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SET COVERING-BASED TOPSIS METHOD FOR SLOVING SUP-*T* EQUATION CONSTRAINED MULTI-OBJECTIVE OPTIMIZATION PROBLEMS

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Abstract

This paper considers solving a multi-objective optimization problem with sup-*T* equation constraints. A set covering-based technique for order of preference by similarity to the ideal solution is proposed for solving such a problem. It is shown that a compromise solution of the sup-*T* equation constrained multi-objective optimization problem can be obtained by solving an associated set covering problem. A surrogate heuristic is then applied to solve the resulting optimization problem. Numerical experiments on solving randomly generated multi-objective optimization problems with sup-*T* equation constraints are included. Our computational results confirm the efficiency of the proposed method and show its potential for solving large scale sup-*T* equation constrained multi-objective optimization problems.

Keywords: Fuzzy relational equations, fuzzy optimization, set covering problems

1. Introduction

Fuzzy relational equations have played an important role in many applications of fuzzy sets and systems (Li and Fang 2008, Peeva and Kyosev 2004). Simultaneously optimizing several objective functions in the presence of fuzzy relational equation constraints can be formulated as the following multi-objective optimization problem:

$$\begin{aligned} \text{Max/Min} \quad & F(\mathrm{x}) = [f_1(\mathrm{x}), \cdots, f_k(\mathrm{x}), \cdots, f_K(\mathrm{x})] \\ \text{s.t.} \quad & A \circ \mathrm{x} = \mathrm{b}, \ \mathrm{x} \in [0,1]^n, \end{aligned} \tag{1}$$

where $f_k:[0,1]^n \to \mathrm{R}$, $k = 1,2,...,K$, is a real-valued function, $A = (a_{ij})_{m\times n} \in [0,1]^{m\times n}$, $\mathrm{x}=(x_j)_{n\times 1} \in \mathrm{R}^n, \mathrm{b} = (b_i)_{m\times 1} \in [0,1]^m$ and " $\circ$ " stands for the specific sup-*T* composition with *T* being a continuous *t*-norm. In this way, $A \circ \mathrm{x} = \mathrm{b}$ represents a system of fuzzy relational equations with sup-*T* composition (or a system of sup-*T* equations for short). The most frequently used *t*-norm in applications of fuzzy relational equations is the minimum operator

T_M, i.e., $T_M(x,y)=\min(x,y)$. Another two important triangular norms are the product operator $T_P(x,y)=xy$ and the Lukasiewicz t-norm $T_L(x,y)=\max(x+y-1,0)$.

The problem (1) was first studied by Wang (1995) for medical applications, with f_k being linear, for $k\in\{1,2,...,K\}$. The properties of efficient solutions were investigated and the necessary and sufficient conditions for identifying efficient solutions were provided. Some investigations of the problem (1) can also be found in the area called the "inverse problem" (Bourke and Fisher 1998). Typically, the constraint part of the problem (1) is one form of the inverse problem, where matrix A stands for the relation between symptoms and causes and vector b stands for the symptom. Each variable x_j may represent one cause for the problem. Solving the problem means finding a combination of causes to yield the given symptom. In general, we may associate some measures such as cost, time to completion, etc., to a combination of causes. The problem (1) then represents selecting the best combinations among all feasible combinations of causes which yield the given symptom (Guu et al. 2011).

The resolution of the system of sup-T equations is to determine the unknown vector X for a given coefficient matrix A and a right hand side vector b such that $A\circ$x=b. The set of all solutions, when it is non-empty, is a finitely generated root system which can be fully determined by a unique maximum solution and a finite number of minimal solutions (Li and Fang 2009). For a finite system of fuzzy relational equations with sup-T composition, its consistency can be verified by constructing and checking a potential maximum solution. However, the detection of all minimal solutions is closely related to the set covering problem and remains a challenging problem (Pedrycz 1991, Markovskii 2005, Li and Fang 2009). Overviews of fuzzy relational equations and their applications can be found in Li and Fang (2009) and Peeva and Kyosev (2004).

It is well known that many decision making problems have multiple objectives which cannot be optimized simultaneously due to the inherent incommensurability and conflict among these objectives. Thus, making a trade off between these objectives becomes a major subject of finding the "best compromise" solution. A variety of methodologies for solving the multi-objective decision making (MODM) problems have been proposed (Hwang and Masud 1981, Zeleny 1982, Steuer 1986, Sakawa 1993, Ramezani et al. 2011). Among them, the goal programming and global criterion methods are the popular approaches. These methods consider only one criterion based on the shortest distance from the given goal or the positive ideal solution. However, in practice, such single criterion may not be enough for a decision maker (Hwang and Masud 1981). Instead, the technique for order of preference by similarity to ideal solution (TOPSIS) was first developed by Hwang and Yoon (1981) to solve a multiple attribute decision making (MADM) problem. It provides the principle of compromise saying that the chosen alternative should have "the shortest distance from the positive ideal solution" and "the farthest distance from the negative ideal solution." Lately, this principle of compromise was also suggested by Hwang et al. (1993) for solving MODM problems. Abo-Sinna (2000) extended the TOPSIS to solve multi-objective dynamic programming problems. Lin and Yeh (2012) considered solving stochastic computer network optimization problems by employing the TOPSIS and genetic algorithms. Since vague concepts frequently appeared in decision data for modeling real-life situations, multi-objective decision makings in a fuzzy environment is of theoretical and practical importance. Chen (2000) extended the concept of TOPSIS to develop a methodology for solving multi-person multi-criterion decision-making problems in a fuzzy environment. In this paper, we show that by applying the basic principle of compromise of TOPSIS, the fuzzy relational equation constrained multi-objective optimization problem (1) can be reduced to a sup-T equation constrained optimization problem.

