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Analysis of Fluid Model Modulated by an M/PH/1 Working Vacation Queue

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Abstract. We propose a fluid model driven by the queue length process of a working vacation queue with PH service distribution, which can be applied to the Ad Hoc network with every data group. We obtain the stationary distribution of the queue length in driving process based on a quasi-birth-and-death process. Then, we analyze the fluid model, and derive the differential equations satisfied by the stationary joint distribution of the fluid queue based on the balance equation. Moreover, we obtain some performance indices, such as, the average throughput, server utilization and the mean buffer content. These indices are relevant to pack transmission in the network, and they can be obtained by using the Laplace Transform (LT) and the Laplace-Stieltjes Transform (LST). Finally, some numerical examples have been discussed with respect to the effect of several parameters on the system performance indices.

Keywords: Fluid model, M/PH/1 queue, average throughput, server utilization, buffer content, Ad Hoc network

1. Introduction

Recently, with the development of science technology and the progress of the times, the studies of the discrete systems are being more and more complex. It's limited and inconvenient to study the complicate system by using the traditional discrete queue model. The fluid model (fluid queue) is an input-output system, in which continuous fluid flowing into or out of a storage device, called a buffer. It is not sensitive to the change of the fluid transmission rate and the size of buffer in the system. And, it is easy to handle problems in the analysis of that system. Therefore, the fluid model has attracted considerable attention of many researchers. Virtamo and Norros (1994) studied the fluid queue driven by the M/M/1 queue using the spectral decomposition method in the finite-state for the first time. Adan and Resing (1996) presented an embedded point method to analyzed the fluid queue driven by the Markovian queue with an alternating renewal process, and the stationary buffer content distribution was given. Barbot and Sericola (2002) considered an infinite capacity buffer depend on M/M/1 queue by using generating function, and obtained the expression for the joint stationary distribution of the buffer level. Parthasarathy et al. (2002) researched the fluid queue driven by an M/M/1 queue using a continued fraction method, and obtained the buffer content distribution. Li and Zhao (2005) analyzed block-structured fluid model based on means of the RG-factorization method, and the stationary probability distribution of the buffer content was expressed in terms of the R-measure. Mao et al. (2011) discussed fluid model driven by an M/G/1 queue with multiple exponential vacations, and the express of mean buffer content was given by using Laplace transform method. Xu et al. (2013) presented the fluid model driven by the M/M/c queue with working vacation queue, the probability of empty buffer content and the mean of the buffer content was obtained. Ammar (2014) analyzed an M/M/1 driven fluid queue with multiple exponential vacations, obtained the stationary distribution of the buffer content by modified Bessel function of first kind. Economou and Manou (2016) considered the strategic behavior in an observable fluid queue with an alternating service process, and obtained symmetric equilibrium strategy profiles. Xu et al. (2017) researched the fluid model driven by a PH/M/1 queue, the expected buffer content and the probability of empty buffer content were obtained by using the matrix function method.

Expected for the theoretical study, researchers have applied fluid models to practical problems in the real life, such as, inventory, communication networks, medical treatment, transportation systems and so on. Yan (2006) researched fluid models for production-inventory system, and derived the limiting distribution of the bivariate process such as fluid level, environment state. Barron (2016) considered a fluid production/inventory model operating in a stochastic environment, and explicit formulas for the cost functions obtained by using a matrix analytic approach. Irnich and Stuckmann (2003) studied fluid-flow modeling of internet traffic in GSM/GPRS networks, and calculated the mean equilibrium buffer content and the sojourn time in the fluid-flow model. Liu and Whitt (2013) presented some algorithms to calculate the performance functions for a time varying open network of many-server fluid queues, and analyzed the variety of these performance indices. Zhou et al. (2015) proposed stability analysis of wireless network with improved fluid model, conducted the classical fluid model and the convex optimization model. They obtained WTFM in the perspectives of delay, dropping probability, throughput, sliding window size and queue length. However, the model here is to provide a matrix function method to research the fluid model based on the M/PH/1 working vacation queue and its application in an Ad Hoc network.

Ad Hoc network is a mobile network that is quick organization network, and each packet has a large amount of information to be processed during the transmission. Thus, we can consider the process of transmission of each data group as a fluid queueing system, make it more convenient for information processing. The structure of this paper is presented as follows. In Section 2, we introduce the model description. In Section 3, we analyze the application based on the fluid model. In Section 4, we show some numerical examples to demonstrate the effect of some parameters on the system performance indices. Finally, we give some conclusions in Section 5.

