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DUAL HESITANT FUZZY INFORMATION AGGREGATION WITH EINSTEIN T-CONORM AND T-NORM

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Abstract

Dual hesitant fuzzy set (DHFS) is a new generalization of fuzzy set (FS) consisting of two parts (i.e., the membership hesitancy function and the non-membership hesitancy function), which confronts several different possible values indicating the epistemic degrees whether certainty or uncertainty. It encompasses fuzzy set (FS), intuitionistic fuzzy set (IFS), and hesitant fuzzy set (HFS) so that it can handle uncertain information more flexibly in the process of decision making. In this paper, we propose some new operations on dual hesitant fuzzy sets based on Einstein *t*-conorm and *t*-norm, study their properties and relationships and then give some dual hesitant fuzzy aggregation operators, which can be considered as the generalizations of some existing ones under fuzzy, intuitionistic fuzzy and hesitant fuzzy environments. Finally, a decision making algorithm under dual hesitant fuzzy environment is given based on the proposed aggregation operators and a numerical example is used to demonstrate the effectiveness of the method.

Keywords: Dual hesitant fuzzy set, Einstein *t*-conorm and *t*-norm, operation, aggregation operator

1. Introduction

Fuzzy set (FS) (Zadeh 1965) is a useful tool for dealing with the information being imprecise or vague (Zadeh 1975, 1978), allowing the degree of an element to a set denoted by a fuzzy number. Yet, in actual situations, someone may have a hesitation about the membership degree of x in A. Aiming at this problem, Atanassov proposed the concept of intuitionistic fuzzy set (IFS) (Atanassov 1986), assigning a membership degree and a non-membership degree of each element to a given set. IFS has been shown to be so flexible in dealing with fuzzy data that it has been used in many fields, such as medical diagnosis (De et al. 2001), pattern recognition (Hung and Yang 2004), decision making (Tan and Chen 2010, Wang and Liu 2011, Wei 2010, Xu 2007, 2011, Xu and Yager 2006, Zhao et al. 2010, Xu et al. 2014, Zeng et al. 2014) and so on. Atanassov and Gargov (1989) further generalized IFS into interval-valued intuitionistic fuzzy set (IVIFS). Chen et al. (2011), Ye (2010, 2011), and Yu et al. (2011) proposed some decision making methods for interval-valued intuitionistic fuzzy data. Later, Torra and Narukawa (2009, 2010) extended fuzzy set to another kind of set, called hesitant fuzzy set (HFS), allowing the membership degree to have a set of possible values. They also discussed the relationships between HFSs and IFSs, showing that the envelope of a HFS is an IFS. Xia and Xu (2011), Xu and Xia (2011a, 2011b), and Zhu et al. (2012) investigated the aggregation techniques, distance, correlation and similarity measures for HFSs, and gave their applications to decision making. Gu et al. (2011) aggregated the hesitant fuzzy information in the evaluation model for risk investment with the hesitant fuzzy weighted averaging operator. Wei (2012) considered the multi-attribute decision making problems with hesitant fuzzy data in which the attributes are in different priority levels and developed some prioritized aggregation operators to deal with hesitant fuzzy information. Liao et al. (2015) introduced the correlation measures and the correlation coefficients of the hesitant fuzzy linguistic term set (HFLTS), discussed their properties and then used them in the qualitative decision making. Recently, Zhu et al. (2012) generalized the HFS and proposed a new kind of set, called dual hesitant fuzzy set (DHFS), which encompasses fuzzy set, IFS, HFS and fuzzy multiset (FMS) (Miyamoto 2000, 2001, 2005) as special cases. DHFS consists of two parts, i.e., the membership hesitancy function and the non-membership hesitancy function, which confront several different possible values indicating the cognitive degrees whether certainty or uncertainty. As we all know, when the decision makers provide their judgments in decision making, the more the information they take into account, the more the values we obtain from the decision makers. Because DHFS can reflect the original information given by the decision makers as much as possible, and thus, compared to the existing sets mentioned above, DHFS can be regarded as a more comprehensive set, supporting a more flexible approach. Furthermore, Zhu et al. (2012) studied some basic operations of DHFSs, proposed an extension principle of DHFSs, and then they gave an example to illustrate the application of DHFSs in group forecasting. Up to now, how to aggregate dual hesitancy fuzzy information is still a new research direction, which is also the focus of this paper.

Another important problem in fuzzy set theory is that of triangular norms and conorms, that is, *t*-norms and *t*-conorms which are very useful to deal with “and” and “or” operations in decision making problems (Wang and Liu 2011, Xia et al. 2012, Zhao and Wei 2013). Wang and Liu (2011) introduced some operations on IFSs, such as Einstein sum, Einstein product and Einstein exponentiation, and developed some new geometric aggregation operators for IFSs. Xia et al. (2012) gave a further study about the application of Archimedean *t*-conorm and *t*-norm under intuitionistic fuzzy environment, and gave some new operational laws for intuitionistic fuzzy numbers, studied their properties and correlations, based on which, some specific intuitionistic fuzzy aggregation operators have been developed to solve the decision making problems. Based on the Einstein sum, Einstein product and Einstein exponentiation (Wang and Liu 2011), Zhao and Wei (2013) developed some new Einstein hybrid aggregation operators, such as the intuitionistic fuzzy Einstein hybrid averaging (IFEHA) operator and the intuitionistic fuzzy Einstein hybrid geometric (IFEHG) operator, and applied them to decision making. We can find that all the aggregation operators above are based on different *t*-conorms and *t*-norms, which can provide more choices for the decision makers in dealing with intuitionistic fuzzy data. Yet, there's almost no research on such issues under dual hesitant fuzzy environment. In this paper, we shall focus on this issue which needs to be explored in depth. In order to do that, we organize the remainder of the paper as follows. In Section 2, we review the concepts of DHFS and study the dual hesitant fuzzy operations based on the Einstein *t*-conorm and t-norm. In Section 3, we propose some new operators for aggregating dual hesitant fuzzy information. Section 4 gives an application of the new operators to multi-attribute decision making. Section 5 concludes the paper.

2. Dual Hesitant Fuzzy Operations Based on Einstein *T*-conorm and *T*-norm

2.1 The Notions of FS, HFS and DHFS

Now we first introduce the concept of fuzzy set (Zadeh 1965). The characteristic of fuzzy set is that it assigns to each element a membership degree which is in the interval $[0,1]$, that is to say, it accepts partial memberships of an element to a set. So it is more useful in dealing with fuzzy and uncertain phenomenon than the classical set whose membership degree of an element to a set is either 1 or 0. Fuzzy set theory is so useful that it has been widely used in various fields, such as control, decision making, management, and so on.

Definition 1 (Zadeh 1965). Let X be a fixed set, a fuzzy set A on X is represented by a function $\mu_A : X \to [0,1]$, with the condition $0 \le \mu_A(x) \le 1, \forall x \in X$.

However, when giving the membership degree of an element, we usually have several possible values. For such cases, Torra (2010) proposed a generation of fuzzy set:

Definition 2 (Torra 2010). Let X be a fixed set, a hesitant fuzzy set (HFS) on X is in terms of a function that when applied to X returns a subset of $[0,1]$.

Xia and Xu (2011) expressed the HFS by a mathematical symbol:

$$E = \{< x, h_E(x) > | x \in X\},$$

where $h_E(x)$ is a set of some values in $[0,1]$, which denote the possible membership degrees of the element $x \in X$ to the set E and called $h = h_E(x)$ a hesitant fuzzy element (HFE).

As we know, the membership grips with epistemic certainty while the non-membership grips with epistemic uncertainty, and thus can reflect the original information given by the decision makers as much as possible, similar to HFSs, the uncertainty on the possible values should also be considered. Based on this idea, Zhu et al. (2012) defined the notion of dual hesitant fuzzy set (DHFS) in terms of two functions that return two sets of membership values and non-membership values respectively for each element in the domain as follows:

Definition 3 (Zhu et al. 2012). Let X be a fixed set, then a DHFS D on X is described as:

$$D = \{< x, h(x), g(x) > | x \in X\}, \quad (1)$$

in which $h(x)$ and $g(x)$ are two sets of some values in $[0,1]$, denoting the possible membership degrees and non-membership degrees of the element $x \in X$ to the set D

respectively, with the conditions:

$$0 \le \gamma, \eta \le 1, \quad 0 \le \gamma^+ + \eta^+ \le 1, \tag{2}$$

where $\gamma \in h(x), \eta \in g(x)$, $\gamma^+ \in h^+(x) = \bigcup_{\gamma \in h(x)} \max\{\gamma\}$, and $\eta^+ \in g^+(x) = \bigcup_{\eta \in g(x)} \max\{\eta\}$ for all $x \in X$. The pair $d(x) = (h(x), g(x))$ is called a dual hesitant fuzzy element (DHFE) denoted by $d = (h, g)$ for convenience.

Zhu et al. (2012) concluded that the DHFS encompasses several fuzzy sets including the fuzzy set (Zadeh 1965), the IFS (Atanassov 1986) and the HFS (Torra 2010). Furthermore, Zhu et al. (2012) gave some basic operations for DHFEs (See Appendix 1) which are based on the Algebraic *t*-conorm and *t*-norm. In fact, there are various types of *t*-conorm and *t*-norm. If we replace the Algebraic *t*-conorm and *t*-norm in Zhu et al. (2012)'s operations for DHFEs with other forms of *t*-conorm and *t*-norm, we shall get more operational methods for DHFEs and then give different aggregation operators, and thus can provide more choices for decision makers. Therefore we shall focus on discussing these problems in the next sections.

2.2 Operations of DHFEs Based on Einstein *T*-conorm and *T*-norm

In what follows, we review the Einstein *t*-conorm and *t*-norm respectively (Beliakov et al. 2007):

$$S^E(x, y) = \frac{x+y}{1+xy}, \; T^E(x, y) = \frac{xy}{1+(1-x)(1-y)}. \tag{3}$$

Based on the above Einstein *t*-conorm and t-norm, we can define the Einstein sum and the Einstein product of DHFEs as below:

Definition 4 For any two DHFEs $d_1 = (h_1, g_1)$ and $d_2 = (h_2, g_2)$, we have

(1) $d_1 \oplus d_2 = \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1, \gamma_2 \in h_2, \eta_2 \in g_2}$
$$\left(\left\{ \frac{\gamma_1 + \gamma_2}{1 + \gamma_1 \gamma_2} \right\}, \left\{ \frac{\eta_1 \eta_2}{1 + (1-\eta_1)(1-\eta_2)} \right\} \right),$$

(2) $d_1 \otimes d_2 = \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1, \gamma_2 \in h_2, \eta_2 \in g_2}$
$$\left(\left\{ \frac{\gamma_1 \gamma_2}{1 + (1-\gamma_1)(1-\gamma_2)} \right\}, \left\{ \frac{\eta_1 + \eta_2}{1 + \eta_1 \eta_2} \right\} \right),$$

and all the results of the above two operations are also DHFEs.

To get the Einstein scalar multiplication and the Einstein power for DHFEs, we first introduce the following two Theorems:

Theorem 1 If $d = (h, g)$ is a DHFE, then

$$nd = \bigcup_{\gamma \in h, \eta \in g} \left(\left\{ \frac{(1+\gamma)^n - (1-\gamma)^n}{(1+\gamma)^n + (1-\gamma)^n} \right\}, \left\{ \frac{2\eta^n}{(2-\eta)^n + \eta^n} \right\} \right), \tag{4}$$

where $nd = \overbrace{d \oplus d \oplus \cdots \oplus d}^{n}$, the operation $\oplus$ is as defined in Definition 4. Moreover, nd is a DHFE even if n is any positive real number.

Furthermore, nd is a DHFE even if n is any positive real number. Similarly, we can prove the following result:

Theorem 2 If d is a DHFE, then

$$d^n = \bigcup_{\gamma \in h, \eta \in g} \left(\left\{ \frac{2\gamma^n}{(2-\gamma)^n + \gamma^n} \right\}, \left\{ \frac{(1+\eta)^n - (1-\eta)^n}{(1+\eta)^n + (1-\eta)^n} \right\} \right), \tag{5}$$

where $d^n = \overbrace{d \otimes d \otimes \cdots \otimes d}^{n}$, the operation $\otimes$ is defined in Definition 4, and d^n defined above is a DHFE even if n is any positive real number.

From Theorems 1 and 2, we can define the Einstein scalar multiplication and the Einstein power of DHFEs as follows:

Definition 5 Let $d = (h, g)$ be a DHFE, $\lambda > 0$, then

(1)

$$\lambda d = \bigcup_{\gamma\in h,\eta\in g}\left(\left\{\frac{(1+\gamma)^{\lambda}-(1-\gamma)^{\lambda}}{(1+\gamma)^{\lambda}+(1-\gamma)^{\lambda}}\right\},\left\{\frac{2\eta^{\lambda}}{(2-\eta)^{\lambda}+\eta^{\lambda}}\right\}\right),$$

(2)

$$d^{\lambda} = \bigcup_{\gamma\in h,\eta\in g}\left(\left\{\frac{2\gamma^{\lambda}}{(2-\gamma)^{\lambda}+\gamma^{\lambda}}\right\},\left\{\frac{(1+\eta)^{\lambda}-(1-\eta)^{\lambda}}{(1+\eta)^{\lambda}+(1-\eta)^{\lambda}}\right\}\right).$$

Moreover, some relations of the Einstein operations in Definitions 4 and 5 can be discussed as follows:

Theorem 3 Let d, d_1 and d_2 be any three DHFEs, $\lambda > 0$, then the relations of the operations in Definitions 2 and 3 are given as:

(1) $d_1 \oplus d_2 = d_2 \oplus d_1$,

(2) $d_1 \otimes d_2 = d_2 \otimes d_1$,

(3) $\lambda\left(d_1 \oplus d_2\right) = \lambda d_1 \oplus \lambda d_2$,

(4) $\left(d_1 \otimes d_2\right)^{\lambda} = d_1^{\lambda} \otimes d_2^{\lambda}$,

(5) $\lambda_1 d \oplus \lambda_2 d = \left(\lambda_1 \oplus \lambda_2\right) d$,

(6) $d^{\lambda_1} \otimes d^{\lambda_2} = d^{\lambda_1+\lambda_2}$.

