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Technical Note

AN ANALYTIC HIERARCHY PROCESS MODEL OF GROUP CONSENSUS

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Abstract

In group decision making, a certain degree of consensus is necessary to derive a meaningful and valid outcome. This paper proposes a consensus reaching model for a group by using the Analytic Hierarchy Process (AHP). It supports people to improve their group consensus level through an updating of their judgments. In this model, a moderator suggests the most discordant decision maker to update his judgment in each step. The proposed consensus reaching model allows decision makers to accept or reject the suggestion from the moderator. This model ensures that the judgment updating is effective and the final solution will be of acceptable consistency. Finally, a numerical example is given to illustrate the validity of the proposed consensus reaching model.

Keywords: Group decision making, Analytic Hierarchy Process (AHP), consensus, judgment updating

1. Introduction

A group decision making problem can be defined as a decision problem where a group of decision makers express their judgments on a finite set of alternatives to achieve a common solution. There are four essential issues in group decision making: formats for expressing judgments, deriving priority weights for the decision makers, aggregating individual judgments into a representative group judgment and finally obtaining a certain degree of consensus among the judges. Failure in any of these issues would lead to difficulties in group decision making.

The first thing to do in a group decision problem is to choose a suitable method to express judgments. Usually, the judgments of decision makers can be represented through formats as ranking, deriving a utility function (Brock 1980, Keeney 1976, Keeney and Kirkwood 1975, Yu 1973), fuzzy preference relations (F. Chiclana et al. 2007, Hsi-Mei and Chen-Tung 1996, Tan 2011, Tanino 1984), linguistic preference relations (Dong et al. 2009, Herrera-Viedma et al. 2005, Herrera and Martínez 2000, Wang and Chu 2004, Wu and Xu 2012a), and pairwise comparisons. The Analytic Hierarchy Process (AHP), has become a popular decision tool for helping people to deal with complex decision problems involving both tangible and intangible factors (E. H. Forman and Gass 2001, Subramanian and Ramanathan,

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2012, Vaidya and Kumar 2006). It's also been used in group decision making, perhaps because of its flexibility and adaptability.

Determining the importance weights of the decision makers themselves has been discussed in previous studies (E. Forman and Peniwati, 1998, Ramanathan and Ganesh 1994, Saaty 1994a). These weights depend on criteria such as expertise, experience, previous performance, etc. In some cases, however, it is hard to get this kind of evaluation. In that case, one may use equal weight for an approximately optimal solution of an incompatibility minimization problem of each individual from a representative group judgment (Y. Xu et al. 2013).

There are two ways to aggregate individual judgments into an appropriate group judgment (E. Forman and Peniwati 1998, Ramanathan and Ganesh 1994): (1) aggregation of individual priorities (AIP), which transforms individual priority vectors into a group priority vector; and (2) aggregation of individual judgments (AIJ), which transforms individual pairwise comparison matrices (PCMs) into a group PCM from which the group priorities are then derived. Since the decision makers should update their judgment in the consensus process, we adopt the second way in our study to avoid computing the priorities every time. In that case we must use the geometric mean as the merging function whether the judges have equal or different weights. J. Aczél and Saaty (1983), J. Aczél and Alsina (1987), and János Aczél and Roberts (1989) showed that when aggregating the PCMs with the reciprocal property, the geometric mean is the only mathematically valid aggregation function that satisfies both the unanimity and homogeneity conditions.

Despite the use of the geometric mean, the judgments maybe a compromise because the geometric mean involves the product of numerical judgments that can be reciprocals such as 9 and 1/9 yielding the value 1, which can also be obtained by using the values 2 and 1/2. Here the more divergent judgments have been used in the first case. What to do with this divergence especially when using the geometric mean is the purpose of the paper. Therefore a measure of consensus and reaching group consensus are of considerable significance in group decision making (Fu and Yang 2011).

As full consensus, which means total agreement and coincidence of the numerical judgments, is hard to achieve, a soft consensus measure is introduced. Kacprzyk et al. (1992) defined a soft degree of consensus by using the fuzzy majority. Bryson (1996) proposed the group strong agreement quotient and group strong disagreement quotient to estimate the group consensus level. Herrera-Viedma et al. (2002) defined the consensus measure by comparing the position of the alternatives in the solution vector. All of these consensus measurement methods are associated with the closeness of opinions. Because in this paper we adopt aggregation of judgments (AIJ) to collect individual preferences, the compatibility index between two PCMs, which was proposed by Saaty (1994b), is used to measure the consensus level in group decision making.

