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# The Evolution Dynamic and Long-Run Equilibrium in a Stock Market with Heterogeneous Traders

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**Abstract.** This paper uses ideas from biological evolution to analyze the evolution of the securities market in which rational and irrational traders coexist. A market evolutionary model is developed to describe the dynamic trajectories of rational and irrational traders' wealth. The main question is, are irrational traders eliminated as the securities market evolves. The paper considers the impact of new entrants on the security market long-term equilibrium. In addition, it discusses the existence and uniqueness of the long-term equilibrium The paper finds that, under some market conditions, irrational traders could survive in the long run.

**Keywords:** Behavioral Finance, irrational traders, financial evolution theory, random dynamic system

## 1. Introduction

Alchian (1951) and Friedman (1953) proposed that market participants who pursued the maximum return would survive and profit. Based on their ideas, Fama (1965) in his classic work on the efficient-market hypothesis analyzed arbitrage behavior (EMH), and drew the conclusion that financial markets would eventually become efficient because irrational traders would be ultimately eliminated due to long term losses.

Fama's argument relies on the presupposition of market validity and is supported by the assumption that irrational traders would exit losses caused by rational traders' arbitrage behavior. This is virtually a circular argument based on the premise of rational expectation. If the market is not perfectly effective, arbitrage based on rational expectation is not risk-free and rational traders are not a stable market force. However new irrational traders will constantly enter the market even if former irrational traders have left the market due to continual long-term losses. If irrational traders as a group, completely disappeared from the financial market, rigorous analysis would still be required. Therefore, the key to analyzing financial market evolution is the investigation of the dynamics involved between these two types of traders and how these dynamics affect market stability.

This investigation divides traders into rational and irrational traders using the principals of biological population dynamic evolution. The irrational traders and rational traders definition just follow Fama (1970), Fama (1991) and Hirshleifer et al. (2005) etc. It establishes a differential dynamic system model as well as a random dynamic system model to describe a law of stock market evolution. It aims to describe the dynamic trajectories and long-term equilibrium of different types of traders. The investigation begins with a thorough literature review regarding market evolutionary dynamics. Then an evolutionary model based on a differential dynamic system is developed and simulations are performed with subsequent analysis. Finally, the differential dynamic system model is extended to a random dynamic system model which is used to simulate and analyze market evolution.

## 2. Literature Review

Alchian and Friedman proposed that natural selection eliminates market participants who do not pursue the maximum return. This line of reasoning is widely recognized but lacks a formal analysis that encompasses the long-term behavior of the market. This issue was not addressed until the early 1990s when Blume and Easley (1992) deduced the Market Selection Hypothesis, which modeled long term arbitrage behavior and derived trader survival fitness. Since then a series of formal models have appeared that investigate financial market evolution, and these are fundamentally divided based on two assumptions.

One line of reasoning relies on the belief that irrational traders would exit the market under certain conditions. In 2006, Blume and Easley discussed the Market Selection Hypothesis further. Their research results showed that there was doubt in the Market Selection Hypothesis, which relies on the belief that rational traders would be eliminated from market if the market was not perfect. Other scholars have also investigated this hypothesis that the market would naturally eliminate irrational traders. Biais and Shadur (2000) considered the possibility that because irrational traders might overestimate or underestimate future profits and "stubbornly" insist that their judgments are correct they may obtain higher yields than rational traders since their irrational trade behavior could objectively improve their bargaining power. Based on that research, Biais and Shadur analyzed the evolution dynamic of irrational traders and concluded that irrational traders could survive in the long term with enough assets in the market Chiarella et al. (2006) divided market traders into two types, fundamentalists and chartists, which was a derivation of the rational and irrational traders reasoning. They demonstrated that these two types of traders could coexist in the financial market over the long term. Research conducted by Zhang et al. (2010) also demonstrated that the stock market couldn't eliminate various types of traders, such as noise traders and feedback traders with the conditions of endogenous prices and limits of arbitrage.

Another group of researchers believe that market evolution can achieve a rational equilibrium. Sandroni (2005) proposed that traders with accurately beliefs could eventually control the market. Evstigneev et al. (2006) developed a discretetime model wherein stock markets can evolve to a stable ideal state. The necessary and sufficient conditions for stable market evolution include the assumptions that the stock price would be equal to the mathematical expectation of the present value of future dividends Yang and Qin (2006) studied stock market evolution using a continuous time model based on Evstigneev's work. Their research show that if one assumes a positive investor's initial wealth, using a strategy of buying shares of relative assets with dividends greater than the relative price would be an effective strategy that would eventually control the whole of stock market wealth. The research conclusions of Evstigneev et al., Zhao-jun Yang and others were a response to the research results of Blume and Easley (1992), Blume and Easley (2006). They concluded that rational strategies are the fittest trading strategies to adapt to the market environment. Some empirical research does show that smart money can correct mispricing in stock market (Akbas, Armstrong et al. 2015).

The question of long-term equilibrium in stock market evolution is still a pending and controversial problem with two schools of thought that have emerged Neoclassical Finance and Behavioral Finance. First, previous studies viewed the stock market as a closed system without considering the impact of new entrants into each group. In fact even if some groups could be eliminated gradually due to their own unsuitable trading strategies they would be replaced by new entrants and their type would continue to impact the market. This means stock market efficiency could be disrupted with irrational traders pouring into the market and this would change the market evolution process.