The problem of minimizing a linear objective function subject to a system of fuzzy relational equations with max-min composition was first investigated by Fang and Li (1999). Most recently, it was shown (Li and Fang 2008) that the problem of minimizing an objective function subject to a system of fuzzy relational equations can be reduced to a 0-1 mixed integer programming problem in polynomial time. If the objective function is linear, or more generally, separable and monotone in each of the variables, then it can be further reduced to a set covering problem (Hu and Fang 2011). The set covering problem is known to be one of Karp's 21 NP-complete problems. Many algorithms have been developed to solve it. Exact algorithms are mostly based on brand-and-bound and branch-and-cut (Balas and Carrera 1996, Fisher and Kedia 1990). However, these algorithms are rather time consuming and can only solve instances of very limited size. For this reason, many research efforts have been focused on the development of heuristics to find good or near-optimal solutions within a reasonable period of time. The most effective heuristics for solving set covering problems are those based on Lagrangian relaxation with subgradient optimization (Beasley 1990, Fisher and Kedia 1990). Some meta-heuristics methods, including genetic algorithm (Beasley and Chu 1996), simulated annealing algorithm (Brusco 1999) and tabu search algorithm (Caserta 2007), have also been applied to the set covering problems. For a deeper understanding of the effective algorithms for set covering problems in the literature, readers may refer to the survey in Caprara (2000). In this paper, we consider applying an effective surrogate heuristic studied in Lorena and Lopes (1994) for solving the resulting sup-T equation constrained optimization problem in view of the associated set covering problem. The surrogate heuristic presents comparable results in terms of bounds to the Lagrangian heuristic of Beasley (1990), one of the best known heuristics for the set covering problem, but with about half the computational time for the same test problems (Lorena and Lopes 1994).

The rest of this paper is organized as follows. In Section 2, we show that a compromise solution of the fuzzy relational equation constrained multi-objective optimization problem (1) can be obtained by solving a sup-T equation constrained optimization problem. A surrogate heuristic is introduced in Section 3 for solving the resulting sup-T equation constrained optimization problem. Numerical experiments on solving randomly generated multi-objective optimization problems with sup-T equation constraints are included in Section 4 to show the prominent performance of the proposed method. Conclusions are provided in Section 5.

2. Solving Fuzzy Relational Equation Constrained Multi-objective Optimization Problem

To solve the sup-T equation constrained multi-objective optimization problem (1), we adopt the principle of compromise, i.e., the chosen solution should have "the shortest distance from the positive ideal solution" and "the farthest distance from the negative ideal solution" (Hwang and Yoon 1981). An analogical discussion in Hwang et al. (1993) for solving MODM problems by the principle of compromise is provided in this section.

Let $X := \{\mathrm{x} \in [0,1]^n \mid A \circ \mathrm{x}=\mathrm{b}\}$ be the feasible domain and I, J be two index sets. For each $j \in J$, $f_i(\mathrm{x})$ is an objective function to be maximized. Similarly, for each $i \in I$, $f_i(\mathrm{x})$ is an objective function to be minimized.

To define the positive ideal solution and negative ideal solution of the problem (1), for each k, $k = 1,2,\cdots,K,$ we consider

$$f_k^* \triangleq \begin{cases} \max_{x\in \mathrm{X}} f_k(X), \text{ if } k \in J, \\ \min_{x\in \mathrm{X}} f_k(X), \text{ if } k \in I, \end{cases} \tag{2}$$

and

$$f_k^* \triangleq \begin{cases} \max\limits_{x\in \mathrm{X}} f_k(X), \text{ if } k \in J, \\ \min\limits_{x\in \mathrm{X}} f_k(X), \text{ if } k \in I. \end{cases} \tag{3}$$

Let $f^* \triangleq (f_1^*, f_2^*, ..., f_K^*) \in \mathrm{R}^K$ be the solution vector of equation (2) which consists of individual best feasible solutions for all objectives. f^* is then called the positive ideal solution (PIS). Similarly, let $f^* \triangleq (f_1^*, f_2^*, ..., f_K^*) \in \mathrm{R}^K$ be the solution vector of equation (3) which consists of individual worst feasible solutions for all objectives. f^- is then called the negative ideal solution (NIS).

To measure the distances from PIS and NIS to all objectives, the Minkowski's L_P-metric is employed, i.e., the distance between two points $f_k(x)$ and f_k^* (or f_k^-), $k = 1,2,\cdots,K,$ is defined by the L_P-norm with $p \geq 1$. Moreover, because of the incommensurability among objectives, the component distance from PIS or NIS for each objective is normalized. The following distance functions are then considered:

$$d_p^{PIS}(X) = \left\{ \sum_{j\in J} w_j^p \left[\frac{f_j^* - f_j(X)}{f_j^* - f_j^-} \right]^p + \sum_{i\in I} w_i^p \left[\frac{f_i(X) - f_i^*}{f_j^- - f_i^*} \right]^p \right\}^{1/p}, \tag{4}$$

and

$$d_p^{NIS}(X) = \left\{ \sum_{j\in J} w_j^p \left[\frac{f_j(X) - f_j^*}{f_j^* - f_j^-} \right]^p + \sum_{i\in I} w_i^p \left[\frac{f_i^- - f_i(X)}{f_i^- - f_i^*} \right]^p \right\}^{1/p}, \tag{5}$$

where d_P^{PIS} and d_P^{NIS} are the distances from the PIS and NIS to all objectives, respectively, $w_k \in [0,1], k = 1,2,\cdots,K,$ is the relative importance (weight) of objective function k, and $p \geq 1$ is the parameter of a norm function.

To consider the objectives of "minimize the distance from PIS or d_P^{PIS}" and "maximize the distance from NIS or d_P^{NIS}" instead of the original K objectives in problem (1), we have the following bi-objective programming problem:

$$\begin{aligned} &\min\ d_P^{PIS}(\mathrm{x}) \\ &\max\ d_P^{NIS}(\mathrm{x}) \\ &\text{s.t. } \mathrm{A} \circ \mathrm{x}=\mathrm{b}, \\ &\quad \mathrm{x} \in [0,1]^n. \end{aligned} \tag{6}$$

Among all p values, the case of p = 1 is operationally and practically important, which provides better credibility than others in the measuring concept and emphasizes the sum of individual distances (regrets for d_P^{PIS} and rewards for d_P^{NIS}) in the utility concept (Hwang et al. 1993). Our work adopts p=1 for finding the compromise solution to the sup-T equation constrained multi-objective optimization problem (1). For the rest of the paper, p = 1 is chosen, although other values may be applicable.

Lemma 2.1 *The compromise solution of problem (1) can be obtained by solving the following sup-T equation constrained optimization problem:*

$$\begin{aligned} &\min\ d_P^{PIS}(\mathrm{x}) \\ &\text{s.t. } \mathrm{A} \circ \mathrm{x}=\mathrm{b}, \\ &\quad \mathrm{x} \in [0,1]^n \end{aligned} \tag{7}$$

or

$$\begin{aligned} &\max\ d_P^{NIS}(\mathrm{x}) \\ &\text{s.t. } \mathrm{A} \circ \mathrm{x}=\mathrm{b}, \\ &\quad \mathrm{x} \in [0,1]^n. \end{aligned} \tag{8}$$

Proof. Since $d_P^{PIS} = 1 - d_P^{NIS}$ for $p=1$, "min d_1^{PIS}" and "max d_1^{NIS}" are subjected to the same system of sup-T equation constraints and have the same solution whether the weights of the objectives are the same or not. Thus, solving the bi-objective programming problem (6) is equivalent to solving either problem (7) or problem (8). Consequently, the compromise solution of problem (1) can be obtained by solving the sup-T equation constrained optimization problem (7) or (8).

