathbb{N}$ there exist $c(n,m)$ and $d(n)$ with

$$M_n \cdot M_m \subseteq M_{c(n,m)} \quad \text{and} \quad M_n^{-1} \subseteq M_{d(n)},$$

a situation which occurs in free constructions. Similar situations, with infinite-dimensional M, occur in the ind-variety description of Kac–Moody groups (*cf.* [Kum02], [BiPi02]). □

VIII. Linear Lie groups

In this section, we take a closer look at linear Lie groups, i.e., Lie subgroups of CIAs. The point of departure is that the unit group of a Mackey complete CIA A is a BCH–Lie group (Theorem IV.1.11). This permits us to use the full machinery described Section IV for linear Lie groups.

Definition VIII.1. *A* linear Lie group *is a Lie group which can be realized as a locally exponential Lie subgroup of the unit group of some unital CIA.* □

We collect some of the basic tools in the following theorem.

Theorem VIII.2. *The following assertions hold:*

(1) Linear Lie groups are BCH.
(2) Continuous homomorphisms of linear Lie groups are analytic.

(3) *If G is a linear Lie group, then each closed Lie subalgebra $\mathfrak{h} \subseteq \mathbf{L}(G)$ integrates to a linear Lie group.*
(4) *For each morphism $\varphi\colon G \to H$ of linear Lie groups the kernel is a linear Lie group.*
(5) *For each $n \in \mathbb{N}$, the algebra $M_n(A)$ also is a CIA and $\mathrm{GL}_n(A) = M_n(A)^\times$ is a linear Lie group.*

Proof. (1)-(4) follow from the fact that $A^\times$ is BCH (Theorem IV.1.11), Theorem IV.1.8, and the corresponding assertions on BCH–Lie groups in Section IV.

For (5), we refer to [Gl02b] (see also [Sw77]). □

Linear Lie groups traditionally play an important role as groups of operators on Hilbert spaces, where they mostly occur as Banach–Lie subgroups (*cf.* [PS86], [Ne02b]). The connection between Lie theory and CIAs is more recent. The first systematic investigation of CIAs from a Lie theoretic perspective has been undertaken by Glöckner in [Gl02b]. Originally, complex CIAs came up in the 1950s as a natural class of locally convex associative algebras still permitting a powerful holomorphic functional calculus (*cf.* [Wae54a/b], [Al65]; see also [Hel93], and [Gram84] for Fréchet algebras of pseudo-differential operators).

In K-theory, the condition on a topological ring R that its unit group $R^\times$ is open and that the inversion map is continuous is quite natural because it is a crucial assumption for the analysis of idempotents in matrix algebras, resp., finitely generated projective modules, and the natural equivalence classes ([Swa62]; Section VI.1).

To get an impression of the variety of linear Lie groups, we describe some examples of CIAs:

Examples VIII.3. (a) Unital Banach algebras are CIAs.

(b) If M is a compact smooth manifold (with boundary) and A is a CIA over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, then for each $r \in \mathbb{N}_0 \cup \{\infty\}$, the algebra $C^r(M,A)$ of A-valued C^r-functions on M is a CIA ([Gl02b]). If M is non-compact, but σ-compact, then $C^\infty_c(M,A)$, endowed with the direct limit topology of the subalgebras $C^\infty_X(M,A)$, is a non-unital CIA (Definition II.1.3(b)).

(c) For $A = M_n(\mathbb{C})$, the preceding construction leads in particular to the CIAs

$$C^r(M, M_n(\mathbb{C})) \cong M_n(C^r(M,\mathbb{C})),$$

whose unit groups are the mapping groups $C^r(M, \mathrm{GL}_n(\mathbb{C}))$.

(d) Let X be a compact subset of $\mathbb{C}^n$ and $(U_n)_{n\in\mathbb{N}}$ a sequence of compact neighborhoods of X with $\bigcap_n U_n = X$. In [Wae54b], Waelbroeck shows that the algebra $\mathscr{O}(X,\mathbb{C})$ of germs of holomorphic functions on X is a CIA if it is endowed with the locally convex direct limit topology of the Banach algebras $C_{\mathscr{O}}(U_n,\mathbb{C})$ of those continuous functions on U_n which are holomorphic on the interior of U_n (the *van Hove topology*). This defines on $\mathscr{O}(X,\mathbb{C})$ the structure of a Silva space. The continuity of the multiplication and the completeness of this algebra is due to van Hove ([vHo52a]).

(e) If A is a Banach algebra, M a smooth manifold, $\alpha\colon M \to \mathrm{Aut}(A)$ a map and $\alpha^a(m) := \alpha(m)(a)$, then the subspace $A^\infty := \{a \in A\colon \alpha^a \in C^\infty(M,A)\}$ is a CIA ([Gram84]). □

Part (d) of the preceding example shows in particular that for each compact subset $X \subseteq \mathbb{C}^n$ the unit group $\mathscr{O}(X,\mathbb{C}^\times)$ of the CIA $\mathscr{O}(X,\mathbb{C})$ is a Lie group. In [Gl04b], Glöckner generalizes this Lie group construction as follows:

Theorem VIII.4. *Let X be a compact subset of a metrizable topological vector space, $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ and K a Banach–Lie group over $\mathbb{K}$. Then the group $\mathscr{O}(X,K)$ of germs of K-valued analytic maps on open neighborhoods of X is a $\mathbb{K}$-analytic BCH–Lie group.* □

Examples VIII.5. The following examples are Fréchet algebras with continuous inversion which are not CIAs because their unit groups are not open:

(1) $A = C^\infty(M,\mathbb{C})$, where M is a non-compact σ-compact finite-dimensional manifold (*cf.* Remark II.2.10).

(2) $A = \mathscr{O}(M,\mathbb{C})$, where M is a complex submanifold of some $\mathbb{C}^n$, i.e., a Stein manifold.

(3) $A = \mathbb{R}^{\mathbb{N}}$ with componentwise multiplication. □

In finite dimensions, a connected Lie group is called *linear* if it is isomorphic to a Lie subgroup of some $\mathrm{GL}_n(\mathbb{R})$. Not all connected finite-dimensional Lie groups are linear. Typical examples of non-linear Lie groups are the universal covering $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$ and the quotient H/Z, where

$$H = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix}$$

is the 3-dimensional Heisenberg group (Example V.3.5(c)) and $Z \subseteq Z(H)$ is a non-trivial cyclic subgroup of its center ([Wie49]). It is a natural question whether the linearity condition on a connected finite-dimensional Lie group becomes weaker if we only require that it is a Lie subgroup of the unit group of some Banach algebra or even a CIA. According to the following theorem, this is not the case ([BelNe06]). Its Banach version is due to Luminet and Valette ([LV94]).

Theorem VIII.6. *For a connected finite-dimensional Lie group G, the following are equivalent:*

(1) The continuous homomorphisms $\eta\colon G \to A^\times$ into the unit groups of Mackey complete CIAs separate the points of G.

(2) G is linear in the classical sense. □

Remark VIII.7. Let us call a Banach–Lie algebra $\mathfrak{g}$ *linear* if it has a faithful homomorphism into some Banach algebra A.