However most market evolutionary research has used only one kind of utility function to build models and use utility function to analyze the trading behavior of traders (see Biais and Shadur (2000), Blume and Easley (1992), Blume and Easley (2006), Chiarella et al. 2006, Evstigneev et al. (2006), etc). Then but as the research of Bottazzi (2014 2017ab) showed: the optimality of a portfolio rule derived from utility maximization relies on agents having perfect foresight on future prices. But the hypothesis of the rational expectation and optimal decision have been demonstrated to not be reliable (Kahneman and Tversky 1979 1991). Even if the rational trader's behavior can be descripted by optimal utility function, the irrational traders' behavior shouldn't use such functions to analyze (see Guo and He (2017)). So we didn't adopt the utility function method to analyze the evolutionary dynamic of stock market. For simplicity, just assume that rational trader can have higher returns than irrational traders. It's just the points of Alchain and Friedman, and also confirmed by empirical researchs (see Grinblatt and Titman (1993)).

A differential dynamic system model to describe stock market evolution has been developed in this study. The model considers the impact of the new entrants on long-term market evolution. At the same time, it considers the impact of endogenous price on short-term dynamics and the long-term equilibrium of the stock market by the setting of specific parameters. Parameters such as traders' wealth at any time, stock market return, and net market entry rate are always random in a real market. Therefore, this study extends the model to a random dynamic model to simulate and analyze dynamic trajectories and the long-term equilibrium in stock market evolution.

## 3. A Stock Market Evolution Model Based on a Differential Dynamic System

Consider a market with heterogeneous traders in which rational traders wealth at time $t$ is $x_t$ and irrational traders wealth at time $t$ is $y_t$ Yields are $r_x$ and $r_y$ respectively. Market yield is $r_x$ when all participants are rational traders. Similarly, market yield is $r_y$ when all participants are irrational traders. It was Friedman's argument that when the yield of rational traders was higher than the average market yield rational traders would gradually expel irrational traders when the stock market is composed of heterogeneous traders. Thus it is apparent that the yields of rational traders should decrease progressively to the average market yield with the reduction of irrational traders to extinction. That is, $r_x$ is a function of market structure, namely $r_x(x, y; \theta_1)$ and in the same way $r_y$ is also a function of market structure namely $r_y(x, y; \theta_2)$ in which, $\theta_1, \theta_2$ are parameter vectors. The evolution dynamic function can be described as follows.

$$\begin{cases} \frac{dx_t}{dt} = f(x_t, r_x(x_t, y_t; \theta_1)) \\ \frac{dy_t}{dt} = g(y_t, r_y(x_t, y_t; \theta_2)) \end{cases} \tag{1}$$

### 3.1 A Stock Market Evolution Model as a Closed System

To analyze the dynamic trajectory and long-term equilibrium of market evolution, the model needs to find the appropriate function to model the form of the trajectory. To simplify the situation, an initial assumption was made that there are no new entrants to the stock market Namely the market will be analyzed as a closed system. It can be seen from function (1) that to analyze the above differential dynamic system it should take the form of $f(\cdot)$ and $g(\cdot)$ and the yield functions $r_x(\cdot)$ and $r_y(\cdot)$. The following will attempt to find the functions $f(\cdot)$ and $g(\cdot)$ first and then analyze the yield function.

### 3.1.1 Basic Framework of Closed Model

We posit the growth of rational traders' wealth at time $t \frac{dx_t}{dt}$ equals the product of rational traders' wealth yield and rational traders' wealth at time t according to the above analysis. This is expressed as follows.

$$\frac{dx_t}{dt} = x_t \cdot r_x \tag{2}$$

In the same way, the growth of irrational traders' wealth $\frac{dy_t}{dt} = y_t \cdot r_y$ can be described using the differential function below.

$$\frac{dy_t}{dt} = y_t \cdot r_y \tag{3}$$

Equations (2) and (3) constitute the differential functions that describe the wealth change dynamic of the two types of traders.

$$\begin{cases} \frac{dx_t}{dt} = x_t \cdot r_x \\ \frac{dy_t}{dt} = y_t \cdot r_y \end{cases} \tag{4}$$

### 3.1.2 Parameter Analysis on Closed Market Evolution Model

The functions $r_x(\cdot)$ and $r_y(\cdot)$ are analyzed in the following manner. Assuming stock market yield is $r_m$ the following equation can be deduced:

$$x_t \cdot r_x + y_t \cdot r_y = (x_t + y_t) \cdot r_m \tag{5}$$

Based on the proposition of Alchian and Friedman, irrational traders will be seen as prey in the market. Therefore, we assumed that the yield of irrational traders would be described as follows:

$$r_y = r_m - \alpha \tag{6}$$

Function (7) can be obtained by synthesizing functions (5) and (6):

$$\begin{cases} r_x = \alpha \cdot \frac{y_t}{x_t} + r_m \\ r_y = r_m - \alpha \end{cases} \tag{7}$$

From the function (7) we can obtain the yield of rational traders and the yield of irrational traders. These are related to market structure, and are correlated as described above. Namely $\frac{x_t}{x_t+y_t} \rightarrow 1$ decreases with the increase of rational traders' proportion in the market Specifically, the properties are the following.

(1) The yield of rational traders decreases with the enhanced proportion of rational traders' yield to the market and this curve is convex to the origin.

(2) When $\frac{x_t}{x_t+y_t} \rightarrow 1$ rational traders' yield is market yield when rational traders completely dominate the market. The function properties are shown in Figure 1.

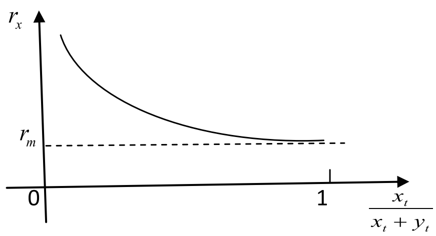


**Figure 1** Characteristic Chart of the Rational Traders Yield

$r_x = r_m$ if $x_t = x_t + y_t$ when $y_t \rightarrow 0$ In addition, the function curve is convex to the origin.

