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Multiplicative Integral Theory of Generalized Orthopair Fuzzy Sets and Its Applications

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Abstract. There are two main issues of fuzzy multi-attribute decision-making: determine the weight of each attribute and choose an appropriate aggregation method to integrate the evaluation information of different attributes. In order to solve the multi-attribute decision-making problem in generalized orthopair fuzzy environment with unknown attribute weights more effectively, we give a decision-making method based on generalized orthopair fuzzy definite integrals. To be specific, we first introduce the complement operations of q-rung orthopair fuzzy numbers, and then investigate the multiplicative q-rung orthopair fuzzy calculus. Through the complement operations, we establish the mutual conversion formula between additive and multiplicative q-rung orthopair fuzzy calculus theory. Then, we give a multiplicative integral-based q-rung orthopair fuzzy multi-attribute decision-making method, and discuss the relationship between the q-rung orthopair fuzzy definite integrals and the q-rung orthopair fuzzy weighted geometric operator. Compared with traditional decision-making methods, this method does not rely on subjective weight information, which is especially important when dealing with large sample data. Finally, the application of election is studied to verify the feasibility and effectiveness of the proposed method. With the introduction of generalized orthopair fuzzy sets, the expression form of election evaluation information has been expanded. We also provide some examples to compare the obtained results with the results generated by the addition operation and reveal the correlation between them.

Keywords: Fuzzy sets, decision making, aggregation operators

1. Introduction

Due to the limitations of decision makers' (DMs') cognitions and the complexity of the decision-making environment, uncertainty widely exists in actual decision-making problems. It is a very necessary and urgent problem to deal with uncertainty with appropriate theories and methods. Bellman and Zadeh (1970) first applied the method of fuzzy mathematics to multi-attribute decision-making problems. Yager (1977), Baas and Kwakernaak (1977) gave the earliest classical methods of fuzzy multi-attribute decision-making. Fuzzy sets provide an effective quantitative representation tool for dealing with the cognitive uncertain information faced in the decision-making process. Generalized orthopair fuzzy sets, also known as the q-rung orthopair fuzzy sets (q-ROFSs), are useful extensions of fuzzy sets (Yager 2017, Yager and Alajlan 2017). Because they can dynamically adjust the degree of hesitation when describing uncertain information through the parameter q, thus having wider constraints and stronger modeling capabilities, and are successfully applied to solve practical uncertain decision-making problems (Liu et al. 2020, Wei et al. 2018). Chen and Hwang (1992) systematically summarized the decision-making method in the fuzzy environment and pointed out that the multi-attribute decision-making method in the fuzzy environment can usually be divided into two steps: 1) The first step is to determine the importance degree of each attribute, that is, the attribute weight; 2) The second step is to select an information aggregation operator suitable for the given fuzzy data, and integrate the evaluation information and attribute weight information of different attributes. After that, we can use the fuzzy ranking method to sort and compare the fuzzy utility values, and finally obtain the optimal decision result.

As we know, the most important thing in the first step is to determine the attribute weight. The attribute weight and the fuzzy information corresponding to each attribute will be integrated into the fuzzy utility value of each scheme. Therefore, whether the weights are accurate and reliable will have an important impact on the decision-making results. How to scientifically, reasonably and objectively determine the weights of attributes in decision-making problems is an important and difficult issue in the research of fuzzy multi-attribute decision-making (Liu et al. 2018, Krishankumar et al. 2020). In the existing research, the methods for solving attribute weights can be roughly divided into three categories: 1) Subjective methods assume that the weights of attributes are determined using subjective preferences or empirical information provided by DMs. The most commonly used methods include AHP (Saaty 1977, Kou and Chang 2014) and Delphi (Hwang and Yoon 1981) methods. 2) Contrary to the subjective weight solution method, the weight calculation in the objective method completely relies on the existing decision data and the use of various mathematical models to model and solve the attribute weights, mainly including multi-objective programming models (Xu 2010 2011) , the information entropy-based weight solution methods (Chen and Li 2010, Xia and Xu 2012) and attribute weight solving method based on ideal points (Xu 2007, Zhao 2016), etc. 3)Hybrid weight solving methods, such as TOPSIS-based hybrid weighting method (Wang and Lee 2009), entropy-based hybrid weighting method(Li et al. 2015), PSO-based weighting method (Nabavi et al. 2010) and sparse set method (Zhang and Zhou 2011), etc. However, the above-mentioned methods still have some shortcomings. For example, the subjective weighting method relies heavily on the subjective judgment and empirical information of the DMs, and the weighting result is subjective and easily affected by DMs' personal evaluations. Most objective weighting methods and hybrid weighting methods rely on multi-objective programming models, which will increase the computational complexity and reduce efficiency and timeliness. These defects are more prominent when dealing with large-scale data.

Aiming at the second step of fuzzy multi-attribute decision-making problems, most of the existing researches on q-rung orthopair fuzzy multi-attribute decision-making methods focus on how to use various q-rung orthopair fuzzy aggregation operators for information integration. Such as the q-rung orthopair fuzzy weighted averaging operator and the q-rung orthopair fuzzy weighted geometric operator (Liu and Wang 2018); the q-rung orthopair fuzzy ordered weighted averaging aggregation operator (Yager et al. 2018); the fuzzy aggregation operator based on Choquet integral (Yager et al. 2018); the q-rung orthopair fuzzy Muirhead mean operator (Wang et al. 2019); the q-rung orthopair fuzzy dual Maclaurin symmetric mean operator(Wei et al. 2019); the q-rung orthopair fuzzy Bonferroni mean operator (Liu and Liu 2018); the q-rung orthopair fuzzy Archimedean Bonferroni mean operator (Liu and Wang 2018); the q-rung orthopair fuzzy partitioned Bonferroni mean operator (Yang and Pang 2019), etc. However, since the attribute weight and the q-rung orthopair fuzzy information corresponding to each attribute will be directly fused during the integration process, the decision results obtained by these information integration methods are greatly affected by the weight information. Besides, these studies have focused only on the aggregation of relatively few and discrete data. Although some scholars have conducted research on the integration of continuous information in an intuitionistic fuzzy environment (Lei and Xu 2015 2016, Ai and Xu 2017), these studies rarely involve q-rung orthopair fuzzy information (Shu et al. 2019, Shao and Zhuo 2020), especially in q-rung orthopair fuzzy multi-attribute decision-making problems under multiplicative operations.

Therefore, this study focuses on overcoming the shortcomings mentioned in the above two steps, and the contributions of this paper can be summarized as follows:

1) We introduce some complement operations of q-rung orthopair fuzzy numbers (q-ROFNs) and properties of q-rung orthopair fuzzy functions (q-ROFFs)' derivatives. They are the basis for further research on the multiplicative integrals of q-ROFFs.

2) We propose the integral theory of q-ROFFs under their multiplication operations, which is parallel of the additive integrals (Gao et al. 2020). We also intend to study the correlations between them, which will make the q-rung orthopair fuzzy calculus (q-ROFC) theory more systematic and complete. Compared with the previous research in the intuitionistic fuzzy environment (Lei and Xu 2015 2016, Ai and Xu 2017), our work is completely nonlinear, so it has different operations and properties in mathematics. In addition, because q-ROFF is applicable to all ranges of $q \in [1, \infty)$, it has a wider range of applications and is more suitable for describing multi-attribute decision-making problems.

3) We are trying to propose an information integration method based on multiplicative integrals, which has a strong mathematical and theoretical foundation. In addition, the relationship between q-ROFF's definite integral and q-rung orthopair fuzzy weighted geometric (q-ROFWG) operator is discussed. The former one focuses on processing and collecting the position distribution information of q-rung orthopair fuzzy information, so it is more suitable for processing information integration problems with unknown weights. Actually, by defining appropriate integrands and selecting appropriate integral limits, we can use multiplicative q-ROFIs to fuse continuous and large-scale q-rung orthopair fuzzy information, which greatly improves the time efficiency. This is extremely important in multi-attribute decision-making problems dealing with big data.

In order to do that, the remainder of this paper is set up as follows: We give some preparations for the whole work in Section 2. In Section 3, we investigate the multiplication q-ROFIs, including the indefinite and definite integrals. And then we focus on studying the correlation between the multiplicative and additive integrals of the q-ROFFs. In Section 4, we discuss the relationship between the q-rung orthopair fuzzy definite integral and the q-ROFWG operator, and investigate information integration based on q-rung orthopair fuzzy multiplicative integrals. In Section 5, in order to verify the effectiveness and superiority of the proposed method, it is applied to the voting decision-making, and a set of mutual conversion methods between additive calculus and multiplication calculus is established to show the relationship between them. Finally, we end the paper with some concluding remarks in Section 6.

2. Some Concepts Related to the q-ROFS

In this section, we review and define some related concepts, which will be frequently used throughout this paper.

Definition 1 (Yager 2017). *Let X be a given non-empty set, then a q-ROFS $\mathbb{A}$ has the form:*

$$\mathbb{A} = \{\langle x, \mu_{\mathbb{A}}(x), v_{\mathbb{A}}(x)\rangle \mid x \in X\}$$

where $\mu_{\mathbb{A}} : X \longrightarrow [0,1]$ *is defined as the membership function and the function* $v_{\mathbb{A}} : X \longrightarrow [0,1]$ *is defined as the nonmembership function. For any* $x \in X$, *we have*

$$0 \leq \left(\mu_{\mathbb{A}}(x)\right)^q + (v_{\mathbb{A}}(x))^q \leq 1, \quad q \geq 1$$

Moreover, $\mu_{\mathbb{A}}(x)$ and $v_{\mathbb{A}}(x)$ respectively represent the membership degree and the nonmembership degree of x in X. For convenience, we consider $\langle \mu, v \rangle$ as a q-ROFN.

Definition 2 (Liu and Wang 2018). *Let* $q \geq 1$ *and* $\alpha = \langle \mu_\alpha, v_\alpha \rangle$, $\beta = \langle \mu_\beta, v_\beta \rangle$ *be two q-ROFNs. Then, the addition and multiplication operations between them are defined as:*

$$\alpha \oplus \beta = \left\langle \left(\mu_\alpha^q + \mu_\beta^q - \mu_\alpha^q \mu_\beta^q\right)^{\frac{1}{q}}, v_\alpha v_\beta \right\rangle \quad (1)$$

$$\alpha \otimes \beta = \left\langle \mu_\alpha \mu_\beta, \left(v_\alpha^q + v_\beta^q - v_\alpha^q v_\beta^q\right)^{\frac{1}{q}} \right\rangle \quad (2)$$

Based on (1)-(2), we give the following definition:

Definition 3 (Gao et al. 2019) . *Let* α *and* β *be as mentioned in Definition 2. We have*

$$\begin{aligned} \beta \ominus \alpha &= \left\langle \left(\frac{\mu_\beta^q - \mu_\alpha^q}{1 - \mu_\alpha^q}\right)^{\frac{1}{q}}, \frac{v_\beta}{v_\alpha} \right\rangle, \\ \text{where } 0 &\leq \frac{v_\beta^q}{v_\alpha^q} \leq \frac{(1 - \mu_\beta^q)}{(1 - \mu_\alpha^q)} \leq 1 \end{aligned} \quad (3)$$

$$\begin{aligned} \beta \oslash \alpha &= \left\langle \frac{\mu_\beta}{\mu_\alpha}, \left(\frac{v_\beta^q - v_\alpha^q}{1 - v_\alpha^q}\right)^{\frac{1}{q}} \right\rangle, \\ \text{where } 0 &\leq \frac{\mu_\beta^q}{\mu_\alpha^q} \leq \frac{(1 - v_\beta^q)}{(1 - v_\alpha^q)} \leq 1 \end{aligned} \quad (4)$$

Furthermore, for $\lambda \geq 0$,

$$\begin{cases} \lambda \alpha = \left\langle (1 - (1 - \mu_\alpha^q)^\lambda)^{\frac{1}{q}}, v_\alpha^\lambda \right\rangle \\ \alpha^\lambda = \left\langle \mu_\alpha^\lambda, (1 - (1 - v_\alpha^q)^\lambda)^{\frac{1}{q}} \right\rangle \end{cases} \quad (5)$$

The restrictions imposed in (3) and (4) guarantee the legitimacy of the difference and quotient operations.

In order to compare and rank q-ROFNs, we next introduce some partial orders of the q-ROFNs:

Definition 4 (Gao et al. 2019). *Let* $\alpha_i = \langle \mu_{\alpha_i}, v_{\alpha_i} \rangle$ $i = 1, 2$ *be q-ROFNs and* $\mathbb{S}$ *be a set, which consists of all the q-ROFNs. Then there are partial orders defined in the set* $\mathbb{S}$ *as follows:*

- $\alpha_1 \leq \alpha_2$ *if* $\mu_{\alpha_1} \leq \mu_{\alpha_2}$ *and* $v_{\alpha_1} \geq v_{\alpha_2}$
- $\alpha_1 \geq \alpha_2$ *if* $\mu_{\alpha_1} \geq \mu_{\alpha_2}$ *and* $v_{\alpha_1} \leq v_{\alpha_2}$

Definition 5 *Let* $\alpha_i = \langle \mu_{\alpha_i}, v_{\alpha_i} \rangle$ *be given as in Definition 4.*

(i). If $\alpha_1 \otimes \alpha_3 = \alpha_2$, *then we denote by* $\alpha_1 \gtrapprox \alpha_2$. *Particularly,* $\alpha_1 \gtrsim \alpha_2$ *when* $\alpha_3 \neq \langle 1, 0 \rangle$.

(ii). If $\alpha_1 \oplus \alpha_3 = \alpha_2$, *then we denote by* $\alpha_1 \trianglelefteq \alpha_2$. *Particularly,* $\alpha_1 \triangleleft \alpha_2$ *when* $\alpha_3 \neq \langle 0, 1 \rangle$.