According to Ado's Theorem ([Ado36]), each finite-dimensional Lie algebra is linear, but the situation becomes more interesting, and also more complicated, for Banach–Lie algebras.

In view of Corollary IV.4.10, enlargeability is necessary for linearity, but it is not sufficient. In fact, if the Lie algebra $\mathfrak{g}$ of a 1-connected Banach–Lie group G contains elements p,q for which $[p,q]$ is a non-zero central element with $\exp_G([p,q]) = \mathbf{1}$, then $\mathfrak{g}$ is not linear, because any morphism $\mathfrak{g} \to A$ would lead to a linear representation of the quotient H/Z of the 3-dimensional Heisenberg group modulo a cyclic central subgroup Z. Such elements exist in the Lie algebra $\widehat{\mathfrak{g}}$ of the central extension of the Banach–Lie algebra $C^1(\mathbb{S}^1, \mathfrak{su}_2(\mathbb{C}))$ by $\mathbb{R}$ (Example VI.1.16; [ES73]).

Since for each Banach–Lie algebra $\mathfrak{g}$ the quotient $\mathfrak{g}_{\rm ad} = \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is linear, the intersection $\mathfrak{n}$ of all kernels of linear representations of $\mathfrak{g}$ is a central ideal of $\mathfrak{g}$. This links the linearity problem intimately with central extensions: When is a central extension of a linear Banach–Lie algebra linear? As the enlargeability is necessary, the discreteness of the corresponding period group is necessary (Theorem VI.1.6), but what else?

In [ES73], van Est and Świerczkowski describe a condition on the cohomology class of a central extension which is sufficient for linearity. They apply this in particular to show that, under some cohomological condition involving the center, for a Banach–Lie algebra $\mathfrak{g}$, the Banach–Lie algebra $C^1_*([0,1],\mathfrak{g})$ of C^1-curves γ in $\mathfrak{g}$ with $\gamma(0) = 0$ is linear. It is remarkable that their argument does not work for C^0-curves. Closely related to this circle of ideas is van Est's proof of Ado's theorem, based on the vanishing of π_2 for each finite-dimensional Lie group ([Est66]).

It is also interesting to note that for a real Banach–Lie algebra $\mathfrak{g}$, linearity implies the linearity of the complexification $\mathfrak{g}_{\mathbb{C}}$, which in turn implies that $\mathfrak{g}_{\mathbb{C}}$ is enlargeable, which is crucial for the existence of universal complexifications of the corresponding groups (*cf.* Theorem VI.1.23). In view of Corollary IV.4.10, we thus have the implications

$$\mathfrak{g} \quad \text{linear} \quad \Rightarrow \quad \mathfrak{g}_{\mathbb{C}} \quad \text{enlargeable} \quad \Rightarrow \quad \mathfrak{g} \quad \text{enlargeable}. \qquad \square$$

Remark VIII.8. [GN07] (a) Let A be a unital CIA and $n \in \mathbb{N}$. Further let $\mathfrak{sl}_n(A) \trianglelefteq \mathfrak{gl}_n(A)$ denote the closed commutator algebra (*cf.* the end of Section VI.1). As this is a closed subalgebra, it generates some integral subgroup $S \to \mathrm{GL}_n(A)$ with $\mathbf{L}(S) = \mathfrak{sl}_n(A)$. But in general S will not be a Lie subgroup. This problem is caused by the fact that $\mathrm{GL}_n(A)$ need not be simply connected.

Let $q\colon \widetilde{\mathrm{GL}}_n(A) \to \mathrm{GL}_n(A)_0$ denote the universal covering group of the identity component of $\mathrm{GL}_n(A)$. Then the Lie algebra morphism

$$\mathrm{Tr}\colon \mathfrak{gl}_n(A) \to A/\overline{[A,A]}, \quad (a_{ij}) \mapsto \Big[\sum_{i=1}^{n} a_{ii}\Big]$$

satisfies $\ker \mathrm{Tr} = \mathfrak{sl}_n(A)$. Let $HC_0(A)$ denote the completion of $A/\overline{[A,A]}$. Then $\mathrm{Tr}\colon \mathfrak{gl}_n(A) \to HC_0(A)$ integrates to a morphism of BCH–Lie groups

$$\widetilde{D}\colon \widetilde{\mathrm{GL}}_n(A) \to HC_0(A),$$

and $\widehat{S} := \ker D \trianglelefteq \widetilde{\mathrm{GL}}_n(A)$ is a BCH–Lie subgroup whose identity component $\widehat{S}_0$ is a covering group of S. If the image of the induced period homomorphism

$$\mathrm{per}_{\mathrm{Tr}}\colon \pi_1(\mathrm{GL}_n(A)) \to HC_0(A) \tag{8.1}$$

is discrete, then $Z := HC_0(A)/\operatorname{im}(\mathrm{per}_{\mathrm{Tr}})$ is a Lie group and D factors through a homomorphism $D\colon \mathrm{GL}_n(A)_0 \to Z$, which can be considered as a generalization of the determinant. Now $\ker D$ is a BCH–Lie subgroup of $\mathrm{GL}_n(A)$ with Lie algebra $\mathfrak{sl}_n(A)$, which implies that $(\ker D)_0 = S$. It is interesting to compare this situation with the one in Remark V.2.14(c), where the group of Hamiltonian diffeomorphisms of a symplectic manifolds plays a similar role.

If A is commutative, then the determinant $\det\colon \mathrm{GL}_n(A) \to A^\times$ is a morphism of Lie groups and $\mathrm{SL}_n(A) \trianglelefteq \mathrm{GL}_n(A)$ is a normal BCH–Lie subgroup with Lie algebra $\mathfrak{sl}_n(A)$.

Since the period maps (8.1) are compatible for different n, they lead to a homomorphism

$$\mathrm{per}_A^0\colon K_2(A) = \varinjlim \pi_1(\mathrm{GL}_n(A)) \to HC_0(A)$$

(*cf.* the end of Section VI.1). If A is complex, we may compose with the Bott isomorphism $\beta_A^0\colon K_0(A) \to K_2(A)$ to get a natural transformation

$$T_A := \mathrm{per}_A^0 \circ \beta_A^0\colon K_0(A) \to HC_0(A),$$

which is unique and therefore given by $T_A([e]) = \mathrm{Tr}(e)$. It follows that the image of per_A^0 is discrete if and only if the image of the trace map

$$\mathrm{Tr}\colon \bigcup_{n=1}^{\infty} \mathrm{Idem}(M_n(A)) \to HC_0(A)$$

generates a discrete subgroup.

If A is commutative, then $HC_0(A) = A$, and the image of the trace map lies in the discrete subgroup $\frac{1}{2\pi i}\ker(\exp_{A_{\mathbb{C}}})$ of $A_{\mathbb{C}}$. Hence the image of the trace map is discrete for each commutative CIA. $\square$

Remark VIII.9. The set $\mathrm{Idem}(A)$ of idempotents of a CIA plays a central role in (topological) K-theory. In [Gram84], Gramsch shows that this set always carries a natural manifold structure, which implies in particular that its connected components are open subsets. The key point is to use rational methods to obtain charts on this set.