### 3.1.3 The Differential Dynamic Model of Closed Market Evolution

The differential dynamic system used to describe the market evolution dynamic was obtained from functions (7) and (4).

$$\begin{cases} \frac{dx_t}{dt} = \alpha \cdot y_t + x_t \cdot r_m \\ \frac{dy_t}{dt} = y_t(r_m - \alpha) \end{cases} \tag{8}$$

In function (8), $x_t$ represents rational traders' wealth and $y_t$ represents the irrational traders' wealth at time t $r_x$ and $r_y$ represent their yields respectively. $r_m$ represents the market yield. $\alpha$ represents the predation rate of rational traders on irrational traders.

### 3.2 The Stock Market Evolution Dynamic Model as an Open System

The assumptions used in the above model are too strict. New traders enter the stock market constantly and traders continue to leave since the market is an open system. Simplifying the impact of entrants and quitters into a net entry rate in the model is a more comprehensive and more precise method to describe the dynamic trajectory of stock market evolution. Intuitively, the net entry rate has several potential values. One is the constant entry rate which means a fixed increase of rational traders and irrational traders at any time $t$. The second is linear entry which means rational traders and irrational traders enter the market with a fixed proportion of stock of wealth. The third potential value for net entry rate is complex, as it includes other factors, such as entry rates related to market boom status and other factors.

Take the Chinese Shanghai Stock Exchange as an example It assumes that institutional investors are rational traders and individual investors are irrational traders. The new annual account numbers of the two types of investors shows an increasingly progressive linear characteristic and this is affected by the Shanghai Stock Exchange boom status. The figure below shows the correlation between the annual new account numbers and total account numbers of individual investors in the Shanghai Stock Exchange from 1995 to 2015. The result of regression analysis showed a regression coefficient of new account numbers to total account numbers of 0.062 ($R^2 = 0.16$, $t = 1.84$). The regression coefficient of annual new account numbers to total account numbers of institutional investors in this market was 0.00023 ($R^2 = 0.17$, $t = 1.92$). Figure 2 shows the correlation between annual new account numbers and total account numbers of individual investors in the Shanghai Stock Exchange.

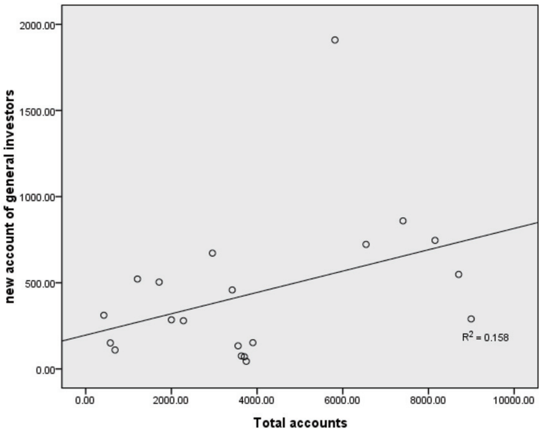


**Figure 2** The Correlation between New Account Number and Total Account Number in the Shanghai Stock Exchange from 1995 to 2015. (Data Source: Shanghai Stock exchange Yearbook,421-422, 2016, )

It could be assumed from the above analysis that there is a linear correlation between the rate with which rational traders and irrational traders enter the market at time $t$ and the stock of wealth of institutional and individual investors Namely, rational traders' wealth is $e_x(x_t + y_t)$ in which $e_x$ is the net rational traders entry rate when they enter the market at time t. Irrational traders' wealth is $e_y(x_t + y_t)$ in which $e_y$ is the net irrational traders entry rate when they enter the market at time $t$. The above assumptions could be called the Linear Entry Hypothesis which describes the entry process dynamics of the two types of traders during the initial stages of stock market development.

Based on the above analysis, this model utilizes the following assumptions:

**Assumption 1** There is no conversion between rational traders and irrational traders.

**Assumption 2** The market investment strategies of rational traders and irrational traders are exogenous from asset prices. In addition, the returns of rational trader are always higher than irrational trader returns.

**Assumption 3** The net entry ratio of rational traders to irrational traders is a linear function of stock market assets.

Thus, the differential function that describes stock market evolution as an open system is as follows:

$$\begin{cases} \frac{dx_t}{dt} = \alpha y_t + r_m x_t + e_x\left(x_t + y_t\right) \\ \frac{dy_t}{dt} = \left(r_m - \alpha\right) y_t + e_y\left(x_t + y_t\right) \end{cases} \tag{9}$$

Considering the meaning of the parameters $r_m, \alpha, e_x, e_y$ obviously $r_m, \alpha, e_x, e_y \geq 0$ in the true stock market.

## 3.3 Analysis of the Deterministic Evolution Model

The following section describes how solutions of function (9) are calculated and how this expresses the differential dynamic system. Then the economic meanings of the solutions are discussed Subsequently, we discuss the solution obtained under the condition of no new entrants to show the impact of new entrants on evolution. In addition, Matlab software was used to simulate the model.

### 3.3.1 Solution to the Model

The solution to Function (9) was obtained as follows.

$$\begin{cases} x_t = c_1 \exp\left\{\left(r_m - \alpha\right) t\right\} \\ \qquad + c_2 \exp\left\{\left(e_x + e_y + r_m\right) t\right\} \\ y_t = c_2 \frac{e_y}{e_x + \alpha} \exp\left\{\left(e_x + e_y + r_m\right) t\right\} \\ \qquad - c_1 \exp\left\{\left(r_m - \alpha\right) t\right\} \end{cases} \tag{10}$$

When $t = 0$, $x_t = x_0$, $y_t = y_0$. $x_0$ represents the rational traders' wealth at the beginning of the simulation Meanwhile, $y_0$ represents the irrational traders' wealth at the beginning. The undetermined coefficients, $C_1, C_2$ were obtained as follows:

$$C_1 = \frac{e_y x_0 - (\alpha + e_x) y_0}{e_x + e_y + \alpha}$$

$$C_2 = \frac{(\alpha + e_x)(x_0 + y_0)}{e_x + e_y + \alpha}$$

Equation (11) was obtained by taking the parameters $C_1, C_2$ into the model solution (10) and simplifying.