Then, we propose the following lemma which is concerned with the complement operations of q-ROFNs:

Lemma 1 *Let* α *and* α_i *be q-ROFNs and* $\mathbb{S}$ *be the set of all the q-ROFNs, and let* $\mathbb{S}_{\oplus}(\alpha) = \left\{\beta \oplus \alpha \in \mathbb{S} \,|\, \beta \in \mathbb{S}\right\}$ *and* $\mathbb{S}_{\otimes}(\alpha) = \left\{\beta \otimes \alpha \in \mathbb{S} \,|\, \beta \in \mathbb{S}\right\}$ *be the additive set and the multiplicative set. Also we introduce*

$$\overline{\alpha} = \overline{\langle \mu_\alpha, v_\alpha \rangle} := \langle v_\alpha, \mu_\alpha \rangle \quad (6)$$

Then,

(i). $\overline{\alpha_1 \oplus \alpha_2} = \overline{\alpha_1} \otimes \overline{\alpha_2}$, $\overline{\alpha_1 \ominus \alpha_2} = \overline{\alpha_1} \oslash \overline{\alpha_2}$, $\overline{\lambda \alpha} = \overline{\alpha}^\lambda$,

(ii). $\overline{\mathbb{S}_{\oplus}(\alpha)} = \mathbb{S}_{\otimes}(\overline{\alpha})$, $\overline{\mathbb{S}_{\ominus}(\alpha)} = \mathbb{S}_{\oslash}(\overline{\alpha})$.

Proof of Lemma 1. The first part in (i). It follows from Definition 2 and (6) that

$$\begin{aligned} \overline{\alpha_1 \oplus \alpha_2} &= \overline{\left\langle \left(\mu_{\alpha_1}^q + \mu_{\alpha_2}^q - \mu_{\alpha_1}^q \mu_{\alpha_2}^q\right)^{\frac{1}{q}}, v_{\alpha_1} v_{\alpha_2} \right\rangle} \\ &= \left\langle v_{\alpha_1} v_{\alpha_2}, \left(\mu_{\alpha_1}^q + \mu_{\alpha_2}^q - \mu_{\alpha_1}^q \mu_{\alpha_2}^q\right)^{\frac{1}{q}} \right\rangle \\ &= \langle v_{\alpha_1}, \mu_{\alpha_1} \rangle \otimes \langle v_{\alpha_2}, \mu_{\alpha_2} \rangle = \overline{\alpha_1} \otimes \overline{\alpha_2} \end{aligned}$$

The middle equality is similar. Let us turn to the last one in (i). Again using Definition 3 and (6), we have

$$\begin{aligned}\overline{\lambda\alpha} &= \overline{\left\langle (1-(1-\mu_\alpha^q)^\lambda)^{\frac{1}{q}}, v_\alpha^\lambda \right\rangle}\\ &= \left\langle v_\alpha^\lambda, (1-(1-\mu_\alpha^q)^\lambda)^{\frac{1}{q}} \right\rangle\\ &= \langle v_\alpha, \mu_\alpha\rangle^\lambda = \overline{\alpha}^\lambda\end{aligned}$$

The proof of (ii). For any $\beta \in \overline{\mathbb{S}_\oplus(\alpha)}$, there exists $\gamma \in \mathbb{S}$ such that $\beta = \overline{\alpha \oplus \gamma} = \overline{\alpha} \otimes \overline{\gamma}$. Hence $\beta \in \mathbb{S}_\otimes(\overline{\alpha})$, and then

$$\overline{\mathbb{S}_\oplus(\alpha)} \subseteq \mathbb{S}_\otimes(\overline{\alpha}) \tag{7}$$

Moreover, if $\beta \in \mathbb{S}_\otimes(\overline{\alpha})$, then we have $\beta = \overline{\alpha} \otimes \overline{\gamma} = \overline{\alpha \oplus \gamma}$ for some $\overline{\gamma} \in \mathbb{S}$. Thus, $\overline{\beta} = \alpha \oplus \gamma$, which implies $\overline{\beta} \in \mathbb{S}_\oplus(\alpha)$, and clearly, $\beta \in \overline{\mathbb{S}_\oplus(\alpha)}$. Therefore,

$$\mathbb{S}_\otimes(\overline{\alpha}) \subseteq \overline{\mathbb{S}_\oplus(\alpha)} \tag{8}$$

The combination of (7) and (8) yields the first part of (ii). The argument of latter one runs similarly. ■

For the readers to understand more intuitively, we recall the addition region of α_0 and the multiplication region of β_0 , as shown in Figure 1 (Gao et al. 2019). Next, with complement concept in Lemma 1, we explain connection between them from a geometric perspective. We see that, by (6), $\overline{\alpha}$ is the reflection of α along the line $\mu = v$. See Figure 2 below. This implies that the region $\mathbb{S}_\oplus(\alpha)$ is perfectly symmetric with $\overline{\mathbb{S}_\oplus(\bar{\alpha})}$ along the line $\mu = v$.

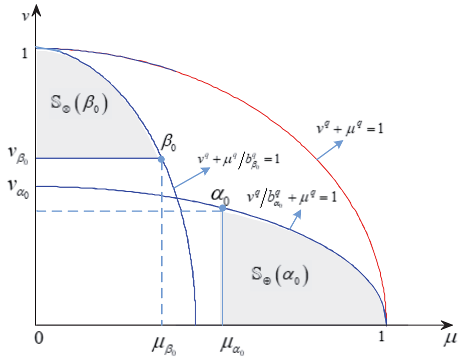

Figure 1 $\mathbb{S}_\oplus(\alpha_0)$ with $b_{\alpha_0}^q = \frac{v_{\alpha_0}^q}{1-\mu_{\alpha_0}^q}$ and $\mathbb{S}_\otimes(\beta_0)$ with $b_{\beta_0}^q = \frac{\mu_{\beta_0}^q}{1-v_{\beta_0}^q}$

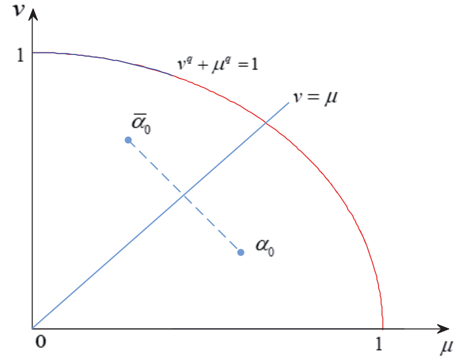

Figure 2 Complement of α_0

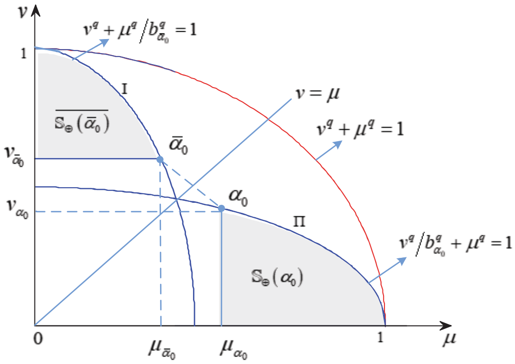

Figure 3 Complement of $\mathbb{S}_\oplus(\alpha_0)$

The segments of the ellipses I and II, namely $v^q/b_{\alpha_0}^q + \mu^q = 1$ and $v^q + \mu^q/b_{\bar{\alpha}_0}^q = 1$, where $b_{\alpha_0}^q = v_{\alpha_0}^q/(1-\mu_{\alpha_0}^q)$, $b_{\bar{\alpha}_0}^q = v_{\bar{\alpha}_0}^q/(1-\mu_{\bar{\alpha}_0}^q)$, are also symmetric with each other along the line $\mu = v$. In other words, if we switch the components v and μ, then I becomes II. Therefore, we have $\mathbb{S}_\otimes(\beta) = \mathbb{S}_\otimes(\bar{\alpha}) = \overline{\mathbb{S}_\oplus(\alpha)}$, as long as $\beta = \bar{\alpha}$, can be shown in Figure 3. The same argument explains the relationship of $\overline{\mathbb{S}_\ominus(\alpha)} = \mathbb{S}_\oslash(\overline{\alpha})$.

Definition 6 (Gao et al. 2019). *Let the continuous functions of multi-variables be*

$$\begin{aligned} f(\mu_\alpha, v_\alpha) &: [0,1]\times[0,1] \mapsto [0,1]\\ g(\mu_\alpha, v_\alpha) &: [0,1]\times[0,1] \mapsto [0,1]\end{aligned} \tag{9}$$

Then the function

$$\begin{aligned}&\varphi(\alpha) = \langle f(\mu_\alpha, v_\alpha), g(\mu_\alpha, v_\alpha)\rangle\\ &with \quad 0 \le f^q + g^q \le 1\end{aligned} \tag{10}$$

is called a continuous q-ROFF in terms of f and g.

3. Multiplicative Derivative and Differential Operations of q-ROFFs

First, we analyze the derivative operations of q-ROFFs based on the multiplication. For the sake of convenience, we explain some terminologies used in this section. For a given function of two variables $f(x,y)$, we denote by $\frac{\partial f(x,y)}{\partial x}$ its partial derivative with respect to x. In addition, we use $\frac{\beta}{\alpha} := \beta \oslash \alpha$ for simplicity. When it comes to a limit of q-ROFF $\varphi(\alpha) = \langle f(\mu_\alpha, v_\alpha), g(\mu_\alpha, v_\alpha)\rangle$ in multiplication (quotient), we mean

$$\lim_{\beta\oslash\alpha\to\langle 1,0\rangle} \varphi(\alpha) = \left\langle \lim_{\mu_\beta\to\mu_\alpha, v_\beta\to v_\alpha} f(\mu_\beta, v_\beta), \lim_{\mu_\beta\to\mu_\alpha, v_\beta\to v_\alpha} g(\mu_\beta, v_\beta)\right\rangle$$

where the right-hand side of quantity is the standard limits in calculus.

Definition 7 *Let the q-ROFF $\varphi(\alpha)$ be given as in Definition 6. If the quantity*

$$\lim_{\beta\oslash\alpha\to\langle 1,0\rangle} \frac{\varphi(\beta)}{\varphi(\alpha)} \ominus \frac{\beta}{\alpha}$$

is still a q-ROFN, then we say that φ admits a derivative at α, which is denoted by

$$\frac{\mathcal{D}\varphi(\alpha)}{\mathcal{D}\alpha} := \lim_{\beta\oslash\alpha\to\langle 1,0\rangle} \frac{\varphi(\beta)}{\varphi(\alpha)} \ominus \frac{\beta}{\alpha} \tag{11}$$

The following theorem shows in what conditions a q-ROFF admits its derivative under multiplication arithmetic.

Theorem 1 *Assume that the q-ROFF*

$$\varphi(\alpha) = \langle f(\mu_\alpha, v_\alpha),\ g(\mu_\alpha, v_\alpha)\rangle = \langle f_\alpha,\ g_\alpha\rangle$$

as defined in Definition 7. Then $\varphi(\alpha)$ has a derivative of the form

$$\frac{\mathcal{D}\varphi(\alpha)}{\mathcal{D}\alpha} = \left\langle \left(1 - \frac{\mu_\alpha}{f_\alpha}\frac{\partial f_\alpha}{\partial \mu_\alpha}\right)^{\frac{1}{q}}, \left(\frac{(1-v_\alpha^q)g_\alpha^{q-1}}{(1-g_\alpha^q)v_\alpha^{q-1}}\frac{\partial g_\alpha}{\partial v_\alpha}\right)^{\frac{1}{q}}\right\rangle \tag{12}$$

if and only if $\frac{\partial f_\alpha}{\partial \mu_\alpha}$, $\frac{\partial g_\alpha}{\partial v_\alpha}$ exist, and the following conditions hold true

$$\begin{aligned} &\frac{\partial f_\alpha}{\partial v_\alpha} = 0 = \frac{\partial g_\alpha}{\partial \mu_\alpha},\\ &0 \le \frac{(1-v_\alpha^q)}{(1-g_\alpha^q)}\frac{\partial g_\alpha^q}{\partial v_\alpha^q} \le \frac{\mu_\alpha}{f_\alpha}\frac{\partial f_\alpha}{\partial \mu_\alpha} \le 1 \end{aligned} \tag{13}$$

Remark 1. *The latter part of (13) is to guarantee that if $\frac{\mathcal{D}\varphi(\alpha)}{\mathcal{D}\alpha}$ exists, it is still a q-ROFF.*

Remark 2. *In particular, if $\varphi(\alpha) = \langle \mu_\alpha,\ v_\alpha\rangle$. The formula (12) becomes*

$$\frac{\mathcal{D}\varphi(\alpha)}{\mathcal{D}\alpha} = \langle 0, 1\rangle$$

Proof of Theorem 1. According to (3) and (4), we have

$$\begin{aligned} \frac{\mathcal{D}\varphi(\alpha)}{\mathcal{D}\alpha} &= \lim_{\beta\oslash\alpha\to\langle 1,0\rangle} \frac{\varphi(\beta)}{\varphi(\alpha)} \ominus \frac{\beta}{\alpha}\\ &= \lim_{\beta\oslash\alpha\to\langle 1,0\rangle} \left\langle \frac{f_\beta}{f_\alpha}, \left(\frac{g_\beta^q - g_\alpha^q}{1-g_\alpha^q}\right)^{\frac{1}{q}}\right\rangle \ominus \left\langle \frac{\mu_\beta}{\mu_\alpha}, \left(\frac{v_\beta^q - v_\alpha^q}{1-v_\alpha^q}\right)^{\frac{1}{q}}\right\rangle\\ &= \lim_{\beta\oslash\alpha\to\langle 1,0\rangle} \left\langle \left(\frac{\frac{f_\beta^q}{f_\alpha^q} - \frac{\mu_\beta^q}{\mu_\alpha^q}}{1-\frac{\mu_\beta^q}{\mu_\alpha^q}}\right)^{\frac{1}{q}}, \left(\frac{(g_\beta^q - g_\alpha^q)(1-v_\alpha^q)}{(1-g_\alpha^q)(v_\beta^q - v_\alpha^q)}\right)^{\frac{1}{q}}\right\rangle\\ &= \left\langle \lim_{\beta\oslash\alpha\to\langle 1,0\rangle} I, \lim_{\beta\oslash\alpha\to\langle 1,0\rangle} II\right\rangle \end{aligned} \tag{14}$$