In the implementation of TOPSIS for solving the sup-T equation constrained multi-objective optimization problem (1), we face the challenge of solving the fuzzy relational equation constrained optimization problems (2), (3), (7) or (8). This work considers solving the fuzzy relational equation constrained optimization problem in view of a set covering problem.

3. Solving Sup-*T* Equation Constrained Optimization Problem

To solve the fuzzy relational equation constrained optimization problem, we recall some basic concepts and theoretical results associated with fuzzy relational equations in Li and Fang (2008). For the convenience of description, two index sets are defined by $M = \{1, 2, ..., m\}$ and $N = \{1, 2, ..., n\}$. The relationship between the sup-T equation constrained optimization problem and its associated set covering problem is provided in Section 3.1.

3.1. Resolution of Systems of Sup-*T* Equations

As mentioned in Section 1, the solution set of a finite system of sup-T equations, when it is non-empty, can be determined by a maximum solution and a finite number of minimal solutions. To characterize the solution set of a system of sup-T equations, two residual operators are defined with respect to a continuous t-norm T.

Definition 3.1 Given a t-norm T, the binary residual operators $I_T:[0,1]^2\to[0,1]$ and $J_T:[0,1]^2\to[0,1]$ are defined, respectively, by

$$I_T(x,y)=\sup\{z\in[0,1]\,|\,T(x,z)\le y\}$$

and

$$J_T(x,y)=\inf\{z\in[0,1]\,|\,T(x,z)\ge y\}.$$

The residual operators I_T and J_T of the three most important continuous t-norms are listed in Table 1.

Table 1 Residual Operators of T_M, T_P, and T_L

T	$I_T(x,y)$	$J_T(x,y)$
T_M	1, if $x\le y$ y, otherwise	1, if $x<y$ y, otherwise.
T_P	1, if $x\le y$ y/x, otherwise.	1, if $x<y$ y/x, if $0<y\le x$, 0, otherwise.
T_L	$\min(1-x+y,1)$	1, if $x<y$ $1-x+y$, if $0<y\le x$ 0, otherwise.

Given a system of sup-T equations $A\circ$x=b with a continuous t-norm T, the set of all solutions to $A\circ$x=b is called its complete solution set and denoted by $S(A,\mathrm{b})=\{\mathrm{x}\in[0,1]^n\,|\,A\circ\mathrm{x}=\mathrm{b}\}$. A partial order can be defined on $S(A,\mathrm{b})$ by extending the natural order such that for any $\mathrm{x}^1,\mathrm{x}^2\in S(A,\mathrm{b})$, $\mathrm{x}^1\le\mathrm{x}^2$ if and only if $\mathrm{x}^1_j,\mathrm{x}^2_j$ for all $j\in N$. A system of sup-T equations $A\circ$x=b is called consistent if $S(A,\mathrm{b})\ne\phi$. Otherwise, it is inconsistent. Due to the monotonicity of the t-norm involved in the composition, if $\mathrm{x}^1,\mathrm{x}^2\in S(A,\mathrm{b})$, and $\mathrm{x}^1\le\mathrm{x}^2$, any x satisfying $\mathrm{x}^1\le\mathrm{x}\le\mathrm{x}^2$ is also in $S(A,\mathrm{b})$. Therefore, the attention could be focused on the so-called extremal solutions as defined below.

Definition 3.2 A solution $\breve{\mathrm{x}}\in\mathrm{S(A,b)}$ is called a minimal or lower solution if, for any $\mathrm{x}\in\mathrm{S(A,b)}$, the relation $\mathrm{x}\le\breve{\mathrm{x}}$ implies $\mathrm{x}=\breve{\mathrm{x}}$. A solution $\hat{\mathrm{x}}\in\mathrm{S(A,b)}$ is called the maximum or greatest solution if $\mathrm{x}\le\hat{\mathrm{x}},\ \forall\mathrm{x}\in\mathrm{S(A,b)}$.

Theorem 3.1 *(see, e.g., Li and Fang 2008) Let $A\circ$x=b be a system of sup-T equations with a continuous t-norm T. The system is consistent if and only if the vector $A^T\varphi_t\mathrm{b}$ with its components defined by*

$$(A^T\varphi_t\mathrm{b})_j=\inf_{i\in M}I_T(a_{ij},b_i),\ \forall\, j\in N, \qquad (9)$$

is a solution to $A\circ$x=b. Moreover, if the system is consistent, the solution set $S(A,\mathrm{b})$ can be determined by a unique maximum solution and a finite number of minimal solutions, i.e.,

$$S(A,\mathrm{b})=\bigcup_{\breve{\mathrm{x}}\in\breve{S}(A,\mathrm{b})}\{\breve{\mathrm{x}}\le\mathrm{x}\le\hat{\mathrm{x}}\}, \qquad (10)$$

where $\breve{S}(A,\mathrm{b})$ is the set of all minimal solutions to $A\circ$x=b and $A^T\varphi_t\mathrm{b}$ is the maximum solution defined in (9).

Clearly, the consistency of a system $A\circ$x=b can be detected by constructing and checking the potential maximum solution $\hat{\mathrm{x}}=A^T\varphi_t\mathrm{b}$ in a time complexity of $O(mn)$.

With the potential maximum solution $\hat{\mathrm{x}}$, the characteristic matrix $\tilde{Q}=(\tilde{q}_{ij})_{m\times n}$ of a system $A\circ$x=b can be defined by

$$\tilde{q}_{ij}=\begin{cases}[J_T(a_{ij},b_i),\hat{x}_j], & \text{if } T(a_{ij},\hat{x}_j)=b_i,\\ \phi, & \text{otherwise,}\end{cases}$$

and obtained in a time complexity of $O(mn)$. When T is a continuous Archimedean t-norm of which the product operator T_P and the Lukasiewicz t-norm T_L are typical representatives, the nonempty elements in $\tilde{Q}$ are always singletons with their values determined by the potental maximum solution (Li and Fang 2008). The characteristic matrix $\tilde{Q}$ in this case can be further simplified as $Q=(q_{ij})_{m\times n}$ with

$$q_{ij}=\begin{cases}1, & \text{if } \tilde{q}_{ij}\neq 0,\\ 0, & \text{otherwise,}\end{cases}$$

Definition 3.3 Let $Q=(q_{ij})_{mxn}\in\{0,1\}^{mn}$ be a binary matrix. A column j is said to cover a row i if $q_{ij}=1$. A set of nonzero columns P forms a covering of Q if each row of Q is covered by some column from P. A column j in a covering P is called redundant if the set of columns $P\setminus\{j\}$ remains to be a covering of Q. A covering P is irredundant if it has no redundant columns. The set of all coverings of Q is denoted by $P(Q)$ while the set of all irredundant coverings of Q is denoted by $\breve{P}(Q)$.