In a similar spirit, it is explained in [BerN04/05] how Jordan methods can be used in an infinite-dimensional context to obtain manifold structures on geometrically defined manifolds generalizing symmetric spaces and Graßmann manifolds. □

Open Problems for Section VIII

Problem VIII.1. Show that the completion of a CIA A is again a CIA or give a counterexample. □

Problem VIII.2. Characterize those Banach–Lie algebras which are linear in the sense that they have an injective homomorphism into some Banach algebra (Remark VIII.7).

Not much seems to be known about this problem, which is partly related to the non-existence of Lie's theorem on the representation of solvable Banach–Lie algebras. In view of this connection, the class of Banach–Lie algebras of the form $\mathfrak{g} = E \rtimes_D \mathbb{R}$, where D is a continuous linear operator on the Banach space E should be a good testing ground.

As each real Banach space E is isomorphic to $\mathscr{L}(\mathbb{R}, E)$, which can be embedded as a Banach–Lie algebra into the Banach algebra $\mathscr{L}(E \oplus \mathbb{R})$, each abelian Banach–Lie algebra is linear. What about nilpotent ones? □

Problem VIII.3. Show that each linear Lie group is regular. We know that this is the case for unit groups of CIAs. If, in addition, A is μ-regular in the sense of Definition III.2.4, then Theorem III.2.10 implies the μ-regularity of each Lie subgroup. □

Problem VIII.4. Is the tensor product $A \otimes B$ of two CIAs, endowed with the projective tensor topology a CIA? This is true for $B = M_n(\mathbb{K})$, $n \in \mathbb{N}$, where $A \otimes B \cong M_n(A)$. Is it also true if B is the algebra of rapidly decreasing matrices or the direct limit algebra $M_\infty(\mathbb{K}) := \varinjlim M_n(\mathbb{K})$? □

Problem VIII.5. Let $\mathfrak{g}$ be a locally convex Lie algebra. Does the enveloping algebra $U(\mathfrak{g})$ carry a natural topology for which the multiplication is continuous and the natural map $\mathfrak{g} \to U(\mathfrak{g})$ is continuous?

More generally, let E be a locally convex space and endow its tensor algebra $\mathscr{T}(E) = \bigoplus_{n=0}^{\infty} E^{\otimes n}$ with the locally convex direct limit topology, where the subspaces $E^{\otimes n}$ carry the projective tensor topology. Is the multiplication on $\mathscr{T}(E)$ continuous? □

Problem VIII.6. (a) Does every locally convex Lie algebra $\mathfrak{g}$ have a faithful topological module E? If Problem VIII.5 has a positive solution, then we may simply take $E := U(\mathfrak{g})$.

(b) Does every nilpotent locally convex Lie algebra have a faithful nilpotent topological module? Is this true in the Banach category? □

IX. Lie transformation groups

One of the fundamental references on topological transformation groups is the monograph [MZ55] by Montgomery and Zippin. Since it also deals with differentiability properties of transformation groups on manifolds, some of the techniques described there have interesting applications in the context of infinite-dimensional Lie theory.

IX.1. Smooth Lie group actions

Theorem IX.1.1. ([BoMo45], [MZ55, p.212]) *Any continuous action $\sigma\colon G\times M \to M$ of a finite-dimensional Lie group on a finite-dimensional smooth manifold M by diffeomorphisms is smooth.* □

For compact manifolds we obtain the following "automatic smoothness" result on homomorphisms of Lie groups (see also [CM70] for one-parameter groups; and [Gl02d] for the non-compact case).

Corollary IX.1.2. *If M is a σ-compact finite-dimensional manifold and G a finite-dimensional Lie group, then any continuous homomorphism $\varphi\colon G \to \mathrm{Diff}_c(M)$ is smooth.* □

The following result provides a positive answer to (FP9) for diffeomorphism groups.

Theorem IX.1.3. ([MZ55, Th. 5.2.2, p.208]) *If a locally compact group G acts faithfully on a smooth finite-dimensional manifold M by diffeomorphisms, then G is a finite-dimensional Lie group. If M is compact, then each locally compact subgroup of* $\mathrm{Diff}(M)$ *is a Lie group.* □

The preceding results take care of the actions of locally compact groups on manifolds. As the work of de la Harpe and Omori ([OdH71/72]) shows, the situation for Banach–Lie groups is more subtle:

Theorem IX.1.4. ([OdH72]) *Let G be a Banach–Lie group. If $\mathbf{L}(G)$ has no proper finite-codimensional closed ideals, then $\mathbf{L}(G)$ has no proper finite-codimensional closed subalgebra and each smooth action of G on a finite-dimensional manifold is trivial.* □

If $\alpha\colon \mathfrak{g} \to \mathcal{V}(M)$ is an injective map, then for each $p \in M$ the subspace

$$\mathfrak{g}_p := \{x \in \mathfrak{g}\colon \alpha(x)(p) = 0\}$$

is a finite-codimensional subalgebra with $\bigcap_p \mathfrak{g}_p = \{0\}$. Therefore the existence of many finite-codimensional subalgebras is necessary for Lie algebras to be realizable by vector fields on a finite-dimensional manifold.

Theorem IX.1.5. ([OdH72]) *If a Banach–Lie group G acts smoothly, effectively, amply (for each $m \in M$ the evaluation map $\mathfrak{g} \to T_m(M)$ is surjective), and primitively (it leaves no foliation invariant) on a finite-dimensional manifold M, then it is finite-dimensional.* □

Theorem IX.1.6. ([Omo78, Th. B/C]) *Let G be a connected Banach–Lie group acting smoothly, effectively and transitively on a finite-dimensional manifold M.*

(1) If M is compact, then G is finite-dimensional.
(2) If M is non-compact, then $\mathbf{L}(G)$ contains a finite-codimensional closed solvable ideal. □

Since $\mathrm{Diff}(M)$ acts smoothly, effectively and transitively on M, this implies:

Corollary IX.1.7. $\mathrm{Diff}(M)$ *cannot be given a Banach-Lie group structure for which the natural action on M is smooth.* □

In Section 4 of [OdH72], Omori and de la Harpe construct an example of a Banach–Lie group G acting smoothly and amply, but not primitively on $\mathbb{R}^2$.

The preceding discussion implies in particular that Banach–Lie groups rarely act on finite-dimensional manifolds. As the gauge groups of principal bundles $q\colon P \to M$ over compact manifolds M show, the situation is different for locally exponential Lie groups (*cf.* Theorem IV.1.12). Therefore it is of some interest to have good criteria for the integrability of infinitesimal actions of locally exponential Lie algebras on finite-dimensional manifolds (*cf.* (FP7)).