A consensus reaching process is usually defined as an interactive process with several negotiation rounds, in which a moderator suggests to the decision makers to update their judgments. The moderator does not participate in the discussion but supervises and leads the consensus process toward success, i.e., to achieve a certain level of agreement (Francisco Chiclana et al. 2008, Herrera-Viedma et al. 2002, Herrera et al. 1996). Usually the moderator should be a distinguished person responsible for this decision making problem. In this paper, however, we do not need a distinguished person or even a person, as the suggestion a moderator makes to individuals can be automatically calculated in our consensus reaching process. Thus given the algorithm for reaching consensus, a computer as a judgment processing center can be viewed as a moderator.

A considerable amount of literature has been published on consensus reaching models. Francisco Chiclana et al. (2008) proposed a consensus framework to improve both consistency and consensus in group decision making. Xu (2009) developed a convergent algorithm for changing the decision makers' opinion to achieve group consensus. Considering the consistency control in group decision making, Dong et al. (2010) proposed two AHP consensus models under a row geometric mean prioritization method. Wu and Xu (2012b) provided a consistency and consensus based group decision making model which is independent of the method of prioritization.

The consensus models described above focused on revising or updating the decision makers' judgments. Generally, these models improve the group consensus level by forcing a reluctant member to update his judgment as the moderator suggests. The methods of updating judgments may make decision makers uncomfortable. There are two ways of forming judgments. Those developed in an exploratory way is based on hypothetical assumptions without testing them in reality. There are also judgments developed on the basis of both theory and experience and are less likely to change. The consensus model presented in this paper focuses on the exploratory judgments case and does not force them to change their mind. In our model, a decision maker is able to either use his/her own judgment or update it based on the moderator's suggestion.

This paper is structured as follows. We briefly introduce some basic knowledge of group AHP in section 2. In section 3, we explain the consensus reaching model and analyze its properties. In section 4, a numerical example is provided. Conclusions are made in section 5.

2. Theoretical Background

In this section, we first introduce some basic knowledge about group decision making with the AHP. Then we discuss consistency of both individual and group PCMs in group AHP. Finally, we use a group consensus index to measure the consensus level of a decision maker.

2.1 Group AHP aggregation with the weighted geometric mean

For simplicity, we use $N=\{1,2,\cdots,n\}$, $M=\{1,2,\cdots,m\}$ to denote elements in sets. For a finite set of alternatives $X=\{x_1,x_2,\cdots,x_n\}$, we use $\boldsymbol{w}=\left(w_1,w_2,\cdots,w_n\right)^{\mathrm{T}}$ to denote the priority vector, where $w_i \geq 0$, and $\sum_{i=1}^{n} w_i = 1$. Then the judgment information is represented as an $n \times n$ pairwise comparison matrix (PCM) $\boldsymbol{A}=\left(a_{ij}=w_i/w_j\right)_{n\times n}$, where $a_{ij}=1/a_{ji}$ and a_{ij} belongs to the AHP 1-9 fundamental scale and represents the relative importance or better, the dominance of x_i over x_j .We assume that there are m decision makers $DM_1, DM_2, \cdots, DM_m$ with PCMs $\boldsymbol{A}_k = \left(a_{ij(k)}\right)_{n\times n}$, for $k \in M$, and let $\boldsymbol{\rho} = (\rho_1, \rho_2, \cdots, \rho_m)^{\mathrm{T}}$ be the weight or importance vector of the decision makers, where $\rho_k \geq 0$, $\sum_{k=1}^{m} \rho_k = 1, \ldots, m$ Then by aggregating with the weighted geometric mean (J. Aczél and Alsina 1987, János Aczél and Roberts 1989, J. Aczél and Saaty 1983), the group PCM $\boldsymbol{G} = \left(g_{ij}\right)_{n\times n}$ can be calculated as

$$g_{ij} = \prod_{k=1}^{m} \left(a_{ij(k)}\right)^{\rho_k} \quad i, j \in N. \tag{1}$$

2.2 Consistency of both individual and group PCMs

Consistency is a concept defined to describe and reflect the quality of a PCM, which is a critical issue of AHP and has been discussed over a long period of time (Aguarón and Moreno-Jiménez 2003, Aull-Hyde et al. 2006, Chen et al. 2002). As defined by Saaty (1977), PCM is consistent if

$$a_{hi} = a_{hj} a_{ji}, \quad h, i, j \in N. \tag{2}$$

However, in a real life decision situation, consistency is hard to achieve because a modicum of inconsistency reflects that people are making a decision with new information that may not be fully understood and is not consistent with what was already known. Also when a set of comparisons is too inconsistent one could just as well have used random entries and the information from the comparisons would not be useful. In order to provide a balance, Saaty (1980) defined the consistency index as

$$CI_A = \frac{\lambda_{\max} - n}{n-1}, \tag{3}$$

where $\lambda_{\max}$ is the largest or principal eigenvalue of $\boldsymbol{A}$. To measure the inconsistency of $\boldsymbol{A}$, we have the consistency ratio

$$CR_A = \frac{CI_A}{RI_n}, \tag{4}$$

where RI_n is the average random consistency index derived from randomly generated $n \times n$ PCMs. In general, if CR_A is less than 0.10, we say that $\boldsymbol{A}$ is acceptably consistent (Saaty, 1990).