Example 1 Consider the binary matrix

$$Q=\begin{pmatrix}0&0&0&1&1\\1&0&0&1&0\\1&0&0&0&1\\0&1&1&0&0\\1&1&1&0&0\end{pmatrix}$$

$P_1=\{1,2,4,5\}$ is a covering of Q. Since $P_1\setminus\{5\}$ remains a covering of Q, P_1 is redundant. The set of all irredundant coverings of Q is $\breve{P}(Q)$ = {{1, 2, 4}, {1, 3, 4}, {1, 2, 5}, {1, 3, 5}, {2,4, 5}, {3, 4, 5}}.

It is well-known (Li and Fang 2008) that the set of all coverings $P(Q)$ of a binary matrix Q can be well represented by the feasible solution set of a set covering problem, i.e., $\{u\in[0,1]^n \mid Qu\geq e\}$ where $e=(1,1,...,1)^n\in\{0,1\}^m$, while the irredundant coverings of Q correspond to the minimal elements in $\{u\in[0,1]^n \mid Qu\geq e\}$.

When the system of sup-T equations $A\circ x=b$ with T being a continuous non-Archimedean t-norm, the situation turns out to be a little bit complicated. Denote r_j the numbers of different values in $\{J_T(a_{ij},b_i)\mid T(a_{ij},\hat{x}_j)=b_i, i\in M\}$ for each $j\in N$, $r=\sum_{j\in J}r_j$, $K_j=\{1,2,...,r_j\}$ and $\breve{v}_{jk}$, for k $\in K_j$, the different values in $\{J_T(a_{ij},b_i)\mid T(a_{ij},\hat{x}_j)=b_i, i\in M\}$ for every $j\in N$. Let $\breve{v}=(\breve{v}_{11},...,\breve{v}_{1r_1},...,\breve{v}_{n1},...,\breve{v}_{nr_n})^T\in[0,1]^r$ and

$$x_j=\sum_{k\in K_j}\breve{v}_{jk}u_{jk},\ j\in N,$$

where $u_{jk}\in\{0,1\}, \forall\ k\in K_j, j\in N$. Obviously, for each $j\in N$, at most one of u_{jk}, $k\in K_j$ can be 1, i.e., $\sum_{k\in K}u_{jk}\leq 1, j\in N$. These restrictions are called the innervariable incompatibility constraints and can be represented by $G\mathbf{u}\leq e^n$, where $e^n=(1,1,\cdots,1)^T\in\{0,1\}^n$, u=$(u_{11},\cdots,u_{1r_1},\cdots,u_{n1},\cdots,u_{nr_n})^T\in\{0,1\}^r$ and $G=(g_{jk})_{n\times r}$ with

$$g_{jk}=\begin{cases}1, \text{ if } \sum_{s=1}^{j-1}r_s<k\leq\sum_{s=1}^{j}r_s,\\ 0, \quad \text{otherwise.}\end{cases}$$

The characteristic matrix $\tilde{Q}$ in this case can be further converted to its augmented characteristic matrix $Q=(q_{ik})_{m\times r}\in\{0,1\}^{mr}$ with

$$q_{ik} = \begin{cases} 1, \text{ if } \sum_{s=1}^{j-1} r_s < k \le \sum_{s=1}^{j} r_s,\ \breve{v}_{jk} \in \tilde{q}_{ij},\ j \in N, \\ 0, \text{otherwise.} \end{cases}$$

It was shown that the minimal solutions of a system of sup-*T* equations correspond one-to-one to the irredundant coverings of a set covering problem when the involved t-norm is Archimedean and correspond to a subset of constrained irredundant coverings of a set covering problem when the t-norm is non-Archimedean (Li and Fang 2008).

The problem of minimizing a linear objective function subject to a consistent system of sup-T_M equations was first investigated by Fang and Li (1999). Most recently, it was shown (Li and Fang 2008) that the problem of minimizing a linear objective function subject to a system of sup-*T* equations can be reduced to a 0-1 mixed integer programming problem in polynomial time. The procedure for solving sup-*T* equation constrained linear optimization problems can be directly extended to the case where the objective function is separable and monotone in each of the variables, i.e., $z = \sum_{j\in J} f_j(x_j)$ with $f_j : [0,1] \to R$ being a monotone function for every $j \in N$. Without loss of generality, we may assume that $f_j(0) = 0$ for every $j \in N$. A systematic and unified result for solving the sup-*T* equation constrained optimization problem can be described as follows:

Theorem 3.2 *Let* $A \circ \text{x=b}$ *be a consistent system of sup-T equations with T being a continuous t-norm. Denote* $\hat{\text{x}}$ *its maximum solution. For a given function* $z = \sum_{j\in J} f_j(x_j)$ *with* f_j *: [0, 1] →* ***R*** *being a monotone function for every* $j \in N$, *let* $N^- \triangleq \{j \in N \mid f_j$ *is a decreasing function}* *and* $N^+ \triangleq N \setminus N^-$. *Consider the following sup-T equation constrained optimization problem:*

$$\begin{aligned} &\min\ d_P^{PIS}(\text{x}) \\ &\text{s.t. } A \circ \text{x=b}, \\ &\text{x} \in [0,1]^n. \end{aligned} \tag{11}$$

(i)When T is a continuous Archimedean t-norm, any optimal solution $\text{u}^* = (u_1^*, u_2^*, ..., u_n^*)^T$ *to the set covering problem*

$$\begin{aligned} &\min z_u^+ = \sum_{j\in N^+} f_j(\hat{x}_j)u_j \\ &(\text{SCP}-\text{Ar})\, s.t.\ Qu \ge e^m, \\ &u \in \{0,1\}^n, \end{aligned}$$

defines an optimal solution $\breve{\text{x}}^* = (\hat{x}_1 u_1^*, \hat{x}_2 u_2^*, ..., \hat{x}_n u_n^*)^T$ *to problem (11), where Q is the associated simplified characteristic matrix of* $A \circ \text{x=b}$.