We first consider I. By the continuity of f and μ, and the simple fact

$$a^q - b^q = (a-b)\left(\sum_{i=0}^{q-1} a^{q-1-i}b^i\right) \tag{15}$$

we compute

$$\lim_{\beta\oslash\alpha\to\langle 1,0\rangle}\frac{\frac{f_\beta^q}{f_\alpha^q}-\frac{\mu_\beta^q}{\mu_\alpha^q}}{1-\frac{\mu_\beta^q}{\mu_\alpha^q}}$$

$$=\frac{1}{f_\alpha^q}\lim_{\beta\oslash\alpha\to\langle 1,0\rangle}\left(\frac{f_\beta^q\mu_\alpha^q-f_\alpha^q\mu_\beta^q}{\mu_\alpha^q-\mu_\beta^q}\right)$$

$$=\frac{\mu_\alpha^q}{f_\alpha^q}\lim_{\beta\oslash\alpha\to\langle 1,0\rangle}\left(\frac{f_\beta^q-f^q(\mu_\beta,v_\alpha)}{\mu_\alpha^q-\mu_\beta^q}\right)$$

$$+\frac{\mu_\alpha^q}{f_\alpha^q}\lim_{\beta\oslash\alpha\to\langle 1,0\rangle}\left(\frac{f^q(\mu_\beta,v_\alpha)-f_\alpha^q}{\mu_\alpha^q-\mu_\beta^q}\right)+1$$

$$=-\frac{\mu_\alpha}{f_\alpha}\frac{\partial f_\alpha}{\partial v_\alpha}\cos^{-1}(\theta,\alpha)-\frac{\mu_\alpha}{f_\alpha}\frac{\partial f_\alpha}{\partial\mu_\alpha}+1$$

with $\cos(\theta,\alpha):=\lim_{\beta\oslash\alpha\to\langle 1,0\rangle}\frac{v_\beta-v_\alpha}{\mu_\beta-\mu_\alpha}$. Hence,

$$\lim_{\beta\oslash\alpha\to\langle 1,0\rangle} I=\left(1-\frac{\mu_\alpha}{f_\alpha}\left(\frac{\partial f_\alpha}{\partial\mu_\alpha}+\frac{\partial f_\alpha}{\partial v_\alpha}\cos^{-1}(\theta,\alpha)\right)\right)^{\frac{1}{q}}$$

$$=\left(1-\frac{\mu_\alpha}{f_\alpha}\frac{\partial f_\alpha}{\partial\mu_\alpha}\right)^{\frac{1}{q}} \tag{16}$$

where the last equality sign owes to (13).

Utilizing (15) once more,

$$\lim_{\beta\oslash\alpha\to\langle 1,0\rangle}\frac{g_\beta^q-g_\alpha^q}{v_\beta^q-v_\alpha^q}$$

$$=\frac{g_\alpha^{q-1}}{v_\alpha^{q-1}}\lim_{\beta\oslash\alpha\to\langle 1,0\rangle}\frac{g_\beta-g_\alpha}{v_\beta-v_\alpha}$$

$$=\frac{g_\alpha^{q-1}}{v_\alpha^{q-1}}\lim_{\beta\oslash\alpha\to\langle 1,0\rangle}\frac{g_\beta-g(\mu_\alpha,v_\beta)}{\mu_\beta-\mu_\alpha}\frac{\mu_\beta-\mu_\alpha}{v_\beta-v_\alpha} \tag{17}$$

$$+\frac{g_\alpha^{q-1}}{v_\alpha^{q-1}}\lim_{\beta\oslash\alpha\to\langle 1,0\rangle}\frac{g(\mu_\alpha,v_\beta)-g_\alpha}{v_\beta-v_\alpha}$$

$$=\frac{g_\alpha^{q-1}}{v_\alpha^{q-1}}\left(\frac{\partial g_\alpha}{\partial\mu_\alpha}\cos^{-1}(\theta,\alpha)+\frac{\partial g_\alpha}{\partial v_\alpha}\right)$$

With (17) in hand, one has

$$\lim_{\beta\oslash\alpha\to\langle 1,0\rangle} II$$

$$=\left(\frac{(1-v_\alpha^q)}{(1-g_\alpha^q)}\lim_{\beta\oslash\alpha\to\langle 1,0\rangle}\frac{g_\beta^q-g_\alpha^q}{v_\beta^q-v_\alpha^q}\right)^{\frac{1}{q}}$$

$$=\left(\frac{(1-v_\alpha^q)}{(1-g_\alpha^q)}\frac{g_\alpha^{q-1}}{v_\alpha^{q-1}}\left(\frac{\partial g_\alpha}{\partial v_\alpha}+\frac{\partial g_\alpha}{\partial\mu_\alpha}\cos^{-1}(\theta,\alpha)\right)\right)^{\frac{1}{q}}$$

$$=\left(\frac{(1-v_\alpha^q)g_\alpha^{q-1}}{(1-g_\alpha^q)v_\alpha^{q-1}}\frac{\partial g_\alpha}{\partial v_\alpha}\right)^{\frac{1}{q}} \tag{18}$$

where in the last equality sign we again used (13).

Inserting the expressions (16) and (18) into (14), we get the desired (12). Observe that all the deductions above are inverse, we complete the proof of Theorem 1. ■

Theorems 2-3 below are for the basic operations on the derivatives of q-ROFFs.

Theorem 2 *Assume that the q-ROFFs*

$$\varphi_i(\alpha)=\langle f_i(\mu_\alpha,v_\alpha),\ g_i(\mu_\alpha,v_\alpha)\rangle$$
$$=\langle f_i,g_i\rangle\quad(i=1,2)$$

satisfy the assumptions listed in Theorem 1. Then,

$$\frac{\mathcal{D}}{\mathcal{D}\alpha}(\varphi_1(\alpha)\otimes\varphi_2(\alpha))$$
$$=\left\langle\left(1-\frac{\mu_\alpha}{f_1}\frac{\partial f_1}{\partial\mu_\alpha}-\frac{\mu_\alpha}{f_2}\frac{\partial f_2}{\partial\mu_\alpha}\right)^{\frac{1}{q}},\right. \tag{19}$$
$$\left.\left(\frac{(1-v_\alpha^q)}{(1-g_1^q)}\frac{\partial g_1^q}{\partial v_\alpha^q}+\frac{(1-v_\alpha^q)}{(1-g_2^q)}\frac{\partial g_2^q}{\partial v_\alpha^q}\right)^{\frac{1}{q}}\right\rangle$$

and

$$\frac{\mathcal{D}}{\mathcal{D}\alpha}(\varphi_1(\alpha)\oslash\varphi_2(\alpha))$$
$$=\left\langle\left(1-\frac{\mu_\alpha}{f_1}\frac{\partial f_1}{\partial\mu_\alpha}+\frac{\mu_\alpha}{f_2}\frac{\partial f_2}{\partial\mu_\alpha}\right)^{\frac{1}{q}},\right. \tag{20}$$
$$\left.\left(\frac{(1-v_\alpha^q)}{(1-g_1^q)}\frac{\partial g_1^q}{\partial v_\alpha^q}-\frac{(1-v_\alpha^q)}{(1-g_2^q)}\frac{\partial g_2^q}{\partial v_\alpha^q}\right)^{\frac{1}{q}}\right\rangle$$

Proof of Theorem 2. Define $\mathbb{G} = 1 - g_1^q - g_2^q + g_1^q g_2^q$. By Definition 2 and Theorem 1, one has

$$\begin{aligned}
&\frac{\mathcal{D}}{\mathcal{D}\alpha}(\varphi_1(\alpha) \otimes \varphi_2(\alpha)) \\
&= \frac{\mathcal{D}}{\mathcal{D}\alpha}\left\langle f_1 f_2, (1-\mathbb{G})^{\frac{1}{q}} \right\rangle \\
&= \left\langle \left(1 - \frac{\mu_\alpha}{f_1 f_2}\frac{\partial(f_1 f_2)}{\partial \mu_\alpha}\right)^{\frac{1}{q}}, \right. \\
&\qquad \left. \left(\frac{1-v_\alpha^q}{\mathbb{G}}\left(\frac{1-\mathbb{G}}{v_\alpha^q}\right)^{\frac{q-1}{q}} \frac{\partial}{\partial v_\alpha}(1-\mathbb{G})^{\frac{1}{q}}\right)^{\frac{1}{q}} \right\rangle \\
&= \left\langle \left(1 - \frac{\mu_\alpha}{f_1 f_2}\frac{\partial(f_1 f_2)}{\partial \mu_\alpha}\right)^{\frac{1}{q}}, \right. \\
&\quad \left(\frac{1-v_\alpha^q}{\mathbb{G}v_\alpha^{q-1}}\left((1-g_2^q)g_1^{q-1}\frac{\partial g_1}{\partial v_\alpha}\right.\right. \\
&\quad \left.\left.\left. +(1-g_1^q)g_2^{q-1}\frac{\partial g_2}{\partial v_\alpha}\right)\right)^{\frac{1}{q}} \right\rangle \\
&= \left\langle \left(1 - \frac{\mu_\alpha}{f_1}\frac{\partial f_1}{\partial \mu_\alpha} - \frac{\mu_\alpha}{f_2}\frac{\partial f_2}{\partial \mu_\alpha}\right)^{\frac{1}{q}}, \right. \\
&\qquad \left. \left(\frac{(1-v_\alpha^q)}{(1-g_1^q)}\frac{\partial g_1^q}{\partial v_\alpha^q} + \frac{(1-v_\alpha^q)}{(1-g_2^q)}\frac{\partial g_2^q}{\partial v_\alpha^q}\right)^{\frac{1}{q}} \right\rangle
\end{aligned}$$

Now let us turn to (20). A tedious computation gives

$$\begin{aligned}
&\frac{\mathcal{D}}{\mathcal{D}\alpha}(\varphi_1(\alpha) \oslash \varphi_2(\alpha)) \\
&= \frac{\mathcal{D}}{\mathcal{D}\alpha}\left\langle \frac{f_1}{f_2}, \left(\frac{g_1^q - g_2^q}{1-g_2^q}\right)^{\frac{1}{q}} \right\rangle \\
&= \left\langle \left(1 - \frac{\mu_\alpha f_2}{f_1}\frac{\partial}{\partial \mu_\alpha}\left(\frac{f_1}{f_2}\right)\right)^{\frac{1}{q}}, \right. \\
&\qquad \left(\frac{1-v_\alpha^q}{1-\frac{g_1^q-g_2^q}{1-g_2^q}}\left(\frac{g_1^q-g_2^q}{(1-g_2^q)v_\alpha^q}\right)^{\frac{q-1}{q}}\right. \\
&\qquad \left.\left. \frac{\partial}{\partial v_\alpha}\left(\frac{g_1^q-g_2^q}{1-g_2^q}\right)^{\frac{1}{q}}\right)^{\frac{1}{q}} \right\rangle \\
&= \left\langle \left(1 - \frac{\mu_\alpha}{f_1}\frac{\partial f_1}{\partial \mu_\alpha} + \frac{\mu_\alpha}{f_2}\frac{\partial f_2}{\partial \mu_\alpha}\right)^{\frac{1}{q}}, \right. \\
&\quad \left(\frac{(1-v_\alpha^q)}{1-g_1^q}\cdot\frac{1}{v_\alpha^{q-1}}(g_1^{q-1}\frac{\partial g_1}{\partial v_\alpha} - g_2^{q-1}\frac{\partial g_2}{\partial v_\alpha}) + \right. \\
&\qquad \left.\left. \frac{(1-v_\alpha^q)}{1-g_1^q}\frac{g_1^q-g_2^q}{v_\alpha^{q-1}}\frac{g_2^{q-1}\frac{\partial g_2}{\partial v_\alpha}}{1-g_2^q}\right)^{\frac{1}{q}} \right\rangle \\
&= \left\langle \left(1 - \frac{\mu_\alpha}{f_1}\frac{\partial f_1}{\partial \mu_\alpha} + \frac{\mu_\alpha}{f_2}\frac{\partial f_2}{\partial \mu_\alpha}\right)^{\frac{1}{q}}, \right. \\
&\qquad \left. \left(\frac{(1-v_\alpha^q)}{(1-g_1^q)}\frac{\partial g_1^q}{\partial v_\alpha^q} - \frac{(1-v_\alpha^q)}{(1-g_2^q)}\frac{\partial g_2^q}{\partial v_\alpha^q}\right)^{\frac{1}{q}} \right\rangle
\end{aligned}$$

This is (20). The proof of Theorem 2 is finished. ■

Corollary 1 *For any constant q-ROFN $C = \langle c_1, c_2 \rangle$ with $c_i (i = 1, 2)$ being the constants, one has*

$$\frac{\mathcal{D}}{\mathcal{D}\alpha}(\varphi(\alpha) \otimes C) = \frac{\mathcal{D}}{\mathcal{D}\alpha}(\varphi(\alpha) \oslash C) = \frac{\mathcal{D}\varphi(\alpha)}{\mathcal{D}\alpha}$$

Proof of Corollary 1. The proof follows directly from Theorem 2 and Remark 2. ■

Theorem 3 (Compound q-ROFFs derivation law) *Assume that the q-ROFFs $\varphi(\alpha)$ and $\phi(\alpha)$ are defined as in Definition 6. ,Then the following equality sign holds true*

$$\frac{\mathcal{D}\varphi(\phi(\alpha))}{\mathcal{D}\alpha} = \frac{\mathcal{D}\varphi}{\mathcal{D}\phi} \oplus \frac{\mathcal{D}\phi}{\mathcal{D}\alpha}$$

provided that the derivatives exist.

Proof of Theorem 3. The proof of Theorem 3 is from Definition 2 and Theorem 1. ■

4. Indefinite Integrals of q-ROFFs under Multiplication

In the light of elementary calculus, we present the definition of primitive of q-ROFFs.

Definition 8 *Suppose that $\mathbb{S}$ is the collection of all q-ROFNs, and the q-ROFFs $\phi(\alpha)$ and $\Phi(\alpha)$ are defined as in Definition 6. If*

$$\frac{\mathcal{D}}{\mathcal{D}\alpha}\Phi(\alpha) = \phi(\alpha), \quad \alpha \in \mathbb{S}$$

then Φ is called a primitive of ϕ in $\mathbb{S}$.