2. External Stochastic Environment Description

Assume that the external control environment is the queue length process of an M/PH/1 queue with multiple working vacation. In the system, the inter-arrival times of an customer is based on a Poisson process with rate λ. The server begins with a working vacation when the queue becomes empty, and the vacation duration is an exponential distribution with parameter θ. In working vacation period, the server not completely stop operate, but in a lower service rate μ_v to continue. When a vacation is end, if there are customers in the queue, the queue system will switch to a regular busy period and the server rate will be $\mu_b\left(\mu_b > \mu_v\right)$. Moreover, the service time follows a phase type distribution with representation $(\boldsymbol{\alpha}, \mathbf{T})$ of order m, and $\mathbf{Te}+\mathbf{T}^0 = \mathbf{0}$, where $\boldsymbol{\alpha}=(\alpha_1, \alpha_2, \dots, \alpha_m)$, $\mathbf{T} = (T_{ij})$ is invertible matrix of order m, $\mathbf{T}^0 = \left(T_1^0, T_2^0, \dots, T_m^0\right)^{\mathrm{T}}$ is a non-negative column vector. And, for ease of notation, we assume that $\alpha_{m+1} = 0$, then $\boldsymbol{\alpha}\mathbf{e} = 1$. Hereinafter, we suppose that $\mathbf{e}$ is an appropriate dimensional vector of ones and $\mathbf{I}$ is an appropriate order identity matrix. Furthermore, the mean of the service time is $\mu_b^{-1} = -\boldsymbol{\alpha}\mathbf{T}^{-1}\mathbf{e}$. Otherwise, the server begins with another working vacation.

Let $L(t)$ represents the number of customers in the system at time t, $J(t)$ denotes the state in the system at time t, and we have

$$J(t)=\begin{cases}0, & \text{the system is in a working vacation period at time } t\\ j, & \text{the system is in a busy period and in the service phase } j\,(1\le j\le m) \text{ at time } t\end{cases}$$

Then $\{L(t), J(t), t\ge 0\}$ is a Markov process with the state space

$$\Omega=\{(0,0)\}\cup\left\{(k,j), k\ge 1, 0\le j\le m\right\}$$

By using the lexicographical sequence for the states, the infinitesimal generator of QBD process can be written as follows

$$\mathbf{Q}=\begin{bmatrix}A_0 & \mathbf{C}_0 & & & \\ \mathbf{B}_1 & \mathbf{A} & \mathbf{C} & & \\ & \mathbf{B} & \mathbf{A} & \mathbf{C} & \\ & & \mathbf{B} & \mathbf{A} & \mathbf{C}\\ & & & \ddots & \ddots & \ddots\end{bmatrix}$$

where

$$A_0=-\lambda,\ \mathbf{C}_0=(\lambda,\mathbf{0}),\ \mathbf{B}_1=\left(\mu_v, T^0\right)^{\mathrm{T}}$$

$$\mathbf{B}=\begin{pmatrix}\mu_v & \mathbf{0}\\ \mathbf{0} & \mathbf{T}^0\boldsymbol{\alpha}\end{pmatrix},\ \mathbf{C}=\begin{pmatrix}\lambda & \mathbf{0}\\ \mathbf{0} & \lambda\mathbf{I}\end{pmatrix}$$

$$\mathbf{A}=\begin{pmatrix}-(\lambda+\mu_v+\theta) & \theta\boldsymbol{\alpha}\\ \mathbf{0} & -\lambda\mathbf{I}+\mathbf{T}\end{pmatrix}$$

If $\rho=\lambda/\mu_b<1$, let (L,J) represents the stationary random vector $\{L(t), J(t), t\ge 0\}$, and its distribution is

$$\begin{aligned}\pi_{kj}&=P\left\{L=k, J=j\right\}\\ &=\lim_{t\to\infty}P\left\{L(t)=k, J(t)=j\right\},\ (k,j)\in\Omega\end{aligned}$$

Denote by $\boldsymbol{\pi}_k=(\pi_{k1},\pi_{k2},\ldots,\pi_{km}), k\ge 1$. Similar to Yang (2008), we have the following results

Lemma 1 *If $\rho=\lambda/\mu_b<1$, the matrix equation $\mathbf{R}^2\mathbf{B}+\mathbf{R}\mathbf{A}+\mathbf{C}=\mathbf{0}$ has the minimal non-negative solution*

$$\mathbf{R}=\begin{bmatrix}r & \dfrac{\theta r}{\lambda(1-r)}\boldsymbol{\alpha}\mathbf{R}_0\\ \mathbf{0} & \mathbf{R}_0\end{bmatrix}$$

where

$$r=\frac{1}{2\mu_v}\left(\lambda+\theta+\mu_v-\sqrt{(\lambda+\theta+\mu_v)^2-4\lambda\mu_v}\right)$$

$$\mathbf{R}_0=\lambda(\lambda\mathbf{I}-\mathbf{T}-\lambda\mathbf{e}\boldsymbol{\alpha})^{-1}$$