In the following, we shall propose some novel aggregation operators for DHFSs on the basis of the above new operations.

3. Dual Hesitant Fuzzy Operators Based on Einstein *T*-conorm and *T*-norm

The operational laws defined in Section 2.2 will be used to aggregate the dual hesitant fuzzy information, which is the core of this section. In the following, we present the Einstein dual hesitant fuzzy weighted averaging (EDHFWA) operator based on the ordinary weighted arithmetic mean.

Definition 6 Let $d_i (i = 1, 2, \cdots, n)$ be a collection of DHFEs, then we define the Einstein dual hesitant fuzzy weighted averaging (EDHFWA) operator as follows:

$$\mathrm{EDHFWA}\left(d_1, d_2, \cdots d_n\right) = \overset{n}{\underset{i=1}{\oplus}} w_i d_i, \qquad (6)$$

where $w = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of $d_i (i = 1, 2, \cdots, n)$, w_i indicates the importance degree of d_i, satisfying $w_i \geq 0$ ($i = 1, 2, \cdots, n$) and $\sum_{i=1}^{n} w_i = 1$, the operation $\oplus$ is the Einstein sum in Definition 4.

Note 1 If the weight vector $w = (w_1, w_2, \cdots, w_n)^T = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$, then the EDHFWA operator reduces to the Einstein dual hesitant fuzzy arithmetic averaging (EDHFA) operator, which is defined as:

$$\mathrm{EDHFWA}\left(d_1, d_2, \cdots d_n\right) = \frac{1}{n}\overset{n}{\underset{i=1}{\oplus}} d_i. \qquad (7)$$

The following theorem tells us how to obtain the aggregated results by the EDHFWA operator specifically.

Theorem 4 Let $d_i (i = 1, 2, \cdots, n)$ be a collection of DHFEs, and $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $d_i (i = 1, 2, \cdots, n)$, where w_i indicates the importance degree of d_i, satisfying $w_i \geq 0$ ($i = 1, 2, \cdots, n$) and $\sum_{i=1}^{n} w_i = 1$, then the aggregated value by using the EDHFWA operator is also a DHFE, and

$$\begin{aligned}
&\mathrm{EDHFWA}\left(d_1, d_2, \cdots, d_n\right) \\
&= \overset{n}{\underset{i=1}{\oplus}} w_i d_i \\
&= \bigcup_{\substack{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n \\ \eta_1\in g_1,\eta_2\in g_2,\cdots,\eta_n\in g_n}}\left(\left\{\frac{\prod_{i=1}^{n}(1+\gamma_i)^{w_i}-\prod_{i=1}^{n}(1-\gamma_i)^{w_i}}{\prod_{i=1}^{n}(1+\gamma_i)^{w_i}+\prod_{i=1}^{n}(1-\gamma_i)^{w_i}}\right\},\right. \\
&\qquad\left.\left\{\frac{2\prod_{i=1}^{n}\eta_i^{w_i}}{\prod_{i=1}^{n}(2-\eta_i)^{w_i}+\prod_{i=1}^{n}\eta_i^{w_i}}\right\}\right).
\end{aligned} \qquad (8)$$

Because the DHFS encompasses the fuzzy set, IFS and HFS, the following special operators can be obtained subsequently:

Note 2 Let $d_i (i=1,2,\cdots,n)$ be a collection of DHFEs, where $d_i=(h_i,g_i), i=1,2,\cdots,n$, then

(1) If h_i and g_i have only one value γ_i and η_i respectively, and $\gamma_i+\eta_i=1$, or h_i owns one value, and $g_i=\varnothing$, then the EDHFWA operator reduces to:

$$\begin{aligned}&\mathrm{EFWA}(d_1,d_2,\cdots,d_n)\\&=\overset{n}{\underset{i=1}{\oplus}} w_i d_i\\&=\frac{\prod_{i=1}^{n}(1+\gamma_i)^{w_i}-\prod_{i=1}^{n}(1-\gamma_i)^{w_i}}{\prod_{i=1}^{n}(1+\gamma_i)^{w_i}+\prod_{i=1}^{n}(1-\gamma_i)^{w_i}},\end{aligned} \tag{9}$$

which is called the Einstein fuzzy weighted averaging (EFWA) operator.

(2) If h_i and g_i contain only one value γ_i and η_i respectively, and $\gamma_i+\eta_i<1$, then the EDHFWA operator becomes:

$$\begin{aligned}&\mathrm{EIFWA}(d_1,d_2,\cdots,d_n)\\&=\overset{n}{\underset{i=1}{\oplus}} w_i d_i\\&=\left(\frac{\prod_{i=1}^{n}(1+\gamma_i)^{w_i}-\prod_{i=1}^{n}(1-\gamma_i)^{w_i}}{\prod_{i=1}^{n}(1+\gamma_i)^{w_i}+\prod_{i=1}^{n}(1-\gamma_i)^{w_i}},\ \frac{2\prod_{i=1}^{n}\eta_i^{w_i}}{\prod_{i=1}^{n}(2-\eta_i))^{w_i}+\prod_{i=1}^{n}\eta_i^{w_i}}\right),\end{aligned} \tag{10}$$

which is called the Einstein intuitionistic fuzzy weighted averaging (EIFWA) operator (Xia et al. 2012).

(3) If $h\neq\varnothing$ but $g=\varnothing$, then the EDHFWA operator reduces to:

$$\begin{aligned}&\mathrm{EHFWA}(d_1,d_2,\cdots,d_n)\\&=\overset{n}{\underset{i=1}{\oplus}} w_i d_i\\&=\bigcup\nolimits_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\frac{\prod_{i=1}^{n}(1+\gamma_i)^{w_i}-\prod_{i=1}^{n}(1-\gamma_i)^{w_i}}{\prod_{i=1}^{n}(1+\gamma_i)^{w_i}+\prod_{i=1}^{n}(1-\gamma_i)^{w_i}}\right\},\end{aligned} \tag{11}$$

which is called the Einstein Hesitant fuzzy weighted averaging (EHFWA) operator.

In the following, we shall discuss the properties of the EDHFWA operator:

Property 1 Let $d_i=(h_i,g_i)(i=1,2,\cdots,n)$ be a collection of DHFEs, where $w=(w_1,w_2,\cdots,w_n)^T$ is the weight vector of $d_i(i=1,2,\cdots,n)$, and w_i indicates the importance degree of d_i, satisfying $w_i\geq 0$ ($i=1,2,\cdots,n$) and $\sum_{i=1}^{n} w_i=1$, then

(1) (Idempotency): If $d_i(i=1,2,\cdots,n)$ are equal, i.e. $d_i=d,(i=1,2,\cdots,n)$, then

$$\mathrm{EDHFWA}\left(d_1,d_2,\cdots,d_n\right)=d. \tag{12}$$

(2) (Boundedness):

$$d^-\leq \mathrm{EDHFWA}(d_1,d_2,\ldots,d_n)\leq d^+, \tag{13}$$

where $d^-=(\gamma_{\min},\eta_{\max}), d^+=(\gamma_{\max},\eta_{\min})$, and $\gamma_{\min}=\min\limits_{\gamma_i\in h_i}\{\gamma_i\}$, $\gamma_{\max}=\max\limits_{\gamma_i\in h_i}\{\gamma_i\}$, $\eta_{\min}=\min\limits_{\eta_i\in g_i}\{\eta_i\}$, and $\eta_{\max}=\max\limits_{\eta_i\in g_i}\{\eta_i\}$.

(3) (Monotonicity): Let $d_i^*=(h_i^*,g_i^*)(i=1,2,\cdots,n)$ be another collection of DHFEs, and

$$\max_{\gamma_i\in h_i}\{\gamma_i\}\leq\min_{\gamma_i^*\in h_i^*}\{\gamma_i^*\},\min_{\eta_i\in g_i}\{\eta_i\}\geq\max_{\eta_i^*\in g_i^*}\{\eta_i^*\},$$
$$\#h_i\geq\#h_i^*,\ \#g_i\leq\#g_i^*,\ i=1,2,\cdots,n,$$

then

$$\mathrm{EDHFWA}(d_1,d_2,\cdots,d_n)\leq\mathrm{EDHFWA}(d_1^*,d_2^*,\cdots,d_n^*). \tag{14}$$

Motivated by the idea of the ordinary geometric mean, the following definition can be given:

Definition 7 Let $d_i (i = 1, 2, \cdots, n)$ be a collection of DHFEs, we define the Einstein dual hesitant fuzzy weighted geometric mean (EDHFWG) operator as:

$$\text{EDHFWG}\,(d_1, d_2, \cdots, d_n) = \overset{n}{\underset{i=1}{\otimes}} d_i^{\,w_i}, \qquad (15)$$

where $w = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of $d_i (i = 1, 2, \cdots, n)$, w_i indicates the importance degree of d_i, satisfying $w_i \geq 0$ ($i = 1, 2, \cdots, n$) and $\sum_{i=1}^{n} w_i = 1$, the operation $\otimes$ in Equation (15) is the Einstein product in Definition 4.

Note 3 If the weight vector $w = (w_1, w_2, \cdots, w_n)^T = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^{\mathrm{T}}$, then the EDHFWG operator reduces to the Einstein dual hesitant fuzzy geometric averaging (EDHFG) operator, which is defined as:

$$\text{EDHFG}(d_1, d_2, \cdots, d_n) = (\overset{n}{\underset{i=1}{\otimes}} d_i)^{\frac{1}{n}}.$$

Based on the operations of the DHFEs given in Definitions 4 and 5, we can derive the following theorem which provides a specific calculating equation to aggregate the dual hesitant fuzzy information with the EDHFWG operator:

Theorem 5 Let $d_i (i = 1, 2, \cdots, n)$ be a collection of DHFEs, and $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $d_i (i = 1, 2, \cdots, n)$, where w_i indicates the importance degree of d_i, satisfying $w_i \geq 0$ ($i = 1, 2, \cdots, n$) and $\sum_{i=1}^{n} w_i = 1$, then the aggregated value by using the EDHFWG operator is also a DHFE, and

$$\begin{aligned}
&\text{EDHFWG}(d_1, d_2, \cdots, d_n) \\
&= \overset{n}{\underset{i=1}{\otimes}} d_i^{w_i} \\
&= \bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n \\ \eta_1 \in g_1, \eta_2 \in g_2, \cdots, \eta_n \in g_n}} \left(\left\{ \frac{2\prod_{i=1}^{n} \gamma_i^{w_i}}{\prod_{i=1}^{n} (2-\gamma_i)^{w_i} + \prod_{i=1}^{n} \gamma_i^{w_i}} \right\}, \right. \\
&\qquad \left. \left\{ \frac{\prod_{i=1}^{n} (1+\eta_i)^{w_i} - \prod_{i=1}^{n} (1-\eta_i)^{w_i}}{\prod_{i=1}^{n} (1+\eta_i)^{w_i} + \prod_{i=1}^{n} (1-\eta_i)^{w_i}} \right\} \right).
\end{aligned} \qquad (16)$$

Also, in view of the relations among the DHFS, FS, IFS and HFS, some special operators can be found as follows:

Note 4. Suppose that $d_i (i = 1, 2, \cdots, n)$ are a collection of DHFEs, in which $d_i = (h_i, g_i)$, $i = 1, 2, \cdots, n$, then

(1) If h_i and g_i have only one value γ_i and η_i respectively, and $\gamma_i + \eta_i = 1$, or h_i contains one element, and $g_i = \varnothing$, then the EDHFWG operator reduces to:

$$\text{EFWG}(d_1, d_2, \cdots, d_n) = \overset{n}{\underset{i=1}{\otimes}} d_i^{w_i} = \frac{2\prod_{i=1}^{n} \gamma_i^{w_i}}{\prod_{i=1}^{n} (2-\gamma_i)^{w_i} + \prod_{i=1}^{n} \gamma_i^{w_i}}, \qquad (17)$$

which is called the Einstein fuzzy weighted geometric (EFWG) operator (Wang and Liu 2011).

(2) If h_i and g_i own only one value γ_i and η_i respectively, and $\gamma_i + \eta_i < 1$, then the EDHFWG operator reduces to:

$$\mathrm{EIFWG}(d_1,d_2,\cdots,d_n)$$
$$=\overset{n}{\underset{i=1}{\otimes}} d_i^{w_i}$$
$$=\left(\frac{2\prod_{i=1}^{n}\gamma_i^{w_i}}{\prod_{i=1}^{n}(2-\gamma_i)^{w_i}+\prod_{i=1}^{n}\gamma_i^{w_i}},\frac{\prod_{i=1}^{n}(1+\eta_i)^{w_i}-\prod_{i=1}^{n}(1-\eta_i)^{w_i}}{\prod_{i=1}^{n}(1+\eta_i)^{w_i}+\prod_{i=1}^{n}(1-\eta_i)^{w_i}}\right), \tag{18}$$

which is called the Einstein intuitionistic fuzzy weighted geometric (EIFWG) operator (Wang and Liu 2011, Xia et al. 2012).

(3) If $h\neq\varnothing$ but $g=\varnothing$, then the EDHFWG operator becomes

$$\mathrm{EHFWG}(d_1,d_2,\cdots,d_n)$$
$$=\overset{n}{\underset{i=1}{\otimes}} d_i^{w_i}$$
$$=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\frac{2\prod_{i=1}^{n}\gamma_i^{w_i}}{\prod_{i=1}^{n}(2-\gamma_i)^{w_i}+\prod_{i=1}^{n}\gamma_i^{w_i}}\right\}, \tag{19}$$

which is called the Einstein Hesitant fuzzy weighted geometric (EHFWG) operator.