By Corollary 1, it has

$$\frac{\mathcal{D}}{\mathcal{D}\alpha}(\Phi(\alpha)\otimes C)=\frac{\mathcal{D}}{\mathcal{D}\alpha}\Phi(\alpha)$$

Clearly, if Φ is a primitive of ϕ, so does $\Phi(\alpha)\otimes C$. This implies that Φ is not unique.

Definition 9 *We define all the primitives of ϕ as $\int\phi(\alpha)\mathcal{D}\alpha$, the indefinite integral of ϕ. In particular,*

$$\int\phi(\alpha)\mathcal{D}\alpha=\Phi(\alpha)\otimes C \tag{21}$$

where $\Phi(\alpha)$ is one of primitives of $\phi(\alpha)$.

By previous derivative formula and Definition 8, we easily check

$$\int\langle 0,1\rangle\mathcal{D}\alpha=\langle\mu_\alpha,v_\alpha\rangle\otimes C$$
$$\int\langle(1-\lambda)^{\frac{1}{q}},\lambda\rangle\mathcal{D}\alpha=\alpha^\lambda\otimes C$$

Next, it is desirable to provide a rigorous mathematical proof for the derivation of primitives, which will help us in dealing with continuous q-rung orthopair fuzzy information.

Theorem 4 *Assume that the q-ROFF ϕ is defined as in Definition 6. Assume in addition that all hypotheses in Theorem 1 hold true. Then,*

$$\begin{aligned}&\int\phi(\alpha)\mathcal{D}\alpha\\&=\Bigg\langle c_1\exp\left\{\int\frac{1-f_\alpha^q}{\mu_\alpha}d\mu_\alpha\right\},\\&\left(1-c_2\exp\left\{-\int\frac{qv_\alpha^{q-1}g_\alpha^q}{1-v_\alpha^q}dv_\alpha\right\}\right)^{\frac{1}{q}}\Bigg\rangle\end{aligned} \tag{22}$$

where the constant c_1 and c_2 are arbitrarily given.

Proof of Theorem 4. Assume that

$$\int\varphi(\alpha)\mathcal{D}\alpha=\langle F_\alpha,G_\alpha\rangle$$

The remaining task is to prove

$$F_\alpha=c_1\exp\left\{\int\frac{1-f_\alpha^q}{\mu_\alpha}d\mu_\alpha\right\} \tag{23}$$

and

$$\begin{aligned}&G_\alpha\\&=\left(1-c_2\exp\left\{-\int\frac{qv_\alpha^{q-1}g_\alpha^q}{1-v_\alpha^q}dv_\alpha\right\}\right)^{\frac{1}{q}}\end{aligned} \tag{24}$$

In fact, to prove (23)-(24), from Definition 8 we see that

$$\begin{aligned}&\frac{\mathcal{D}}{\mathcal{D}\alpha}\langle F_\alpha,G_\alpha\rangle\\&=\frac{\mathcal{D}}{\mathcal{D}\alpha}\int\varphi(\alpha)\mathcal{D}\alpha=\langle f_\alpha,g_\alpha\rangle\end{aligned} \tag{25}$$

On the other hand, by Theorem 1 we have

$$\begin{aligned}&\frac{\mathcal{D}}{\mathcal{D}\alpha}\langle F_\alpha,G_\alpha\rangle\\&=\left\langle\left(1-\frac{\mu_\alpha}{F_\alpha}\frac{\partial F_\alpha}{\partial\mu_\alpha}\right)^{\frac{1}{q}},\left(\frac{(1-v_\alpha^q)}{(1-G_\alpha^q)}\frac{\partial G_\alpha^q}{\partial v_\alpha^q}\right)^{\frac{1}{q}}\right\rangle\end{aligned} \tag{26}$$

In accordance with (25)-(26), we deduce

$$\begin{aligned}&\frac{\mu_\alpha}{F_\alpha}\frac{\partial F_\alpha}{\partial\mu_\alpha}=1-f_\alpha^q\quad\text{and}\\&\frac{(1-v_\alpha^q)G_\alpha^{q-1}}{(1-G_\alpha^q)v_\alpha^{q-1}}\frac{\partial G_\alpha}{\partial v_\alpha}=g_\alpha^q\end{aligned} \tag{27}$$

Owing to (13), the latter part in (27) is equals to

$$\frac{\mathcal{D}G_\alpha^q}{1-G_\alpha^q}=\frac{g_\alpha^q}{1-v_\alpha^q}\mathcal{D}v_\alpha^q$$

Solving the ordinary differential equation gives birth to the desired (24). The same argument yields (23). The proof is done. ■

The theorem below states some algebraic manipulations of q-ROFIs.

Theorem 5 *Assume that the q-ROFFs*

$$\begin{aligned}&\phi_i=\langle f_i(\mu_\alpha,v_\alpha),\\&g_i(\mu_\alpha,v_\alpha)\rangle=\langle f_i,g_i\rangle\ \ (i=1,2)\end{aligned} \tag{28}$$

are defined as in Definition 6. Then

$$\begin{aligned}&\int\phi_1\mathcal{D}\alpha\otimes\int\phi_2\mathcal{D}\alpha\\&=\int\left\langle(1-(1-f_1^q)-(1-f_2^q))^{\frac{1}{q}},(g_1^q+g_2^q)^{\frac{1}{q}}\right\rangle\mathcal{D}\alpha\end{aligned} \tag{29}$$

and

$$\begin{aligned}&\int\phi_1\mathcal{D}\alpha\oslash\int\phi_2\mathcal{D}\alpha\\&=\int\left\langle(1-f_2^q+f_1^q)^{\frac{1}{q}},(g_1^q-g_2^q)^{\frac{1}{q}}\right\rangle\mathcal{D}\alpha\end{aligned} \tag{30}$$

Proof of Theorem 5. For simplicity, we only prove (29). Observe from (22) that

$$\begin{aligned}
&\int \phi_1 \mathcal{D}\alpha \otimes \int \phi_2 \mathcal{D}\alpha \\
&= \left\langle c_1^1 \exp\left\{\int \frac{1-f_1^q}{\mu_\alpha} d\mu_\alpha\right\}, A_1 \right\rangle \\
&\qquad \otimes \left\langle c_1^2 \exp\left\{\int \frac{1-f_2^q}{\mu_\alpha} d\mu_\alpha\right\}, A_2 \right\rangle \\
&= \left\langle c_1^1 c_1^2 \exp\left\{\int \frac{(1-f_1^q)+(1-f_2^q)}{\mu_\alpha} d\mu_\alpha\right\}, A \right\rangle
\end{aligned} \tag{31}$$

in which

$$\begin{aligned}
&A_i^q \\
&= 1 - c_2^i \exp\left\{-\int \frac{q v_\alpha^{q-1} g_i^q}{1-v_\alpha^q} dv_\alpha\right\} \quad (i=1,2)
\end{aligned}$$

and

$$\begin{aligned}
A^q &= A_1^q + A_2^q - A_1^q A_2^q \\
&= 1 - c_2^1 c_2^2 \exp\left\{-\int \frac{q v_\alpha^{q-1}(g_1^q + g_2^q)}{1-v_\alpha^q} dv_\alpha\right\}
\end{aligned}$$

is due to the multiplying operation of q-ROFFs. The combination of (22) with (31) yields (29). The proof is done. ■

Corollary 2 *Let $\lambda \in [0,1]$. Then,*

$$\begin{aligned}
&\left(\int \langle f_\alpha, g_\alpha\rangle \mathcal{D}\alpha\right)^\lambda \\
&= \int \left\langle (1-\lambda(1-f_\alpha^q))^{\frac{1}{q}}, \lambda^{\frac{1}{q}} g_\alpha \right\rangle \mathcal{D}\alpha
\end{aligned} \tag{32}$$

Proof of Corollary 2. Operating $\mathcal{D}$ to the right hand side of (32) yields

$$\begin{aligned}
&\frac{\mathcal{D}}{\mathcal{D}\alpha} \int \langle (1-\lambda(1-f_\alpha^q))^{\frac{1}{q}}, \lambda^{\frac{1}{q}} g_\alpha\rangle \mathcal{D}\alpha \\
&= \langle (1-\lambda(1-f_\alpha^q))^{\frac{1}{q}}, \lambda^{\frac{1}{q}} g_\alpha\rangle
\end{aligned}$$

For the left side, from Theorem 3 we obtain

$$\begin{aligned}
&\frac{\mathcal{D}}{\mathcal{D}\alpha}\left(\int \langle f_\alpha, g_\alpha\rangle \mathcal{D}\alpha\right)^\lambda \\
&= \frac{\mathcal{D}}{\mathcal{D}\int \langle f_\alpha, g_\alpha\rangle \mathcal{D}\alpha}\left(\int \langle f_\alpha, g_\alpha\rangle \mathcal{D}\alpha\right)^\lambda \\
&\quad \oplus \frac{\mathcal{D}}{\mathcal{D}\alpha}\int \langle f_\alpha, g_\alpha\rangle \mathcal{D}\alpha \\
&= \langle (1-\lambda)^{\frac{1}{q}}, \lambda^{\frac{1}{q}}\rangle \oplus \langle f_\alpha, g_\alpha\rangle \\
&= \left\langle (1-\lambda(1-f_\alpha^q))^{\frac{1}{q}}, \lambda^{\frac{1}{q}} g_\alpha \right\rangle
\end{aligned}$$

This, along with the properties of primitive, leads to

$$\begin{aligned}
&\int \left\langle (1-\lambda(1-f_\alpha^q))^{\frac{1}{q}}, \lambda^{\frac{1}{q}} g_\alpha \right\rangle \mathcal{D}\alpha \\
&\oslash \left(\int \langle f_\alpha, g_\alpha\rangle \mathcal{D}\alpha\right)^\lambda = C
\end{aligned} \tag{33}$$

Selecting $\lambda = 1$ in (33) yields $C = \langle 1, 0\rangle$, and thus, generates (32). The proof is done. ■

Remark 3. *In view of* (29)*, one should expect that* (32) *is still valid for integrals $\lambda = 2, 3, \cdots$.*

5. Definite Integrals of q-ROFFs under Multiplication

To proceed, we need to introduce the concept of q-rung orthopair fuzzy integral line (q-ROFIL) under multiplication operations circumstance.

Definition 10 *Let $\alpha, \beta \in \mathbb{S}$, and let $\mathcal{L}(\alpha,\beta) \subset \mathbb{S}$ be a line which connects α and β. If for any $t_1, t_2 \in \mathcal{L}(\alpha,\beta)$, it has*

$$t_2 \in \mathbb{S}_\otimes(t_1) \quad \forall \ \alpha \succeq t_1 \succeq t_2 \succeq \beta$$

Then, the line $\mathcal{L}(\alpha,\beta)$ is called an q-ROFIL from α to β.

Let us present several examples to assist the readers' understanding.

Example 1 Fixed q-ROFN $\alpha = \langle \mu_\alpha, v_\alpha\rangle$, and let another q-ROFN $\beta = \langle \mu_\beta, v_\beta\rangle \in \mathbb{S}_\otimes(\alpha)$. Then, according to Definition 10, we might as well construct the following q-ROFILs:

(i) Segment

$$\mathcal{L}_1 := \left\{\langle \mu_\xi, v_\xi\rangle \;\middle|\; \frac{v_\xi - v_\alpha}{\mu_\xi - \mu_\alpha} = \frac{v_\beta - v_\alpha}{\mu_\beta - \mu_\alpha}, \quad \mu_\xi \in [\mu_\beta, \mu_\alpha]\right\}$$

(ii) Convex curve

$$\mathcal{L}_2 := \left\{\langle \mu_\xi, v_\xi\rangle \;\middle|\; \begin{array}{l} v_\xi = \psi(\mu_\xi), \quad \mu_\xi \in [\mu_\beta, \mu_\alpha], \\ v_\beta = \psi(\mu_\beta), \quad v_\alpha = \psi(\mu_\alpha), \\ \psi' \leq 0, \; \psi'' \leq 0 \end{array}\right\}$$

(iii) Concave curve

$$\mathcal{L}_3 := \left\{\langle \mu_\xi, v_\xi\rangle \;\middle|\; \begin{array}{l} v_\xi = \varphi(\mu_\xi), \quad \mu_\xi \in [\mu_\alpha, \mu_\beta], \\ v_\alpha = \varphi(\mu_\alpha), \quad v_\beta = \varphi(\mu_\beta), \\ \varphi' \leq 0, \; \varphi'' \geq 0 \end{array}\right\}$$

With Definition 10, we now turn to the integrals of q-ROFF in $\mathcal{L}(\alpha, \beta)$.

Definition 11 *Assume that q-ROFNs $\alpha, \beta \in \mathbb{S}$ and q-ROFF ϕ is well defined in $\mathcal{L}(\alpha, \beta)$.*

(i). Insert arbitrarily finite point $\alpha_i = \langle \mu_i, v_i\rangle$ $(i = 0, 1, \cdots, n)$ in $\mathcal{L}(\alpha, \beta)$, satisfying

$$\alpha = \alpha_0 \gtrapprox \alpha_1 \gtrapprox \cdots \gtrapprox \alpha_{n-1} \gtrapprox \alpha_n = \beta$$

(ii). Select arbitrarily $\xi_i \in \mathcal{L}(\alpha_{i-1}, \alpha_i)$ and compute

$$\otimes_{i=1}^{n} \left(\phi(\xi_i) \oplus (\alpha_i \oslash \alpha_{i-1})\right) := \langle F_n, G_n\rangle \quad (34)$$

(iii). Define

$$\lambda_1 = \min\left\{\frac{\mu_1}{\mu_0}, \cdots, \frac{\mu_i}{\mu_{i-1}}, \cdots, \frac{\mu_n}{\mu_{n-1}}\right\} \quad (35)$$

and

$$\lambda_2 = \max\{v_1 - v_0, \cdots, v_i - v_{i-1}, \cdots, v_n - v_{n-1}\} \quad (36)$$

Then, if there exists a constant q-ROFF $\langle F, G\rangle$ which relies only on α, β, f and g, such that

$$\lim \otimes_{i=1}^{n} \left(\phi(\xi_i) \oplus (\alpha_i \oslash \alpha_{i-1})\right) := \langle \lim_{\lambda_1 \to 1} F_n, \lim_{\lambda_2 \to 0} G_n\rangle := \langle F, G\rangle$$

we call that ϕ is summable over $\mathcal{L}(\alpha, \beta)$, denoted by

$$\int_{\mathcal{L}(\alpha,\beta)} \phi(\xi)\mathcal{D}\xi = \int_{\mathcal{L}(\alpha,\beta)} \langle f_\xi, g_\xi\rangle \mathcal{D}\xi = \langle F, G\rangle \quad (37)$$

The theorem below gives the explicit expression of $\langle F, G\rangle$, as long as some extra assumptions are made.