(ii)When T is a continuous non-Archimedean t-norm, any optimal solution to the set covering problem

$$\begin{aligned} &\min z_u^+ = \sum_{j\in N^+} \sum_{j\in K_j} f_i(\hat{v}_{jk}) u_{jk} \\ &(\text{SCP}-\text{nAr})\, s.t.\ Qu \ge e^m, \\ &u \in \{0,1\}^r, \end{aligned}$$

defines an optimal solution

$$\breve{\text{x}}^* = \Big(\sum_{k\in K_1} \breve{v}_{1k} u_{1k}^*, \sum_{k\in K_2} \breve{v}_{2k} u_{2k}^*, ..., \sum_{k\in K_n} \breve{v}_{nk} u_{nk}^*\Big)^T$$

to problem (11), where Q is the associated augmented characteristic matrix of $A \circ \text{x=b}$.

Once the optimal solution $\breve{\text{x}}^* = (\breve{x}_1^*, \breve{x}_2^*, ..., \breve{x}_n^*)^T$ *to problem (11) is obtained, the corresponding optimal solution that minimizes the objective function* $\sum_{j\in J} f_j(x_j)$ *over S(A,* ***b****), i.e.,* $\text{x}^* = (x_1^*, x_2^*, \cdots, x_n^*)^T$ *can be obtained by*

$$x_j^* = \begin{cases} \breve{x}_j^*, & \text{if } j \in N^+, \\ \hat{x}_j, & \text{if } j \in N^-. \end{cases}$$

Proof. The results are direct consequences of Theorems 3.4, 3.7 and 4.2 of Li and Fang (2008).

The following example studied in Hu and Fang (2011) illustrates the result of Theorem 3.2.

Example 2. Consider the system of sup-T_M equations with a separable and monotone objective function in each of the variables described as below.

$$\min\ z_x = x_1(x_1-4)+(x_2)^2+(x_3)^2+x_4+(x_5)^2+x_6$$

$$\text{s.t.} \begin{pmatrix} 0.6 & 0.5 & 0.6 & 0.6 & 0.6 & 0.2 \\ 0.1 & 0.6 & 0.8 & 0.5 & 0.6 & 0.7 \\ 0.8 & 0.8 & 0.5 & 0.8 & 0.2 & 0.8 \\ 0.8 & 0.95 & 0.1 & 0.3 & 0.9 & 0.9 \\ 0.9 & 0.8 & 0.4 & 0.95 & 0.4 & 1 \\ 1 & 0.8 & 0.4 & 1 & 1 & 0.5 \end{pmatrix} \circ \mathrm{x} = \begin{pmatrix} 0.6 \\ 0.7 \\ 0.8 \\ 0.9 \\ 0.95 \\ 1 \end{pmatrix},$$

$$x_j \in [0,1],\ j=1,2,\dots,6. \tag{12}$$

According to Theorem 3.1, the system of sup-T_M equations has the maximum solution $\hat{\mathrm{x}} = (1, 0.9, 0.7, 1, 1, 0.95)^T$. Its associated characteristic matrix is

$$\tilde{Q} = \begin{pmatrix} [0.6,1] & \phi & [0.6,0.7] & [0.6,1] & [0.6,1] & \phi \\ \phi & \phi & 0.7 & \phi & \phi & [0.7,0.95] \\ [0.8,1] & [0.8,0.9] & \phi & [0.8,1] & \phi & [0.8,0.95] \\ \phi & 0.9 & \phi & \phi & [0.9,1] & [0.9,0.95] \\ \phi & \phi & \phi & [0.95,1] & \phi & 0.95 \\ 1 & \phi & \phi & 1 & 1 & \phi \end{pmatrix}.$$

In this case, $N^- = \{1\}$, $N^+ = \{2,3,4,5,6\}$ and

$$\breve{v} = (\breve{v}_{11}, \breve{v}_{12}, \breve{v}_{13}, \breve{v}_{21}, \breve{v}_{22}, \breve{v}_{31}, \breve{v}_{32}, \breve{v}_{41}, \breve{v}_{42}, \breve{v}_{43}, \breve{v}_{44}, \breve{v}_{51}, \breve{v}_{52}, \breve{v}_{53}, \breve{v}_{61}, \breve{v}_{62}, \breve{v}_{63}, \breve{v}_{64})^T$$
$$= (0.6, 0.8, 1, 0.8, 0.9, 0.6, 0.7, 0.6, 0.8, 0.95, 1, 0.6, 0.9, 1, 0.7, 0.8, 0.9, 0.95)^T.$$

According to Theorem 3.2, the associated set covering problem of (12) can be described as below.

$$\min z_u^+ = 0.64u_{21} + 0.81u_{22} + 0.36u_{31} + 0.49u_{32} + 0.6u_{41} + 0.8u_{42} + 0.95u_{43} + u_{44} + 0.36u_{51} + 0.81u_{52} + u_{53} + 0.7u_{61} + 0.8u_{62} + 0.9u_{63} + 0.95u_{64}.$$

$$\text{s.t.} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} u \ge \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

$$u = (u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{31}, u_{32}, u_{41}, u_{42}, u_{43}, u_{44}, u_{51}, u_{52}, u_{53}, u_{61}, u_{62}, u_{63}, u_{64})^T \in \{0,1\}^{18}.$$

Based on the above discussion, the problem of minimizing a separable and monotone objective function in each of the variables subject to a system of sup-T equations, with T being a continuous t-norm, can be polynomially reduced to a set covering problem. The set covering problem is a well-known NP-hard problem and, hence, difficult to solve. Consequently, large set covering problems are usually solved by means of greedy type heuristics for quickly identifying a near optimal solution. However, classical greedy algorithms seldom provide high quality solutions. An effective heuristic approach based upon continuous surrogate relaxations and subgradient optimization has been proposed for solving set covering problems (Lopes and Lorena 1994). The computational results in Lopes and Lorena (1994) indicate that the surrogate heuristic is faster and more stable than the known heuristics based on Lagrangian relaxations.

3.2. A Surrogate Heuristic for Solving Set Covering Problems

In this section, an analogical discussion of the surrogate heuristic in Lopes and Lorena (1994) for solving the resulting sup-T equation con- strained optimization problem in view of the set covering problem is introduced.