Similarly, we can prove that the EDHFWG operator also has the following three properties:

Property 2 Let $d_i=(h_i,g_i)\,(i=1,2,\cdots,n)$ be a collection of DHFEs, where $w=(w_1,w_2,\cdots,w_n)^T$ is the weight vector of $d_i(i=1,2,\cdots,n)$, and w_i indicates the importance degree of d_i, with $w_i\geq 0$ $(i=1,2,\cdots,n)$ and $\sum_{i=1}^{n}w_i=1$, then

(1) (Idempotency): If $d_i(i=1,2,\cdots,n)$ are equal, i.e. $d_i=d(i=1,2,\cdots,n)$, then

$$\mathrm{EDHFWG}\left(d_1,d_2,\cdots,d_n\right)=d.$$

(2) (Boundedness):

$$d^-\leq \mathrm{EDHFWG}(d_1,d_2,\cdots,d_n)\leq d^+.$$

where $d^-=(\gamma_{\min},\eta_{\max}),\,d^+=(\gamma_{\max},\eta_{\min})$, and $\gamma_{\min}=\min\limits_{\gamma_i\in h_i}\{\gamma_i\}$, $\gamma_{\max}=\max\limits_{\gamma_i\in h_i}\{\gamma_i\}$, $\eta_{\min}=\min\limits_{\eta_i\in g_i}\{\eta_i\}$, and $\eta_{\max}=\max\limits_{\eta_i\in g_i}\{\eta_i\}$.

(3) (Monotonicity): Let $d_i^*=(h_i^*,g_i^*)\,(i=1,2,\cdots,n)$ be a collection of DHFEs, and

$$\max_{\gamma_i\in h_i}\{\gamma_i\}\leq \min_{\gamma_i^*\in h_i^*}\{\gamma_i^*\},\ \min_{\eta_i\in g_i}\{\eta_i\}\geq \max_{\eta_i^*\in g_i^*}\{\eta_i^*\},$$
$$\#h_i\geq \#h_i^*,\ \#g_i\leq \#g_i^*,\ i=1,2,\cdots,n,$$

then

$$\mathrm{EDHFWG}(d_1,d_2,\cdots,d_n)\leq \mathrm{EDHFWG}(d_1^*,d_2^*,\cdots,d_n^*).$$

The EDHFWA and EDHFWG operators have the same problem that they only consider the importance degrees of the given attribution values but ignore the ordered positions in the aggregation. To reflect the ordered positions, based on the idea of the OWA operator (Yager 1988, 2004) and the OWG operator (Xu and Da 2002), we now introduce the other two Einstein dual hesitant fuzzy aggregating operators:

Definition 8 Let $d_i(i=1,2,\cdots,n)$ be a collection of DHFEs, then we define the Einstein dual hesitant fuzzy ordered weighted averaging (EDHFOWA) operator as:

$$\mathrm{EDHFOWA}\,(d_1,d_2,\cdots,d_n)=\overset{n}{\underset{i=1}{\oplus}} w_i d_{\sigma(i)}, \tag{20}$$

where $d_{\sigma(i)}$ is the *ith* largest of $d_i(i=1,2,\cdots,n)$, $w=(w_1,w_2,\cdots,w_n)^T$ is the associated vector of the EDHFOWA operator, w_i indicates the importance degree of the position of $d_{\sigma(i)}$, such that $w_i\geq 0$ $(i=1,2,\cdots,n)$ and $\sum_{i=1}^{n}w_i=1$, the operation $\oplus$ in Equation (20) is the Einstein sum in Definition 4.

To know the detailed computing method of the EDHFOWA operator, we give the following theorem:

Theorem 6 Let $d_i=(h_i,g_i)\,(i=1,2,\cdots,n)$ be a collection of DHFEs, $d_{\sigma(i)}$ be the *ith* largest of $d_i(i=1,2,\cdots,n)$, and

$w=(w_1,w_2,\cdots,w_n)^T$ be the weight vector of the EDHFOWA operator, where w_i indicates the importance degree of the position of $d_{\sigma(i)}$, with $w_i \ge 0$ $(i=1,2,\cdots,n)$ and $\sum_{i=1}^n w_i=1$, then the aggregated value by using the EDHFOWA operator is also a DHFE, and

$$\begin{aligned}
&\mathrm{EDHFOWA}(d_1,d_2,\cdots,d_n)\\
&=\overset{n}{\underset{i=1}{\oplus}} w_i d_{\sigma(i)}\\
&=\bigcup\nolimits_{\substack{\gamma_{\sigma(1)}\in h_{\sigma(1)},\gamma_{\sigma(2)}\in h_{\sigma(2)},\cdots,\gamma_{\sigma(n)}\in h_{\sigma(n)}\\ \eta_{\sigma(1)}\in g_{\sigma(1)},\eta_{\sigma(2)}\in g_{\sigma(2)},\cdots,\eta_{\sigma(n)}\in g_{\sigma(n)}}}\\
&\qquad\left(\left\{\frac{\prod_{i=1}^n(1+\gamma_{\sigma(i)})^{w_i}-\prod_{i=1}^n(1-\gamma_{\sigma(i)})^{w_i}}{\prod_{i=1}^n(1+\gamma_{\sigma(i)})^{w_i}+\prod_{i=1}^n(1-\gamma_{\sigma(i)})^{w_i}}\right\},\right.\\
&\qquad\left.\left\{\frac{2\prod_{i=1}^n\eta_{\sigma(i)}^{w_i}}{\prod_{i=1}^n(2-\eta_{\sigma(i)})^{w_i}+\prod_{i=1}^n\eta_{\sigma(i)}^{w_i}}\right\}\right).
\end{aligned} \tag{21}$$

Below let's discuss some special cases of the EDHFOWA operator:

Note 5 Assume that $d_i(i=1,2,\cdots,n)$ are a set of DHFEs, where $d_i=(h_i,g_i)(i=1,2,\cdots,n)$, then

(1)If h_i and g_i own only one value γ_i and η_i respectively, and $\gamma_i+\eta_i=1$, or h_i contains one value, and $g_i=\varnothing$, then the EDHFOWA operator reduces to:

$$\begin{aligned}
&\mathrm{EFOWA}(d_1,d_2,\cdots,d_n)\\
&=\overset{n}{\underset{i=1}{\oplus}} w_i d_{\sigma(i)}\\
&=\frac{\prod_{i=1}^n(1+\gamma_{\sigma(i)})^{w_i}-\prod_{i=1}^n(1-\gamma_{\sigma(i)})^{w_i}}{\prod_{i=1}^n(1+\gamma_{\sigma(i)})^{w_i}+\prod_{i=1}^n(1-\gamma_{\sigma(i)})^{w_i}},
\end{aligned} \tag{22}$$

which is called the Einstein fuzzy order weighted averaging (EFOWA) operator.

(2) If h_i and g_i own only one value γ_i and η_i respectively, and $\gamma_i+\eta_i<1$, then the EDHFOWA operator changes

$$\begin{aligned}
&\mathrm{EIFOWA}(d_1,d_2,\cdots,d_n)\\
&=\overset{n}{\underset{i=1}{\oplus}} w_i d_{\sigma(i)}\\
&=\left(\frac{\prod_{i=1}^n(1+\gamma_{\sigma(i)})^{w_i}-\prod_{i=1}^n(1-\gamma_{\sigma(i)})^{w_i}}{\prod_{i=1}^n(1+\gamma_{\sigma(i)})^{w_i}+\prod_{i=1}^n(1-\gamma_{\sigma(i)})^{w_i}},\right.\\
&\qquad\left.\frac{2\prod_{i=1}^n\eta_{\sigma(i)}^{w_i}}{\prod_{i=1}^n(2-\eta_{\sigma(i)}))^{w_i}+\prod_{i=1}^n\eta_{\sigma(i)}^{w_i}}\right),
\end{aligned} \tag{23}$$

which is called the Einstein intuitionistic fuzzy order weighted averaging (EIFOWA) operator.

(3)If $h\ne\varnothing$ but $g=\varnothing$, then the EDHFOWA operator turns into

$$\begin{aligned}
&\mathrm{EHFOWA}(d_1,d_2,\cdots,d_n)\\
&=\overset{n}{\underset{i=1}{\oplus}} w_i d_{\sigma(i)}\\
&=\bigcup\nolimits_{\gamma_{\sigma(1)}\in h_{\sigma(1)},\gamma_{\sigma(2)}\in h_{\sigma(2)},\cdots,\gamma_{\sigma(n)}\in h_{\sigma(n)}}\\
&\qquad\left\{\frac{\prod_{i=1}^n(1+\gamma_{\sigma(i)})^{w_i}-\prod_{i=1}^n(1-\gamma_{\sigma(i)})^{w_i}}{\prod_{i=1}^n(1+\gamma_{\sigma(i)})^{w_i}+\prod_{i=1}^n(1-\gamma_{\sigma(i)})^{w_i}}\right\},
\end{aligned} \tag{24}$$

which is called the Einstein Hesitant fuzzy Order weighted averaging (EHFOWA) operator.

Another ordered dual hesitant fuzzy operator is shown as follows:

Definition 9 Let $d_i(i=1,2,\cdots,n)$ be a collection of DHFEs, then we define the Einstein dual hesitant fuzzy ordered weighted geometric mean (EDHFOWG) operator as:

$$\text{EDHFOWG}(d_1, d_2, \cdots, d_n) = \overset{n}{\underset{i=1}{\otimes}} d_{\sigma(i)}{}^{w_i}, \tag{25}$$

where $d_{\sigma(i)}$ is the *ith* largest of $d_i (i = 1, 2, \cdots, n)$, $w = (w_1, w_2, \cdots, w_n)^T$ is the associated vector of the EDHFOWG operator, w_i indicates the importance degree of the position of $d_{\sigma(i)}$, such that $w_i \geq 0$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^n w_i = 1$, the operation $\otimes$ in Equation (25) is the Einstein product in Definition 4.

To know how to aggregate the DHFEs by the EDHFOWG operator in detail, we introduce the following theorem:

Theorem 7 Let $d_i = (h_i, g_i)(i = 1, 2, \cdots, n)$ be a collection of DHFEs, and $d_{\sigma(i)}$ be the i th largest of $d_i (i = 1, 2, \cdots, n)$, $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of the EDHFOWG operator, w_i indicates the importance degree of the position of $d_{\sigma(i)}$, satisfying $w_i \geq 0$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^n w_i = 1$, then the aggregated value by using the EDHFOWG operator is also a DHFE, and

$$\begin{aligned}
&\text{EDHFOWG}(d_1, d_2, \cdots, d_n) \\
&= \overset{n}{\underset{i=1}{\otimes}} d_{\sigma(i)}^{w_i} \\
&= \bigcup_{\substack{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)} \\ \eta_{\sigma(1)} \in g_{\sigma(1)}, \eta_{\sigma(2)} \in g_{\sigma(2)}, \cdots, \eta_{\sigma(n)} \in g_{\sigma(n)}}} \\
&\left(\left\{ \frac{2\prod_{i=1}^n \gamma_{\sigma(i)}^{w_i}}{\prod_{i=1}^n (2 - \gamma_{\sigma(i)})^{w_i} + \prod_{i=1}^n \gamma_{\sigma(i)}^{w_i}} \right\}, \right. \\
&\left. \left\{ \frac{\prod_{i=1}^n (1 + \eta_{\sigma(i)})^{w_i} - \prod_{i=1}^n (1 - \eta_{\sigma(i)})^{w_i}}{\prod_{i=1}^n (1 + \eta_{\sigma(i)})^{w_i} + \prod_{i=1}^n (1 - \eta_{\sigma(i)})^{w_i}} \right\} \right).
\end{aligned} \tag{26}$$

Similarly, some special cases of the EDHFOWG operator are discussed as follows:

Note 6 Let $d_i (i = 1, 2, \cdots, n)$ be a set of DHFEs, where $d_i = (h_i, g_i)(i = 1, 2, \cdots, n)$, then

(1) If h_i and g_i contain only one value η_i and γ_i respectively, and $\gamma_i + \eta_i = 1$, or h_i owns one element, and $g_i = \varnothing$, then the EDHFOWG operator reduces to:

$$\begin{aligned}
&\text{EFOWG}(d_1, d_2, \cdots, d_n) \\
&= \overset{n}{\underset{i=1}{\otimes}} d_{\sigma(i)}{}^{w_i} \\
&= \frac{2\prod_{i=1}^n \gamma_{\sigma(i)}^{w_i}}{\prod_{i=1}^n (2 - \gamma_{\sigma(i)})^{w_i} + \prod_{i=1}^n \gamma_{\sigma(i)}^{w_i}},
\end{aligned} \tag{27}$$

which is called the Einstein fuzzy ordered weighted geometric (EFOWG) operator (Wang and Liu 2011).

(2) If h_i and g_i have only one value η_i and γ_i respectively, and $\gamma_i + \eta_i < 1$, then the EDHFOWG operator becomes

$$\begin{aligned}
&\text{EIFOWG}(d_1, d_2, \cdots, d_n) \\
&= \overset{n}{\underset{i=1}{\otimes}} d_{\sigma(i)}{}^{w_i} \\
&= \left(\frac{2\prod_{i=1}^n \gamma_{\sigma(i)}^{w_i}}{\prod_{i=1}^n (2 - \gamma_{\sigma(i)})^{w_i} + \prod_{i=1}^n \gamma_{\sigma(i)}^{w_i}}, \right. \\
&\left. \frac{\prod_{i=1}^n (1 + \eta_{\sigma(i)})^{w_i} - \prod_{i=1}^n (1 - \eta_{\sigma(i)})^{w_i}}{\prod_{i=1}^n (1 + \eta_{\sigma(i)})^{w_i} + \prod_{i=1}^n (1 - \eta_{\sigma(i)})^{w_i}} \right),
\end{aligned} \tag{28}$$

which is called the Einstein intuitionistic fuzzy ordered weighted geometric (EIFOWG) operator (Wang and Liu 2011).