$$\begin{cases} x_t = \frac{e_y x_0 - (\alpha + e_x) y_0}{e_x + e_y + \alpha} \exp\left\{\left(r_m - \alpha\right) t\right\} \\ \qquad + \frac{(\alpha + e_x)(x_0 + y_0)}{e_x + e_y + \alpha} \exp\left\{\left(e_x + e_y + r_m\right) t\right\} \\ y_t = \frac{e_y (x_0 + y_0)}{e_x + e_y + \alpha} \exp\left\{\left(e_x + e_y + r_m\right) t\right\} \\ \qquad - \frac{e_y x_0 - (\alpha + e_x) y_0}{e_x + e_y + \alpha} \exp\left\{\left(r_m - \alpha\right) t\right\} \end{cases} \tag{11}$$

Equation (9) is the model, and Equation (11) is the analytic solution. The change of rational traders' wealth and irrational traders' wealth are related to a series of factors, such as market yield, predatory rate of rational traders on irrational traders and the net entry rate of the two types of traders. The following discussion analyzes the model and model's analytic solution in detail.

### 3.3.2 Model Solution Analysis

In the analysis of stock market evolution, the choice of whether the argument is based on Behavioral Finance or Neoclassical Financial is determined by whether or not the researcher believes the stock market could be perfectly effective by eliminating irrational traders in longterm. Namely whether or not $\frac{y_t}{x_t + y_t} \to 0$ when $t \to \infty$ Another concern is model solution stability. More specifically, whether small perturbations during the evolutionary beginning could lead to drastic changes later on. The two cases are discussed below.

***-Model solution stability analysis***

Initially, model solution stability analysis (in the Lyapunov sense) and the analysis of small perturbations at the beginning of evolution could lead to drastic changes requires analytical investigation. Function (9) is a typical linear model and is expressed as follows.

$$\frac{df(\boldsymbol{x})}{dt} = A\boldsymbol{x} \tag{12}$$

in which:

$$A = \begin{bmatrix} r_m + e_x & \alpha + e_x \\ e_y & r_m + e_y - \alpha \end{bmatrix}$$

$$\boldsymbol{x} = (x, y)^{\mathsf{T}}$$

For such differential function types as equation (12), zero solution stability discrimination can be attributed to matrix eigenvalue analysis. The characteristic function of Matrix $A$ in equation (12) is as follows:

$$\lambda^2 - (2r_m + e_y + e_x - \alpha)\lambda + (r_m - \alpha)(r_m + e_x + e_y) \tag{13}$$

There are two real roots of characteristic function.

$$\lambda_1 = r_m - \alpha, \quad \lambda_2 = r_m + e_x + e_y$$

For $\lambda_1, \lambda_2$, at least one is positive in a general sense. Thus, the solution to equation (9) is unstable at the initial value (Lyapunov sense). Specifically, small perturbations at the beginning of evolution can lead to drastic changes.

***-Long-term equilibrium analysis of the model***

The long-term equilibrium of stock market evolution with heterogeneous traders is described as $\lim_{t\to\infty} \frac{y_t}{x_t+y_t}$ which is the proportion of irrational traders in the market when $t \to \infty$ Equation (14) is obtained from function (11).

$$\lim_{t\to\infty} \frac{y_t}{x_t+y_t} = \lim_{t\to\infty} \frac{\frac{e_y(x_0+y_0)}{e_x+e_y+\alpha}\exp\{(e_x+e_y+r_m)t\}}{(x_0+y_0)\exp\{(e_x+e_y+r_m)t\}} - \frac{\frac{e_y x_0 - (\alpha+e_x)y_0}{e_x+e_y+\alpha}\exp\{(r_m-\alpha)t\}}{(x_0+y_0)\exp\{(e_x+e_y+r_m)t\}} \tag{14}$$

where $e_x, e_y > 0$ when the stock market grows. Equation (15) is obtained from equation (14).

$$\lim_{t\to\infty} \frac{y_t}{x_t + y_t} = \frac{e_y}{e_x + e_y + \alpha} \tag{15}$$

Equation (15) shows the result of stock market evolution when there are new entrants. In this situation, the market could not eliminate irrational traders and obtain a long-term equilibrium of heterogeneous trader coexistence even when the rational traders' yield was always higher than the irrational traders' yield.

***-Simulation analysis of model***

Simulation analysis was performed based on function (11). The dynamic trajectory of stock market evolution was intuitively observed by setting different levels of $r_m, \alpha, x_0, y_0$, by using the basic assumptions used in this study. MATLAB 7.0 was used as a simulation tool to analyze the long-term equilibrium of stock market evolution as an open system.

The annual yield of the Shanghai Composite Index (SCI) is 4.67% and daily yield of the SCI is approximately 0.022%, according to data obtained from the Chinese stock market from the dates of December 31, 1992 to December31, 2013. Therefore, was set to 0.022% and the original rational traders' wealth and original irrational traders' wealth were set to 20 and 80[1] respectively. According to function (10), was set to 0.01%, $e_x$ was set to 0.005%, $e_y$ was set to 0.02%, and the period was set to 8,000 working days (in the last 40 years). The dynamic trajectories of the irrational traders' wealth as a proportion of the total market wealth were respectively calculated for the two cases of new entrants and no new entrants.