Theorem 6 *Under the same assumptions listed in Theorem 1, we have*

$$\int_{\mathcal{L}(\alpha,\beta)} \phi(\xi)\mathcal{D}\xi = \left\langle \exp\left\{\int_{\mu_\alpha}^{\mu_\beta} \frac{1 - f_\xi^q}{\mu_\xi} d\mu_\xi\right\}, \left(1 - \exp\left\{-\int_{v_\alpha}^{v_\beta} \frac{q v_\xi^{q-1} g_\xi^q}{1 - v_\xi^q} dv_\xi\right\}\right)^{\frac{1}{q}} \right\rangle \quad (38)$$

Proof of Theorem 6. By Definition 11, it suffices to show

$$\lim_{\lambda_1 \to 1} F_n = \exp\left\{\int_{\mu_\alpha}^{\mu_\beta} \frac{1 - f_\xi^q}{\mu_\xi} d\mu_\xi\right\} \quad (39)$$

and

$$\lim_{\lambda_2 \to 0} G_n^q = 1 - \exp\left\{-\int_{v_\alpha}^{v_\beta} \frac{q v_\xi^{q-1} g_\xi^q}{1 - v_\xi^q} dv_\xi\right\} \quad (40)$$

Due to the basic operations in Definitions 2-3, we infer

$$\begin{aligned}
&\otimes_{i=1}^{n} \left(\phi(\xi_i) \oplus (\alpha_i \oslash \alpha_{i-1})\right) \\
&= \otimes_{i=1}^{n} \left(\langle f_i, g_i\rangle \oplus \left\langle \frac{\mu_i}{\mu_{i-1}}, \left(\frac{v_i^q - v_{i-1}^q}{1 - v_{i-1}^q}\right)^{\frac{1}{q}} \right\rangle\right) \\
&= \otimes_{i=1}^{n} \left\langle \left(f_i^q + \frac{\mu_i^q}{\mu_{i-1}^q} - f_i^q \frac{\mu_i^q}{\mu_{i-1}^q}\right)^{\frac{1}{q}}, g_i \left(\frac{v_i^q - v_{i-1}^q}{1 - v_{i-1}^q}\right)^{\frac{1}{q}} \right\rangle \\
&= \left\langle \prod_i \left(1 - (1 - f_i^q)(1 - \frac{\mu_i^q}{\mu_{i-1}^q})\right)^{\frac{1}{q}}, \left(1 - \prod_i \left(1 - g_i^q \frac{v_i^q - v_{i-1}^q}{1 - v_{i-1}^q}\right)\right)^{\frac{1}{q}} \right\rangle
\end{aligned} \quad (41)$$

Combining with (34) brings us to

$$F_n^q = \prod_i \left(1-(1-f_i^q)(1-\frac{\mu_i^q}{\mu_{i-1}^q})\right) \quad \text{and} \qquad G_n^q = 1-\prod_i \left(1-g_i^q\frac{v_i^q-v_{i-1}^q}{1-v_{i-1}^q}\right) \tag{42}$$

Let us first deal with (40). Notice from (42) that

$$\begin{aligned} 1-G_n^q &= \prod_i \left(1-g_i^q\frac{v_i^q-v_{i-1}^q}{1-v_{i-1}^q}\right) \\ &= \exp\left\{\ln\prod_i \left(1-g_i^q\frac{v_i^q-v_{i-1}^q}{1-v_{i-1}^q}\right)\right\} \\ &= \exp\left\{\sum_i \ln\left(1-g_i^q\frac{v_i^q-v_{i-1}^q}{1-v_{i-1}^q}\right)\right\} \end{aligned}$$

which implies that if we pass $\lambda_2 \to 0$

$$\begin{aligned} &1-\lim_{\lambda_2\to 0} G_n^q \\ &= \exp\left\{\lim_{\lambda_2\to 0}\sum_i \ln\left(1-g_i^q\frac{v_i^q-v_{i-1}^q}{1-v_{i-1}^q}\right)\right\} \\ &= \exp\left\{-\int_{v_\alpha}^{v_\beta} g_\xi^q\frac{qv_\xi^{q-1}}{1-v_\xi^q}dv_\xi\right\} \end{aligned}$$

where in the equalities we have used Proposition 2.1 in (Gao et al. 2020), the continuity of exponential function, g and μ. This proves (40). (39) can be dealt in a similar method. So, we complete the proof of Theorem 6. ■

As a direct application of Theorem 6, we have the following

$$\begin{aligned} &\int_{\mathcal{L}(\alpha,\alpha)} \phi(\xi)\mathcal{D}\xi = \langle 1,0\rangle \\ &\int_{\mathcal{L}(\alpha,\beta)} \langle 0,1\rangle\,\mathcal{D}\xi = \beta\oslash\alpha \\ &\int_{\mathcal{L}(\alpha,\beta)} \langle 1,0\rangle\,\mathcal{D}\xi = \langle 1,0\rangle \end{aligned} \tag{43}$$

Remark 4. *Under the same assumptions listed in Theorem 1, we check that the value of* $\int_{\mathcal{L}(\alpha,\beta)}\phi\mathcal{D}\xi$ *is independent of the path of* $\mathcal{L}$*, but only relies on the location of the endpoints* α *and* β*.*

The following several theorems provide us some basic properties of multiplication integrals.

Theorem 7 (Multiplication of Integrals)

$$\begin{aligned} &\int_{\mathcal{L}(\alpha,\beta)}\langle f_1,g_1\rangle\mathcal{D}\xi\otimes\int_{\mathcal{L}(\alpha,\beta)}\langle f_2,g_2\rangle\mathcal{D}\xi \\ &= \int_{\mathcal{L}(\alpha,\beta)}\Big\langle\left(1-(1-f_1^q)-(1-f_2^q)\right)^{\frac{1}{q}}, \\ &\qquad (g_1^q+g_2^q)^{\frac{1}{q}}\Big\rangle\mathcal{D}\xi \end{aligned} \tag{44}$$

Proof of Theorem 7. Utilizing Theorem 6 and the basic operation of q-ROFFs, we compute

$$\begin{aligned} &\int_{\mathcal{L}(\alpha,\beta)}\langle f_1,g_1\rangle\mathcal{D}\xi\otimes\int_{\mathcal{L}(\alpha,\beta)}\langle f_2,g_2\rangle\mathcal{D}\xi \\ &= \left\langle\exp\left\{\int_{\mu_\alpha}^{\mu_\beta}\frac{1-f_1^q}{\mu}d\mu\right\},B_1\right\rangle \\ &\quad\otimes\left\langle\exp\left\{\int_{\mu_\alpha}^{\mu_\beta}\frac{1-f_2^q}{\mu}d\mu\right\},B_2\right\rangle \\ &= \left\langle\exp\left\{\int_{\mu_\alpha}^{\mu_\beta}\frac{(1-f_1^q)+(1-f_2^q)}{\mu}d\mu\right\},\right. \\ &\qquad\left.\left(1-\exp\left\{-\int_{v_\alpha}^{v_\beta}\frac{qv^{q-1}(g_1^q+g_2^q)}{1-v^q}dv\right\}\right)^{\frac{1}{q}}\right\rangle \\ &= \int_{\mathcal{L}(\alpha,\beta)}\Big\langle\left(1-(1-f_1^q)-(1-f_2^q)\right)^{\frac{1}{q}}, \\ &\qquad \left(g_1^q+g_2^q\right)^{\frac{1}{q}}\Big\rangle\mathcal{D}\xi \end{aligned} \tag{45}$$

where

$$B_i^q = 1-\exp\left\{-\int_{v_\alpha}^{v_\beta}\frac{qv^{q-1}g_i^q}{1-v^q}dv\right\} \quad (i=1,2)$$

and the last equality sign owes to (38). The proof is done. ■

Theorem 8 (Power Algorithm of Integral) *Let* $\lambda\in[0,1]$ *or* $\lambda\in\mathbb{N}_+$*. Then it satisfies*

$$\begin{aligned} &\left(\int_{\mathcal{L}(\alpha,\beta)}\langle f,g\rangle\mathcal{D}\xi\right)^\lambda \\ &= \int_{\mathcal{L}(\alpha,\beta)}\Big\langle(1-\lambda(1-f_\xi^q))^{\frac{1}{q}},\lambda^{\frac{1}{q}}g_\xi\Big\rangle\mathcal{D}\xi \end{aligned} \tag{46}$$

Proof of Theorem 8. In case of $\lambda \in [0,1]$. Utilizing (38), one deduces

$$\begin{aligned}&\left(\int_{\mathcal{L}(\alpha,\beta)} \langle f,g\rangle \mathcal{D}\xi\right)^{\lambda}\\&=\left\langle \exp\left\{\int_{\mu_\alpha}^{\mu_\beta} \frac{\lambda(1-f^q)}{\mu}d\mu\right\}, A^{\frac{1}{q}}\right\rangle\\&=\int_{\mathcal{L}(\alpha,\beta)} \langle (1-\lambda(1-f^q))^{\frac{1}{q}}, \lambda^{\frac{1}{q}} g\rangle \mathcal{D}\xi\end{aligned}$$

where the last equality sign is due to the basic operations of q-ROFFs, and

$$\begin{aligned}A&=1-\left(1-\left(1-\exp\left\{-\int_{v_\alpha}^{v_\beta}\frac{qv^{q-1}g^q}{1-v^q}dv\right\}\right)\right)^{\lambda}\\&=1-\exp\left\{-\int_{v_\alpha}^{v_\beta}\frac{qv^{q-1}\lambda g^q}{1-v^q}dv\right\}\end{aligned}$$

While in case of $\lambda \in \mathbb{N}_+$, (46) follows from (44) and deduction argument. We complete the proof. ■

Theorem 9 (Algebraic Operation of Integral Line) *Assume that*

$$\gamma \in \mathbb{S}_{\otimes}(\beta), \quad \beta \in \mathbb{S}_{\otimes}(\alpha), \quad \alpha \in \mathbb{S}$$

Then

$$\begin{aligned}&\int_{\mathcal{L}(\alpha,\beta)} \langle f,g\rangle \mathcal{D}\xi \otimes \int_{\mathcal{L}(\beta,\gamma)} \langle f,g\rangle \mathcal{D}\xi\\&=\int_{\mathcal{L}(\alpha,\gamma)} \langle f,g\rangle \mathcal{D}\xi\end{aligned} \tag{47}$$

Proof of Theorem 9. By (38), we compute

$$\begin{aligned}&\int_{\mathcal{L}(\alpha,\beta)} \langle f,g\rangle \mathcal{D}\xi \otimes \int_{\mathcal{L}(\beta,\gamma)} \langle f,g\rangle \mathcal{D}\xi\\&=\left\langle \exp\left\{\int_{\mu_\alpha}^{\mu_\beta}\frac{1-f^q}{\mu}d\mu\right\}, B(\alpha,\beta)\right\rangle\\&\quad\otimes\left\langle \exp\left\{\int_{\mu_\beta}^{\mu_\gamma}\frac{1-f^q}{\mu}d\mu\right\}, B(\beta,\gamma)\right\rangle\\&=\left\langle \exp\left\{\int_{\mu_\alpha}^{\mu_\gamma}\frac{1-f^q}{\mu}d\mu\right\}, B(\alpha,\gamma)\right\rangle\\&=\int_{\mathcal{L}(\alpha,\gamma)} \langle f,g\rangle \mathcal{D}\xi\end{aligned} \tag{48}$$

where

$$B^q(\alpha,\beta)=1-\exp\left\{-\int_{v_\alpha}^{v_\beta}\frac{qv^{q-1}g^q}{1-v^q}dv\right\}$$

Thus, we complete the proof. ■

At last we discuss the relationship between two different types integrals which are based on the addition and multiplication operations respectively.

In the light of (6) and Definition 6, we define

$$\begin{cases}\phi(\overline{\alpha}) := \langle f(v_\alpha,\mu_\alpha),\ g(v_\alpha,\mu_\alpha)\rangle\\ \overline{\phi}(\overline{\alpha}) := \langle g(v_\alpha,\mu_\alpha),\ f(v_\alpha,\mu_\alpha)\rangle\end{cases} \tag{49}$$

To achieve the above purpose, we first recall some known theorems developed in (Gao et al. 2020).

Lemma 2 (Gao et al. 2020) *Let the same assumptions in Theorem 1 hold true. Then,*

$$\begin{aligned}&\int \phi(\alpha)d\alpha\\&=\left\langle\left(1-c_1\exp\left\{-\int\frac{q\mu_\alpha^{q-1}f_\alpha^q}{1-\mu_\alpha^q}d\mu_\alpha\right\}\right)^{\frac{1}{q}},\right.\\&\qquad\left. c_2\exp\left\{\int\frac{1-g_\alpha^q}{v_\alpha}dv_\alpha\right\}\right\rangle\end{aligned}$$

where c_1 and c_2 are given constant.

Lemma 3 (Gao et al. 2020) *The following equality holds true*

$$\begin{aligned}&\int\langle f_1,g_1\rangle d\alpha \oplus \int\langle f_2,g_2\rangle d\alpha\\&=\int\left\langle (f_1^q+f_2^q)^{\frac{1}{q}},\ (1-(1-g_1^q)-(1-g_2^q))^{\frac{1}{q}}\right\rangle d\alpha\end{aligned}$$

provided the quantities on both sides make sense.