We start with a more general setup for infinite-dimensional manifolds which need extra smoothness assumptions:

Theorem IX.1.8. (Integration of locally exponential Lie algebras of vector fields; [AbNe06]) *Let M be a smooth manifold modeled on a locally convex space, $\mathfrak{g}$*

a locally exponential Lie algebra and $\alpha\colon \mathfrak{g} \to \mathcal{V}(M)$ *a homomorphism of Lie algebras whose range consists of complete vector fields. Suppose further that the map*

$$\mathrm{Exp}\colon \mathfrak{g} \to \mathrm{Diff}(M), \quad x \mapsto \mathrm{Fl}_1^{\alpha(x)}$$

is smooth in the sense of Definition II.3.1 and that 0 *is isolated in* $\mathfrak{z}(\mathfrak{g}) \cap \mathrm{Exp}^{-1}(\mathrm{id}_M)$*. Then there exists a locally exponential Lie group G and a smooth action* $\sigma\colon G \times M \to M$ *whose derived action* $\dot{\sigma}\colon \mathfrak{g} \to \mathcal{V}(M)$ *coincides with* α*.* □

In the finite-dimensional case, the smoothness assumptions in Theorem IX.1. 8 follows from the smooth dependence of solutions of ODEs on parameters and initial values, and the condition on the exponential function can be verified with methods to be found in [MZ55]. This leads to the following less technical generalization of the Lie–Palais Theorem which subsumes in particular Omori's corresponding results for Banach–Lie algebras ([Omo80, Th. A], [Pe95b, Th.4.4]).

Theorem IX.1.9. *Let M be a smooth finite-dimensional manifold,* $\mathfrak{g}$ *a locally exponential Lie algebra and* $\alpha\colon \mathfrak{g} \to \mathcal{V}(M)$ *a continuous homomorphism of Lie algebras whose range consists of complete vector fields. Then there exists a locally exponential Lie group G and a smooth action* $\sigma\colon G \times M \to M$ *with* $\dot{\sigma} = \alpha$*.* □

The following result is a generalization of Palais' Theorem in another direction. Since Diff(M) is μ-regular (Theorem III.3.1), it also follows from Theorem III.2.8.

Theorem IX.1.10. ([Les68]) *If M is compact, then a subalgebra* $\mathfrak{g} \subseteq \mathcal{V}(M)$ *is integrable to an integral subgroup if* $\mathfrak{g}$ *is finite-dimensional or closed and finite-codimensional.* □

IX.2. Groups of diffeomorphisms as automorphism groups

In this subsection, we simply collect some results stating that automorphism groups of certain algebra, Lie algebra or groups associated to geometric structure on manifolds are what one expects. Most of the results formulated below for automorphisms of structures attached to a manifold M generalize to results saying that if M_1 and M_2 are two manifolds and two objects of the same kind attached to M_1 and M_2 are isomorphic, then this isomorphism can be implemented by a diffeomorphism $M_1 \to M_2$, compatible with the geometric structures under consideration.

Theorem IX.2.1. *Let M be a* σ*-compact finite-dimensional smooth manifold. Then the following assertions hold:*

(1) For the Fréchet algebra $C^\infty(M,\mathbb{R})$, each homomorphism to $\mathbb{R}$ is a point evaluation.
(2) $\mathrm{Aut}(C^\infty(M,\mathbb{R})) \cong \mathrm{Diff}(M)$.
(3) $\mathrm{Aut}(\mathcal{V}_c(M)) \cong \mathrm{Aut}(\mathcal{V}(M)) \cong \mathrm{Diff}(M)$.
(4) If, in addition, M is complex and $\mathcal{V}^{(1,0)}(M) \subseteq \mathcal{V}(M)_{\mathbb{C}}$ is the Lie algebra of complex vector fields of type $(1,0)$, then $\mathrm{Aut}(\mathcal{V}^{(1,0)}(M)) \cong \mathrm{Aut}_{\mathcal{O}}(M)$ is the group of biholomorphic automorphisms of M.
(5) For each finite-dimensional σ-compact manifold M and each simple (real or complex) finite-dimensional Lie algebra $\mathfrak{k}$, the natural homomorphism

$$C^\infty(M,\mathrm{Aut}(\mathfrak{k})) \rtimes \mathrm{Diff}(M) \to \mathrm{Aut}(C^\infty(M,\mathfrak{k}))$$

is surjective.
(6) If M is a Stein manifold and $\mathfrak{k}$ is a finite-dimensional complex simple Lie algebra, then $\mathrm{Aut}(\mathcal{O}(M,\mathfrak{k})) \cong \mathcal{O}(M,\mathrm{Aut}(\mathfrak{k})) \rtimes \mathrm{Aut}_{\mathcal{O}}(M)$, where $\mathrm{Aut}_{\mathcal{O}}(M)$ denotes the group of biholomorphic diffeomorphisms of M.
(7) If $K \subseteq \mathbb{C}^n$ is a polyhedral domain and $\mathcal{O}(K,\mathbb{C})$ the algebra of germs of holomorphic $\mathbb{C}$-valued functions in K, then the group $\mathrm{Aut}(\mathcal{O}(K,\mathbb{C}))$ consists of the germs of biholomorphic maps of K and $\mathrm{der}(\mathcal{O}(K,\mathbb{C}))$ consists of the germs holomorphic vector fields on K.

Proof. (1) (*cf.* [My54] for the compact case; [Pu52]; [Co94]).

(2) follows easily from (1) because each automorphism of the algebra $C^\infty(M,\mathbb{R})$ acts on $\mathrm{Hom}(C^\infty(M,\mathbb{R}),\mathbb{R}) \cong M$.

(3) The representability of each isomorphism of $\mathcal{V}_c(M)$ by a diffeomorphism is due to Pursell and Shanks ([PuSh54]), and the other assertion follows from Theorem 2 in [Ame75]. It is based on the fact that the maximal proper subalgebras of finite codimension are all of the form $\mathcal{V}(M)_m := \{X \in \mathcal{V}(M) \colon X(m) = 0\}$ for some $m \in M$, hence permuted by each automorphism; resp. the fact that all maximal ideals consist of all vector fields whose jet vanishes in some $m \in M$.

(4) follows from Theorem 1 in [Ame75].

(5) [PS86, Prop. 3.4.2]. A central point is that every non-zero endomorphism of $\mathfrak{k}$ is an automorphism. Further, it is used that $[\mathfrak{k}, C^\infty(M,\mathfrak{k})] = C^\infty(M,\mathfrak{k})]$ and that distributions supported by one point are of finite order.

(6) [NeWa06b].

(7) This is [vHo52b, Th. III], where it is first shown that the maximal ideals in the Silva CIA $\mathcal{O}(K,\mathbb{C})$ (Example VIII.3(d)) are the kernels of the point evaluations ([vHo52b, Th. I]). □

Remark IX.2.2. Let $K \subseteq \mathbb{C}^n$ be a compact subset and $\mathrm{Aut}_{\mathcal{O}}(K)$ the group of germs of bihomolorphic maps, defined on some neighborhood of K, mapping K onto itself. In [vHo52a], van Hove introduces a group topology on this group as the topology for which the map

$$\mathrm{Aut}_{\mathcal{O}}(K) \to \mathcal{O}(K,\mathbb{C}^n) \times \mathcal{O}(K,\mathbb{C}^n), \quad g \mapsto (g, g^{-1})$$

is an embedding. He shows that, under certain geometric conditions on the set K, this group is complete and contains no small subgroups. Moreover, its natural action on $\mathscr{O}(K,\mathbb{C})$ is continuous. $\square$

Theorem IX.2.3. ([Omo74], §10]) *Let M be a σ-compact finite-dimensional smooth manifold. For a differential form α on M we put $\mathscr{V}(M,\alpha) := \{X \in \mathscr{V}(M) \colon \mathscr{L}_X\alpha = 0\}$. Then the following assertions hold:*