(3) If $h \neq \varnothing$ but $g = \varnothing$, then the EDHFOWG operator turns into:

$$\mathrm{EHFOWG}(d_1, d_2, \cdots, d_n)$$
$$= \overset{n}{\underset{i=1}{\otimes}} d_{\sigma(i)}{}^{w_i}$$
$$= \bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}} \left\{ \frac{2\prod_{i=1}^{n} \gamma_{\sigma(i)}^{w_i}}{\prod_{i=1}^{n} (2-\gamma_{\sigma(i)})^{w_i} + \prod_{i=1}^{n} \gamma_{\sigma(i)}^{w_i}} \right\}, \tag{29}$$

which is called the Einstein hesitant fuzzy ordered weighted geometric (EHFOWG) operator.

The EDHFOWA and EDHFOWG operators also have the properties of idempotency, boundedness and monotonicity. Moreover, they satisfy the following property:

Property 3 Let $d_1', d_2', \cdots, d_n'$ be any permutation of $d_1, d_2, \cdots, d_n$, then

$$\mathrm{EDHFOWA}(d_1, d_2, \cdots, d_n) = \mathrm{EDHFOWA}(d_1', d_2', \cdots, d_n'),$$
$$\mathrm{EDHFOWG}(d_1, d_2, \cdots, d_n) = \mathrm{EDHFOWG}(d_1', d_2', \cdots, d_n')$$

for every weight vector w.

4. An Application of the Operators to Multi-Attribute Decision Making

As a preparation for this section, we first introduce the comparison laws for the HFEs (Xia and Xu 2011) and DHFEs (Zhu et al. 2012):

Definition 10 (Xia and Xu 2011). For a HFE h, $s(h) = \frac{1}{\#h}\sum_{\gamma \in h} \gamma$ is called the score function of h, where $\#h$ is the number of the elements in h. For two HFEs h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

Definition 11 (Zhu et al. 2012). Let $d = \{h_d, g_d\}$, $d_i = \{h_{d_i}, g_{d_i}\}$ $(i = 1, 2)$ be any three DHFEs, $s(d) = \frac{1}{\#h}\Sigma_{\gamma \in h} \gamma - \frac{1}{\#g}\Sigma_{\eta \in g} \eta$ is called the score function of d, and $p(d) = \frac{1}{\#h}\Sigma_{\gamma \in h} \gamma + \frac{1}{\#g}\Sigma_{\eta \in g} \eta$ the accuracy function of d, where $\#h$ and $\#g$ are the numbers of the elements in h and g respectively, then

if $s(d_1) > s(d_2)$, then d_1 is superior to d_2, denoted by $d_1 > d_2$;

if $s(d_1) = s(d_2)$, then

(1) if $p(d_1) = p(d_2)$, then d_1 is equivalent to d_2, denoted by $d_1 = d_2$;

(2) If $p(d_1) > p(d_2)$, then d_1 is superior than d_2, denoted by $d_1 > d_2$.

In the following, we apply the operators to multi-attribute decision making under the dual hesitant fuzzy environment.

Let $A = \{A_1, A_2, \cdots, A_m\}$ be a set of alternatives, $G = \{G_1, G_2, \cdots, G_n\}$ a set of attributes, and $w = (w_1, w_2, \cdots, w_n)^T$ the weight vector associated with the EDHFWA, EDHFWG, EDHFOWA or EDHFOWG operator with $w_i \geq 0$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} w_i = 1$. Assume that the characteristic information of the alternatives A_i $(i = 1, 2, \cdots, m)$ is represented by the DHFEs:

$$d_{ij} = \bigcup_{\gamma_{ij} \in h_{ij}, \eta_{ij} \in g_{ij}} \left(\{\gamma_{ij}\}, \{\eta_{ij}\}\right), \tag{30}$$

where γ_{ij} indicates the degree that the alternative A_i satisfies the attribute G_j, η_{ij} indicates the degree that the alternative A_i does not satisfy the attribute G_j, where $\gamma_{ij} \in [0,1]$, $\eta_{ij} \in [0,1]$, and $0 \leq \gamma_{ij}^{+} + \eta_{ij}^{+} \leq 1$. So we can get the dual hesitant fuzzy decision matrix $D=(d_{ij})_{m\times n}$.

Now we give a method for multi-attribute decision making based on the Einstein dual hesitant fuzzy aggregating operators:

Step 1 Utilize the four operators above:

$$d_i=\text{EDHFWA}\left(d_{i1},d_{i2},\cdots,d_{in}\right),i=1,2,\cdots,m, \quad (31)$$

$$d_i=\text{EDHFWG}\left(d_{i1},d_{i2},\cdots,d_{in}\right),i=1,2,\cdots,m, \quad (32)$$

$$d_i=\text{EDHFOWA}\left(d_{i1},d_{i2},\cdots,d_{in}\right),i=1,2,\cdots,m, \quad (33)$$

or

$$d_i=\text{EDHFOWG}\left(d_{i1},d_{i2},\cdots,d_{in}\right),i=1,2,\cdots,m, \quad (34)$$

to get the overall DHFEs $d_i\left(i=1,2,\cdots,m\right)$ of the alternatives $A_i\left(i=1,2,\cdots,m\right)$.

Step 2 Calculate the scores $S\left(d_i\right)\left(i=1,2,\cdots,m\right)$ and the accuracy degrees $p\left(d_i\right)\left(i=1,2,\cdots,m\right)$ of the overall values $d_i\left(i=1,2,\cdots,m\right)$ by using the comparison laws of Definition 11.

Step 3 Rank the alternatives $A_i\left(i=1,2,\cdots,m\right)$, and then select the best one(s) by using the comparison laws of Definition 11.

In the following, we solve a supplier selection problem in a supply chain by the proposed dual hesitant fuzzy aggregating operators.

Table 1 The dual hesitant fuzzy decision matrix D

	G_1	G_2	G_3
A_1	({0.3,0.5},{0.2})	({0.4,0.6},{0.3})	({0.4,0.5},{0.4})
A_2	({0.2,0.3,0.5},{0.4})	({0.5, 0.6},{0.3})	({0.5},{0.3,0.4})
A_3	({0.5, 0.6 },{0.1,0.3})	({0.3},{0.4,0.6})	({0.4,0.6},{0.2})
A_4	({0.4,0.7},{0.2})	({0.6,0.7},{0.2})	({0.4},{0.2,0.4,0.5})
A_5	({0.3,0.4,0.5},{0.2,0.3})	({0.4, 0.5},{0.4})	({0.6},{0.2,0.3})
A_6	({0.2,0.4,0.6},{0.3})	({0.3, 0.6},{0.2})	({0.7},{0.1,0.2})
A_7	({0.3,0.5,0.7},{0.2})	({0.2, 0.7},{0.2})	({0.5},{0.2,0.4})
A_8	({0.3, 0.4 },{0.4,0.5})	({0.3,0.5},{0.4})	({0.2,0.5},{0.3})

Example (Chen 2011). In a small enterprise, several decision makers try to reduce the supply chain risk and uncertainty to improve customer service, inventory levels, and cycle times, which results in increased competitiveness and profitability. The decision makers evaluate eight suppliers: $A_1,A_2,\cdots,A_8$, considering three criteria involving: (1) G_1: performance (e.g., delivery, quality, price); (2) G_2: technology (e.g., manufacturing capability, design capability, ability to cope with technology changes); and (3) G_3: organizational culture and strategy (e.g., feeling of trust, internal and external integration of suppliers, compatibility across levels and functions of the buyer and supplier). Assume that the weight vector of the attribution G_j ($j=1,2,3$) is $w=(0.3,0.4,0.3)^T$, and all the possible evaluated values under the criteria G_j

($j=1,2,3$) for the alternatives $A_i\left(i=1,2,\cdots,8\right)$ are given by the DHFEs $d_{ij}=\bigcup_{\gamma_{ij}\in h_{ij},\eta_{ij}\in g_{ij}}\{\{\gamma_{ij}\},\{\eta_{ij}\}\}$ $\left(i=1,2,\cdots,8,\ j=1,2,3\right)$ where γ_{ij} indicates the degree that the alternative A_i satisfies the attribute G_j, η_{ij} indicates the degree that the alternative A_i does not satisfy the attribute G_j, with $\gamma_{ij}\in[0,1]$, $\eta_{ij}\in[0,1]$, $0\le\gamma_{ij}^{+}+\eta_{ij}^{+}\le 1$, and the decision matrix $D=(d_{ij})_{8\times 3}$ is constructed as shown in Table 1.

To get the optimal supplier(s), the following steps are given:

Step 1 Utilize one of the proposed dual hesitant fuzzy aggregation operators, for example, the EDHFWA operator to get the overall values $d_i\left(i=1,2,\cdots,8\right)$ of the alternatives $A_i\left(i=1,2,\cdots,8\right)$:

$$\begin{aligned}
&\mathrm{EDHFWA}(d_{i1},d_{i2},d_{i3}) \\
&=\mathop{\oplus}_{j=1}^{3} w_j d_{ij} \\
&=\bigcup_{\substack{\gamma_{i1}\in h_{i1},\gamma_{i2}\in h_{i2},\gamma_{i3}\in h_{i3}\\ \eta_{i1}\in g_{i1},\eta_{i2}\in g_{i2},\eta_{i3}\in g_{i3}}} \\
&\left(\left\{\frac{\prod_{k=1}^{3}(1+\gamma_{ik})^{w_k}-\prod_{k=1}^{3}(1-\gamma_{ik})^{w_k}}{\prod_{i=1}^{3}(1+\gamma_{ik})^{w_k}+\prod_{i=1}^{3}(1-\gamma_{ik})^{w_k}}\right\},\right. \\
&\left.\left\{\frac{2\prod_{k=1}^{3}\eta_{ik}^{w_k}}{\prod_{k=1}^{3}(2-\eta_{ik})^{w_k}+\prod_{k=1}^{3}\eta_{ik}^{w_k}}\right\}\right),
\end{aligned} \tag{35}$$

to get the overall DHFEs of the alternatives $A_i\left(i=1,2,\cdots,8\right)$:

$d_1=(\{0.37,0.40,0.43,0.46,0.49,0.52,0.54\},\{0.29\})$,
$d_2=(\{0.42,0.44,0.46,0.49,0.50,0.54\},\{0.33,0.36\})$,
$d_3=(\{0.39,0.43,0.46,0.49\},\{0.22,0.27,0.30,0.36\})$,
$d_4=(\{0.49,0.54,0.58,0.63\},\{0.20,0.25,0.27\})$,
$d_5=(\{0.44,0.47,0.48,0.49,0.50,0.53\},\{0.27,0.30,0.34\})$,
$d_6=(\{0.42,0.47,0.53,0.54,0.58,0.63\},\{0.19,0.23\})$,
$d_7=(\{0.33,0.39,0.47,0.54,0.59,0.65\},\{0.20,0.25\})$,
$d_8=(\{0.26,0.29,0.33,0.36,0.38,0.41,0.44,0.47\},\{0.37,0.39\})$.

Step 2 By the comparison laws of Definition 11, we calculate the scores $S\left(d_i\right)\left(i=1,2,\cdots,8\right)$ of the overall values of $d_i\left(i=1,2,\cdots,8\right)$:

$s(d_1)=0.169$, $s(d_2)=0.13$, $s(d_3)=0.156$, $s(d_4)=0.32$, $s(d_5)=0.182$, $s(d_6)=0.318$, $s(d_7)=0.27$, $s(d_8)=-0.012$.

Step 3 Because $s(d_4)>s(d_6)>s(d_7)>s(d_5)>s(d_1)>s(d_3)>s(d_2)>s(d_8)$, we have

$$A_4\succ A_6\succ A_7\succ A_5\succ A_1\succ A_3\succ A_2\succ A_8,$$

and thus, the alternative A_4 is the best one.

If we only consider the membership degree and neglect the non-membership degree or we don't know the non-membership information at all, then the decision making matrix in Table 1 will turn into the following form:

Table 2 The hesitant fuzzy decision matrix D

	G_1	G_2	G_3
A_1	{0.3,0.5}	{0.4,0.6}	{0.4,0.5}
A_2	{0.2,0.3,0.5}	{0.5, 0.6}	{0.5}
A_3	{0.5, 0.6 }	{0.3}	{0.4,0.6}
A_4	{0.4,0.7}	{0.6,0.7}	{0.4}
A_5	{0.3,0.4,0.5}	{0.4, 0.5}	{0.6}
A_6	{0.2,0.4,0.6}	{0.3, 0.6}	{0.7}
A_7	{0.3,0.5,0.7}	{0.2, 0.7}	{0.5}
A_8	{0.3, 0.4 }	{0.3,0.5}	{0.2,0.5}

Because the data in Table 2 are HFEs, to aggregate the information, we have to utilize the Einstein hesitant fuzzy aggregating operators, for example, the EHFWA operator (11) and the sorting steps are given as follows:

Step 1 Utilize the EHFWA operator to get the overall HFE h_i of the alternatives $A_i\ (i=1,2,\cdots,8)$:

$$h_1=\{0.37,0.40,0.43,0.46,0.49,0.52,0.54\},$$
$$h_2=\{0.42,0.44,0.46,0.49,0.50,0.54\},$$
$$h_3=\{0.39,0.43,0.46,0.49\},$$
$$h_4=\{0.49,0.54,0.58,0.63\},$$
$$h_5=\{0.44,0.47,0.48,0.49,0.50,0.53\},$$
$$h_6=\{0.42,0.47,0.53,0.54,0.58,0.63\},$$
$$h_7=\{0.33,0.39,0.47,0.54,0.59,0.65\},$$
$$h_8=\{0.26,0.29,0.33,0.36,0.38,0.41,0.44,0.47\}.$$

Step 2 By the comparison laws of Definition 10, we calculate the scores $s(h_i)\ (i=1,2,\cdots,8)$ of the overall values of $h_i\ (i=1,2,\cdots,8)$:

$$s(h_1)=0.459, s(h_2)=0.475, s(h_3)=0.443, s(h_4)=0.562,$$
$$s(h_5)=0.485, s(h_6)=0.528, s(h_7)=0.495, s(h_8)=0.368.$$

Step 3 Because $s(h_4)>s(h_6)>s(h_7)>s(h_5)>s(h_2)>s(h_1)>s(h_3)>s(h_8)$, we have

$$A_4\succ A_6\succ A_7\succ A_5\succ A_2\succ A_1\succ A_3\succ A_8,$$

and thus, the alternative A_4 is the best one.