Definition 12 (Gao et al. 2020) *The line $L(\alpha,\beta) \subset \mathbb{S}$ from α to β is called an integral line, which satisfies for any $t_1,\ t_2 \in L(\alpha,\beta)$,*

$$t_2 \in \mathbb{S}_{\oplus}(t_1) \quad \forall\ \ \alpha \preceq t_1 \preceq t_2 \preceq \beta$$

Definition 13 (Gao et al. 2020) *Insert finitely many IFNs $\alpha_i = \langle \mu_i,\ v_i\rangle\,(i = 0 \sim n)$ into the line $L(\alpha,\beta)$ such that*

$$\alpha = \alpha_0 \lhd \alpha_1 \lhd \cdots \lhd \alpha_{n-1} \lhd \alpha_n = \beta$$

Select arbitrarily $\xi_i \in L(\alpha_{i-1},\alpha_i)$ and define

$$\oplus_{i=1}^{n}\ \phi(\xi_i) \otimes (\alpha_i \ominus \alpha_{i-1}) := \langle F_n,\ G_n\rangle \tag{50}$$

Let

$$\lambda_1 = \max_{1\le i\le n}\{\mu_1-\mu_0,\cdots,\mu_i-\mu_{i-1},\cdots,\mu_n-\mu_{n-1}\} \tag{51}$$

and

$$\lambda_2 = \min_{1\le i\le n}\left\{\frac{v_1}{v_0},\cdots,\frac{v_i}{v_{i-1}},\cdots,\frac{v_n}{v_{n-1}}\right\} \tag{52}$$

If there exists a constant q-ROFF $\langle F,G\rangle$ which relies only on α, β, f and g, and satisfies

$$\begin{aligned}&\lim \oplus_{i=1}^{n}\phi(\xi_i)\otimes(\alpha_i\ominus\alpha_{i-1})\\&=\langle \lim_{\lambda_1\to 0}F_n,\ \lim_{\lambda_2\to 1}G_n\rangle=\langle F,G\rangle\end{aligned}$$

Then, we say that ϕ is summable over $L(\alpha,\beta)$, and denote by

$$\begin{aligned}\int_{L(\alpha,\beta)}\phi(\xi)d\xi&=\int_{L(\alpha,\beta)}\langle f_\xi,g_\xi\rangle d\xi\\&=\langle F,G\rangle\end{aligned} \tag{53}$$

Having the above preparation obtained, we are willing to compare the integral values based on the addition and multiplication.

Theorem 10 *Under the assumptions made in Theorems 4 and 6 , it has*

$$\overline{\int\phi(\alpha)d\alpha}=\int\overline{\phi}(\overline{\alpha})\mathcal{D}\overline{\alpha} \tag{54}$$

Proof of Theorem 10. Owing to Theorem 6 ,

$$\begin{aligned}&\overline{\int\phi(\alpha)d\alpha}\\&=\overline{\left\langle\mathcal{F},c_2\exp\left\{\int\frac{1-g^q}{v}dv\right\}\right\rangle}\\&=\left\langle c_2\exp\left\{\int\frac{1-g^q}{v}dv\right\},\mathcal{F}\right\rangle\end{aligned}$$

where

$$\mathcal{F}^q=1-c_1\exp\left\{-\int\frac{q\mu^{q-1}f^q}{1-\mu^q}d\mu\right\}$$

and the last equality sign owes to (6).

Due to (22) and (49), we have

$$\begin{aligned}&\int\overline{\phi}(\overline{\alpha})\mathcal{D}\overline{\alpha}\\&=\left\langle c_1\exp\left\{\int\frac{1-g^q}{v}dv\right\},\right.\\&\left.\left(1-c_2\exp\left\{-\int\frac{q\mu^{q-1}f^q}{1-\mu^q}d\mu\right\}\right)^{\frac{1}{q}}\right\rangle\end{aligned}$$

The last two equality quantities, along with the arbitrariness of c_1 and c_2, generates the desired (54). The proof is done. ■

Theorem 11 *Under the assumptions made in Theorems 5 and 7, it has*

$$\begin{aligned}&\int\langle f_1,g_1\rangle d\alpha\oplus\int\langle f_2,g_2\rangle d\alpha\\&=\int\overline{\langle f_1,g_1\rangle}\mathcal{D}\overline{\alpha}\otimes\int\overline{\langle f_2,g_2\rangle}\mathcal{D}\overline{\alpha}\end{aligned}$$

Theorem 12 *Under the assumptions made in Definitions 11 and 13, it has*

$$\overline{\int_{L(\alpha,\beta)}\phi(\xi)d\xi}=\int_{\mathcal{L}(\overline{\alpha},\overline{\beta})}\overline{\phi}(\overline{\xi})\mathcal{D}\overline{\xi} \tag{55}$$

Proof of Theorem 12. By (6), (49), and Definition 11, we deduce

$$\begin{aligned}&\int_{\mathcal{L}(\overline{\alpha},\overline{\beta})}\overline{\phi}(\overline{\xi})\mathcal{D}\overline{\xi}\\&=\overline{\lim\otimes_{i=1}^{n}\left(\overline{\phi}(\overline{\xi_i})\oplus(\overline{\alpha_i}\oslash\overline{\alpha_{i-1}})\right)}\\&=\lim\oplus_{i=1}^{n}\left(\phi(\xi_i)\otimes(\alpha_i\ominus\alpha_{i-1})\right)\\&=\int_{L(\alpha,\beta)}\phi(\xi)d\xi\end{aligned} \tag{56}$$

in which the limits are specified in Definitions 11 and 13. Again using (6), we immediately get (55) from (56). ■

6. Information Integration Based on q-rung Orthopair Fuzzy Multiplicative Integrals

Based on the q-rung orthopair fuzzy multiplicative integral theory, this section mainly discusses the relationship between the q-rung orthopair fuzzy definite integral and q-ROFWG operator, and proposes a multiplicative integral-based orthopair fuzzy multi-attribute decision-making method.

Refer to Liu and Wang (2018), we have the classical q-ROFWG:

$$q\text{-}ROFWG(\alpha_1,\alpha_2,\cdots,\alpha_n)=\otimes_{i=1}^{n}\alpha_i^{w_i} \tag{57}$$

We claim that the formula (57) can be regarded as a special case of the Theorem 7-9. To see this, let us select $\langle f, g\rangle = \langle 0, 1\rangle$ in (46) and utilize (43), to discover

$$(\beta \oslash \alpha)^{\lambda} = \left(\int_{\mathcal{L}(\alpha,\beta)} \langle 0,1\rangle \mathcal{D}\xi\right)^{\lambda} = \int_{\mathcal{L}(\alpha,\beta)} \langle (1-\lambda)^{\frac{1}{q}}, \lambda^{\frac{1}{q}}\rangle \mathcal{D}\xi \tag{58}$$

If we denote by $\beta_0 = 0$ and $\beta_{i+1} = \beta_i \otimes \alpha_{i+1}$ $(i = 0, \cdots, n-1)$, and choose

$$\langle f, g\rangle = \begin{cases} \langle (1-w_1)^{1/q}, w_1^{1/q}\rangle, & \beta_0 \gtrapprox \xi \gtrapprox \beta_1 \\ \vdots & \\ \langle (1-w_i)^{1/q}, w_i^{1/q}\rangle, & \beta_{i-1} \gtrapprox \xi \gtrapprox \beta_i \\ \vdots & \\ \langle (1-w_n)^{1/q}, w_n^{1/q}\rangle, & \beta_{n-1} \gtrapprox \xi \gtrapprox \beta_n \end{cases} \tag{59}$$

we deduce that, by virtue of (59) and Theorem 9,

$$\begin{aligned} \int_{\mathcal{L}(\beta_0,\beta_n)} \langle f, g\rangle d\xi &= \otimes_{i=0}^{n-1} \int_{\mathcal{L}(\beta_0,\beta_n)} \langle f, g\rangle d\xi \\ &= \otimes_{i=0}^{n-1} \left(\beta_{i+1} \oslash \beta_i\right)^{w_{i+1}} \\ &= \otimes_{i=0}^{n-1} \alpha_{i+1}^{w_{i+1}} \\ &= \otimes_{i=1}^{n} \alpha_i^{w_i} \end{aligned} \tag{60}$$

Therefore, combining (60) with (57), we conclude

$$\int_{\mathcal{L}(\beta_0,\beta_n)} \langle f, g\rangle d\xi = q\text{-}ROFWG(\alpha_1, \alpha_2, \cdots, \alpha_n) \tag{61}$$

The above analysis implies that the information integration method can be manipulated by using q-ROFF multiplicative integrals, as follows:

Step 1 Suppose that there are n DMs using q-ROFNs to give their evaluation value $\alpha_i = \langle \mu_i, v_i\rangle (i = 1, 2, \cdots, n)$ to the evaluation object, as shown in Figure 4. In addition, the starting point $O = \alpha_0 = \langle 1, 0\rangle$ and ending point $\mathcal{E} = \alpha_{n+1} = \langle 0, 1\rangle$ are introduced. For the convenience of constructing the latter integration function, we arrange the monotone sequences $\{\mu^{(i)}\}_{i=1}^{n}$ and $\{v^{(i)}\}_{i=1}^{n}$ as below:

$$0 \le \mu^{(1)} \le \cdots \le \mu^{(i)} \le \cdots \le \mu^{(n)} \le 1$$

and

$$0 \le v^{(1)} \le \cdots\cdots \le v^{(i)} \le \cdots\cdots \le v^{(n)} \le 1$$

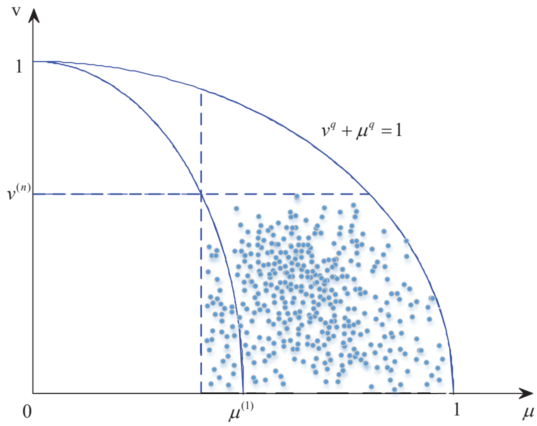

Figure 4 Decision Maker's Evaluation

Step 2 Let the positive integer $n \in \mathbb{N}_+$ be the number of evaluation values determined by all DMs. For each $i \in \{0, 1, 2, \cdots, n\}$, introducing function as:

$$\frac{|f(\mu)|}{n} = \begin{cases} 1, & \mu \le \mu^{(1)} \\ \dfrac{n-(i-1)}{n}, & \mu^{(i-1)} < \mu \le \mu^{(i)} \\ 0, & \mu^{(n)} < \mu \end{cases} \tag{62}$$

and

$$\frac{|g(v)|}{n} = \begin{cases} 0, & v^{(n)} \le v \\ \dfrac{n-(i-1)}{n}, & v^{(i-1)} \le v < v^{(i)} \\ 1, & v < v^{(1)} \end{cases} \tag{63}$$

where $i = 2, 3, \cdots, n$, and $\{\mu^{(i)}\}_{i=1}^{n}$ and $\{v^{(i)}\}_{i=1}^{n}$ are the same as in previous **Step 1.**

In fact, if the value of f at point α_i is very large, it means that μ_i is too small for most DMs, so it can be increased reasonably. Similarly, if g is too large at point α_i, it means that most DMs think that v_i of α_i is very small and should not continue to decrease the value. Therefore, we can consider that f and g contain some uncertain data.

Step 3 Inspired by Lei and Xu (2015), we can get the q-ROFF, $Count(\alpha) =$

$\left\langle\left(\frac{|f(\mu)|}{n}\right),\left(\frac{|g(v)|}{n}\right)\right\rangle$. However, the value of q-rung orthopair fuzzy line integral based on the $Count(\alpha)$ will be compressed, so that the obtained integration result may be very small, which is not ideal for comparative analysis. Thus, according to the distribution of the above evaluation, and the above-mentioned functions $|f(\mu)|$ and $|g(v)|$, we can construct the q-rung orthopair fuzzy line integral aggregation function (q-ROFLIAF) as:

$$Count_q(\alpha)=\left\langle\left(\frac{|f(\mu)|}{n}\right)^{1/q},\left(\frac{|g(v)|}{n}\right)^{1/q}\right\rangle$$

Obviously, $0\le\left(\frac{|f(\mu)|}{n}\right)^{1/q}\le 1$ and $0\le\left(\frac{|g(v)|}{n}\right)^{1/q}\le 1$ are correct. In order to make $0\le\left(\frac{|f(\mu)|}{n}\right)+\left(\frac{|g(v)|}{n}\right)\le 1$ always hold, we can find an integral line, as shown in Figure 5. In this way, the function $Count_q(\alpha)$ we constructed will always meet the conditions in Definition 6.

Remark 5. *For Figure 5, it should be noted that: 1) When the evaluation value points are finite, we can find lines (a) or (c) as the suitable integration lines; 2) As the evaluation value points in the area increase to infinite, the above-mentioned integral lines may become lines (b) or (c). That is, when the number of evaluation value points $N\longrightarrow+\infty$, we have $|\mathcal{L}_a-\mathcal{L}_b|\longrightarrow 0$, which represents that the integral line $\mathcal{L}(\mathcal{O},\mathcal{E})$ in Fig.5(a) and Fig.5(b) overlap.*

Step 4 In view of (61) and Theorem 6, we have the following formula of the information integration method

$$\begin{aligned}&q\text{-}ROFLIA(\alpha_1,\alpha_2,\cdots,\alpha_n)\\&=\int_{\mathcal{L}(\mathcal{O},\mathcal{E})}Count_q(\alpha)\mathcal{D}\alpha\\&=\left\langle\exp\left\{\int_1^{\mu^{(1)}}\frac{1-\frac{|f(\mu)|}{n}}{\mu}\right\},\right.\\&\left.\left(1-\exp\left\{-\int_0^{v^{(n)}}\frac{qv^{q-1}|g(v)|}{n(1-v^q)}\right\}\right)^{1/q}\right\rangle\end{aligned}\tag{64}$$

Step 5 Using the above steps to integrate the evaluation decision matrix R of the corresponding attributes of each candidate, and establish the consensus of the voters, the overall evaluation value Val of each candidate can be obtained. Next, the final ranking results can be obtained by calculating the scoring values of candidates.