Lemma 2 *If $\rho=\lambda/\mu_b<1$, the stationary probability distribution of (L,J) is*

$$\begin{cases}\pi_{k0}=Kr^k,\ k\ge 0\\ \boldsymbol{\pi}_k=K\dfrac{\theta r}{\lambda(1-r)}\boldsymbol{\alpha}\displaystyle\sum_{j=0}^{k-1}r^j\mathbf{R}_0^{k-j},\ k\ge 1\end{cases}$$

where

$$K=(1-\rho)(1-r)\left(1-\rho+\frac{\theta r}{\mu_b(1-r)}\right)^{-1}$$

3. Analysis of the Fluid Model

In this section, we present the fluid model and discuss the stationary distribution of fluid model. And, we obtain some theoretical results, and some performance indices are derived.

3.1 Fluid Queue Modeling

The customers are viewed as fluid deposited in a buffer in a first-in-first-out fashion. Let $X(t)$ represents the content of buffer with infinity room at time t, and $X(t)$ is a non-negative random vector. We suppose that the effective input rate of fluid (i.e., the input rate minus the output rate) to the buffer is a function of stochastic process $\{(X(t), L(t), J(t)), t\ge 0\}$, that is

$$\eta[L(t), J(t), X(t)]=\frac{\mathrm{d}}{\mathrm{d}t}X(t)=\begin{cases}0, & (L(t),J(t))=(0,0), X(t)=0\\ \sigma, & (L(t),J(t))=(0,0), X(t)>0\\ \sigma_0, & (L(t),J(t))=(k,0), k\ge 1\\ \sigma_1, & (L(t),J(t))=(k,j), k\ge 1,\ 1\le j\le m\end{cases}\tag{1}$$

where $\sigma<0$, $\sigma_1>\sigma_0>0$. This means that, when there are customers in the external driving system during working vacations period, the effective input rate of the buffer content is linear increasing at the rate of σ_0. In regular busy period, the effective input rate of the buffer content is linear increasing at the rate of σ_1, and $\sigma_1 > \sigma_0 > 0$. Moreover, the buffer content is linear decreasing at the rate of σ_0 when the driving system is empty, and the buffer content decrease with empty driving system. And, the model is involved to a 3-dimensional Markov process $\{L(t), J(t), X(t), t \geq 0\}$ with the effective input rate $\eta[L(t), J(t), X(t)]$.

3.2 Stationary Distribution of the Fluid Model

Next, we present a matrix function method using Laplace Transform, and research the stationary joint distribution of the fluid queue. Compared with the general Markov-Modulate fluid model, the calculation here is explicit and the results are elegant.

Note that Eq.(1) and consider Lemma 2, we can derive the mean drift of the fluid model as follows

$$
\begin{aligned}
d &= \sigma\pi_{00} + \sigma_0 \sum_{k=1}^{\infty} \pi_{k0} + \sigma_1 \sum_{k=1}^{\infty} \pi_k \mathbf{e} \\
&= K\left(\sigma + \sigma_0 \sum_{k=1}^{\infty} r^k + \frac{\sigma_1 \theta r}{\lambda(1-r)} \sum_{k=1}^{\infty} \boldsymbol{\alpha} \sum_{j=0}^{k-1} r^j \mathbf{R}_0^{k-j} \mathbf{e}\right) \\
&= K\left(\sigma + \sigma_0 \frac{r}{1-r} + \sigma_1 \frac{\theta r}{\mu_b(1-\rho)(1-r)^2}\right)
\end{aligned}
$$

If $d < 0, \rho < 1$, it can be proved that the fluid model is stable (Kulkarni 1997). We denote the stationary random vector of stochastic process $\{L(t), J(t), X(t), t \geq 0\}$ as (L, J, X), where X represents the stationary content of buffer. The stationary joint distribution of fluid model and the stationary probability distribution of the buffer content are as follows, respectively.

$$
\begin{aligned}
F_{kj}(x) &= P\left\{L = k, J = j, X \leq x\right\} \\
&= \lim_{t\to\infty} P\left\{L(t) = k, J(t) = j, X(t) \leq x\right\}, \\
&\quad x \geq 0, \ (k, j) \in \Omega
\end{aligned}
$$

$$
F(x) = P\{X \leq x\} = F_{00}(x) + \sum_{k=1}^{\infty} \sum_{j=0}^{m} F_{kj}(x)
$$