Consider a set covering problem in the following form:

$$\min z = \sum_{j=1}^{n'} c_j u_j$$
$$s.t.\ Qu \geq e^m, \qquad (13)$$
$$u_j \in \{0,1\}, \forall j \in \{1,2,...,n'\},$$

where $c_j \geq 0$ represents the weight associated with the variable $u_j, j \in \{1,2,...,n'\}$, Q is an $m \times n'$ matrix of zeros and ones, and e^m is the m-vector of 1's.

The continuous surrogate relaxation of the set covering problem (13) can be defined as

$$\min z = \sum_{j=1}^{n'} c_j u_j$$
$$(S_w) \quad s.t.\ w^T Qu \geq w^T e^m,$$
$$u_j \in \{0,1\}, \forall j \in \{1,2,...,n'\},$$

where $\mathbf{w} = (w_1, w_2, ..., w_m)^T \in \mathrm{R}_+^m$ is the surrogate multiplier vector. It is easy to see that the continuous surrogate relaxation, (S_w), corresponds to a very particular case of a classical knapsack problem. In this case, its optimal solution can be achieved resorting to the well known properties established for such problems. To find the optimal solution of (S_w), for a given $\mathbf{w} \in \mathrm{R}_+^m$, the efficiency of the j-th variable of (S_w) is defined by

$$d_j \triangleq \frac{c_j}{w^T q_j}, \forall j \in \{1,2,...,n'\},$$

where q_j is the j-th column of matrix Q. If the optimal solution $u_w = (u_{w1}, u_{w2}, \cdots, u_{wn'})^T$ of (S_w) is assumed to be ordered according to efficiency, i.e., $d_{w1} \leq d_{w2} \leq ... d_{wn'}$, then

$$u_w = \left(1,1,...,1, \frac{w^T e^m - \sum_{j=1}^{j*-1} w^T q_{wj}}{w^T q_{wj*}}, 0,0,...,0 \right),$$

where $\sum_{j=1}^{j^*-1} \mathrm{w}^T q_{wj} \leq \mathrm{w}^T e^m \leq \sum_{j=1}^{j^*} \mathrm{w}^T q_{wj}$ and wj^* is the index of the fractional variable.

Let $v(\cdot)$ denote the optimal value of a given problem (·). The problem to find a surrogate multiplier vector $\mathbf{w} \in R_+^m$ that maximizes v (S_w) is called the surrogate dual problem. We assume the reader is familiar with the duality theory in the combinatorial optimization (see, e.g., Parker 1988 for details). The surrogate heuristic is a procedure that approximates the solution of the surrogate dual and provides a lower bound for the set covering problem (13). A subgradient method is employed in the surrogate heuristic to determine the optimal (near optimal) surrogate multipliers for the surrogate dual. The use of subgradient procedure in the context of Lagrangian duality to solve structured combinatorial optimization problems has been shown to be very effective and to work better than classical gradient procedures or column generation techniques (Held et al. 1974, Nemhauser and Wolsey 1988). For a given $\mathbf{w} \in \mathrm{R}_+^m$, the subgradient procedure in the surrogate heuristic uses the direction

$$G(w) = \left(G_1(w) G_2(w), ..., G_m(w)\right)^T \triangleq e^m - Qu_w,$$

where u_w is the optimal solution of (S_w) and u_{wj^*} is set to be 0. It generates a sequence of nonnegative surrogate multiplier vectors

$\{w^{(0)}, w^{(1)}, \cdots\}$, where $w^{(0)}$ is a given initial vector and $w^{(l+1)}$ is updated from $w^{(l)}$, $l = 0,1,2,...,$ by the following formula:

$$w_i^{(l+1)} \leftarrow \max\left\{0, w_i^{(l)} + \rho\frac{f_{ub}-f_{lb}}{\| G(w^{(l)})\|^2}G_i(w_i^{(l)})\right\},$$
$$i = 1,2,...,m,$$

with f_{ub} and f_{lb} being the upper and lower bounds of the set covering problem (13), respectively, and ρ being a parameter associated with the step size. It has been shown (Lorena and Plateau 1988) that the direction $G(\mathbf{w})$, for $\mathbf{w} \in R_+^m$, is a subgradient for the Lagrangian function of the problem (L_λ)

$$L_\lambda \quad \min z = \sum_{j=1}^{n'} c_j u_j + \lambda^T(e^m - Qu)$$
$$s.t.\ u_j \in \{0,1\}, \forall j \in \{1,2,...,n'\},$$

by setting $\lambda = (c_{wj*} / w^T q_{wj*}) \cdot w$ and $v(L_\lambda) = v(S_w)$. As a consequence, it can be conjectured that the surrogate heuristic is also a Lagrangian heuristic. Computational tests in Lopes and Lorena (1994) for large scale set covering problems (up to 1,000 rows and 12,000 columns) indicate the surrogate heuristic produces better-quality results than algorithms based on Lagrangian relaxations in terms of final solutions and mainly in computing time.

Based on the above discussion, a set covering-based TOPSIS algorithm with surrogate heuristic for finding the compromise solution of the sup-T equation constrained multi-objective optimization problem (1) can be organized as below.

Set Covering-based TOPSIS Algorithm

Step 1. Decision maker provides the relative importance w_k of the K objective functions. (There are various methods including the eigenvector, weighted least square, entropy and LINMAP methods for assessing w_k (Hwang and Yoon 1981).

Step 2. Determine the positive ideal solution (f^*) by solving equation (2).

Step 2.1. Construct the associated set covering problem of (2) and set $C = 1$.

Step 2.2. Solve the associated set covering problem (SCP) with the weight vector $(c_1, c_2, ..., c_{n'})^T$ and an $m \times n'$ matrix Q of zeros and ones using the surrogate heuristic.

Step 2.2.1. Initialize. Let $f_{ub} = +\infty$, $f_{lb} = -\infty$ and $\mathbf{w} = e^m$, the m-vector of 1*'s*, ($\mathbf{w} \geq 0$ and $\mathbf{w} \neq 0$).

Step 2.2.2. Solve the associated surrogate relaxation (S_w) of SCP and let the solution be $u_w = (u_{w1}, u_{w2}, \cdots, u_{wn'})^T$ with an optimal value $v(S_w)$ and $wj*$ being the index of the fractional variable.

Step 2.2.3. Construct a feasible solution for SCP using u_w. Set $u_{wj*} = 1$ and construct a feasible solution $u_f = (u_{f_1}, u_{f_2}, ..., u_{f_{n'}})^T$ for SCP with value $z(\mathbf{w}) = \sum_{j=1}^{n'} c_j u_{f_j}$.

Step 2.2.4. Update f_{ub} and f_{lb}. Let f_{ub} = min(f_{ub}, $z(\mathbf{w})$) and f_{lb} = max(f_{lb}, $v(S_w)$).