Although the best alternatives derived by the two methods are the same, the orderings of the alternatives A_1, A_2 and A_3 are different. In the first method (the case under dual hesitant fuzzy environment), the ordering of them is $A_1\succ A_3\succ A_2$, while in the second method (the one over hesitant fuzzy environment), the ordering is $A_2\succ A_1\succ A_3$. The fundamental reason is that the HFS neglects the non-membership degree and loses some of the information, while the DHFS considers not only the membership degree, but also the non-membership degree and thus can capture more information than the HFS.

5. Concluding Remarks

The triangular norms and conorms, that is, the *t*-norms and *t*-conorms play an important role in dealing with "and" and "or" operations in decision making problems. In this paper, we have deeply studied the applications of the Einstein *t*-conorm and t-norm under dual hesitant fuzzy environment, and given some new operational laws for DHFEs, studied their relations, and then given some Einstein dual hesitant fuzzy aggregation operators, including the EDHFWA, EDHFWG, EDHFOWA and EDHFOWG operators. The first two operators emphasize the importance degrees of the given attribution values, neglecting the ordered positions in the aggregation. The last two ones can reflect the ordered positions but ignore the importance of the attributions. The two kinds of operators can be chosen to aggregate the information in different occasions on the basis of their characteristics. In the final part of this paper, we have also presented a decision making method using the developed operators based on the Einstein *t*-conorm and *t*-norm.

In fact, there are various kinds of forms of *t*-conorm and *t*-norm. As space is limited, we have only discussed the aggregation method based on the Einstein *t*-conorm and *t*-norm under dual hesitant fuzzy environment. How to combine the other *t*-conorms and t-norms with the dual hesitant fuzzy aggregation technique is still valuable and interesting research topic in future work.

All the proofs of the paper can be found in the following Appendices.

Appendices

1. The Operations of the DHFEs

Definition 12 (Zhu et al. 2012)**.** Let X be a fixed set, d_1 and d_2 two DHFEs, n a positive real number, then the following operations are valid:

(1) $d_1 \oplus d_2$

$$= (h_{d_1} \oplus h_{d_2}, g_{d_1} \otimes g_{d_2})$$
$$= \bigcup_{\gamma_{d_1}\in h_{d_1},\eta_{d_1}\in g_{d_1},\gamma_{d_2}\in h_{d_2},\eta_{d_2}\in g_{d_2}} \left(\{\gamma_{d_1}+\gamma_{d_2}-\gamma_{d_1}\gamma_{d_2}\},\{\eta_{d_1}\eta_{d_2}\}\right),$$

(2) $d_1 \otimes d_2$

$$= (h_{d_1} \otimes h_{d_2}, g_{d_1} \oplus g_{d_2})$$
$$= \bigcup_{\gamma_{d_1}\in h_{d_1},\eta_{d_1}\in g_{d_1},\gamma_{d_2}\in h_{d_2},\eta_{d_2}\in g_{d_2}} \left(\{\gamma_{d_1}\gamma_{d_2}\},\{\eta_{d_1}+\eta_{d_2}-\eta_{d_1}\eta_{d_2}\}\right),$$

(3) $nd = \bigcup_{\gamma_d\in h_d,\eta_d\in g_d}\left(\{1-(1-\gamma_d)^n\},\{(\eta_d)^n\}\right),$

(4) $d^n = \bigcup_{\gamma_d\in h_d,\eta_d\in g_d}\left(\{(\gamma_d)^n\},\{1-(1-\eta_d)^n\}\right),$

where all the results are also DHFEs.

2. The Proof of Theorem 1

Proof. We use mathematical induction to prove that Equation (4) holds for the positive integer n:

For $n=1$, we shall prove

$$1d = \bigcup_{\gamma\in h,\eta\in g}\left(\left\{\frac{(1+\gamma)^1-(1-\gamma)^1}{(1+\gamma)^1+(1-\gamma)^1}\right\},\left\{\frac{2\eta^1}{(2-\eta)^1+\eta^1}\right\}\right).$$

In the left-hand side of the equation above, $1d = d = \bigcup_{\gamma\in h,\eta\in g}(\{\gamma\},\{\eta\})$, and in the right-hand side of it,

$$\bigcup_{\gamma\in h,\eta\in g}\left(\left\{\frac{(1+\gamma)^1-(1-\gamma)^1}{(1+\gamma)^1+(1-\gamma)^1}\right\},\left\{\frac{2\eta^1}{(2-\eta)^1+\eta^1}\right\}\right)$$
$$= \bigcup_{\gamma\in h,\eta\in g}\left(\{\gamma\},\{\eta\}\right),$$

which indicates that the two sides are equal, so Equation (4) holds for $n=1$.

Assume Equation (4) holds for $n=k$, it must be proven that Equation (4) holds for $n=k+1$, i.e.,

$$(k+1)d$$
$$= \bigcup_{\gamma\in h,\eta\in g}\left(\left\{\frac{(1+\gamma)^{k+1}-(1-\gamma)^{k+1}}{(1+\gamma)^{k+1}+(1-\gamma)^{k+1}}\right\},\left\{\frac{2\eta^{k+1}}{(2-\eta)^{k+1}+\eta^{k+1}}\right\}\right). \tag{36}$$

Because the left-hand side of Equation (36) can be rewritten as $(k+1)d = kd \oplus d$, using the induction hypothesis that Equation (4) is correct for $n=k$, and based on the Einstein sum operation of two DHFEs, we have

$$(k+1)d$$
$$= kd \oplus d$$
$$= \bigcup_{\gamma\in h,\eta\in g}\left(\left\{\frac{(1+\gamma)^{k}-(1-\gamma)^{k}}{(1+\gamma)^{k}+(1-\gamma)^{k}}\right\},\left\{\frac{2\eta^{k}}{(2-\eta)^{k}+\eta^{k}}\right\}\right)$$
$$\oplus \bigcup_{\gamma\in h,\eta\in g}\{\gamma,\eta\}$$
$$= \bigcup_{\gamma\in h,\eta\in g}\left(\left\{\frac{\frac{(1+\gamma)^{k}-(1-\gamma)^{k}}{(1+\gamma)^{k}+(1-\gamma)^{k}}+\gamma}{1+\frac{(1+\gamma)^{k}-(1-\gamma)^{k}}{(1+\gamma)^{k}+(1-\gamma)^{k}}\gamma}\right\},\left\{\frac{\frac{2\eta^{k}}{(2-\eta)^{k}+\eta^{k}}\eta}{1+(1-\frac{2\eta^{k}}{(2-\eta)^{k}+\eta^{k}})(1-\eta)}\right\}\right)$$
$$= \bigcup_{\gamma\in h,\eta\in g}\left(\left\{\frac{(1+\gamma)^{k+1}-(1-\gamma)^{k+1}}{(1+\gamma)^{k+1}+(1-\gamma)^{k+1}}\right\},\left\{\frac{2\eta^{k+1}}{(2-\eta)^{k+1}+\eta^{k+1}}\right\}\right).$$

So we have proven that indeed Equation (4) holds for $n=k+1$. Since both the basis and the inductive step are correct, we complete the proof of the theorem.

3. The Proof of Theorem 1 When n Is Any Positive Real Number

Proof. Because $0\le\gamma\le 1, 0\le\eta\le 1, 1\le 2-\eta\le 2$, and $1-\gamma\ge\eta\ge 0, 1-\eta\ge\gamma\ge 0$, obviously, we have

$$0 \le \frac{(1+\gamma)^n-(1-\gamma)^n}{(1+\gamma)^n+(1-\gamma)^n} \le 1, \tag{37}$$

and

$$0 \le \frac{2\eta^n}{(2-\eta)^n+\eta^n} \le 1, \tag{38}$$

$$0 \le \frac{(1+\gamma)^n-(1-\gamma)^n}{(1+\gamma)^n+(1-\gamma)^n} \le \frac{(1+\gamma)^n-(1-\gamma)^n}{(1+\gamma)^n+\eta^n} \le \frac{(1+\gamma)^n-\eta^n}{(1+\gamma)^n+\eta^n}, \tag{39}$$

$$0 \le \frac{2\eta^n}{(2-\eta)^n+\eta^n} = \frac{2\eta^n}{(1+(1-\eta))^n+\eta^n} \le \frac{2\eta^n}{(1+(1-\eta))^n+\eta^n} \le \frac{2\eta^n}{(1+\gamma)^n+\eta^n}. \tag{40}$$

From Equations (39) and (40), we have

$$0 \le \frac{(1+\gamma)^n-(1-\gamma)^n}{(1+\gamma)^n+(1-\gamma)^n} + \frac{2\eta^n}{(2-\eta)^n+\eta^n} \le 1. \tag{41}$$

So from Equations (37), (38), and (41), we know that the DHFE nd defined above is a DHFE for any positive real number.

4. The proof of Theorem 3

Proof. (1) and (2) are obvious, below we prove the others:

(3) $\lambda(d_1 \oplus d_2)$

$$= \lambda\left(\bigcup_{\gamma_1\in h_1,\eta_1\in g_1,\gamma_2\in h_2,\eta_2\in g_2}\left(\left\{\frac{\gamma_1+\gamma_2}{1+\gamma_1\gamma_2}\right\},\left\{\frac{\eta_1\eta_2}{1+(1-\eta_1)(1-\eta_2)}\right\}\right)\right)$$

$$= \bigcup_{\gamma_1\in h_1,\eta_1\in g_1,\gamma_2\in h_2,\eta_2\in g_2}\left(\left\{\frac{(1+\frac{\gamma_1+\gamma_2}{1+\gamma_1\gamma_2})^\lambda-(1-\frac{\gamma_1+\gamma_2}{1+\gamma_1\gamma_2})^\lambda}{(1+\frac{\gamma_1+\gamma_2}{1+\gamma_1\gamma_2})^\lambda+(1-\frac{\gamma_1+\gamma_2}{1+\gamma_1\gamma_2})^\lambda}\right\},\right.$$

$$\left.\left\{\frac{2(\frac{\eta_1\eta_2}{1+(1-\eta_1)(1-\eta_2)})^\lambda}{(2-\frac{\eta_1\eta_2}{1+(1-\eta_1)(1-\eta_2)})^\lambda+(\frac{\eta_1\eta_2}{1+(1-\eta_1)(1-\eta_2)})^\lambda}\right\}\right)$$

$$= \bigcup_{\gamma_1\in h_1,\eta_1\in g_1,\gamma_2\in h_2,\eta_2\in g_2}\left(\left\{\frac{(1+\gamma_1)^\lambda(1+\gamma_2)^\lambda-(1-\gamma_1)^\lambda(1-\gamma_2)^\lambda}{(1+\gamma_1)^\lambda(1+\gamma_2)^\lambda+(1-\gamma_1)^\lambda(1-\gamma_2)^\lambda}\right\},\right.$$

$$\left.\left\{\frac{2(\eta_1\eta_2)^\lambda}{(2-\eta_1)^\lambda(2-\eta_2)^\lambda+(\eta_1\eta_2)^\lambda}\right\}\right)$$