For convenience, we introduce the following vectors

$$
\mathbf{F}_k(x) = (F_{k0}(x), F_{k1}(x), \dots, F_{km}(x)), k \geq 1
$$

$$
\mathbf{F}(x) = (F_{00}(x), \mathbf{F}_1(x), \mathbf{F}_2(x), \dots)
$$

From the balance equation, we can prove that $\mathbf{F}(x)$ is satisfied with the following matrix differential equation and the boundary condition

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F}(\mathrm{x})\boldsymbol{\Lambda} = \mathbf{F}(\mathrm{x})\mathbf{Q} \tag{2}
$$

with initial condition

$$
\mathbf{F}(0) = (a, \mathbf{0}, \mathbf{0}, \mathbf{0}\dots)
$$

where

$$
a = F_{00}(0) = P\{L = 0, J = 0, X = 0\}
$$

$$
\boldsymbol{\Lambda} = \mathrm{diag}(\sigma, \boldsymbol{\Sigma}, \boldsymbol{\Sigma}, \dots), \ \boldsymbol{\Sigma} = \mathrm{diag}(\sigma_0, \sigma_1 \mathbf{I})
$$

Then, introducing the Laplace Transform of the stationary joint distribution of fluid model and the stationary probability distribution of the buffer content, we have

$$
\hat{F}_{kj}(s) = \int_0^{\infty} e^{-sx} F_{kj}(x)\,\mathrm{d}x, s > 0, (k, j) \in \Omega
$$

$$
\hat{F}(s) = \int_0^{\infty} e^{-sx} F(x)\,\mathrm{d}x, s > 0
$$

Similar, we introduce the following vectors

$$
\hat{\mathbf{F}}_k(s) = \left(\hat{F}_{k0}(s), \hat{F}_{k1}(s), \dots, \hat{F}_{km}(s)\right), k \geq 1
$$

$$
\hat{\mathbf{F}}(s) = \left(\hat{F}_{00}(s), \hat{\mathbf{F}}_1(s), \hat{\mathbf{F}}_2(s), \dots\right)
$$

Taking the LT on both sides in Eq.(2) and using the boundary conditions, we obtain the equation as follows

$$
\hat{\mathbf{F}}(s)(\mathbf{Q} - s\boldsymbol{\Lambda}) = -\mathbf{F}(0)\boldsymbol{\Lambda} = (-a\sigma, \mathbf{0}, \mathbf{0}, \dots) \tag{3}
$$

Then, we have the following conclusions from the above analysis.

Theorem 1 *For any $s > 0$, the quadratic matrix equation*

$$
\mathbf{R}^2(s)\mathbf{B} + \mathbf{R}(s)(\mathbf{A} - s\boldsymbol{\Sigma}) + \mathbf{C} = \mathbf{0} \tag{4}
$$

has the minimal non-negative solution

$$
\mathbf{R}(s) = \begin{bmatrix} r_0(s) & \dfrac{r_0(s)\theta\boldsymbol{\alpha}}{\lambda \mathbf{R}_{22}^{-1}(s) - r_0(s)\mathbf{T}^0\boldsymbol{\alpha}} \\ \mathbf{0} & \mathbf{R}_{22}(s) \end{bmatrix}
$$

where

$$r_0(s) = \frac{1}{2\mu_v}\left(\lambda + \theta + \mu_v + s\sigma_0\right) - \frac{1}{2\mu_v}\sqrt{\left(\lambda + \theta + \mu_v + s\sigma_0\right)^2 - 4\lambda\mu_v}$$

and $\mathbf{R}_{22}(s)$ *is decided by*

$$\mathbf{R}_{22}^2(s)\mathbf{T}^0\boldsymbol{\alpha} + \mathbf{R}_{22}(s)\left[-(\lambda + s\sigma_1)\mathbf{I} + \mathbf{T}\right] + \lambda\mathbf{I} = \mathbf{0}$$

Proof. Note that the matrix $\mathbf{B}$, $\mathbf{A}-s\boldsymbol{\Sigma}$ and $\mathbf{C}$ are all upper triangular matrices, we suppose that the solution $\mathbf{R}(s)$ of has the same structure. It's given by

$$\mathbf{R}(s) = \begin{bmatrix} R_{11}(s) & \mathbf{R}_{12}(s) \\ \mathbf{0} & \mathbf{R}_{22}(s) \end{bmatrix}, s > 0$$