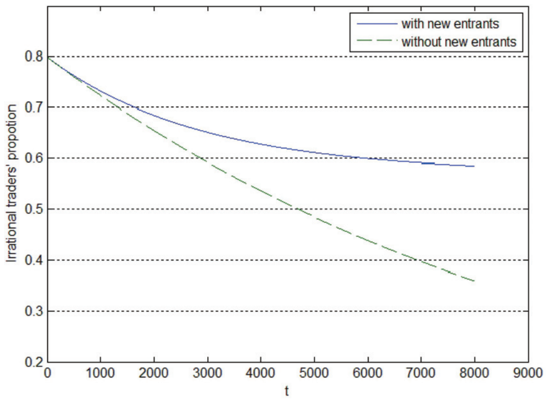


**Figure 3** The Impact of New Entrants on Stock Market Evolution

New entrants changed the stock market evolution significantly, as this situation is an open system. This can be clearly seen in Figure 3. On the one hand, new entrants slowed the decline of the proportion of irrational traders' wealth. On the other hand, the market ultimately failed to eliminate irrational traders but it did converge to a long-term equilibrium of heterogeneous trader coexistence even when the rational traders' yield was always higher than the irrational traders' yield.

## 4. The Random Evolution Dynamic Model

The above differential dynamic model assumes the stock market is an open system of heterogeneous traders in coexistence to describe market evolution dynamics and long-term equilibrium.

Because these two types of traders' total wealth is obviously random, the above model was extended to a random model. At the same time parameter assumptions include a continuous as asset return for both rational traders and irrational traders. To simplify the problem, only $x_t, y_t$ in Function (9) were changed to random parameters. Additionally, the random impact of the market was reduced in the model. Assuming white noise, the random dynamic system model was established to analyze short-term dynamics and the long-term equilibrium of stock market evolution.

The market evolution model, with the addition of white noise, was calculated by consolidating Function (9) and is as follows:

$$\begin{cases} d\xi_t = [(r_m + e_x)\xi_t + (\alpha + e_x)\eta_t]dt + \xi_t \sigma_x dB_{1t} \\ d\eta_t = [e_y \xi_t + \left(r_m - \alpha + e_y\right) \eta_t]dt + \eta_t \sigma_y dB_{2t} \end{cases} \tag{16}$$

In Function (16), the economic meanings of $r_m, \alpha, e_x, e_y$ are the same as in Function (9). To simplify the problem, they are assumed to be constants. The random variables of $\xi_t, \eta_t$ represent the rational traders' and irrational traders' total wealth respectively. $\sigma_x, \sigma_y$ represent the total wealth diffusion coefficients of the two types of traders. $B_{it}, i = 1, 2$ represents the relatively independent standard Brownian motion.

### 4.1 Analysis of The Random Dynamic System Model

The following is the analysis of the random evolution dynamic described by function (16). This function should solve the following questions: 1) Does the random model have a global and unique solution? 2) What is the random evolution long-term equilibrium of the stock market as an open system? 3) Will irrational traders become extinct under the random condition? If not, what are the conditions for them to exist long-term?

#### 4.1.1 The Existence and Uniqueness of the Global Positive Solution

**Theorem 1** *for any given initial values of $\xi_0, \eta_0$, function (16) has a unique solution described by $\xi_t, \eta_t \in R^+$, a.s. when $t \geq 0$.*

**Proof.** Theorem 1 could be proved directly according to theorem 5.2.1 in Oksendal (2003) because function (16) satisfies the linear growth and the local Lipshitz condition. In addition, inequality (17) can be obtained.

$$E\left[\int_0^T |X_t|^2 \, dt\right] < \infty \tag{17}$$

in which $|X_t| = \left|\xi_t + \eta_t\right| = \sqrt{\xi_t^2 + \eta_t^2}$. The above proof shows that the random evolution model of the stock market as an open system has a unique global solution. In addition, inequality (17) shows that the mean square expectation of the model is limited in any limited time $T$. ■

#### 4.1.2 The Long-term Evolution Dynamic of the Random System

The boundary problem of long-term market dynamic evolution is discussed in this section. The total wealth growth trend of the two types of traders is analyzed when time tends to infinity.

**Theorem 2** *The total wealth expectation of the stock market as described in function (16) displays a progressively ex-*

*ponential growth trend and the* $\lim_{t\to\infty} E \cdot (\xi_t + \eta_t) = \infty$.

**Proof.** $K = r_m + e_x + e_y$ and $V(X(t)) = \xi_t + \eta_t$ are defined for the convenience of this analysis. The following is obtained through Ito's formula

$$\begin{aligned} dV(X(t)) =& \frac{\partial V}{\partial \xi} d\xi + \frac{\partial V}{\partial \eta} d\eta \\ =& K(\xi_t + \eta_t)\, dt + \xi_t \sigma_x dB_{1t} + \eta_t \sigma_y dB_{2t} \end{aligned}$$

The stopping time is defined as the following for every integer $k \geq |X(0)|$.

$$\tau_k = \inf\{t \in \mathrm{R}^+; |X(t)| \geq k\}$$

The mathematical expectation of $V(X(t))$ is calculated as follows:

$$\begin{aligned} E(V(X(\tau_k \wedge T))) =& V(X(0)) \\ &+ K \int_0^T E(V(X(\tau_k \wedge t)))\, dt \end{aligned}$$

Equation (18) is obtained by setting $t \to \infty$ due to $V(X(0)) > 0, K := r_m + e_x + e_y > 0$.

$$\lim_{t\to\infty} E(V(X(t))) = \lim_{t\to\infty} V(X(0))\, E\left(e^{Kt}\right) = \infty \tag{18}$$

Thus theorem 2 is proved which shows that the total wealth growth trend of the two types of traders tends to be infinite when time tends to infinity. ■

### 4.1.3 The Long-term Existence of Irrational Traders

The long-term existence of irrational traders is the core issue in market evolution modeling. The long-term existence of irrational traders should be clarified by the ability to prove Theorem 3 when $\eta_t \to \infty$ and $t \to \infty$ under some given conditions.