$\lambda d_1 \oplus \lambda d_2$

$$= \bigcup_{\gamma_1\in h_1,\eta_1\in g_1}\left(\left\{\frac{(1+\gamma_1)^\lambda-(1-\gamma_1)^\lambda}{(1+\gamma_1)^\lambda+(1-\gamma_1)^\lambda}\right\},\left\{\frac{2\eta_1^\lambda}{(2-\eta_1)^\lambda+\eta_1^\lambda}\right\}\right)$$

$$\oplus \bigcup_{\gamma_2\in h_2,\eta_2\in g_2}\left(\left\{\frac{(1+\gamma_2)^\lambda-(1-\gamma_2)^\lambda}{(1+\gamma_2)^\lambda+(1-\gamma_2)^\lambda}\right\},\left\{\frac{2\eta_2^\lambda}{(2-\eta_2)^\lambda+\eta_2^\lambda}\right\}\right)$$

$$= \bigcup_{\gamma_1\in h_1,\eta_1\in g_1,\gamma_2\in h_2,\eta_2\in g_2}$$

$$\left(\left\{\frac{\frac{(1+\gamma_1)^\lambda-(1-\gamma_1)^\lambda}{(1+\gamma_1)^\lambda+(1-\gamma_1)^\lambda}+\frac{(1+\gamma_2)^\lambda-(1-\gamma_2)^\lambda}{(1+\gamma_2)^\lambda+(1-\gamma_2)^\lambda}}{1+\frac{(1+\gamma_1)^\lambda-(1-\gamma_1)^\lambda}{(1+\gamma_1)^\lambda+(1-\gamma_1)^\lambda}\frac{(1+\gamma_2)^\lambda-(1-\gamma_2)^\lambda}{(1+\gamma_2)^\lambda+(1-\gamma_2)^\lambda}}\right\},\right.$$

$$\left.\left\{\frac{\frac{2\eta_1^\lambda}{(2-\eta_1)^\lambda+\eta_1^\lambda}\frac{2\eta_2^\lambda}{(2-\eta_2)^\lambda+\eta_2^\lambda}}{1+(1-\frac{2\eta_1^\lambda}{(2-\eta_1)^\lambda+\eta_1^\lambda})(1-\frac{2\eta_2^\lambda}{(2-\eta_2)^\lambda+\eta_2^\lambda})}\right\}\right)$$

$$= \bigcup_{\gamma_1\in h_1,\eta_1\in g_1,\gamma_2\in h_2,\eta_2\in g_2}$$

$$\left(\left\{\frac{(1+\gamma_1)^\lambda(1+\gamma_2)^\lambda-(1-\gamma_1)^\lambda(1-\gamma_2)^\lambda}{(1+\gamma_1)^\lambda(1+\gamma_2)^\lambda+(1-\gamma_1)^\lambda(1-\gamma_2)^\lambda}\right\},\right.$$

$$\left.\left\{\frac{2(\eta_1\eta_2)^\lambda}{(2-\eta_1)^\lambda(2-\eta_2)^\lambda+(\eta_1\eta_2)^\lambda}\right\}\right),$$

and thus,

$$\lambda(d_1 \oplus d_2) = \lambda d_1 \oplus \lambda d_2.$$

(5) $\lambda_1 d \oplus \lambda_2 d$

$$= \bigcup_{\gamma\in h,\eta\in g}\left(\left\{\frac{(1+\gamma)^{\lambda_1}-(1-\gamma)^{\lambda_1}}{(1+\gamma)^{\lambda_1}+(1-\gamma)^{\lambda_1}}\right\},\left\{\frac{2\eta^{\lambda_1}}{(2-\eta)^{\lambda_1}+\eta^{\lambda_1}}\right\}\right)$$

$$\oplus \bigcup_{\gamma\in h,\eta\in g}\left(\left\{\frac{(1+\gamma)^{\lambda_2}-(1-\gamma)^{\lambda_2}}{(1+\gamma)^{\lambda_2}+(1-\gamma)^{\lambda_2}}\right\},\left\{\frac{2\eta^{\lambda_2}}{(2-\eta)^{\lambda_2}+\eta^{\lambda_2}}\right\}\right)$$

$$=\bigcup_{\gamma\in h,\eta\in g}\left(\left\{\frac{\frac{(1+\gamma)^{\lambda_1}-(1-\gamma)^{\lambda_1}}{(1+\gamma)^{\lambda_1}+(1-\gamma)^{\lambda_1}}+\frac{(1+\gamma)^{\lambda_2}-(1-\gamma)^{\lambda_2}}{(1+\gamma)^{\lambda_2}+(1-\gamma)^{\lambda_2}}}{1+\frac{(1+\gamma)^{\lambda_1}-(1-\gamma)^{\lambda_1}}{(1+\gamma)^{\lambda_1}+(1-\gamma)^{\lambda_1}}\frac{(1+\gamma)^{\lambda_2}-(1-\gamma)^{\lambda_2}}{(1+\gamma)^{\lambda_2}+(1-\gamma)^{\lambda_2}}}\right\},\left\{\frac{\frac{2\eta^{\lambda_1}}{(2-\eta)^{\lambda_1}+\eta^{\lambda_1}}\frac{2\eta^{\lambda_2}}{(2-\eta)^{\lambda_2}+\eta^{\lambda_2}}}{1+(1-\frac{2\eta^{\lambda_1}}{(2-\eta)^{\lambda_1}+\eta^{\lambda_1}})(1-\frac{2\eta^{\lambda_2}}{(2-\eta)^{\lambda_2}+\eta^{\lambda_2}})}\right\}\right)$$

$$=\bigcup_{\gamma\in h,\eta\in g}\left(\left\{\frac{(1+\gamma)^{\lambda_1+\lambda_2}-(1-\gamma)^{\lambda_1+\lambda_2}}{(1+\gamma)^{\lambda_1+\lambda_2}+(1-\gamma)^{\lambda_1+\lambda_2}}\right\},\left\{\frac{2\eta^{\lambda_1+\lambda_2}}{(2-\eta)^{\lambda_1+\lambda_2}+\eta^{\lambda_1+\lambda_2}}\right\}\right)$$

$$=(\lambda_1\oplus\lambda_2)d,$$

and thus, $\lambda_1 d\oplus\lambda_2 d=(\lambda_1\oplus\lambda_2)d$.

Similarly, (4) and (6) can be proven, which completes the proof of the theorem.

5. The Proof of Theorem 4

Proof. We shall prove the equation by using mathematical induction on n.

For $n=2$, we have

$$\text{EDHFWA}(d_1,d_2)=\overset{2}{\underset{i=1}{\oplus}}w_i d_i=w_1d_1\oplus w_2d_2$$

$$=\bigcup_{\gamma_1\in h_1,\eta_1\in g_1}\left(\left\{\frac{(1+\gamma_1)^{w_1}-(1-\gamma_1)^{w_1}}{(1+\gamma_1)^{w_1}+(1-\gamma_1)^{w_1}}\right\},\left\{\frac{2\eta_1^{w_1}}{(2-\eta_1)^{w_1}+\eta_1^{w_1}}\right\}\right)$$

$$\oplus\bigcup_{\gamma_2\in h_2,\eta_2\in g_2}\left(\left\{\frac{(1+\gamma_2)^{w_2}-(1-\gamma_2)^{w_2}}{(1+\gamma_2)^{w_2}+(1-\gamma_2)^{w_2}}\right\},\left\{\frac{2\eta_2^{w_2}}{(2-\eta_2)^{w_2}+\eta_2^{w_2}}\right\}\right)$$

$$=\bigcup_{\gamma_1\in h_1,\eta_1\in g_1,\gamma_2\in h_2,\eta_2\in g_2}\left(\left\{\frac{\frac{(1+\gamma_1)^{w_1}-(1-\gamma_1)^{w_1}}{(1+\gamma_1)^{w_1}+(1-\gamma_1)^{w_1}}+\frac{(1+\gamma_2)^{w_2}-(1-\gamma_2)^{w_2}}{(1+\gamma_2)^{w_2}+(1-\gamma_2)^{w_2}}}{1+\frac{(1+\gamma_1)^{w_1}-(1-\gamma_1)^{w_1}}{(1+\gamma_1)^{w_1}+(1-\gamma_1)^{w_1}}\frac{(1+\gamma_2)^{w_2}-(1-\gamma_2)^{w_2}}{(1+\gamma_2)^{w_2}+(1-\gamma_2)^{w_2}}}\right\},\left\{\frac{\frac{2\eta_1^{w_1}}{(2-\eta_1)^{w_1}+\eta_1^{w_1}}\frac{2\eta_2^{w_2}}{(2-\eta_2)^{w_2}+\eta_2^{w_2}}}{1+(1-\frac{2\eta_1^{w_1}}{(2-\eta_1)^{w_1}+\eta_1^{w_1}})(1-\frac{2\eta_2^{w_2}}{(2-\eta_2)^{w_2}+\eta_2^{w_2}})}\right\}\right)$$

$$=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\eta_1\in g_1,\eta_2\in g_2}\left(\left\{\frac{\prod_{i=1}^{2}(1+\gamma_i)^{w_i}-\prod_{i=1}^{2}(1-\gamma_i)^{w_i}}{\prod_{i=1}^{2}(1+\gamma_i)^{w_i}+\prod_{i=1}^{2}(1-\gamma_i)^{w_i}}\right\},\left\{\frac{2\prod_{i=1}^{2}\eta_i^{w_i}}{\prod_{i=1}^{2}(2-\eta_i)^{w_i}+\prod_{i=1}^{2}\eta_i^{w_i}}\right\}\right),$$

which indicates that Equation (8) holds for $n=2$.

Suppose Equation (8) holds for $n=k$, that is

$$\text{EDHFWA}(d_1,d_2,\cdots,d_k)=\overset{k}{\underset{i=1}{\oplus}}w_i d_i$$

$$=\bigcup_{\substack{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_k\in h_k\\ \eta_1\in g_1,\eta_2\in g_2,\cdots,\eta_k\in g_k}}\left(\left\{\frac{\prod_{i=1}^{k}(1+\gamma_i)^{w_i}-\prod_{i=1}^{k}(1-\gamma_i)^{w_i}}{\prod_{i=1}^{k}(1+\gamma_i)^{w_i}+\prod_{i=1}^{k}(1-\gamma_i)^{w_i}}\right\},\left\{\frac{2\prod_{i=1}^{k}\eta_i^{w_i}}{\prod_{i=1}^{k}(2-\eta_i)^{w_i}+\prod_{i=1}^{k}\eta_i^{w_i}}\right\}\right),$$

then

$$\text{EDHFWA}(d_1,d_2,\cdots,d_k,d_{k+1})=\overset{k}{\underset{i=1}{\oplus}}w_i d_i\oplus w_{k+1}d_{k+1}$$

$$=\bigcup_{\substack{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_k\in h_k\\ \eta_1\in g_1,\eta_2\in g_2,\cdots,\eta_k\in g_k}}\left(\left\{\frac{\prod_{i=1}^{k}(1+\gamma_i)^{w_i}-\prod_{i=1}^{k}(1-\gamma_i)^{w_i}}{\prod_{i=1}^{k}(1+\gamma_i)^{w_i}+\prod_{i=1}^{k}(1-\gamma_i)^{w_i}}\right\},\left\{\frac{2\prod_{i=1}^{k}\eta_i^{w_i}}{\prod_{i=1}^{k}(2-\eta_i)^{w_i}+\prod_{i=1}^{k}\eta_i^{w_i}}\right\}\right)$$

$$\oplus\bigcup_{\gamma_{k+1}\in h_{k+1},\eta_{k+1}\in g_{k+1}}\left(\left\{\frac{(1+\gamma_{k+1})^{w_{k+1}}-(1-\gamma_{k+1})^{w_{k+1}}}{(1+\gamma_{k+1})^{w_{k+1}}+(1-\gamma_{k+1})^{w_{k+1}}}\right\},\left\{\frac{2\eta_{k+1}^{w_{k+1}}}{(2-\eta_{k+1})^{w_{k+1}}+\eta_{k+1}^{w_{k+1}}}\right\}\right)$$

$$=\bigcup_{\substack{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_{k+1}\in h_{k+1}\\ \eta_1\in g_1,\eta_2\in g_2,\cdots,\eta_{k+1}\in g_{k+1}}}\left(\left\{\frac{\dfrac{\prod_{i=1}^{k}(1+\gamma_i)^{w_i}-\prod_{i=1}^{k}(1-\gamma_i)^{w_i}}{\prod_{i=1}^{k}(1+\gamma_i)^{w_i}+\prod_{i=1}^{k}(1-\gamma_i)^{w_i}}+\dfrac{(1+\gamma_{k+1})^{w_{k+1}}-(1-\gamma_{k+1})^{w_{k+1}}}{(1+\gamma_{k+1})^{w_{k+1}}+(1-\gamma_{k+1})^{w_{k+1}}}}{1+\dfrac{\prod_{i=1}^{k}(1+\gamma_i)^{w_i}-\prod_{i=1}^{k}(1-\gamma_i)^{w_i}}{\prod_{i=1}^{k}(1+\gamma_i)^{w_i}+\prod_{i=1}^{k}(1-\gamma_i)^{w_i}}\dfrac{(1+\gamma_{k+1})^{w_{k+1}}-(1-\gamma_{k+1})^{w_{k+1}}}{(1+\gamma_{k+1})^{w_{k+1}}+(1-\gamma_{k+1})^{w_{k+1}}}}\right\},\right.$$

$$\left.\left\{\frac{\dfrac{2\prod_{i=1}^{k}\eta_i^{w_i}}{\prod_{i=1}^{k}(2-\eta_i)^{w_i}+\prod_{i=1}^{k}\eta_i^{w_i}}\dfrac{2\eta_{k+1}{}^{w_{k+1}}}{(2-\eta_{k+1})^{w_{k+1}}+\eta_{k+1}{}^{w_{k+1}}}}{1+(1-\dfrac{2\prod_{i=1}^{k}\eta_i^{w_i}}{\prod_{i=1}^{k}(2-\eta_i)^{w_i}+\prod_{i=1}^{k}\eta_i^{w_i}})(1-\dfrac{2\eta_{k+1}{}^{w_{k+1}}}{(2-\eta_{k+1})^{w_{k+1}}+\eta_{k+1}{}^{w_{k+1}}})}\right\}\right)$$

$$=\bigcup_{\substack{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_{k+1}\\ \eta_1\in g_1,\eta_2\in g_2,\cdots,\eta_n\in g_{n+1}}}\left(\left\{\frac{\prod_{i=1}^{k+1}(1+\gamma_i)^{w_i}-\prod_{i=1}^{k+1}(1-\gamma_i)^{w_i}}{\prod_{i=1}^{k+1}(1+\gamma_i)^{w_i}+\prod_{i=1}^{k+1}(1-\gamma_i)^{w_i}}\right\},\left\{\frac{2\prod_{i=1}^{k+1}\eta_i^{w_i}}{\prod_{i=1}^{k+1}(2-\eta_i)^{w_i}+\prod_{i=1}^{k+1}\eta_i^{w_i}}\right\}\right),$$

i.e., Equation (8) holds for $n=k+1$. Thus, Equation (8) holds for all n.

Using the same method as in Theorem 1, we can prove that the aggregated value by using the EDHFWA operator is also a DHFE, which completes the proof of Theorem 4.