Then, Eq.(4) can be rewritten as

$$\begin{cases} \mu_v R_{11}^2(s) - (\lambda + \mu_v) R_{11}(s) - (\theta + s\sigma_0) R_{11}(s) + \lambda = 0 \\ \mathbf{R}_{22}^2(s)\mathbf{T}^0\boldsymbol{\alpha} + \lambda\mathbf{I} + \mathbf{R}_{22}(s)\left[-(\lambda + s\sigma_1)\mathbf{I} + \mathbf{T}\right] = \mathbf{0} \\ R_{11}(s)\theta\boldsymbol{\alpha} - \mathbf{R}_{12}(s)(\lambda + s\sigma_1)\mathbf{I} + \mathbf{R}_{12}(s)\mathbf{T} + \mathbf{R}_{12}(s)R_{11}(s)\mathbf{T}^0\boldsymbol{\alpha} + \mathbf{R}_{12}(s)\mathbf{R}_{22}(s)\mathbf{T}^0\boldsymbol{\alpha} = \mathbf{0} \end{cases} \tag{5}$$

Solving the first equation of Eq.(5), we obtain two different roots $r_0(s)$ and $r_1(s)$, and

$$\begin{aligned} r_0(s) &= \frac{1}{2\mu_v}\left(\lambda + \theta + \mu_v + s\sigma_0\right) - \frac{1}{2\mu_v}\sqrt{\left(\lambda + \theta + \mu_v + s\sigma_0\right)^2 - 4\lambda\mu_v} \\ r_1(s) &= \frac{1}{2\mu_v}\left(\lambda + \theta + \mu_v + s\sigma_0\right) + \frac{1}{2\mu_v}\sqrt{\left(\lambda + \theta + \mu_v + s\sigma_0\right)^2 - 4\lambda\mu_v} \end{aligned} \tag{6}$$

and $0 < r_0(s) < 1, r_1(s) > 1$, so we have $R_{11}(s) = r_0(s)$.

From the second equation of Eq.(5), we obtain

$$\mathbf{R}_{22}^{-1}(s) = \lambda^{-1}\left[(\lambda + s\sigma_1)\mathbf{I} - \mathbf{T} - \mathbf{R}_{22}(s)\mathbf{T}^0\boldsymbol{\alpha}\right]$$

Substituting $R_{11}(s) = r_0(s)$ and $\mathbf{R}_{22}^{-1}(s)$ into the third equation of Eq.(5), we obtain $\mathbf{R}_{12}(s)$. The conclusion is proved. ■

Theorem 2 *If $d < 0$ and $\rho < 1$, the LT of the joint distributions sequence is*

$$\begin{cases} \hat{F}_{00}(s) = \dfrac{a\sigma}{\lambda + s\sigma - \mu_v r_0(s) - \mathbf{R}_{12}(s)\mathbf{T}^0} \\ \hat{\mathbf{F}}_k(s) = \hat{F}_{00}(s)\mathbf{e}_1\mathbf{R}^k(s), k \geq 1 \end{cases} \tag{7}$$

where $\mathbf{e}_1$ is the $m+1$ dimensional vector, that is $\mathbf{e}_1 = (1, 0_m)$.

Proof. According to Eq.(3), we obtain the following equations

$$\begin{cases} \hat{F}_{00}(s)(A_0 - s\sigma) + \hat{\mathbf{F}}_1(s)\mathbf{B}_1 = -a\sigma \\ \hat{F}_{00}(s)\mathbf{C}_0 + \hat{\mathbf{F}}_1(s)(\mathbf{A} - s\boldsymbol{\Sigma}) + \hat{\mathbf{F}}_2(s)\mathbf{B} = \mathbf{0} \\ \hat{\mathbf{F}}_{k-1}(s)\mathbf{C} + \hat{\mathbf{F}}_k(s)(\mathbf{A} - s\boldsymbol{\Sigma}) + \hat{\mathbf{F}}_{k+1}(s)\mathbf{B} = \mathbf{0}, \ k \geq 2 \end{cases} \tag{8}$$

If $d < 0$, $\rho < 1$, the fluid queue has unique stationary probability distribution $\left\{F_{kj}(x), (k,j) \in \Omega\right\}$, there exists unique solution in Eq.(8). Therefore, we only need to testify that Eq.(7) is the solution of Eq.(8).