(1) If μ is a volume form or a symplectic form on M, then every (algebraic) automorphism of $\mathscr{V}(M,\mu)$ is induced by an element of the group

$$\{\varphi \in \mathrm{Diff}(M) \colon \varphi^*\mu \in \mathbb{R}\mu\}.$$

(2) If α is a contact 1-form on M, then every (algebraic) automorphism of $\mathscr{V}(M,\alpha)$ is induced by an element of the group $\{\varphi \in \mathrm{Diff}(M) \colon \varphi^*\alpha \in C^\infty(M,\mathbb{R}^\times)\cdot\alpha\}$. $\square$

In [Omo80], one finds another interesting result of this type. Let V be a germ of an affine variety in $0 \in \mathbb{C}^n$. Two such germs V and V' are said to be *biholomorphically equivalent* if there exists an element $\varphi \in \mathrm{Gh}_n(\mathbb{C})$ of the group of germs of biholomorphic maps fixing 0 (as in Example VI.2.12), such that $\varphi(V) = \varphi(V')$. On the infinitesimal level the automorphisms of a germ V are given by the Lie algebra

$$\mathfrak{g}(V) := \{X \in \mathfrak{gh}_n(\mathbb{C}) \colon X.J(V) \subseteq J(V)\},$$

where $J(V) \subseteq \mathscr{O}(0,\mathbb{C})$ (the germs of holomorphic functions in 0) is the annihilator ideal of V. Let $\mathfrak{g}(V)_k \trianglelefteq \mathfrak{g}(V)$ denote the ideal consisting of all vector fields vanishing of order k in 0 and form the projective limit Lie algebra

$$\overline{\mathfrak{g}}(V) := \varprojlim \mathfrak{g}(V)/\mathfrak{g}(V)_k,$$

which can be viewed as a Fréchet completion of $\mathfrak{g}(V)$.

An element $X \in \mathfrak{gh}_n(\mathbb{C})$ is called *semi-expansive* if it is $\mathrm{Gh}_n(\mathbb{C})$-conjugate to a linear diagonalizable vector field for which all eigenvalues lie in some open halfplane. The germ V is called an *expansive singularity* if $\mathfrak{g}(V)$ contains an expansive vector field.

Theorem IX.2.4. *Two expansive singularities V and V' are biholomorphically equivalent if and only if the pro-finite Lie algebras $\overline{\mathfrak{g}}(V)$ and $\overline{\mathfrak{g}}(V')$ are isomorphic. Moreover, $\mathrm{Aut}(\overline{\mathfrak{g}}(V))$ can be identified with the stabilizer $\mathrm{Gh}_n(\mathbb{C})_V$ of V in the group $\mathrm{Gh}_n(\mathbb{C})$.* $\square$

On the group level, we have the following analog of Theorem IX.2.3 (*cf.* [Fil82] for (1) and [Ban97, Thms. 7.1.4/5/6] for (2)-(4)):

Theorem IX.2.5. *Let M be a σ-compact connected finite-dimensional smooth manifold. Then the following assertions hold:*

(1) *Every (algebraic) automorphism of* $\mathrm{Diff}(M)$ *is inner.*
(2) *If* α *is a contact* 1*-form on* M*, then every (algebraic) automorphism of* $\mathrm{Diff}(M,\alpha)$ *is conjugation with an element of the group* $\{\varphi \in \mathrm{Diff}(M)\colon \varphi^*\alpha \in C^\infty(M,\mathbb{R}^\times)\cdot\alpha\}$.
(3) *If* ω *is a symplectic form and* M *is compact of dimension* ≥ 2*, then every (algebraic) automorphism of* $\mathrm{Diff}(M,\omega)$ *is conjugation by an element of the group*

$$\{\varphi \in \mathrm{Diff}(M)\colon \varphi^*\omega \in \mathbb{R}\omega\}.$$

(4) *If* μ *is a volume form and* M *is of dimension* ≥ 2*, then every (algebraic) automorphism of* $\mathrm{Diff}(M,\mu)$ *is conjugation by an element of the group* $\{\varphi \in \mathrm{Diff}(M)\colon \varphi^*\mu \in \mathbb{R}\mu\}$. □

Open Problems for Section IX

Problem IX.1. Let $\mathcal{V}(M)_{\mathrm{cp}}$ denote the set of complete vector fields on the finite-dimensional manifold M (Remark II.3.8). Then we have an exponential function

$$\mathrm{Exp}\colon \mathcal{V}(M)_{\mathrm{cp}} \to \mathrm{Diff}(M), \quad X \mapsto \mathrm{Fl}_1^X.$$

Is it true that 0 is isolated in $\mathrm{Exp}^{-1}(\mathrm{id}_M)$ with respect to the natural Fréchet topology on $\mathcal{V}(M)$ (*cf.* Definition I.5.2)?

That this is true for compact manifolds follows from Newman's Theorem ([Dr69, Th. 2]). For the proof of Theorem IX.1.9, we show for each continuous homomorphism $\alpha\colon \mathfrak{g} \to \mathcal{V}(M)$ of a locally exponential Lie algebra $\mathfrak{g}$ to $\mathcal{V}(M)$ with range in $\mathcal{V}(M)_{\mathrm{cp}}$ that 0 is isolated in $(\mathrm{Exp}\circ\alpha)^{-1}(\mathrm{id}_M)$, which is a weaker statement.

Since the set $\mathrm{Exp}^{-1}(\mathrm{id}_M)$ is in one-to-one correspondence with the smooth $\mathbb{T}$-actions on M, the problem is to show that the trivial action is isolated in this "space" of all smooth $\mathbb{T}$-actions on M.

If M is the real Hilbert space $\ell^2(\mathbb{N},\mathbb{R})$ with the Hilbert basis e_n, $n \in \mathbb{N}$, then we have linear vector fields $X_n(v) := 2\pi i\langle v, e_n\rangle e_n$ with $\exp(X_n) = \mathrm{id}_M$ and $X_n \to 0$ uniformly on compact subsets of E. Hence the finite-dimensionality of M is crucial. □

Problem IX.2. (Banach symmetric spaces) Let M be a smooth manifold. We say that (M,μ) is a *symmetric space* (in the sense of Loos) (*cf.* [Lo69]) if $\mu\colon M \times M \to M, (x,y) \mapsto x\cdot y$ is a smooth map with the following properties:

(S1) $x\cdot x$ for all $x \in M$.
(S2) $x\cdot(x\cdot y) = y$ for all $x,y \in M$.
(S3) $x\cdot(y\cdot z) = (x\cdot y)\cdot(x\cdot z)$ for all $x,y \in M$.
(S4) $T_x(\mu_x) = -\mathrm{id}_{T_x(M)}$ for $\mu_x(y) := \mu(x,y)$ and each $x \in M$.