**Theorem 3** *there is* $\lim_{t\to\infty} \eta_t \to \infty$ *if* $r_m - \alpha + e_y > \frac{1}{2}\sigma_y^2$.

**Proof.** Define: $H = r_m - \alpha + e_y$. Function (19) is obtained from function (16).

$$\eta_t = \int_0^t e_y \xi_t dt + \int_0^t H \eta_t dt + \int_0^t \eta_t \sigma_y dB_{2t} \tag{19}$$

Define $\zeta_t$ as a stochastic variable that satisfies:

$$\zeta_t = \int_0^t H \zeta_t dt + \int_0^t \zeta_t \sigma_y dB_{2t} \tag{20}$$

For $e_y > 0$ and $\xi_t \in \mathbb{R}^+$ there is $\eta_t > \zeta_t, t \geq 0$ with a probability of 1 due to the random comparison theorem (Ikeda and Watanabe (1977); Seather et al. (2000)). ■

Define: $g(t, \zeta_t) = \ln \zeta_t$. The following is obtained easily from the Ito formula

$$\zeta_t = \zeta_0 \exp\left(\left(H - \frac{1}{2}\sigma_y^2\right) t + \sigma_y B_t\right) \tag{21}$$

The following conclusions were obtained from the analysis of equation (21).

1) if $r_m - \alpha + e_y > \frac{1}{2}\sigma_y^2$ there is $\zeta_t \to \infty$ a.s. when $t \to \infty$.

2) if $r_m - \alpha + e_y < \frac{1}{2}\sigma_y^2$, there is $\zeta_t \to 0$ a.s. when $t \to \infty$.

3) if $r_m - \alpha + e_y = \frac{1}{2}\sigma_y^2$, $\zeta_t$ fluctuates in $(0, \infty)$ a.s. when $t \to \infty$.

At the same time, for $\eta_t > \zeta_t$, a.s., Theorem 3 is proved.

Theorem 3 demonstrated that if the sum of irrational traders return plus the net entry yield was sufficiently large as compared to the perturbation it received, then irrational traders could exist for the long-term in the market and their wealth would keep growing infinitely. Conversely, if the market return, predation rate and irrational traders net entry ratio was smaller than half the variance of the irrational traders' wealth change, then it was possible for irrational traders to become extinct.

On average, the returns of general investors are positive, and the entry rate is positive. The volatility variance of the market is far less than the market return plus entry rate (0.0003% vs 0.024% calculated on a daily basis, see the Shanghai Stock Exchange Statistics Annual 2018). Therefore, the long-term equilibrium of the market will be a state of coexistence of heterogeneous traders.

## 4.2 Simulation Analysis of the Random Evolution Model

It is very difficult to obtain analytic solutions of the random evolution dynamic model (function (16)). Therefore, a simulation method of the stock market random evolution dynamic trajectory was considered. In addition, repeated simulation means of the sample means of traders' wealth long-term expectations were simulated and analyzed. Then function (16) was calculated by the method put forward by Higham (2000).

### 4.2.1 Sample Trajectory of Random Evolution Dynamic

Function (22) was obtained by discretization for function (16) according to the Higham (2000) method.

$$\begin{cases} \xi_{t+1} = \xi_t + [(r_m + e_x)\xi_t + (\alpha + e_x)\eta_t]\Delta t \\ \qquad +\xi_t \sigma_x \sqrt{\Delta t}\,\varepsilon_{1t} \\ \eta_{t+1} = \eta_t + [e_y \xi_t + \left(r_m - \alpha + e_y\right) \eta_t]\Delta t \\ \qquad +\eta_t \sigma_y \sqrt{\Delta t}\,\varepsilon_{2t} \end{cases} \tag{22}$$

In which $\varepsilon_{1t}, \varepsilon_{2t}$ are random variables subject to the standard normal distribution

Using function (22), the setting of parameters is similar to the deterministic model, and they were set $r_m = 0.022\%$, $\alpha = 0.01\%$, $e_x = 0.005\%$, $e_y = 0.02\%$, $\sigma_x = 0.01\%$, $\sigma_y = 0.01\%$, $\xi_0 = 0.2$, $\eta_0 = 0.8$, and a time period of 8000 working days. Figure 4 shows the simulation result produced by Matlab.

It can be seen clearly from Figure 4 that as time progresses, the wealth of rational traders and irrational traders appears as a growth trend during normal market growth progress. Therefore, rational trader wealth grows more rapidly.

### 4.2.2 The Long-term Equilibrium of the Random Evolution Dynamic

Higham (2000) proposed a method of obtaining expectation by using a random dynamic system which estimated variable expectation by obtaining sample means obtained from re-

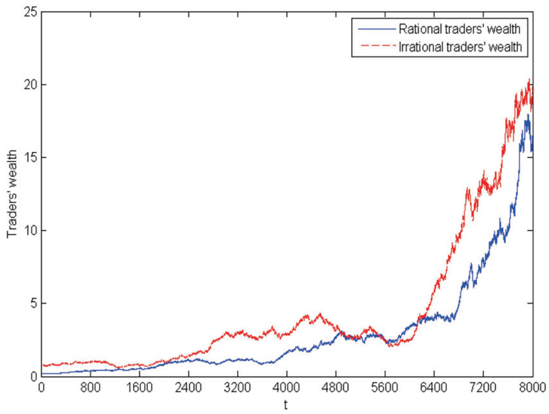


**Figure 4** A Sample trajectory of the Stock Market Random Evolution Dynamic

peated simulation runs of the model. According to this, $\xi_t$ and $\eta_t$ were solved after function (22) was simulated one hundred times. At that time, 100 simulations required approximately five hours of run time due to limited computer performance, therefore it was impossible to reduce the error value between the sample mean and the expectation by the way of greatly increasing the number of samples. Then $\eta_t$ was solved to represent the proportion of irrational traders' wealth in total market wealth. The dynamic trajectory of $\eta_t$ is shown in Figure 5.