For $k \geq 2$, consider the second formula of Eq.(7), we have

$$\begin{aligned} &\hat{\mathbf{F}}_{k-1}(s)\mathbf{C} + \hat{\mathbf{F}}_k(s)(\mathbf{A} - s\boldsymbol{\Sigma}) + \hat{\mathbf{F}}_{k+1}(s)\mathbf{B} \\ &= \hat{F}_{00}(s)\mathbf{e}_1\mathbf{R}^{k-1}(s)\left[\mathbf{C} + \mathbf{R}(s)(\mathbf{A} - s\boldsymbol{\Sigma}) + \mathbf{R}^2(s)\mathbf{B}\right] \\ &= \mathbf{0} \end{aligned}$$

Substituting $\hat{\mathbf{F}}_1(s) = \hat{F}_{00}(s)\mathbf{e}_1\mathbf{R}(s)$ and $\hat{\mathbf{F}}_2(s) = \hat{F}_{00}(s)\mathbf{e}_1\mathbf{R}^2(s)$ into the second formula of Eq.(8), we obtain

$$\begin{aligned} &\hat{F}_{00}(s)\mathbf{C}_0 + \hat{\mathbf{F}}_1(s)(\mathbf{A} - s\boldsymbol{\Sigma}) + \hat{\mathbf{F}}_2(s)\mathbf{B} \\ &= \hat{F}_{00}(s)\mathbf{e}_1\left[\mathbf{C} + \mathbf{R}(s)(\mathbf{A} - s\boldsymbol{\Sigma}) + \mathbf{R}^2(s)\mathbf{B}\right] = \mathbf{0} \end{aligned}$$

Considering $\hat{\mathbf{F}}_1(s) = \hat{F}_{00}(s)\mathbf{e}_1\mathbf{R}(s)$, we obtain $\hat{F}_{00}(s)$. The theorem 2 is proved.

From the theorem 2, the LT can be expressed as the stationary probability distribution of the buffer content. Then, we have

$$\begin{aligned} \hat{F}(s) &= \int_0^\infty e^{-sx}F(x)\,dx \\ &= \hat{F}_{00}(s) + \sum_{k=1}^{\infty}\hat{\mathbf{F}}_k(s)\mathbf{e} \\ &= \hat{F}_{00}(s)\mathbf{e}_1(\mathbf{I} - \mathbf{R}(s))^{-1}\mathbf{e} \end{aligned} \tag{9}$$

Moreover, notice that

$$(\mathbf{I}-\mathbf{R}(s))^{-1} = \begin{bmatrix} \dfrac{1}{1-r_0(s)} & \dfrac{\mathbf{R}_{12}(s)(\mathbf{I}-\mathbf{R}_{22}(s))^{-1}}{1-r_0(s)} \\ \mathbf{0} & (\mathbf{I}-\mathbf{R}_{22}(s))^{-1} \end{bmatrix}$$

Substituting $(\mathbf{I}-\mathbf{R}(s))^{-1}$ and $\hat{F}_{00}(s)$ into Eq.(9), we obtain

$$\hat{F}(s) = \frac{a\sigma}{m}\left(1+\mathbf{R}_{12}(s)(\mathbf{I}-\mathbf{R}_{22}(s))^{-1}\mathbf{e}\right) \quad (10)$$

where

$$m = (1-r_0(s))\left(\lambda + s\sigma - \mu_v r_0(s) - \mathbf{R}_{12}(s)\mathbf{T}^0\right)$$

■

3.3 Performance Indices

To compute the performance indices in steady state, we introduce the LST of the distributions $F_{kj}(s)$ and $F(s)$, respectively

$$F^*_{kj}(s) = \int_0^{+\infty} e^{-sx}\mathrm{d}F_{kj}(x), s>0, (k,j)\in\Omega$$

$$F^*(s) = \int_0^{+\infty} e^{-sx}\mathrm{d}F(x), s>0$$

From the relational expression

$$F^*(s) = \int_0^{+\infty} e^{-sx}\mathrm{d}F(x) = s\hat{F}(s)$$

and Eq.(10), we obtain

$$F^*(s) = \frac{sa\sigma}{m}\left(1+\mathbf{R}_{12}(s)(\mathbf{I}-\mathbf{R}_{22}(s))^{-1}\mathbf{e}\right) \quad (11)$$

In addition, according to the normalization condition $\lim_{s\to 0} F^*(s) = 1$ and L'Hospital rule, taking notice of $r'_0(0) = -\dfrac{r\sigma_0}{\mu_v(r_1-r)}$, $\mathbf{R}_{12}(0)(\mathbf{I}-\mathbf{R}_{22}(0))^{-1}\mathbf{e} = \dfrac{\theta r\rho}{\lambda(1-r)(1-\rho)}$, $\mathbf{R}_{12}(0)\mathbf{T}^0 = \lambda - r\mu_v$. We obtain the probability of empty buffer as follows

$$a = \frac{\xi(1-r)r\sigma_0}{\sigma(r_1-r)} + \frac{\xi(1-r)(r_1-r)\left(\sigma-\mathbf{R}'_{12}(0)\mathbf{T}^0\right)}{\sigma(r_1-r)} \quad (12)$$