(a) Is it true that the automorphism group $\mathrm{Aut}(M,\mu)$ of a Banach symmetric space (M,μ) is a Banach–Lie group? (*cf.* [Ne02c], [La99])

(b) The tangent spaces $T_x(M)$ of a symmetric space carry natural structures of Lie triple systems. Develop a Lie theory for locally exponential, resp., Banach–Lie triple systems, including criteria for the integrability of morphisms and enlargeability (*cf.* Sections IV and VI). □

Problem IX.3. A fundamental problem in the theory of Banach transformation groups is that we do not know if orbits carry natural manifold structures. As in finite dimensions, the main point is to find good criteria for a closed subgroup H of a Banach–Lie group G to ensure that the coset space G/H has a natural manifold structure for which the action of G on G/H is smooth and the quotient map $q\colon G \to G/H$ is a "weak" submersion in the sense that all its differentials are linear quotient maps. In view of Remark IV.4.13, this is true if

(1) H is a split submanifold (the same proof as in finite dimensions works),
(2) H is a normal Banach–Lie subgroup (without any splitting requirements) (Corollary IV.3.6), and
(3) G is a Hilbert–Lie group, which implies the splitting condition (1).

Here are some concrete problems:

(a) Suppose that G/H is a smooth manifold with submersive q and a smooth action of G. Does this imply that H is a Lie subgroup of G?
(b) Are the stabilizer groups G_m for a smooth action of a Banach–Lie group G on a Banach manifold M Lie subgroups? For linear actions this follows from Proposition IV.3.4 and Corollary IV.3.13.
(c) Characterize those Lie subgroups H for which G/H is a smooth manifold.
(d) Let $H \subseteq G$ be a closed subgroup and $\mathfrak{h} := \mathbf{L}^e(H)$ its Lie algebra. Then the normalizer $N_G(\mathfrak{h})$ of $\mathfrak{h}$ is a Lie subgroup (Proposition IV.3.4, Corollary IV.3.12). Is it true that $\mathrm{Ad}(G).\mathfrak{h} \cong G/N_G(\mathfrak{h})$ carries a natural manifold structure?

Note that, if H is connected, it is a normal subgroup of $N_G(\mathfrak{h})$. If H is a Lie subgroup, this implies that $N_G(\mathfrak{h})/H$ carries a Lie group structure and therefore a manifold structure. □

Problem IX.4. Show that for each compact subset $K \subseteq \mathbb{C}^n$ the group $\mathrm{Aut}_{\mathscr{O}}(K)$ from Remark IX.2.2 is a Lie group with respect to the manifold structure inherited from the embedding into $\mathscr{O}(K,\mathbb{C}^n)$ (*cf.* Remark IX.2.2). □

Problem IX.5. (Automorphisms of gauge algebras) Let $q\colon P \to M$ be a smooth K-principal bundle over the (compact) manifold M. Determine the group $\mathrm{Aut}(\mathfrak{gau}(P))$ of automorphisms of the gauge Lie algebra. Does it coincide with the automorphism group $\mathrm{Aut}(\mathrm{ad}(P))$ of the adjoint bundle, whose space of sections $\mathfrak{gau}(P)$ is? If K is a simple complex Lie group, then the results in [Lec80, Th. 16] provide a local description of the automorphisms of this Lie algebra in terms of diffeomorphisms of M and sections of the automorphism bundle $\mathrm{Aut}(\mathrm{ad}(P))$. □

Problem IX.6. Determine the automorphism groups of the Lie algebras $\mathfrak{gf}_n(\mathbb{K})$, $\mathfrak{gs}_n(\mathbb{K})$ and $\mathfrak{gh}_n(\mathbb{C})$. □

Problem IX.7. Describe all connected Banach–Lie groups acting smoothly, effectively and transitively on a finite-dimensional manifold. In view of Theorem IX.1.6, for each Banach–Lie group G, the Lie algebra $\mathbf{L}(G)$ contains a finite-codimensional closed solvable ideal. If, conversely, $\mathfrak{g}$ is a Banach–Lie algebra with a finite-dimensional closed solvable ideal, then Theorem VI.1.19 implies that $\mathfrak{g}$ is enlargeable. Under which conditions do the corresponding groups G act effectively on some finite-dimensional homogeneous space? (see also the corresponding discussion in [Omo97]). □

Problem IX.8. Let G be a Banach–Lie group and $H \subseteq G$ a closed subgroup for which $\mathbf{L}^e(H)$ has finite codimension. Does this imply that G/H is a manifold? □

X. Projective limits of Lie groups

Projective limits play an important role in several branches of Lie theory. Since complete locally convex spaces are nothing but closed subspaces of products of Banach spaces, on the level of the model spaces, the projective limit construction leads us from Banach spaces to the locally convex setting. On the group level, the situation is more involved because, although projective limits of Lie groups are often well-behaved topological groups, in general they are not Lie groups. In this section, we briefly report on some aspects of projective limit Lie theory and the recent theory of Hofmann and Morris of projective limits of finite-dimensional Lie groups.

X.1. Projective limits of finite-dimensional Lie groups

In their recent monograph [HoMo06], Hofmann and Morris approach projective limits of finite-dimensional Lie groups, so-called *pro-Lie groups*, from a topological point of view. We refer to [HoMo06] for details on the results mentioned in this subsection.

Clearly, arbitrary products of finite-dimensional Lie groups, such as

$$\mathbb{R}^J, \quad \mathbb{Z}^J, \quad \mathrm{SL}_2(\mathbb{R})^J$$

for an arbitrary set J, are pro-Lie groups. The following theorem gives an abstract characterization of pro-Lie groups:

Theorem X.1.1. *A topological group G is a pro–Lie group if and only if it is isomorphic to a closed subgroup of a product of finite-dimensional Lie groups. In particular, closed subgroups of pro-Lie groups are pro-Lie groups.* □

A crucial observation is that the class of topological groups with Lie algebra (*cf.* Definition IV.1.23) is closed under projective limits and that

$$\mathfrak{L}(\varprojlim G_j) \cong \varprojlim \mathfrak{L}(G_j)$$

as locally convex Lie algebras. Let us call topological vector spaces isomorphic to $\mathbb{R}^J$ for some set J *weakly complete*. These are the dual spaces of the vector spaces $\mathbb{R}^{(J)}$, endowed with the weak-$*$-topology. This provides a duality between real vector spaces and weakly complete locally convex spaces, which implies in particular that each closed subspace of a weakly complete space is weakly complete and complemented. In particular, weakly complete spaces are nothing but the projective limits of finite-dimensional vector spaces. These considerations lead to:

Theorem X.1.2. *Every pro-Lie group has a Lie algebra which is a a projective limit of finite-dimensional Lie algebras, hence a weakly complete topological Lie algebra. The image of the exponential function*

$$\exp_G\colon \mathfrak{L}(G) \to G, \quad \gamma \mapsto \gamma(1)$$

generates a dense subgroup of the identity component G_0. □

In the following, we call projective limits of finite-dimensional Lie algebras *pro-finite Lie algebras* (called *pro-Lie algebras* in [HoMo06]).