Remark 6. *It should be noted that the value of q is first determined according to the condition of $0\le\mu^q+v^q\le 1$, and we do not change the parameter q during the calculation process.*

At the end of this section, we provide some theoretical results for the proposed q-ROFLI operator $\int_{\mathcal{L}(\mathcal{O},\mathcal{E})}Count_q(\alpha)\mathcal{D}\alpha$, which is also known as the fundamental properties of the aggregation operators.

Theorem 13 (Uniformity) *Assume that all the estimate values by DMs are all the same value $\alpha^*=\langle\mu^*,v^*\rangle$, then*

$$\int_{\mathcal{L}(\mathcal{O},\mathcal{E})}Count_q(\alpha)\mathcal{D}\alpha=\alpha^*$$

Proof of Theorem 13. As in (62)-(63) in *Step 2*, we define

$$\begin{aligned}\frac{|f(\mu)|}{n}&=\begin{cases}1, & \mu\le\mu^{(*)}\\0, & \mu^*<\mu\end{cases}\\\frac{|g(v)|}{n}&=\begin{cases}0, & v>v^{(*)}\\1, & v\le v^{(*)}\end{cases}\end{aligned}\tag{65}$$

Moreover, taking the integral line $\mathcal{L}(\mathcal{O},\mathcal{E})$ with the endpoints $\mathcal{O}=\langle 1,0\rangle$ and $\mathcal{E}=\langle\mu^*,v^*\rangle$. If we select select the functions in (65) in the formula (64) , and utilize Theorem 6, we deduce

$$\begin{aligned}&\int_{\mathcal{L}(\mathcal{O},\mathcal{E})}Count_q(\alpha)\mathcal{D}\alpha\\&=\left\langle\exp\left\{\int_1^{\mu^*}\frac{1}{\mu}d\mu\right\},\right.\\&\left.\left(1-\exp\left\{-\int_0^{v^*}\frac{qv^{q-1}}{1-v^q}dv\right\}\right)^{\frac{1}{q}}\right\rangle\\&=\langle\mu^*,v^*\rangle\end{aligned}\tag{66}$$

The proof is completed. ■

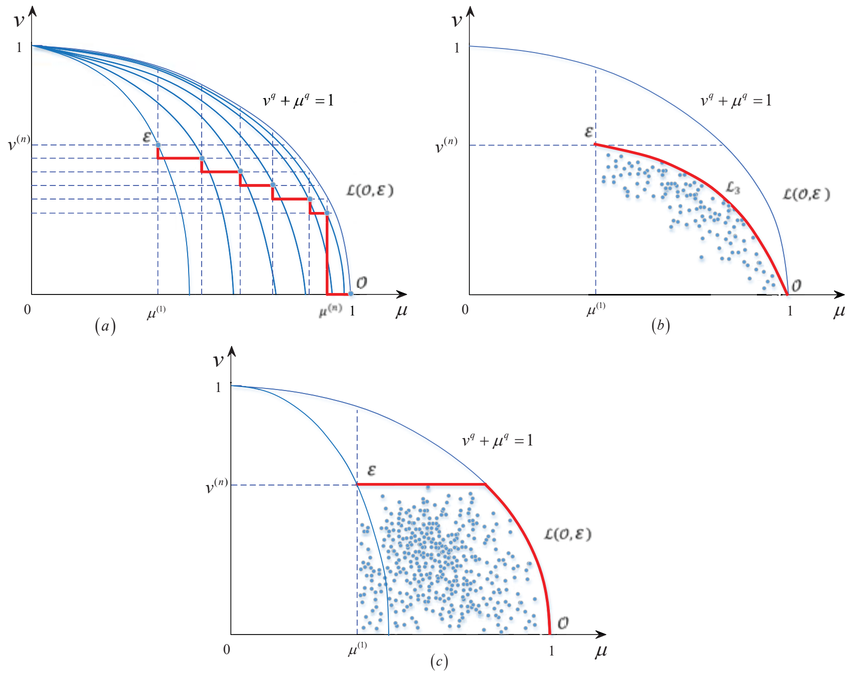

Figure 5 Integral Curve of Function Count(α)

Theorem 14 (Monotony) *Assume that $Count_q(\alpha_1) \geq Count_q(\alpha_2)$, in particular, $\frac{|f(\mu_{\alpha_1})|}{n} \geq \frac{|f(\mu_{\alpha_2})|}{n}$ and $\frac{|g(v_{\alpha_1})|}{n} \leq \frac{|g(v_{\alpha_1})|}{n}$. Then, we have*

$$\int_{\mathcal{L}} Count_q(\alpha_1) \geq \int_{\mathcal{L}} Count_q(\alpha_2) \tag{67}$$

Proof of Theorem 14. Making use of the properties of classical integrations, we easily check the validity of (67) from the formula (38) in Theorem 6. ■

Theorem 15 (Extremum) *For given endpoints $\overline{\alpha} = \langle \mu^{(n)}, v^{(1)} \rangle$ and $\underline{\alpha} = \langle \mu^{(1)}, v^{(n)} \rangle$, as mentioned before, we have the following bounds*

$$\overline{\alpha} \geq \int_{\mathcal{L}(O,\overline{\alpha})} Count_q(\alpha) \geq \underline{\alpha}$$

Proof of Theorem 15. The proof is a direct consequence of Theorem 13 and Theorem 14. ■

7. Applications and Comparative Analysis

7.1 Numerical Examples

In this section, we use an information integration method based on the q-ROFF definite integral for the enumeration. We use two examples to illustrate the finite and infinite evaluation points respectively to verify the feasibility and effectiveness of the method. We also provide examples to compare the results with those generated by the addition operation, and reveal the intrinsic structural differences through the complementary operations.

Example 1 Suppose an election is about to be held in a county, and it decides to choose a voting organization to conduct a poll in order to select the candidates it supports. The organization intends to use the q-ROFNs to fully understand the public opinion of voters. There are two reasons for this choice. Firstly, in a voting election, there are often supporters and opponents, and q-ROFNs can not only show membership information, but also non-membership information, so it can evaluate candidates more fairly and justly. Secondly, q-ROFNs do not need to be restricted by the sum of membership degree and non-membership degree less than 1, and the membership space that can be described is larger. Therefore, it is more suitable for large-scale group decision-making, and can better describe the true preference information of group DMs. The voter evaluation data can be obtained by the following methods:

Suppose there are 20,000 people voting, and they are divided into two groups of $P^g(g = \mu, v)$, each with 10,000 people. Among them, P^μ is called the support group, and P^v is called the disapproval group. Now, there are three candidates $A_i(i = 1, 2, 3)$, and each candidate has four evaluation attributes, including personal charisma (C_1), leadership ability (C_2), political resources (C_3), and democratic orientation (C_4). Voters score each characteristic of the candidate. For example, if voters in the approval group agree with the candidate A_2's evaluation the attribute C_2, then 1 point is scored. There are 2000 people who agree with this attribute, then the score $\mu_{22}^1 = 0.2$, where $\mu_{ij}^k(i,k = 1,2,3; j = 1,2,3,4)$ is the scoring function of μ. On the contrary, if 6000 people disapprove of this attribute, the scored is $v_{22}^1 = 0.6$, where $v_{ij}^k(i,k = 1,2,3; j = 1,2,3,4)$ is the scoring function of v. Through this scoring method, we can get $\alpha_{22}^1 = \langle \mu_{22}^1, v_{22}^1 \rangle = \langle 0.2, 0.6 \rangle$ in Table 1.

Similarly, we performed the above steps three times using non-repeated sampling to get all the evaluation value q-ROFNs $\alpha_{ij}^k = \langle \mu_{ij}^k, v_{ij}^k \rangle$. Then, the q-rung orthopair fuzzy decision matrices $R^k = (\alpha_{ij}^k)_{3\times4}(k = 1, 2, 3)$ can be constructed as shown in Tables 1, 2 and 3.

Table 1 The Decision Matrix R^1

	C_1	C_2	C_3	C_4
A_1	$\langle 0.7, 0.8 \rangle$	$\langle 0.5, 0.7 \rangle$	$\langle 0.3, 0.9 \rangle$	$\langle 0.4, 0.6 \rangle$
A_2	$\langle 0.8, 0.6 \rangle$	$\langle 0.2, 0.6 \rangle$	$\langle 0.2, 0.5 \rangle$	$\langle 0.7, 0.9 \rangle$
A_3	$\langle 0.5, 0.4 \rangle$	$\langle 0.7, 0.3 \rangle$	$\langle 0.4, 0.8 \rangle$	$\langle 0.6, 0.9 \rangle$

Table 2 The Decision Matrix R^2

	C_1	C_2	C_3	C_4
A_1	$\langle 0.6, 0.8 \rangle$	$\langle 0.7, 0.6 \rangle$	$\langle 0.3, 0.7 \rangle$	$\langle 0.6, 0.9 \rangle$
A_2	$\langle 0.4, 0.7 \rangle$	$\langle 0.5, 0.5 \rangle$	$\langle 0.6, 0.8 \rangle$	$\langle 0.6, 0.9 \rangle$
A_3	$\langle 0.5, 0.6 \rangle$	$\langle 0.9, 0.9 \rangle$	$\langle 0.3, 0.5 \rangle$	$\langle 0.7, 0.4 \rangle$

Table 3 The Decision Matrix R^3

	C_1	C_2	C_3	C_4
A_1	$\langle 0.9, 0.7 \rangle$	$\langle 0.7, 0.8 \rangle$	$\langle 0.9, 0.3 \rangle$	$\langle 0.6, 0.9 \rangle$
A_2	$\langle 0.7, 0.9 \rangle$	$\langle 0.8, 0.2 \rangle$	$\langle 0.8, 0.9 \rangle$	$\langle 0.5, 0.9 \rangle$
A_3	$\langle 0.2, 0.6 \rangle$	$\langle 0.7, 0.9 \rangle$	$\langle 0.5, 0.6 \rangle$	$\langle 0.8, 0.8 \rangle$

In order to understand the multiplicative integral-based q-rung orthopair fuzzy multi-attribute decision-making method more clearly, we state it in several steps:

Step 1 Integrate the evaluation values given by the the interviewees. Taking the first integration value α_{11} as an example. The corresponding q-ROFNs in three decision matrices are integrated, namely $\alpha_{11}^1 = \langle 0.7, 0.8 \rangle$ in Table 1, $\alpha_{11}^2 = \langle 0.6, 0.8 \rangle$ in Table 2 and $\alpha_{11}^3 = \langle 0.9, 0.7 \rangle$ in Table 3, and define the endpoints $\mathcal{O} = \langle 1, 0 \rangle$ and $\mathcal{E} = \langle 0, 1 \rangle$. We arrange the following monotone sequences:

$$
0 \le \mu_{11}^{(1)} \le \mu_{11}^{(2)} \le \mu_{11}^{(3)} \le 1
$$
$$
0 \le v_{11}^{(1)} \le v_{11}^{(2)} \le v_{11}^{(3)} \le 1
$$

Step 2 let n be the sample size that consists of $\alpha_{ij}^k, (k = 1, 2, 3)$. We build the q-ROFF

$$
\langle f, g \rangle = \left\langle \left(\frac{|f(\mu)|}{n} \right)^{\frac{1}{q}}, \left(\frac{|g(v)|}{n} \right)^{\frac{1}{q}} \right\rangle
$$

according to the distribution of the above assessments. Specifically,

$$
\frac{|f(\mu)|}{3} = \begin{cases} 1, & 0 \le \mu \le 0.6 \\ \frac{2}{3}, & 0.6 < \mu \le 0.7 \\ \frac{1}{3}, & 0.7 < \mu \le 0.9 \\ 0, & 0.9 < \mu \le 1 \end{cases}
$$

$$
\frac{|g(v)|}{3} = \begin{cases} 1, & 0 \le v < 0.7 \\ \frac{2}{3}, & 0.7 \le v < 0.8 \\ 0, & 0.8 \le v \le 1 \end{cases}
$$

Step 3 According to $0 \le \mu^q + v^q \le 1$, we set $q = 7$ in (64) and calculate directly, to discover,

$$\int_{\mathcal{L}(O,\mathcal{E})}\left\langle\left(\frac{|f(\mu)|}{n}\right)^{1/q},\left(\frac{|g(v)|}{n}\right)^{1/q}\right\rangle\mathcal{D}\alpha$$
$$=\left\langle\exp\left\{\int_{1}^{\mu^{(1)}}\frac{1-\frac{|f(\mu)|}{n}}{\mu}d\mu\right\},\right. \tag{68}$$
$$\left.\left(1-\exp\left\{-\int_{0}^{v^{(3)}}\frac{7v^{6}|g(v)|}{(1-v^{7})n}dv\right\}\right)^{\frac{1}{7}}\right\rangle$$
$$=\langle 0.72, 0.78\rangle$$

Step 4 In the same calculation method, the integrated values corresponding to other items can be calculated. Therefore, the integrated q-rung orthopair fuzzy decision matrix $R=(\alpha_{ij})_{3\times 4}$ is finally obtained, as shown in Table 4.

Table 4 The q-Rung Orthopair Fuzzy Decision Matrix R

	C_1	C_2	C_3	C_4
A_1	$\langle 0.72, 0.78\rangle$	$\langle 0.56, 0.73\rangle$	$\langle 0.43, 0.80\rangle$	$\langle 0.46, 0.86\rangle$
A_2	$\langle 0.61, 0.81\rangle$	$\langle 0.43, 0.53\rangle$	$\langle 0.46, 0.82\rangle$	$\langle 0.59, 0.97\rangle$
A_3	$\langle 0.27, 0.57\rangle$	$\langle 0.76, 0.86\rangle$	$\langle 0.36, 0.70\rangle$	$\langle 0.61, 0.82\rangle$

Step 5 Integrate the evaluation values of each candidates, and the calculation process is the same as Steps 1-3. Utilize the resulting integrated new q-rung orthopair fuzzy decision matrix $R=(\alpha_{ij})_{3\times 4}$, the overall evaluation value $Val[A_i](i=1,2,3)$ of each candidates is as follows:

$$Val[A_1]=\langle 0.53, 0.80\rangle$$
$$Val[A_2]=\langle 0.52, 0.88\rangle$$
$$Val[A_3]=\langle 0.46, 0.79\rangle$$

Step 6 According to Wei et al. (2018), the scores of the three candidates can be obtained: $S[A_1]=0.40, S[A_2]=0.30, S[A_3]=0.41$. Thus we can get the ranking of the candidates as: $A_3\succ A_1\succ A_2$. That is, A_3 is most likely to be elected.