Step 2.2.5. Check the following stopping rules. If (a) $f_{ub} - f_{lb} < \varepsilon$ with a sufficiently small $\varepsilon > 0$, or (b) the value f_{lb} has not increased in the last 10 iterations, then output u_f as the optimal (near optimal) solution for SCP and go to Step 3. Otherwise, continue.

Step 2.2.6. Find the subgradient direction $G(\mathbf{w})$ and the step size t_w. Set $u_{wj*} = 0$ and define $\rho := \alpha / d_{wj^*}$ for the new step size

$$t_w := \rho(f_{ub} - f_{lb}),$$

with $\alpha > 0$ being a given parameter, and update the vector ***w*** with

$$w_i \leftarrow \max\left\{0, w_i + \frac{t_w G_i(w)}{\| G(w) \|^2}\right\}, i = 1,2,...,m,$$

where

$$G(w) = e^m - \sum_{j=1}^{n'} q_{wj} u_{wj}.$$

Return to Step 2.2.2.

Step 3. If $C = 1$, then go to Step 4; else if $C = 2$, then go to Step 5. Otherwise, output the obtained solution u_f as the compromise solution of (1) and go to Step 6.

Step 4. Determine the negative ideal solution (f^-) by solving equation (3).

Step 4.1. Construct the associated set covering problem of (3) and set $C = 2$.

Step 4.2. Go to Step 2.2.

Step 5. Substitute the positive ideal solution and the negative ideal solution obtained in Steps 2 and 4 into problem (7), construct its associated set covering problem and set $C = 3$. Go to Step 2.2.

Step 6. If the compromise solution of (1) obtained by the set covering-based TOPSIS approach is satisfied, stop. Otherwise, the decision maker may like to change w_k or the stopping rules of the surrogate heuristic. Then, go back to Step 1 or modify Step 2.2.5. The solution procedure is then repeated.

4. Numerical Experiments

In this section a numerical example is provided to illustrate the set covering-based TOPSIS for solving the sup-T equation constrained multi-objective optimization problems. Computational experiences on solving randomly generated multi-objective optimization problems with sup-T equation constraints are also reported.

Example 3

Consider the following system of sup-T_M equation constrained multi-objective optimization problem studied in Loetamonphong et al. (2002):

$$\begin{aligned} &\min\ f_1(x) = 0.6x_1 + 0.5x_2 - 0.1x_3 - 0.3x_4 \\ &\min\ f_2(x) = 0.3x_1 - 0.4x_2 - 0.2x_3 - 0.3x_4 \\ &\text{s.t} \begin{pmatrix} 0.1 & 0.9 & 0.8 & 0.1 \\ 0.4 & 0.7 & 1 & 0.3 \\ 0.5 & 0.2 & 0.5 & 0.5 \\ 0.1 & 0 & 0 & 0 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.7 \\ 0.5 \\ 0 \end{pmatrix} \\ &\quad x_j \in [0,1], j = 1,2,3,4. \end{aligned} \tag{14}$$

Let

$$x \in X \triangleq \left\{ x \in [0,1]^4, \begin{pmatrix} 0.1 & 0.9 & 0.8 & 0.1 \\ 0.4 & 0.7 & 1 & 0.3 \\ 0.5 & 0.2 & 0.5 & 0.5 \\ 0.1 & 0 & 0 & 0 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.7 \\ 0.5 \\ 0 \end{pmatrix} \right\}.$$

Applying the basic principle of compromise of TOPSIS, problem (14) can be reduced to the following sup-T_M equation constrained optimization problem:

$$\min_{x \in X} d_1^{PIS}(X) = w_1 \left[\frac{f_1(X) - f_1^*}{f_1^- - f_1^*}\right] + w_2 \left[\frac{f_2(X) - f_2^*}{f_2^- - f_2^*}\right] \tag{15}$$

where

$$f_1^* = \min_{x \in X} f_1(x), \tag{16}$$

$$f_2^* = \min_{x \in X} f_2(x), \tag{17}$$

$$f_1^- = \max_{x \in X} f_1(X), \tag{18}$$

$$f_2^- = \max_{x \in X} f_2(X), \tag{19}$$

and

$$w_1 = w_2 = 1/2.$$

The surrogate heuristic is then applied to solve the problems (16)-(19) in view of the associated set covering problems.

Consider the sup-T_M equation constrained optimization problem (18):

$$\max_{x\in X}\ 0.6x_1+0.5x_2-0.1x_3-0.3x_4.$$

The associated set covering problem becomes

$$-\min 0u_{11}+0u_{21}+0u_{22}+0u_{23}+0u_{31}+0.05u_{32}+0.07u_{33}+0u_{41}+0.15u_{42}+0.3u_{43}$$

$$\text{s.t.}\ \begin{pmatrix}0&0&0&1&0&0&0&0&0&0\\0&0&1&1&0&0&1&0&0&0\\0&0&0&0&0&1&1&0&1&1\\1&1&1&1&1&1&1&1&1&1\end{pmatrix}u\ge\begin{pmatrix}1\\1\\1\\1\end{pmatrix},$$

$$u=\begin{pmatrix}u_{11},u_{21},u_{22},u_{23},u_{31},\\u_{32},u_{33},u_{41},u_{42},u_{43}\end{pmatrix}^T\in\{0,1\}^{10}. \quad (20)$$

Initial reduction tests are conducted for the set covering problem (20) before implementing the surrogate heuristic. Since row 1 is covered only by column 4, we have u_{23}^* = 1 and the rows 1, 2, 4 and columns 1, 2, 3, 4, 5, 8 can be deleted from the problem. The reduced problem becomes

$$-\min 0.05u_{32}+0.07u_{33}+0u_{41}+0.15u_{42}+0.3u_{43}$$
$$\text{s.t.}\ \ u_{32}+u_{33}+u_{42}+u_{43}\ge 1,$$
$$u_{11},u_{21},u_{22},u_{23,}u_{31},u_{32},u_{33},u_{41},u_{42},u_{43}\in\{0,1\} \quad (21)$$

Applying the surrogate heuristic for the set covering problem (21), we first find the solution for the surrogate relaxation (S_w) of (21).