Remark X.1.3. According to Yamabe's Theorem ([MZ55]), each locally compact group G for which G/G_0 is compact is a pro-Lie group. Since the totally disconnected locally compact group G/G_0 contains an open compact subgroup, each locally compact group G contains an open subgroup with a Lie algebra, hence is a topological group with a Lie algebra (*cf.* [Las57], [HoMo05, Prop. 3.5]). □

In view of Theorem X.1.1, the category of pro-Lie groups is closed under products and projective limits, which are remarkable closedness properties which in turn lead to the existence of an adjoint functor Γ for the Lie functor $\mathfrak{L}$:

Theorem X.1.4. (Lie's Third Theorem for Pro-Lie Groups; [HoMo05]) *The Lie functor $\mathfrak{L}$ from the category of pro-Lie groups to the category of pro-Lie algebras* *has a left adjoint* Γ. *It associates with each pro-finite Lie algebra* $\mathfrak{g}$ *a connected pro-Lie group* $\Gamma(\mathfrak{g})$ *and a natural isomorphism* $\eta_{\mathfrak{g}}\colon \mathfrak{g} \to \mathfrak{L}(\Gamma(\mathfrak{g}))$, *such that for every morphism* $\varphi\colon \mathfrak{g} \to \mathfrak{L}(G)$ *there exists a unique morphism* $\varphi'\colon \Gamma(\mathfrak{g}) \to G$ *with* $\mathfrak{L}(\varphi') \circ \eta_{\mathfrak{g}} = \varphi$. □

The first part of the following structure theorem can be found in [HoMo06]. The second part follows from the fact that finite-dimensional tori are the only abelian connected compact Lie groups.

Theorem X.1.5. *A connected abelian pro-Lie group is of the form* $\mathbb{R}^J \times C$ *for a compact connected abelian group* C. *It is a Lie group if and only if* C *is a finite-dimensional torus.* □

It is quite remarkable that the category of pro-finite Lie algebras permits to develop a structure theory which is almost as strong as in finite dimensions. In particular, there is a Levi decomposition. To describe it, we call a pro-finite Lie algebra $\mathfrak{g}$ *pro-solvable* if it is a projective limit of finite-dimensional solvable Lie algebras:

Theorem X.1.6. (Levi decomposition of pro-finite Lie algebras and groups; [HoMo06]) *Each pro-finite Lie algebra* $\mathfrak{g}$ *contains a unique maximal pro-solvable ideal* $\mathfrak{r}$. *There is a family* $(\mathfrak{s}_j)_{j\in J}$ *of finite-dimensional simple Lie algebras such that* $\mathfrak{s} := \prod_{j\in J} \mathfrak{s}_j$ *satisfies*

$$\mathfrak{g} \cong \mathfrak{r} \rtimes \mathfrak{s}.$$

For the corresponding pro-finite Lie group $\Gamma(\mathfrak{g})$ *we then have*

$$\Gamma(\mathfrak{g}) \cong R \rtimes S, \quad \text{where} \quad S \cong \prod_{j\in J} S_j,$$

where S_j *is a* 1*-connected Lie group with Lie algebra* $\mathfrak{s}_j$ *and* R *is diffeomorphic to* $N \times \mathbb{R}^K$ *for some set* K *and some simply connected pro-nilpotent Lie group* $N \cong (\mathbf{L}(N), *)$. □

More concretely, the closed commutator algebra $\mathfrak{n} := \overline{[\mathfrak{r},\mathfrak{r}]}$ is pro-nilpotent, because all images of this subalgebra in finite-dimensional solvable quotients of $\mathfrak{r}$ are nilpotent. If $\mathfrak{e} \subseteq \mathfrak{r}$ is a closed vector space complement of $\mathfrak{n}$ in $\mathfrak{r}$ ([HoMo06, 4.20/21]), then the map

$$\Phi\colon \mathfrak{n} \times \mathfrak{e} \mapsto R, \quad (x,y) \mapsto \exp_R(x)\exp_R(y) \tag{10.1.1}$$

is a homeomorphism ([HoMo06, Th. 8.13]). The point of view of [HoMo06] is purely topological, so that infinite-dimensional Lie group structures are not discussed. We note that (10.1.1) can be viewed as a chart of the topological group R, and it is not hard to see that it defines on R the structure of a smooth Lie group.

Based on the preceding theorem, one can characterize those pro-finite Lie algebras which are integrable to locally convex Lie groups ([HoNe06]):

Theorem X.1.7. *For a pro-finite Lie algebra $\mathfrak{g}$ the following are equivalent:*

(1) $\mathfrak{g}$ is the Lie algebra of a locally convex Lie group G with smooth exponential function.
(2) $\mathfrak{g}$ is the Lie algebra of a regular locally convex Lie group G.
(3) $\mathfrak{g}$ has a Levi decomposition $\mathfrak{g} \cong \mathfrak{r} \rtimes \mathfrak{s}$, where only finitely many factors in $\mathfrak{s} = \prod_{j\in J} \mathfrak{s}_j$ are not isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.
(4) The group $\Gamma(\mathfrak{g})$ is locally contractible.
(5) The group $\Gamma(\mathfrak{g})$ carries the structure of a regular Lie group, compatible with its topology.

Proof. (Sketch) Let R denote the 1-connected group with Lie algebra $\mathfrak{r}$ constructed from the chart (10.1.1) and S_j be the 1-connected Lie group with Lie algebra $\mathfrak{s}_j$. If $\mathfrak{s}_j$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, then $S_j \cong \widetilde{\mathrm{SL}}_2(\mathbb{R})$ is diffeomorphic to $\mathbb{R}^3$, so that $S := \prod_{j\in J} S_j$ carries a natural manifold structure, turning it into a Lie group. One verifies that S acts smoothly on R, so that $G := R \rtimes S$ is a Lie group with Lie algebra $\mathfrak{g}$.

For the converse, let G be a Lie group with Lie algebra $\mathfrak{g}$ and a smooth exponential function $\exp_G\colon \mathfrak{g} \to G$; put $J_0 := \{j \in J\colon \mathfrak{s}_j \not\cong \mathfrak{sl}_2(\mathbb{R})\}$. For each $j \in J_0$, we then have morphisms $\alpha_j\colon S_j \to G$, $\beta_j\colon G \to S_j$ with $\beta_j \circ \alpha_j = \mathrm{id}_{S_j}$. If J_0 is infinite, this contradicts the local contractibility of G. □

Remark X.1.8. To describe all connected regular Lie groups G with a pro-finite Lie algebra $\mathfrak{g}$, we have to describe the discrete central subgroups of the 1-connected ones, which are isomorphic, as topological groups, to some $G := \Gamma(\mathfrak{g})$ (*cf.* [HoMo06]). In all these groups, the exponential function restricts to an isomorphism $\mathfrak{z}(\mathfrak{g}) \to Z(G)_0$ of topological groups, so that $Z(G)_0 \cong \mathbb{R}^X$ for some set X. Based on the information provided in the preceding theorem, it is shown in [HoMo06] that a subgroup $\Gamma \subseteq Z(\Gamma(\mathfrak{g}))_0$ is discrete if and only if it is finitely generated and its intersection with $Z(G)_0$ is discrete. This characterization provides a quite good description of all discrete subgroups of $Z(G)$, hence all non-simply connected regular Lie groups with Lie algebra $\mathfrak{g}$.