6. The Proof of Property 1

Proof. (1) Because $d_i=d\ (i=1,2,\cdots,n)$, and $\sum_{i=1}^{n}w_i=1$, then

$$\mathrm{EDHFWA}(d_1,d_2,\cdots,d_n)=\mathop{\oplus}_{i=1}^{n}w_id_i=\mathop{\oplus}_{i=1}^{n}w_id$$
$$=\sum_{i=1}^{n}w_id=d.$$

(2)Let $f(x)=\dfrac{1-x}{1+x},\ 0\le x\le 1$, and suppose that $x_1\le x_2$, then

$$f(x_1)-f(x_2)=\frac{1-x_1}{1+x_1}-\frac{1-x_2}{1+x_2}=2\frac{x_2-x_1}{(1+x_1)(1+x_2)}\ge 0,$$

and thus, the function $f(x)$ is decreasing. Because $\gamma_{\min}\le\gamma_j\le\gamma_{\max}$ for every $\gamma_j\in h_j$, then

$$\frac{1-\gamma_{\max}}{1+\gamma_{\max}}\le\frac{1-\gamma_j}{1+\gamma_j}\le\frac{1-\gamma_{\min}}{1+\gamma_{\min}},$$
$$j=1,2,\cdots,l,\ l=\#h_1+\#h_2+\cdots+\#h_n,$$

and then, for $w_i\ge 0$, we have

$$(\frac{1-\gamma_{\max}}{1+\gamma_{\max}})^{w_j}\le(\frac{1-\gamma_j}{1+\gamma_j})^{w_j}\le(\frac{1-\gamma_{\min}}{1+\gamma_{\min}})^{w_j},\ j=1,2,\cdots n.$$

Thus,

$$\prod_{j=1}^{n}(\frac{1-\gamma_{\max}}{1+\gamma_{\max}})^{w_j}\le\prod_{j=1}^{n}(\frac{1-\gamma_j}{1+\gamma_j})^{w_j}\le\prod_{j=1}^{n}(\frac{1-\gamma_{\min}}{1+\gamma_{\min}})^{w_j}$$

$$\Leftrightarrow(\frac{1-\gamma_{\max}}{1+\gamma_{\max}})^{\sum_{j=1}^{n}w_j}\le\prod_{j=1}^{n}(\frac{1-\gamma_j}{1+\gamma_j})^{w_j}\le(\frac{1-\gamma_{\min}}{1+\gamma_{\min}})^{\sum_{j=1}^{n}w_j}$$

$$\Leftrightarrow\frac{1-\gamma_{\max}}{1+\gamma_{\max}}\le\prod_{j=1}^{n}(\frac{1-\gamma_j}{1+\gamma_j})^{w_j}\le\frac{1-\gamma_{\min}}{1+\gamma_{\min}}$$

$$\Leftrightarrow\frac{2}{1+\gamma_{\max}}\le 1+\prod_{j=1}^{n}(\frac{1-\gamma_j}{1+\gamma_j})^{w_j}\le\frac{2}{1+\gamma_{\min}}$$

$$\Leftrightarrow\frac{1+\gamma_{\min}}{2}\le\frac{1}{1+\prod_{j=1}^{n}(\frac{1-\gamma_j}{1+\gamma_j})^{w_j}}\le\frac{1+\gamma_{\max}}{2}$$

$$\Leftrightarrow 1+\gamma_{\min}\le\frac{2}{1+\prod_{j=1}^{n}(\frac{1-\gamma_j}{1+\gamma_j})^{w_j}}\le 1+\gamma_{\max}$$

$$\Leftrightarrow\gamma_{\min}\le\frac{2}{1+\prod_{j=1}^{n}(\frac{1-\gamma_j}{1+\gamma_j})^{w_j}}-1\le\gamma_{\max}$$

$$\Leftrightarrow\gamma_{\min}\le\frac{\prod_{j=1}^{n}(1+\gamma_j)^{w_j}-\prod_{j=1}^{n}(1-\gamma_j)^{w_j}}{\prod_{j=1}^{n}(1+\gamma_j)^{w_j}+\prod_{j=1}^{n}(1-\gamma_j)^{w_j}}\le\gamma_{\max}.$$

So we have

$$l\gamma_{\min} \le \sum_{\gamma_j \in h_j} \left(\frac{\prod_{j=1}^{n}(1+\gamma_j)^{w_j} - \prod_{j=1}^{n}(1-\gamma_j)^{w_j}}{\prod_{j=1}^{n}(1+\gamma_j)^{w_j} + \prod_{j=1}^{n}(1-\gamma_j)^{w_j}} \right) \le l\gamma_{\max}$$

$l = \# h_1 \times \# h_2 \times \cdots \times \# h_n$,

then

$$\gamma_{\min} \le \frac{\sum_{\gamma_j \in h_j} \left(\frac{\prod_{j=1}^{n}(1+\gamma_j)^{w_j} - \prod_{j=1}^{n}(1-\gamma_j)^{w_j}}{\prod_{j=1}^{n}(1+\gamma_j)^{w_j} + \prod_{j=1}^{n}(1-\gamma_j)^{w_j}} \right)}{l} \le \gamma_{\max},$$

$$l = \# h_1 \times \# h_2 \times \cdots \times \# h_n. \tag{42}$$

Let $g(x) = \frac{2-x}{x} = \frac{2}{x} - 1,\ 0 < x \le 1$, since $g'(x) = -\frac{2}{x^2} - 1 < 0$, then the function $g(x)$ is decreasing. Because $\eta_{\min} \le \eta_j \le \eta_{\max}$ for every $\eta_j \in g_j$, then

$$\frac{2-\eta_{\max}}{\eta_{\max}} \le \frac{2-\eta_j}{\eta_j} \le \frac{2-\eta_{\min}}{\eta_{\min}},$$

$$j = 1,2,\cdots,L,\ L = \# g_1 + \# g_2 + \cdots + \# g_n,$$

and then

$$\left(\frac{2-\eta_{\max}}{\eta_{\max}}\right)^{w_j} \le \left(\frac{2-\eta_j}{\eta_j}\right)^{w_j} \le \left(\frac{2-\eta_{\min}}{\eta_{\min}}\right)^{w_j},$$

$$\eta_j \in g_j,\ j = 1,2,\cdots,n.$$

Thus,

$$\prod_{j=1}^{n}\left(\frac{2-\eta_{\max}}{\eta_{\max}}\right)^{w_j} \le \prod_{j=1}^{n}\left(\frac{2-\eta_j}{\eta_j}\right)^{w_j} \le \prod_{j=1}^{n}\left(\frac{2-\eta_{\min}}{\eta_{\min}}\right)^{w_j}$$

$$\Leftrightarrow \left(\frac{2-\eta_{\max}}{\eta_{\max}}\right)^{\sum_{j=1}^{n} w_j} \le \prod_{j=1}^{n}\left(\frac{2-\eta_j}{\eta_j}\right)^{w_j} \le \left(\frac{2-\eta_{\min}}{\eta_{\min}}\right)^{\sum_{j=1}^{n} w_j}$$

$$\Leftrightarrow \frac{2-\eta_{\max}}{\eta_{\max}} \le \prod_{j=1}^{n}\left(\frac{2-\eta_j}{\eta_j}\right)^{w_j} \le \frac{2-\eta_{\min}}{\eta_{\min}}$$

$$\Leftrightarrow \frac{2}{\eta_{\max}} \le \prod_{j=1}^{n}\left(\frac{2-\eta_j}{\eta_j}\right)^{w_j} + 1 \le \frac{2}{\eta_{\min}}$$

$$\Leftrightarrow \frac{\eta_{\min}}{2} \le \frac{1}{\prod_{j=1}^{n}\left(\frac{2-\eta_j}{\eta_j}\right)^{w_j} + 1} \le \frac{\eta_{\max}}{2}$$

$$\Leftrightarrow \eta_{\min} \le \frac{2}{\prod_{j=1}^{n}\left(\frac{2-\eta_j}{\eta_j}\right)^{w_j} + 1} \le \eta_{\max}$$

$$\Leftrightarrow \eta_{\min} \le \frac{2\prod_{j=1}^{n}(\eta_j)^{w_j}}{\prod_{j=1}^{n}(2-\eta_j)^{w_j} + \prod_{j=1}^{n}(\eta_j)^{w_j}} \le \eta_{\max}.$$

Therefore, we have

$$L\eta_{\min} \le \sum_{\eta_j \in g_j} \left(\frac{2\prod_{j=1}^{n}(\eta_j)^{w_j}}{\prod_{j=1}^{n}(2-\eta_j)^{w_j} + \prod_{j=1}^{n}(\eta_j)^{w_j}} \right) \le L\eta_{\max},$$

$L = \# g_1 \times \# g_2 \times \cdots \times \# g_n$.

Summarily, we have

$$\eta_{\min} \le \frac{\sum_{\eta_j \in g_j} \left(\frac{2\prod_{j=1}^{n}(\eta_j)^{w_j}}{\prod_{j=1}^{n}(2-\eta_j)^{w_j} + \prod_{j=1}^{n}(\eta_j)^{w_j}} \right)}{L} \le \eta_{\max},$$

$$L = \# g_1 \times \# g_2 \times \cdots \times \# g_n. \tag{43}$$

Let $\text{EDHFWA}(d_1, d_2, \cdots, d_n) = d$, then from Definition 11, and Equations (42) and (43), we know that

$$\gamma_{\min} - \eta_{\max} \le s(d) \le \gamma_{\max} - \eta_{\min},$$

i.e.,

$$s(d^-) \le s(d) \le s(d^+).$$

a) If $s(d^-) < s(d) < s(d^+)$, then we can prove immediately that
$d^- < \text{EDHFWA}(d_1, d_2, \cdots, d_n) < d^+$

b) If $s(d^-) = s(d)$

$$\frac{\sum_{\gamma_j \in h_j} \left(\frac{\prod_{j=1}^{n}(1+\gamma_j)^{w_j} - \prod_{j=1}^{n}(1-\gamma_j)^{w_j}}{\prod_{j=1}^{n}(1+\gamma_j)^{w_j} + \prod_{j=1}^{n}(1-\gamma_j)^{w_j}} \right)}{l} - \frac{\sum_{\eta_j \in g_j} \left(\frac{2\prod_{j=1}^{n}(\eta_j)^{w_j}}{\prod_{j=1}^{n}(2-\eta_j)^{w_j} + \prod_{j=1}^{n}(\eta_j)^{w_j}} \right)}{L}$$

$$= \gamma_{\min} - \eta_{\max},$$

and thus, by Equations (42) and (43), we know that

$$\frac{\sum_{\gamma_j\in h_j}\left(\frac{\prod_{j=1}^{n}(1+\gamma_j)^{w_j}-\prod_{j=1}^{n}(1-\gamma_j)^{w_j}}{\prod_{j=1}^{n}(1+\gamma_j)^{w_j}+\prod_{j=1}^{n}(1-\gamma_j)^{w_j}}\right)}{l}=\gamma_{\min},$$

$$\frac{\sum_{\eta_j\in g_j}\left(\frac{2\prod_{j=1}^{n}(\eta_j)^{w_j}}{\prod_{j=1}^{n}(2-\eta_j)^{w_j}+\prod_{j=1}^{n}(\eta_j)^{w_j}}\right)}{L}=\eta_{\max}.$$

So $h(d^-)=h(d)$, and thus $d^-=d$.

c) If $s(d^+)=s(d)$, then

$$\frac{\sum_{\gamma_j\in h_j}\left(\frac{\prod_{j=1}^{n}(1+\gamma_j)^{w_j}-\prod_{j=1}^{n}(1-\gamma_j)^{w_j}}{\prod_{j=1}^{n}(1+\gamma_j)^{w_j}+\prod_{j=1}^{n}(1-\gamma_j)^{w_j}}\right)}{l}-\frac{\sum_{\eta_j\in g_j}\left(\frac{2\prod_{j=1}^{n}(\eta_j)^{w_j}}{\prod_{j=1}^{n}(2-\eta_j)^{w_j}+\prod_{j=1}^{n}(\eta_j)^{w_j}}\right)}{L}$$
$$=\gamma_{\max}-\eta_{\min}$$

and thus, by Equations (42) and (43), we have

$$\frac{\sum_{\gamma_j\in h_j}\left(\frac{\prod_{j=1}^{n}(1+\gamma_j)^{w_j}-\prod_{j=1}^{n}(1-\gamma_j)^{w_j}}{\prod_{j=1}^{n}(1+\gamma_j)^{w_j}+\prod_{j=1}^{n}(1-\gamma_j)^{w_j}}\right)}{l}=\gamma_{\max},$$

$$\frac{\sum_{\eta_j\in g_j}\left(\frac{2\prod_{j=1}^{n}(\eta_j)^{w_j}}{\prod_{j=1}^{n}(2-\eta_j)^{w_j}+\prod_{j=1}^{n}(\eta_j)^{w_j}}\right)}{L}=\eta_{\min}.$$

So $h(d^+)=h(d)$, and thus $d^-=d$.

From the above discussion, we can get $d^-\le \text{EDHFWA}(d_1,d_2,\cdots,d_n)\le d^+$, which completes the proof.