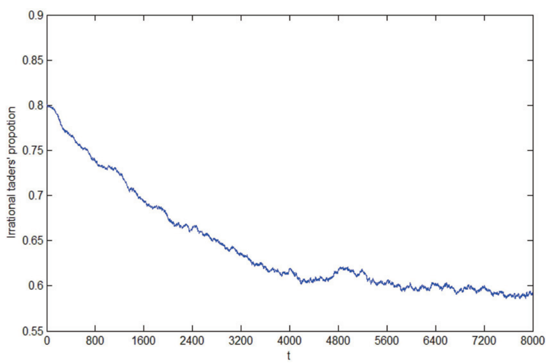


**Figure 5** Dynamic Trajectory of the Proportion of Irrational Traders' Wealth to Total Market Wealth

It can be seen from Figure 5 that with time there was a decrease in the proportion of irrational traders in the market but the amplitude gradually decreased. This is similar to a deterministic model, where the proportion of irrational traders in the market would converge to a non-zero long-term equilibrium value in the long-term in random model.

### 4.3 The Extension of the Random Evolution Model

Research shows that there are different levels of rationality and behaviors attributed to traders, such as loss aversion mental account, and other behaviors. Because the use of the terms rational traders and irrational traders is insufficient to describe the entirety of market characteristics the idea of developing an extending function using a multivariate random differential equation set was considered. This is a more realistic and detailed method to describe market evolution dynamic trajectory and characteristics.

Assuming there exists $n$ different types of traders including rational traders in the market, the return of each type was defined as $r_i, i = 1, 2, \ldots, n$ respectively. The net entry rate of each type was defined as $e_i, i = 1, 2, \ldots, n$. To simplify the problem, linear entry for all types of traders was assumed.

Function (16) could then be extended to the following form under the above assumptions.

$$\mathbf{dX(t) = A_{ij}X(t)dt + \sigma_i X(t)dB}_i\mathbf{(t)} \tag{23}$$

in which

$$\boldsymbol{X}(\boldsymbol{t}) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{pmatrix}$$

$$\boldsymbol{A_{ij}} = \begin{pmatrix} r_1 + e_1 & e_2 & \cdots & e_n \\ e_1 & r_2 + e_2 & \cdots & e_n \\ & \vdots & & \\ e_1 & e_2 & \cdots & r_n + e_n \end{pmatrix}$$

$$\boldsymbol{\sigma_i} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{pmatrix}^{\mathsf{T}}, \quad \boldsymbol{B_i}(t) = \begin{pmatrix} B_{1t} & 0 & \cdots & 0 \\ 0 & B_{2t} & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & B_{nt} \end{pmatrix}$$

Where $\sigma_i, i = 1, 2, \ldots, n$ represents the variances of all the types of traders' wealth change. $B_{it}$is independent standard Brownian motion.

The following results were obtained from analysis of function (23).

1) The system described by function (23) had a unique global solution.

It could be proved directly according to theorem 5.2.1 in Oksendal (2003) because function (23) satisfies the linear growth and the local Lipshitz condition.

2) If $\sum_{i=1}^{n} r_i > 0$, and $\sum_{i=1}^{n} e_i > 0$ total market wealth increased with time and tended to infinity when time tended to infinity.

The proof of this results just the same as the theorem 2. Define the market return as $R_m$, $R_m$ is the weighted average of $r_i$. Define the total entry of traders as $E_T = \sum_i e_i$. The market wealth $W(X(t)) = \sum_i x_i(t)$, Then following is obtained:

$$\begin{aligned} \lim_{t\to\infty} E(W(X(t))) &= \lim_{t\to\infty} W(X(0))\, E\left(e^{(R_m+E_T)t}\right) \\ &= \infty \end{aligned}$$

3) For each type of trader existence or extinction depended on his return and the perturbation received. He could not become extinct as long as the sum of his return plus net entry rate was sufficiently large as compared to his wealth fluctuation.

The proof process of this conclusion is similar to that of Theorem 3. For arbitrary type of traders, there exists:

$$\begin{aligned} x_i(t) &\geq \int_0^t (r_i + e_i) x_i(t)_t\, dt + \int_0^t x_i(t)\, \sigma_i dB_{it} \\ &= x_i(0) \exp\left(\left(r_i + e_i - \frac{1}{2}\sigma_i^2\right) t + \sigma_i B_{it}\right) \end{aligned}$$

This result is similar to the results obtained by Evstigneev et al. (2006) and Yang and Qin (2006). This thesis proves that the market as a closed system could eliminate irrational traders. But in reality the market is an open system. So further results of this thesis indicate that long-term market equilibrium can be obtained with the co-existence of rational traders and irrational traders due to the impact of new entrants.

Lo (1999 2004) proposed an adaptive market hypothesis that considers the scenario of the mixed behaviors of rational traders and irrational traders. In addition, it assumes that traders evolve with and adapt to the market environment. His conclusions support this hypothesis. Based on this, we can draw the conclusion that irrational traders are likely to survive well when the market is booming and tend to be eliminated when the market is in recession (Equation (11)). This conclusion is highly consistent with the conclusion by Andrew Lo (2004):

*If multiple species are competing for rather scarce resources within a single market, that market is likely to be highly efficient, . . . , If, on the other hand, a small number of species are competing for rather abundant resources in a given market, that market will be less efficient.*

## 5. Conclusion

1) Under the condition that the stock price is endogenous, the stock market evolution system is an autonomous system. Evolution results do not depend on starting time. The evolution process has a series of autonomous system characteristics such as uniqueness of the evolution trajectory and additivity of evolution process to the time.