If $S \cong \prod_{j\in J} S_j$ and J is infinite, then infinitely many factors S_j are isomorphic to $\widetilde{\mathrm{SL}}_2(\mathbb{R})$, whose center is isomorphic to $\mathbb{Z}$. The subgroup of index 2 acts trivially in each finite-dimensional representation, which leads to $Z(G) \cap S_j \cong \mathbb{Z}$. Hence $Z(G) \cap S$ contains a subgroup isomorphic to $\mathbb{Z}^{J\setminus J_0}$, which implies in particular that the adjoint group of $\mathfrak{g}$ is *not* a Lie group. □

We have already seen that all pro-nilpotent Lie algebras are exponential, which applies in particular to all pro-nilpotent pro-finite Lie algebras. The following theorem provides a characterization of the locally exponential pro-Lie algebras ([HoNe06]):

Theorem X.1.9. *For a pro-Lie algebra* $\mathfrak{g}$, *the following are equivalent:*

(1) $\mathfrak{g}$ *is locally exponential.*
(2) There exists a 0*-neighborhood* $U \subseteq \mathfrak{g}$, *consisting of* exp*-regular points, i.e.,* $\kappa_{\mathfrak{g}}(x)$ *is invertible for each* $x \in U$.
(3) $\Gamma(\mathfrak{g})$ *is a locally exponential topological group.*

If these conditions are satisfied, then $\mathfrak{g}$ *contains a closed ideal of finite codimension which is exponential. In particular,* $\mathfrak{g}$ *is virtually pro-solvable, i.e.,* $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ *with a finite-dimensional semisimple Lie algebra* $\mathfrak{s}$, *and* $\mathfrak{g}$ *is enlargeable.* □

Recall that we have seen in Example II.5.9(a) an example of a pro-finite Lie algebra $\mathfrak{g} \cong \mathbb{R}^{\mathbb{N}} \rtimes_D \mathbb{R}$ which is not locally exponential. Since this Lie algebra has an abelian closed hyperplane, the existence of an exponential hyperplane ideal is not sufficient for local exponentiality.

X.2. Projective limits of infinite-dimensional Lie groups

As we have seen in the preceding Subsection X.1, the extent to which the structure theory of finite-dimensional Lie groups can be carried forward to projective limits is quite surprising. There are also natural classes of topological groups which are natural projective limits of infinite-dimensional Lie groups. Therefore it would be of some interest to develop a systematic "pro–Lie theory" for such groups.

One of the most natural classes of such groups are the mapping groups. Let M be a σ-compact finite-dimensional smooth manifold, $r \in \mathbb{N}_0 \cup \{\infty\}$, K a Lie group, and $G := C^r(M,K)$, endowed with the compact open C^r-topology (Definition II.2.7).

Then there exists a sequence $(M_n)_{n\in\mathbb{N}}$ of compact subsets of M which is an exhaustion, in the sense that $M_n \subseteq \mathrm{int}(M_{n+1})$. Using the usual Morse theoretic arguments, we may assume that the subsets M_n are compact manifolds with boundary. Then each compact subset of M is contained in some M_n, which implies that

$$G = C^r(M,K) \cong \varprojlim C^r(M_n,K)$$

is a projective limit, where the projection maps are given by restriction. In view of Theorem II.2.8, the groups $C^r(M_n,K)$ are Lie groups, so that the topological group $C^r(M,K)$ is a projective limit of Lie groups.

If K is locally exponential, then each $C^r(M_n,K)$ inherits this property, so that $C^r(M_n,K)$ is a topological group with Lie algebra (Definition IV.1.23), and this implies that $C^r(M,K)$ also is a topological group with Lie algebra, where

$$\mathfrak{L}(C^r(M,K)) \cong \varprojlim \mathfrak{L}(C^r(M_n,K)) = \varprojlim \mathbf{L}(C^r(M_n,K)) = \varprojlim C^r(M_n,\mathbf{L}(K))$$

$= C^r(M, \mathbf{L}(K)).$

As in the case of compact manifolds M, the exponential function of $C^r(M,K)$ is given by

$$\exp\colon C^r(M,\mathbf{L}(K)) \to C^r(M,K), \quad \xi \mapsto \exp_K \circ \xi.$$

If M is a compact complex manifold and K is a linear complex Lie group, then all holomorphic functions $M \to K$ are constant. Therefore the groups $\mathscr{O}(M,K)$ are trivial in this case. If M is non-compact and K is a complex Lie group, we use the compact open topology to turn $\mathscr{O}(M,K)$ into a topological group and observe that we have a topological embedding $\mathscr{O}(M,K) \hookrightarrow C^r(M,K)$ for each $r \in \mathbb{N}_0 \cup \{\infty\}$. Those cases for which we know to have honest Lie group structures on $\mathscr{O}(M,K)$ are quite limited (*cf.* Theorem III.1.9), but it seems that projective limit theory is also a useful tool to study these groups of holomorphic maps. Let $(M_n)_{n\in\mathbb{N}}$, as above, be an exhaustion of M by compact submanifolds with boundary. Then the groups $\mathscr{O}(M_n,K)$, defined appropriately, carry Lie group structures, for which $\mathscr{O}(M_n,\mathbf{L}(K))$ is the corresponding Lie algebra (*cf.* [Wo05b]), so that

$$\mathscr{O}(M,K) \cong \varprojlim \mathscr{O}(M_n,K)$$

is a projective limit of Lie groups.

It would be of considerable interest to find a good categorical framework for such classes of projective limits of Lie groups. Of particular relevance would be to understand the "right class" of central extensions of the groups $\mathscr{O}(M,K)$ and $C^r(M,K)$ in the same spirit as for the groups $C^\infty_c(M,K)$ of compactly supported maps (*cf.* [Ne04c]).

Open Problems for Section X

Problem X.1. Are strong ILB–Lie groups, resp., μ-regular Lie groups, topological groups with Lie algebra? What about diffeomorphism groups? Does it suffice that the Lie group G has a smooth exponential function (*cf.* Problem VII.2)? □

Problem X.2. Let $\mathfrak{g}$ be a pro-finite Lie algebra and $\mathfrak{n} \trianglelefteq \mathfrak{g}$ a closed exponential ideal of finite codimension. Characterize the local exponentiality of $\mathfrak{g}$ in terms of the spectra of the operators $\mathrm{ad}_{\mathfrak{n}} x := \mathrm{ad}\, x|_{\mathfrak{n}}$ (*cf.* Proposition X.1.9). Are all locally exponential pro-finite Lie algebras BCH? □

Problem X.3. For the description of the non-simply connected Lie groups among the projective limits of finite-dimensional Lie groups, it is important to understand the structure of the center of the simply connected ones. Let G be a such a 1-connected group and $Z(G)$ its center. The Lie algebra of $Z(G)$ is $\mathfrak{z}(\mathfrak{g})$, which lies in the pro-solvable radical, so that $Z(G)_0 \cong \mathfrak{z}(\mathfrak{g})$. On the other hand, we have seen in Remark X.1.8 that $Z(G)$ may contain non-discrete subgroups isomorphic to $\mathbb{Z}^{\mathbb{N}}$. Is it possible to determine the structure of $Z(G)$ as a topological group (see [HoMo06] for more details)? □

Problem X.4. Determine the automorphism groups of pro-finite Lie algebras. Under which conditions are they Lie groups? An interesting situation where the automorphism group of a pro-finite Lie algebra is a closed subgroup of a Lie groups is described in Theorem IX.2.4. □

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