(3) Because $f(x)=\frac{1-x}{1+x}, 0\le x\le 1$ is a decreasing function and $\gamma_i\le\gamma_i^*$, $\gamma_i\in h_i$, $\gamma_i^*\in h_i^*$ for every $i=1,2,\cdots,n$, then

$$\frac{1-\gamma_i^*}{1+\gamma_i^*}\le\frac{1-\gamma_i}{1+\gamma_i}\quad,\ i=1,2,\cdots,n,$$

and then

$$\left(\frac{1-\gamma_i^*}{1+\gamma_i^*}\right)^{w_i}\le\left(\frac{1-\gamma_i}{1+\gamma_i}\right)^{w_i},\ i=1,2,\cdots,n.$$

Thus,

$$\prod_{i=1}^{n}\left(\frac{1-\gamma_i^*}{1+\gamma_i^*}\right)^{w_i}\le\prod_{i=1}^{n}\left(\frac{1-\gamma_i}{1+\gamma_i}\right)^{w_i}$$

$$\Leftrightarrow 1+\prod_{i=1}^{n}\left(\frac{1-\gamma_i^*}{1+\gamma_i^*}\right)^{w_i}\le 1+\prod_{i=1}^{n}\left(\frac{1-\gamma_i}{1+\gamma_i}\right)^{w_i}$$

$$\Leftrightarrow\frac{1}{1+\prod_{i=1}^{n}\left(\frac{1-\gamma_i}{1+\gamma_i}\right)^{w_i}}\le\frac{1}{1+\prod_{i=1}^{n}\left(\frac{1-\gamma_i^*}{1+\gamma_i^*}\right)^{w_i}}$$

$$\Leftrightarrow\frac{2}{1+\prod_{i=1}^{n}\left(\frac{1-\gamma_i}{1+\gamma_i}\right)^{w_i}}\le\frac{2}{1+\prod_{i=1}^{n}\left(\frac{1-\gamma_i^*}{1+\gamma_i^*}\right)^{w_i}}$$

$$\Leftrightarrow\frac{2}{1+\prod_{i=1}^{n}\left(\frac{1-\gamma_i}{1+\gamma_i}\right)^{w_i}}-1\le\frac{2}{1+\prod_{i=1}^{n}\left(\frac{1-\gamma_i^*}{1+\gamma_i^*}\right)^{w_i}}-1$$

$$\Leftrightarrow\frac{1-\prod_{i=1}^{n}\left(\frac{1-\gamma_i}{1+\gamma_i}\right)^{w_i}}{1+\prod_{i=1}^{n}\left(\frac{1-\gamma_i}{1+\gamma_i}\right)^{w_i}}\le\frac{1-\prod_{i=1}^{n}\left(\frac{1-\gamma_i^*}{1+\gamma_i^*}\right)^{w_i}}{1+\prod_{i=1}^{n}\left(\frac{1-\gamma_i^*}{1+\gamma_i^*}\right)^{w_i}}$$

$$\Leftrightarrow\frac{\prod_{i=1}^{n}(1+\gamma_i)^{w_i}-\prod_{i=1}^{n}(1-\gamma_i)^{w_i}}{\prod_{i=1}^{n}(1+\gamma_i)^{w_i}+\prod_{i=1}^{n}(1-\gamma_i)^{w_i}}\le\frac{\prod_{i=1}^{n}\left(1+\gamma_i^*\right)^{w_i}-\prod_{i=1}^{n}\left(1-\gamma_i^*\right)^{w_i}}{\prod_{i=1}^{n}\left(1+\gamma_i^*\right)^{w_i}+\prod_{i=1}^{n}\left(1-\gamma_i^*\right)^{w_i}}$$

Therefore, we have

$$\sum_{\gamma_i\in h_i}\frac{\prod_{i=1}^{n}(1+\gamma_i)^{w_i}-\prod_{i=1}^{n}(1-\gamma_i)^{w_i}}{\prod_{i=1}^{n}(1+\gamma_i)^{w_i}+\prod_{i=1}^{n}(1-\gamma_i)^{w_i}}$$

$$\le\sum_{\gamma_i^*\in h_i^*}\frac{\prod_{i=1}^{n}\left(1+\gamma_i^*\right)^{w_i}-\prod_{i=1}^{n}\left(1-\gamma_i^*\right)^{w_i}}{\prod_{i=1}^{n}\left(1+\gamma_i^*\right)^{w_i}+\prod_{i=1}^{n}\left(1-\gamma_i^*\right)^{w_i}}$$

and then

$$\frac{\sum_{\gamma_i\in h_i}\frac{\prod_{i=1}^{n}(1+\gamma_i)^{w_i}-\prod_{i=1}^{n}(1-\gamma_i)^{w_i}}{\prod_{i=1}^{n}(1+\gamma_i)^{w_i}+\prod_{i=1}^{n}(1-\gamma_i)^{w_i}}}{\#h_1\times\#h_2\times\ldots\times\#h_n}\le\frac{\sum_{\gamma_i^*\in h_i^*}\frac{\prod_{i=1}^{n}\left(1+\gamma_i^*\right)^{w_i}-\prod_{i=1}^{n}\left(1-\gamma_i^*\right)^{w_i}}{\prod_{i=1}^{n}\left(1+\gamma_i^*\right)^{w_i}+\prod_{i=1}^{n}\left(1-\gamma_i^*\right)^{w_i}}}{\#h_1^*\times\#h_2^*\times\ldots\times\#h_n^*}. \tag{44}$$

Since $g(x)=\frac{2-x}{x}, 0<x\le 1$, is a decreasing function and $\eta_i^*\le\eta_i$, $\eta_i\in g_i$, $\eta_i^*\in g_i^*$ for every $i=1,2,\cdots,n$, then

$$\frac{2-\eta_i}{\eta_i}\le\frac{2-\eta_i^*}{\eta_i^*},\quad i=1,2,\cdots,n,$$

and then

$$\left(\frac{2-\eta_i}{\eta_i}\right)^{w_i}\le\left(\frac{2-\eta_i^*}{\eta_i^*}\right)^{w_i},\quad i=1,2,\cdots,n.$$

Thus

$$\prod_{i=1}^{n}\left(\frac{2-\eta_i}{\eta_i}\right)^{w_i}\le\prod_{i=1}^{n}\left(\frac{2-\eta_i^*}{\eta_i^*}\right)^{w_i}$$

$$\Leftrightarrow\prod_{i=1}^{n}\left(\frac{2-\eta_i}{\eta_i}\right)^{w_i}+1\le\prod_{i=1}^{n}\left(\frac{2-\eta_i^*}{\eta_i^*}\right)^{w_i}+1$$

$$\Leftrightarrow\frac{1}{\prod_{i=1}^{n}\left(\frac{2-\eta_i^*}{\eta_i^*}\right)^{w_i}+1}\le\frac{1}{\prod_{i=1}^{n}\left(\frac{2-\eta_i}{\eta_i}\right)^{w_i}+1}$$

$$\Leftrightarrow\frac{2}{\prod_{i=1}^{n}\left(\frac{2-\eta_i^*}{\eta_i^*}\right)^{w_i}+1}\le\frac{2}{\prod_{i=1}^{n}\left(\frac{2-\eta_i}{\eta_i}\right)^{w_i}+1}$$

$$\Leftrightarrow\frac{2\prod_{i=1}^{n}\left(\eta_i^*\right)^{w_i}}{\prod_{i=1}^{n}\left(2-\eta_i^*\right)^{w_i}+\prod_{i=1}^{n}\left(\eta_i^*\right)^{w_i}}\le\frac{2\prod_{i=1}^{n}(\eta_i)^{w_i}}{\prod_{i=1}^{n}(2-\eta_i)^{w_i}+\prod_{i=1}^{n}(\eta_i)^{w_i}}.$$

Therefore, we have

$$\sum_{\eta_i^*\in h_i^*}\frac{2\prod_{i=1}^{n}\left(\eta_i^*\right)^{w_i}}{\prod_{i=1}^{n}\left(2-\eta_i^*\right)^{w_i}+\prod_{i=1}^{n}\left(\eta_i^*\right)^{w_i}}\le\sum_{\eta_i\in h_i}\frac{2\prod_{i=1}^{n}(\eta_i)^{w_i}}{\prod_{i=1}^{n}(2-\eta_i)^{w_i}+\prod_{i=1}^{n}(\eta_i)^{w_i}}.$$

Summarily, we get

$$\frac{\sum_{\eta_i^*\in h_i^*}\frac{2\prod_{i=1}^{n}\left(\eta_i^*\right)^{w_i}}{\prod_{i=1}^{n}\left(2-\eta_i^*\right)^{w_i}+\prod_{i=1}^{n}\left(\eta_i^*\right)^{w_i}}}{\#g_1^*\times\#g_2^*\times\cdots\times\#g_n^*}\le\frac{\sum_{\eta_i\in h_i}\frac{2\prod_{i=1}^{n}(\eta_i)^{w_i}}{\prod_{i=1}^{n}(2-\eta_i)^{w_i}+\prod_{i=1}^{n}(\eta_i)^{w_i}}}{\#g_1\times\#g_2\times\cdots\times\#g_n}. \tag{45}$$

Let $\mathrm{EDHFWA}(d_1,d_2,\cdots,d_n)=d$ and $\mathrm{EDHFWA}(d_1^*,d_2^*,\cdots,d_n^*)=d^*$, then from Definition 11 and Equations (44) and (45), we know that $s(d)\le s(d^*)$.

(1) If $s(d)<s(d^*)$, then $\mathrm{EDHFWA}(d_1,d_2,\cdots,d_n)<\mathrm{EDHFWA}(d_1^*,d_2^*,\cdots,d_n^*)$.

(2) If $s(d)=s(d^*)$, that is,

$$\frac{\sum_{\gamma_i \in h_i} \frac{\prod_{i=1}^{n}(1+\gamma_i)^{w_i} - \prod_{i=1}^{n}(1-\gamma_i)^{w_i}}{\prod_{i=1}^{n}(1+\gamma_i)^{w_i} + \prod_{i=1}^{n}(1-\gamma_i)^{w_i}}}{\# h_1 \times \# h_2 \times \cdots \times \# h_n} - \frac{\sum_{\eta_i \in h_i} \frac{2\prod_{i=1}^{n}(\eta_i)^{w_i}}{\prod_{i=1}^{n}(2-\eta_i)^{w_i} + \prod_{i=1}^{n}(\eta_i)^{w_i}}}{\# g_1 \times \# g_2 \times \cdots \times \# g_n}$$

$$= \frac{\sum_{\gamma_i^* \in h_i^*} \frac{\prod_{i=1}^{n}(1+\gamma_i^*)^{w_i} - \prod_{i=1}^{n}(1-\gamma_i^*)^{w_i}}{\prod_{i=1}^{n}(1+\gamma_i^*)^{w_i} + \prod_{i=1}^{n}(1-\gamma_i^*)^{w_i}}}{\# h_1^* \times \# h_2^* \times \cdots \times \# h_n^*} - \frac{\sum_{\eta_i^* \in h_i^*} \frac{2\prod_{i=1}^{n}(\eta_i^*)^{w_i}}{\prod_{i=1}^{n}(2-\eta_i^*)^{w_i} + \prod_{i=1}^{n}(\eta_i^*)^{w_i}}}{\# g_1^* \times \# g_2^* \times \cdots \times \# g_n^*}$$

$$= \frac{\sum_{\gamma_i^* \in h_i^*} \frac{\prod_{i=1}^{n}(1+\gamma_i^*)^{w_i} - \prod_{i=1}^{n}(1-\gamma_i^*)^{w_i}}{\prod_{i=1}^{n}(1+\gamma_i^*)^{w_i} + \prod_{i=1}^{n}(1-\gamma_i^*)^{w_i}}}{\# h_1^* \times \# h_2^* \times \cdots \times \# h_n^*}$$

$$- \frac{\sum_{\eta_i^* \in h_i^*} \frac{2\prod_{i=1}^{n}(\eta_i^*)^{w_i}}{\prod_{i=1}^{n}(2-\eta_i^*)^{w_i} + \prod_{i=1}^{n}(\eta_i^*)^{w_i}}}{\# g_1^* \times \# g_2^* \times \cdots \times \# g_n^*}.$$

From Equations (44) and (45), we know that

$$\frac{\sum_{\gamma_i \in h_i} \frac{\prod_{i=1}^{n}(1+\gamma_i)^{w_i} - \prod_{i=1}^{n}(1-\gamma_i)^{w_i}}{\prod_{i=1}^{n}(1+\gamma_i)^{w_i} + \prod_{i=1}^{n}(1-\gamma_i)^{w_i}}}{\# h_1 \times \# h_2 \times \cdots \times \# h_n}$$

$$= \frac{\sum_{\gamma_i^* \in h_i^*} \frac{\prod_{i=1}^{n}(1+\gamma_i^*)^{w_i} - \prod_{i=1}^{n}(1-\gamma_i^*)^{w_i}}{\prod_{i=1}^{n}(1+\gamma_i^*)^{w_i} + \prod_{i=1}^{n}(1-\gamma_i^*)^{w_i}}}{\# h_1^* \times \# h_2^* \times \cdots \times \# h_n^*}$$

$$\frac{\sum_{\eta_i \in h_i} \frac{2\prod_{i=1}^{n}(\eta_i)^{w_i}}{\prod_{i=1}^{n}(2-\eta_i)^{w_i} + \prod_{i=1}^{n}(\eta_i)^{w_i}}}{\# g_1 \times \# g_2 \times \ldots \times \# g_n}$$

$$= \frac{\sum_{\eta_i^* \in h_i^*} \frac{2\prod_{i=1}^{n}(\eta_i^*)^{w_i}}{\prod_{i=1}^{n}(2-\eta_i^*)^{w_i} + \prod_{i=1}^{n}(\eta_i^*)^{w_i}}}{\# g_1^* \times \# g_2^* \times \ldots \times \# g_n^*},$$

which means $h(d) = h(d^*)$, and it follows from Definition 11 that

$$\mathrm{EDHFWA}(d_1, d_2, \cdots, d_n) = \mathrm{EDHFWA}(d_1^*, d_2^*, \cdots, d_n^*).$$

Therefore,

$$\mathrm{EDHFWA}(d_1, d_2, \cdots, d_n) \leq \mathrm{EDHFWA}(d_1^*, d_2^*, \cdots, d_n^*),$$

which completes the proof.

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