Besides, we also can give an example of the Complement of integrals operations. By the formula (6), (68) takes

$$\overline{\int_{\mathcal{L}(O,\mathcal{E})}\left\langle\left(\frac{|f(\mu)|}{n}\right)^{1/q},\left(\frac{|g(v)|}{n}\right)^{1/q}\right\rangle\mathcal{D}\alpha}$$
$$=\overline{\langle 0.72, 0.78\rangle}$$
$$=\langle 0.78, 0.72\rangle$$
$$=\int_{L(\overline{O},\overline{\mathcal{E}})}\overline{\left\langle\left(\frac{|f(\mu)|}{n}\right)^{1/q},\left(\frac{|g(v)|}{n}\right)^{1/q}\right\rangle}d\overline{\alpha}$$

where the last equality sign is the aggregation under addition operation.

Example 2 Suppose that the voting organization changes its election strategy by conducting the interviewing procedure only once in order to reduce polling expenses, and each respondent gives a specific score from 0-100. Each of kth ($k=1,2,\cdots,10000$) voters from the support group and the disapproval group gave their evaluation information and constituted q-ROFNs α_i^k, where $i=1,2,3$ and $k=1,2,\cdots,10000$). Thus we can obtain the evaluation value with all the q-ROFNs. For the sake of clarity, we use the data of the candidate A_1 for calculation and follow the steps below:

Step 1 According to the distribution of the points, we fit the scatter plot and get the curve $\mathcal{L}$, as shown in Figure 6.

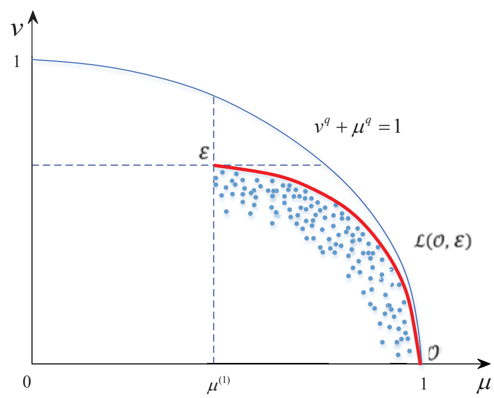

Figure 6 Data Fitting of All the q-ROFNs

$$\mathcal{L}:\ v=2^{-\frac{1}{3}}\left(1-\mu^{3}\right)^{\frac{1}{3}},\quad\forall\ \mu\in[0.53,1] \tag{69}$$

Step 2 The above curve (69) can be regarded as the limit of (62) and (63), we set $q = 3$, then the form of the q-rung orthopair fuzzy line integral aggregation functions can be obtained as:

$$f_\xi^3 = \mu_\xi^3 \quad \text{and} \quad g_\xi^3 = 1 - 2v_\xi^3$$

Step 3 Based on the formula (64), calculating the line integral along $\mathcal{L}$, we can get the following integration result:

$$\begin{aligned}&\int_{\mathcal{L}} \phi(\xi)\mathbb{D}\xi \\ &= \left\langle \exp\left\{\int_1^{0.53} \frac{1-f_\xi^q}{\mu_\xi} d\mu_\xi\right\},\right. \\ &\left.\left(1-\exp\left\{-\int_0^{0.75} \frac{qv_\xi^{q-1} g_\xi^q}{1-v_\xi^q} dv_\xi\right\}\right)^{\frac{1}{q}}\right\rangle \\ &= \langle 0.7039, 0.3512\rangle \end{aligned} \tag{70}$$

Actually, through the above example, it can be shown that by defining a proper integrand and selecting a proper limit of integration, we can use multiplicative q-rung orthopair definite integrals to fuse continuous and large-scale q-rung orthopair fuzzy information.

7.2 Comparison and Analysis

This section uses different methods for comparative analysis to verify the effectiveness and advantages of our approach. The methods include: q-rung orthopair fuzzy weighted averaging (q-ROFWA) operator (Liu and Wang 2018), q-ROFWG operator (Liu and Wang 2018) and q-rung orthopair fuzzy Einstein weighted geometric (q-ROFEWG) operator (Riaz et al. 2020). For generality, we set the weights of each selection in the above operators to be the same, $q = 7$ unchanged, and still build on the case above.

Method 1 q-ROFWA operator (Liu and Wang 2018)

$$\begin{aligned}&q-ROFWA(a_1, a_2, \cdots, a_m) \\ &= \left\langle \left(1-\prod_{k=1}^m \left(1-u_k^q\right)^{w_k}\right)^{\frac{1}{q}}, \prod_{k=1}^m v_k^{w_k}\right\rangle\end{aligned}$$

Step 1 We aggregate the data in Tables 1-3 to obtain the fusion matrix, as shown in Table 5.

Table 5 The q-Rung Orthopair Fuzzy Decision Matrix R'

	C_1	C_2	C_3	C_4
A_1	$\langle 0.81, 0.77\rangle$	$\langle 0.67, 0.70\rangle$	$\langle 0.79, 0.57\rangle$	$\langle 0.57, 0.79\rangle$
A_2	$\langle 0.72, 0.72\rangle$	$\langle 0.69, 0.39\rangle$	$\langle 0.70, 0.71\rangle$	$\langle 0.63, 0.90\rangle$
A_3	$\langle 0.47, 0.52\rangle$	$\langle 0.81, 0.62\rangle$	$\langle 0.44, 0.62\rangle$	$\langle 0.73, 0.66\rangle$

Step 2 By integrating the fusion matrix once, we can obtain the overall evaluation value of each candidate as: $Val[\widetilde{A}_1] = \langle 0.75, 0.70\rangle$, $Val[\widetilde{A}_2] = \langle 0.70, 0.65\rangle$, $Val[\widetilde{A}_3] = \langle 0.71, 0.60\rangle$.

Step 3 Utilize score function to evaluate the three candidates are: $S[\widetilde{A}_1] = 0.52$, $S[\widetilde{A}_2] = 0.51$, $S[\widetilde{A}_3] = 0.53$. Thus, we sort them as $A_3 \succ A_1 \succ A_2$.

Method 2 The q-ROFWG operator (Liu and Wang 2018)

$$\begin{aligned}&q\text{-}ROFWG(a_1, a_2, \cdots, a_m) \\ &= \left\langle \prod_{k=1}^m u_k^{w_k}, \left(1-\prod_{k=1}^m \left(1-v_k^q\right)^{w_k}\right)^{\frac{1}{q}}\right\rangle\end{aligned}$$

Step 1 We aggregate the data in Tables 1-3 to obtain the fusion matrix, as shown in Table 6.

Table 6 The q-Rung Orthopair Fuzzy Decision Matrix R'

	C_1	C_2	C_3	C_4
A_1	$\langle 0.72, 0.78\rangle$	$\langle 0.63, 0.73\rangle$	$\langle 0.43, 0.80\rangle$	$\langle 0.52, 0.86\rangle$
A_2	$\langle 0.61, 0.80\rangle$	$\langle 0.43, 0.53\rangle$	$\langle 0.46, 0.82\rangle$	$\langle 0.60, 0.90\rangle$
A_3	$\langle 0.37, 0.57\rangle$	$\langle 0.76, 0.86\rangle$	$\langle 0.39, 0.70\rangle$	$\langle 0.70, 0.82\rangle$

Step 2 Similarly, we obtain the evaluation value of each candidate as: $Val[\widetilde{A}_1] = \langle 0.56, 0.80\rangle$, $Val[\widetilde{A}_2] = \langle 0.51, 0.82\rangle$, $Val[\widetilde{A}_3] = \langle 0.53, 0.78\rangle$.

Step 3 Utilize the score function to evaluate the three candidates and get: $S[\widetilde{A}_1] = 0.40$, $S[\widetilde{A}_2] = 0.38$, $S[\widetilde{A}_3] = 0.41$. Thus, we sort them as $A_3 \succ A_1 \succ A_2$.

Method 3 The q-ROFEWG operator (Riaz et al. 2020)

$$q-ROFEWG(a_1,a_2,\cdots,a_m)$$
$$=\left\langle \frac{\sqrt[q]{2}\prod_{k=1}^{m}u_k^{w_k}}{\prod_{k=1}^{m}(2-(u_k)^q)^{w_k}+\prod_{k=1}^{m}((u_k)^q)^{w_k}},\right.$$
$$\left.\sqrt[q]{\frac{\prod_{k=1}^{m}(1+(v_k)^q)^{w_k}-\prod_{k=1}^{m}(1-(v_k)^q)^{w_k}}{\prod_{k=1}^{m}(1+(v_k)^q)^{w_k}+\prod_{k=1}^{m}(1-(v_k)-^q)^{w_k}}}\right\rangle$$

Step 1 We aggregate the data in Tables 1-3 to obtain the fusion matrix, as shown in Table 7.

Table 7 The q-Rung Orthopair Fuzzy Ddecision Matrix R'

	C_1	C_2	C_3	C_4
A_1	$\langle 0.73,0.77\rangle$	$\langle 0.63,0.73\rangle$	$\langle 0.44,0.79\rangle$	$\langle 0.52,0.86\rangle$
A_2	$\langle 0.61,0.80\rangle$	$\langle 0.43,0.53\rangle$	$\langle 0.46,0.81\rangle$	$\langle 0.60,0.90\rangle$
A_3	$\langle 0.37,0.56\rangle$	$\langle 0.77,0.85\rangle$	$\langle 0.39,0.70\rangle$	$\langle 0.70,0.81\rangle$

Step 2 Similar to the methods 1 and 2, we can obtain the evaluation values of the candidates as: $Val[\widetilde{A}_1]=\langle 0.57,0.80\rangle$ $Val[\widetilde{A}_2]=\langle 0.52,0.82\rangle$ $Val[\widetilde{A}_3]=\langle 0.53,0.77\rangle$.

Step 3 Utilize the score function to evaluate the three candidates are: $S[\widetilde{A}_1]=0.40, S[\widetilde{A}_2]=0.38, S[\widetilde{A}_3]=0.42$. Thus, we sort them as $A_3 \succ A_1 \succ A_2$.

Obviously, all methods have achieved good results (see Table 8), which shows the rationality of the proposed multiplicative integral-based q-rung orthopair aggregation operator. In addition, the integral-based aggregation operator we proposed here has the following advantages over the above-mentioned methods.

1) The multiplicative integral-based q-rung orthopair aggregation operator has better computational efficiency than q-ROFWA, q-ROFWG, q-ROFEWG and other methods when there are more attributes and alternatives. In particular, discrete aggregation operators often have limitations when dealing with large-scale data. Specifically, consider a large data set with n alternatives and n attributes. The integral-based aggregation operator only needs to consider the maximum and minimum evaluation values of the attributes, and then perform $O(n)$ calculations to obtain the integrated information. Assuming there are two attributes, then the computational complexity is about $O(2n)$ times. In contrast, the other methods need to consider each evaluation value information and weight of each attribute, and its computational complexity will increase significantly. Actually it becomes $O(n^2)$ times when there are only two attributes. Obviously, the integral-based aggregation operator is more time effective in dealing with large-scale multi-attribute decision-making problems.

2) The integration-based aggregation method we proposed here focuses on processing and collecting information on the location and distribution of evaluation information, and does not need to rely on too many variable parameters. As the amount of q-rung orthopair fuzzy information increases, the operational advantages of this aggregation operator will become more obvious.

8. Conclusion

In this paper, the theory of q-rung orthopair fuzzy multiplicative integrals is established on the basis of multiplication algorithm. Specifically, we introduce some basic operations, partial orders, complement operations of q-ROFNs, and related properties of q-ROFFs' derivative operations. They are the basis for further research on q-ROFC. Then, in order to derive the multiplicative q-ROFIs, we start with the concept of primitive of q-ROFFs, and then, come up with the definition of indefinite q-ROFIs. Later, we give the definition of the definite q-ROFIs under the multiplicative operation rules. Furthermore, the relationship between q-ROFF definite integral and q-ROFWG operator is discussed, and an information integration method based on q-ROFF definite integrals is proposed. We also reveal the intrinsic relationship between the additive and

Table 8 Ranking Results from Different Methods

Methods	*The score function*	*Ranking*
q-*ROFWA*	$S[\widetilde{A}_1] = 0.52, S[\widetilde{A}_2] = 0.51, S[\widetilde{A}_3] = 0.53.$	$A_3 > A_1 > A_2$
q-*ROFWG*	$S[\widetilde{A}_1] = 0.40, S[\widetilde{A}_2] = 0.38, S[\widetilde{A}_3] = 0.41.$	$A_3 > A_1 > A_2$
q-*ROFEWG*	$S[\widetilde{A}_1] = 0.40, S[\widetilde{A}_2] = 0.38, S[\widetilde{A}_3] = 0.42.$	$A_3 > A_1 > A_2$
q-*ROFFs(in this paper)*	$S[A_1] = 0.40, S[A_2] = 0.30, S[A_3] = 0.41.$	$A_3 > A_1 > A_2$

the multiplicative q-ROFC. Roughly speaking, through the complement operations, we can establish the mutual transformation relationship between these two different types of q-ROFCs.

Finally, we comment on the novelty of this paper and give possible future research directions. First, please note that our research allows the functions to be completely non-linear, that is, the results are valid for all q-ROFFs in the range of $q \in [1, \infty)$. This is more complicated mathematically, but in fact it has a wider range of applications. Future research can be extended from the following aspects: 1) Further study the related properties and theorems of q-ROFC to simplify its operations. 2) Considering the large-scale heterogeneous fuzzy information environment, by constructing the conversion formula of heterogeneous information, the application scope of the proposed integral model is expanded. 3) Combine the q-ROFC theories with different types of aggregation operators can further solve the practical problems of different properties in real life.

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