where $r_1 = r_1(0)$ is decided by Eq.(6),

$$\xi = \frac{\lambda(1-r)(1-\rho)}{\lambda(1-r)(1-\rho)+\theta r\rho}$$

Besides, based on Eq.(12) and Eq.(11), we obtain

$$F^*(s) = \frac{\xi(1-r)r\sigma_0}{r_1-r}\times \frac{s\left(1+\mathbf{R}_{12}(s)(\mathbf{I}-\mathbf{R}_{22}(s))^{-1}\mathbf{e}\right)}{m}\times \frac{\xi(1-r)(r_1-r)\left(\sigma-\mathbf{R}'_{12}(0)\mathbf{T}^0\right)}{r_1-r} \quad (13)$$

Then, taking the derivatives on both sides of Eq.(13) with respect to s, let $s\to 0$ and using L'Hospital rule, we obtain expression of the mean buffer content in steady state

$$\mathrm{E}(X) = \frac{r\sigma_0}{\mu_v(r_1-r)^2} - \xi\left(\mathbf{R}'_{12}(0)(\mathbf{I}-\mathbf{R}_0)^{-1}\mathbf{e}\right) - \xi\left(\mathbf{R}_{12}(0)(\mathbf{I}-\mathbf{R}_0)^{-1}\mathbf{R}'_{22}(0)(\mathbf{I}-\mathbf{R}_0)^{-1}\mathbf{e}\right) + \frac{(r-1)\left(2rr_1\sigma_0^2+\mu_v(r_1-r)^3\mathbf{R}''_{12}(0)\mathbf{T}^0\right)}{2\mu_v(r_1-r)^3\left((r_1-r)(\sigma-\mathbf{R}'_{12}(0)\mathbf{T}^0)+r\sigma_0\right)}$$

4. Numerical Analysis

In this section, we apply the M/PH/1 fluid queue to analyze the network system of Ad Hoc, and get the system performance indices. And, we demonstrate the effect of several parameters on the system performance indices, such as, the average throughput, the server utilization and the mean buffer content in the steady state.

Assuming that all nodes in the network system have infinite buffer room, and the transmission of data packets follow the rules that first come, first out. The information flow transmitted by the data group in the network system can be regarded as a fluid with effective input rate $\eta[L(t), J(t), X(t)]$. The interarrival time of each group of data arriving system is based on a Poisson process with rate λ. The channel is in idle period when the system is no date transmission, and the idle period is an exponential distribution with parameter θ. In