2) The wealth expectation of the two types of traders displays an exponential growth trend and this total wealth tends to be infinite when time tends to infinity.

3) The stock market whether as a deterministic or random system, failed to thoroughly eliminate irrational traders even when the rational traders' yield was always higher than the irrational traders' yield in condition of new entrants. Under some conditions the market will ultimately stabilize into a status of coexistence of rational traders and irrational traders, with a constant decline in the proportion of irrational traders in the market.

4) The zero solution of the stock market evolution model is unstable in the Lyapunov sense Small perturbations of the original rational traders' wealth and original irrational traders' wealth could be amplified in the evolution and ultimately lead to drastic wealth change But the proportion of the rational to irrational traders' wealth remained stable.

5) The random evolution model of the stock market as an open system had a unique positive global solution. Further, the mean square expectation of total wealth in the market was limited in limited time.

In this investigation market return was set as a constant parameter to simplify the question in the random model. In addition, all random factors were summed to represent white noise and this was included in the analysis of the model. However, in reality, market evolution dynamics can be perturbed by one significant major factor and other secondary factors. Trader return is a major factor which was included in the model as a Markov process A new extension market evolution model that contains Markov impact would likely produce more accurate results.

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### Endnotes

[1] The initial proportion of traders' wealth is arbitrary. It does not affect the final evolution result.

## References

Akbas F, Armstrong W J, Sorescu S, Subrahmanyam A (2015). Smart money, dumb money, and capital market anomalies. *Journal of Finance Economics*. 118(2): 355-382.

Alchian A A (1951). Uncertainty, evolution, and economic theory. *The Journal of Political Economics*. 58(3): 211-221.

Biais B, Shadur R (2000). Darwinian selection does not eliminate irrational traders. *European Economic Review* 44(3): 469-490.

Blume L, Easley D (1992). Evolution and market behavior. *Journal of Economic Theory* 58: 9-40.

Blume L, Easley D (2006). If you're so smart, why aren't you rich? Belief selection in complete and incomplete markets. *Econometrica* 74(4): 929-966.

Bottazzi G, Dindo P (2014). Evolution and market behavior with endogenous investment rules. *Journal of Economic Dynamics and Control* 48(11): 121-146.

Bottazzi G, Giachini D (2017). Wealth and price distribution by diffusive approximation in a repeated prediction market. *Physica A: Statistical Mechanics and Its Applications* 471: 473-479.

Bottazzi G, Dindo P, Giachini D (2017). Long-run heterogeneity in an exchange economy with fixed-mix traders. *Economic Theory* 66(2): 407-447.

Chiarella C, Dieci R, Gardini L(2006). Asset price and wealth dynamics in a financial market with heterogeneous agents. *Journal of Economic Dynamics and Control* 30(9-10): 1755-1786.

Evstigneev IV, Hens T, Schenk-Hoppe KR (2006). Evolutionary stable stock markets. *Journal of Economic Theory* 27(6): 449-468.

Fama EF(1965). The behavior of stock-market prices. *Journal of Business* 38(1): 34-105.

Fama EF (1970). Efficient capital markets: A review of theory and empirical work. *Journal of Finance* 25(3): 383-417.

Fama EF (1991). Efficient capital markets: II. *Journal of Finance* 36(11): 1575-1617.

Friedman M(1953). The case for flexible exchange rates. *Essays in Positive Economics*. University of Chicago Press, Chicago.

Grinblatt M, Titman S (1993). Performance measurement without benchmarks: An examination of mutual fund returns. *The Journal of Business*66.DOI:10.1086/296593

Guo J, He XD (2017). Equilibrium asset pricing with Epstein-Zin and loss-averse investors. *Journal of Economic Dynamics and Control* 76: 86-108.

Hirshleifer D, Subrahmanyamb A, Titman S (2005). Feedback and the success of irrational investors. *Journal of Financial Economics* 32(5): 1-28.

Higham DJ (2000). Mean-square and asymptotic stability of the stochastic theta method. *Siam Journal on Numerical Analysis* 38(3): 753-769.

Ikeda N, Watanabe S (1977). A comparison theorem for solutions of stochastic differential equations and its applications. *Osaka Journal of Math* 14(3): 619-633.

Kahneman D, Tversky A (1991). Loss aversion in riskless choice: A reference-dependent model. *The Quarterly Journal of Economics* 106(4): 1039-1061.

Lin L, Sornette D (2013). Diagnostics of rational expectation financial bubbles with stochastic mean- reverting termination times. *European Journal of Finance* 19(5): 344-365.

Lo AW (1999). The three P's of total risk management. *Financial Analysts Journal* 55(1): 13-26.

Lo AW (2004). The adaptive market hypothesis: markets efficiency from an evolutionary perspective. *Journal of Portfolio Management* 31(1): 21-44.

Oksendal B (2003). *Stochastic Differential Equations*. Springer-Verlag, Berlin.

Sandroni A (2005). Market selection when markets are incomplete. *Journal of Mathematical Economics* 41: 91-104.

Seather BE, Engen SLR (2000). Estimating the time to extinction in an island population of song sparrows. *Proceedings of Theory Society of London. Series B* 267: 621-626.

Tversky A, Kahneman D (1979). Prospect theory: An analysis of decision under risk. *Econometrica* 47(2): 263-292.

Yan H (2008). Natural selection in financial markets: Does it work? *Management Science* 54: 1935-1950.

Yang Z, Qin G (2006). Continuous time evolutionary financial model and globally asymptotically evolutionary stable strategies. *Economic Research Journal* 5: 41-49.

Zhang YJ, Zhang W, Xiong X (2010). Strategies and investment returns: Agent based computational finance perspective. *Journal of Management Science in China* 13(9): 109-118.

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