At iteration #1 :

For a given w = 1, the continuous surrogate relaxation (S_w) of (21) can be described as follows:

$$-\min 0.05u_{32}+0.07u_{33}+0u_{41}+0.15u_{42}+0.3u_{43}$$
$$\text{s.t.}\ \ u_{32}+u_{33}+u_{42}+u_{43}\ge 1,$$
$$u_{11},u_{21},u_{22},u_{23,}u_{31},u_{32},u_{33},u_{41},u_{42},u_{43}\in\{0,1\} \quad (22)$$

Compute

$$d_{32}=\frac{0.05}{1}\le d_{33}=\frac{0.07}{1}\le d_{42}=\frac{0.15}{1}\le d_{43}=\frac{0.3}{1}.$$

Since 0 ≤ 1 ≤ $w^T q_{32}$, we have wj^* = 32. The optimal solution

$$u_w=(u_{11},u_{21},u_{22},u_{23},u_{31},u_{32},u_{33},u_{41},u_{42},u_{43})^T=(0,0,0,1,0,1,0,0,0,0)^T$$

with the optimal value $v(S_w)=-0.05$.

To construct a feasible solution for the set covering problem (21) using u_w, a feasible solution

$$u_f=(u_{11},u_{21},u_{22},u_{23},u_{31},u_{32},u_{33},u_{41},u_{42},u_{43})^T=(0,0,0,1,0,1,0,0,0,0)^T$$

is then obtained with the objective value z(w)=-0.05.

Update $f_{ub}\leftarrow -0.05$ and $f_{lb}\leftarrow -0.05$. Since $f_{ub}=f_{lb}$, an optimal solution to problem (18) can be constructed as $x^*=(0,0.8,0.5,0)^T$ with the optimal value f_1^- = 0.35. The solutions of problems (16)-(19) can be obtained in an analogous manner and are shown in Table 2.

Table 2 The solutions of problems (16)-(19)

	f_1	f_2	x_1	x_2	x_3	x_4
$\min_{x\in X} f_1(x)$	$f_1^*=0.03$	−0.76	0	0.8	0.7	1
$\min_{x\in X} f_2(x)$	0.03	$f_2^*=-0.76$	0	0.8	0.7	1
$\max_{x\in X} f_1(x)$	$f_1^-=0.35$	−0.42	0	0.8	0.5	0
$\max_{x\in X} f_2(x)$	0.35	$f_2^*=-0.42$	0	0.8	0.5	0

Substituting the results of Table 2 into problem (15) with $w_1 = w_2 = 1/2$, we have the following sup- T_M equation constrained optimization problem:

Let $x_j \in [0,1], j = 1,2,3,4$, then

$$\min \ 1.3787x_1 + 0.193x_2 - 0.4504x_3 - 0.9099x_4 + 1.0708$$
$$\text{s.t.} \begin{pmatrix} 0.1 & 0.9 & 0.8 & 0.1 \\ 0.4 & 0.7 & 1 & 0.3 \\ 0.5 & 0.2 & 0.5 & 0.5 \\ 0.1 & 0 & 0 & 0 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.7 \\ 0.5 \\ 0 \end{pmatrix}. \tag{23}$$

Solving problem (23) by the surrogate heuristic, we obtain the optimal solution

$$\mathrm{x}^* = (0, 0.8, 0.7, 1)^T,$$

which is consistent with the result of Loetamonphong et al. (2002). Notice that our result indicates that the surrogate heuristic finds the solutions of problems (15)-(19) at very early iterations.

To investigate the prominent property of the proposed method, numerical experiments on solving randomly generated multi-objective optimization problems with sup-T equation constraints were carried out. The random generator of systems of sup-T equations developed in Hu and Fang (2012) was employed to randomly generate test problems. The algorithm was coded in Matlab and run on the HP Compaq dx2810 MT using the Windows 7 operating system. Table 3 presents the computational results for test problems. 50 test problems were solved for each size. Columns 3 and 4 of Table 3 represent the size of the 0-1 constraint coefficient matrix of the associated set covering problem for test problems after the initial reduction.

For the case of multi-objective optimization problems with sup- T_P equation constraints, after applying the initial reduction in the implementation, the 0-1 constraint coefficient matrix of the associated set covering problem becomes a square matrix with each row being covered by only one column. For example, a 0-1 constraint coefficient matrix of the associated set covering problem of 100 rows and 200 columns could be reduced to a 0-1 matrix of 79 rows and 79 columns with each row being covered by only one column after applying the initial reduction. It makes the surrogate heuristic stop and find the optimization solution at very early iterations. The case of multi-objective optimization problems with sup- T_M equation constraints is more complicated than the one with sup- T_P equation constraints. For example, a problem of sup- T_M case of 100 rows and 200 columns could result in a set covering problem of size 100 × 10, 100. After applying the initial reduction of the surrogate heuristic, the 0-1 constraint coefficient matrix of the associated set covering problem could be reduced to a 0-1 matrix of size 100 × 186. Most of the test problems find the optimization solution in less than 10 iterations. Our computational results validate the property of faster convergence of surrogate heuristic and show that the proposed method is very promising.

Table 3 Computational results for test problems

Problem size	Type of equation constraints	Average number of rows after reduction	Average number of columns after reduction	Average number of iterations	Average computing time (sec.)
20×40	sup- T_P	16	16	1	0.0936
40×80	sup- T_P	32	32	1	0.1404
60×120	sup- T_P	46	46	1	0.1716
80×160	sup- T_P	66	66	1	0.2184
100×200	sup- T_P	79	79	1	0.3096
20×40	sup- T_M	15	16	2	0.2808
40×80	sup- T_M	29	31	3	0.5616
60×120	sup- T_M	45	49	4	0.8312
80×160	sup- T_M	54	59	6	1.3260
100×200	sup- T_M	71	78	7	1.8252

5. Conclusion

This paper studies the compromise solutions to the sup-T equation constrained multi-objective optimization problems, with T being a continuous triangular norm. Taking advantage of the well developed techniques and clarity of exposition in the theory of integer programming, a set covering-based TOPSIS is proposed to solve the sup-T equation constrained multi-objective optimization problem. This study provides, for the first time, a systematic method for solving the sup-T equation constrained multi-objective optimization problems from an integer programming viewpoint. Our computational results confirm the efficiency of the proposed method and show its potential for solving large scale sup-T equation constrained multi-objective optimization problems.

It should be noted that different p values provide decision makers with different compromise solutions. Among various p values, p = 1, 2 and ∞ are operationally and practically important and each exhibits its unique merit. The Euclidean distance (p = 2) is similar to the popular least-square approach and seems to be more acceptable in view of distance aspect. The case of $p = \infty$ emphasizes the maximum of individual regrets and the minimum of individual rewards. Future studies may consider models of different p values to provide compromise solutions for decision makers with specific interests.

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