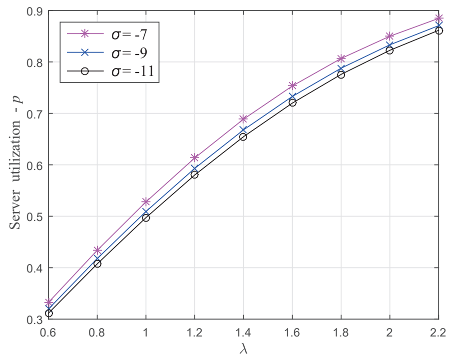

Figure 1 p with Respect to λ and σ

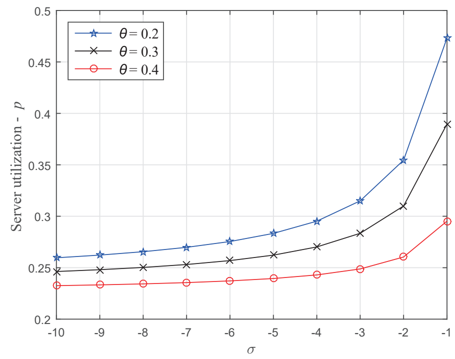

Figure 2 p with Respect to θ and σ

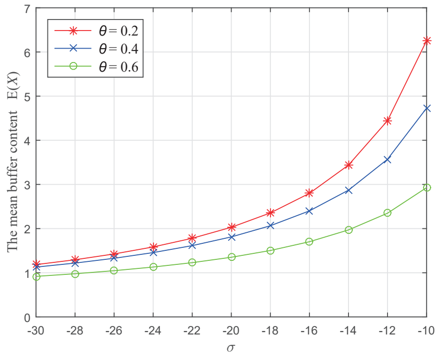

Figure 3 $\mathrm{E}(X)$ with Respect to θ and σ

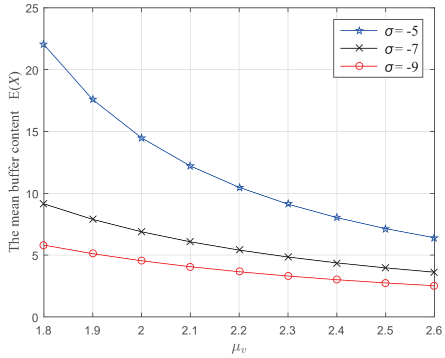

Figure 4 $\mathrm{E}(X)$ with Respect to μ_v and σ

the idle period, the transmission of each data group is not completely stop, but in a lower transmission rate μ_v to continue. When an idle period is end, if there are data groups in the system, the channel will switch to a busy period, and the transmission rate follow that a phase type distribution with parameter $(\boldsymbol{\alpha}, \mathbf{T})$. Moreover, the mean of the transmission time with $\mu_b^{-1} = -\boldsymbol{\alpha}\mathbf{T}^{-1}\mathbf{e}$, and $\mu_b > \mu_v$. Otherwise, the channel begins the idle period.

In Ad Hoc network, the average throughput as follows

$$
\begin{aligned}
\mathrm{E}(T) &= \sigma_0 \sum_{k=1}^{\infty} \pi_{k0} + \sigma_1 \sum_{k=1}^{\infty} \pi_k \mathbf{e} \\
&= K\left(\sigma_0 \frac{r}{1-r} + \sigma_1 \frac{\theta r}{\mu_b (1-\rho)(1-r)^2}\right)
\end{aligned}
$$

Obviously, the average throughput is equal to the arrival rate when the system is in equilibrium.

The server utilization is

$$
\begin{aligned}
p &= 1 - a \\
&= 1 - \frac{\xi (1-r)\left[(r_1 - r)\left(\sigma - \mathbf{R}'_{12}(0)\mathbf{T}^0\right) + r\sigma_0\right]}{\sigma (r_1 - r)}
\end{aligned}
$$

We assume that the transmission of each data group in busy period follows a PH distribution, and there are two stochastic phases, that is, $\boldsymbol{\alpha} = (0.2, 0.6)$,

$$
\mathbf{T} = \begin{bmatrix} -4 & 2 \\ 0 & -3 \end{bmatrix}, \quad \mathbf{T}^0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}
$$

Then $\mu_b = \frac{60}{17}$.

At the beginning, we assume that $\mu_v = 2, \sigma_1 = 3, \sigma_0 = 1.5$, and research the variation of the server utilization p in regard to arrival rate λ, idle times θ and the effective input rate σ. In Fig.1, we can see that when we have a certain σ, the server utilization is increasing about arrive rate λ. When we have a given λ, the server utilization is an increase function about parameter σ. In Fig.2, we can find the server utilization is going up about the growth of the effective input rate σ, when we have a certain θ. The server utilization is an declining function with respect to the idle times θ when we have a fixed parameter σ. Evidently, the server utilization increases with the growth of arrival rate λ, idle times θ and the effective input rate σ. Thus, we can improve the server utilization of network by adjusting the system parameters, which can reduce energy loss in ad hoc network.

Besides, we supposed that $\lambda = 0.5$, $\sigma_1 = 3$, $\sigma_0 = 1.5$, and study the change trend of the mean buffer content $\mathrm{E}(X)$ with respect to idle times θ, the transmission rate μ_v and the effective input rate σ. In Fig.3, the mean buffer content is going up with the growth of the effective input rate σ, when we have a fixed parameter θ. Supposing that given σ, the mean buffer content is falling with respect to the idle times θ. From Fig.4, we can know the mean buffer content is a growing function with the growth of the effective input rate σ. The mean buffer content is a declining function with the growth of server rate μ_v. Obviously, the mean of the buffer content various with the variation of parameters. Therefore, we can increase the energy efficiency of network by adjusting the system parameters.

5. Conclusion

This paper is concentrated on the analysis of the fluid model driven by M/PH/1 queue and its application in ad hoc network. We introduce fluid queue to the network. The buffer content of fluid queue can be used to effectively control input and output of data groups in ad hoc network. Thus, the signal intensity of network tends to be stable. Then, we propose a matrix function method by using LT method. And, we obtain the average throughput, the server utilization and the mean buffer content in steady state. Finally, we analyze the performance indices of the network with parameters change. From this analysis, we can see that if we want to improve the utilization of the channel and reduce the energy consumption, we have to set the appropriate node arrival rate, transmission rate and so on. Moreover, fluid queue can be applied to transportation, medical and other fields. The results of fluid queue also bring a certain theoretical basis for other fields in real life, and the model proves a certain basis for us to extend the distribution of vacation period to a PH distribution in the future research.

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