Table 1 The random consistency index (Saaty, 1990)

n	1	2	3	4	5	6	7	8	9	10
RI	0	0	0.52	0.89	1.11	1.25	1.35	1.40	1.45	1.49

Once individual judgments are aggregated by the geometric mean into group judgments, we face the following question for the inconsistency of the group PCM. Xu (2000) proved that the group judgment matrix derived by using the weighted geometric mean is of acceptable consistency if all individual PCMs are of acceptable consistency. Lin et al. (2008) rejected his proof. Then Saaty and Peniwati (2012) proved that when the importance of the individual judgments are equal, we have the bounded inconsistency theorem: for a group of nearly consistent individuals, the inconsistency of group judgments aggregated by the geometric mean from individual judgments is at most equal to the maximum of the individual inconsistencies.

Grošelj and Zadnik Stirn (2012) gave a more general proof which confirms that if the comparison matrices of all decision makers are of acceptable consistency, then the weighted geometric mean judgment matrix is also of acceptable consistency. Thus, we let $\boldsymbol{\rho}=(\rho_1,\rho_2,\cdots,\rho_m)^{\mathrm{T}}$ be the weights of the m judges, where $\rho_k \ge 0$, $\sum_{k=1}^{m}\rho_k = 1$, $k \in M$, $\boldsymbol{G}=\left(g_{ij}=\prod_{k=1}^{m}(a_{ij(k)})^{\rho_k}\right)_{n\times n}$ be a weighted combination of $\boldsymbol{A}_k$. They proved that if each of the m matrices $\boldsymbol{A}_1,\boldsymbol{A}_2,\cdots,\boldsymbol{A}_m$ is of acceptable consistency, then $\boldsymbol{G}$ is of acceptable consistency.

Thus in a group decision making problem, the consistency of group PCM is related to individual consistency. If we can make sure that every decision maker has acceptable consistency, the group judgment would be of acceptable consistency.

2.3 Consensus measure

Consensus is general agreement or accord in opinion. In a group decision context, an important concern is to measure the consensus or agreement level. Considering that PCM belongs to an absolute and thus also ratio scale, Saaty (1994b) proposed that the closeness of two PCMs can be measured by using the compatibility index.

Definition 1 *(Saaty, 1994b) Let* $\boldsymbol{A}_p=\left(a_{ij(p)}\right)_{n\times n}$ *and* $\boldsymbol{A}_q=\left(a_{ij(q)}\right)_{n\times n}$ *be two PCMs, the compatibility index of* $\boldsymbol{A}_p$ *and* $\boldsymbol{A}_q$ *,* $p,q \in M$ *, can be defined as*

$$c\left(\boldsymbol{A}_p,\boldsymbol{A}_q\right)=\frac{1}{n^2}\boldsymbol{e}^{\mathrm{T}}\boldsymbol{A}_p\circ\boldsymbol{A}_q^{\mathrm{T}}\boldsymbol{e}=\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij(p)}a_{ji(q)} \quad (5)$$

where $\circ$ denotes the Hadamard product of two matrices and $\boldsymbol{e}=(1,1,\cdots,1)^{\mathrm{T}}$. From Eq.(5), we have $c\left(\boldsymbol{A}_p,\boldsymbol{A}_q\right)\ge 1$, where $c\left(\boldsymbol{A}_p,\boldsymbol{A}_q\right)=1$ is attained if and only if $\boldsymbol{A}_p=\boldsymbol{A}_q$. Then by using the compatibility index, we can define the group consensus index (GCI) as Wu and Xu (2012b) did:

Definition 2 *Let* $\boldsymbol{A}_1,\boldsymbol{A}_2,\cdots\boldsymbol{A}_m$ *be the individual PCMs and* $\boldsymbol{G}$ *be the group PCM, then the group consensus index (* GCI *) of* DM_k *is defined as*

$$GCI_k = c\left(\boldsymbol{A}_k,\boldsymbol{G}\right),\ k\in M \quad (6)$$

where GCI_k denotes the closeness between the judgment of decision maker k and that of the group. Thus given a threshold value ε ($\varepsilon \ge 1$), if $GCI_k \le \varepsilon$, we say that decision maker DM_k has acceptable consensus. The value of ε can be set at 1.01 because 1% deviation is usually used as the upper bound of acceptability.

3. Reaching Consensus

We have now arrived at the most important section of this paper that is much needed in practical applications. Since, in a group, there is almost always a diversity of opinions, an acceptable consensus level for every decision maker in the group is usually hard or impossible to achieve. However, a consensus reaching process is still needed to drive decision makers towards consensus. Generally, the entire consensus reaching process would be as shown in Figure 1. In this section, the stop conditions and an algorithm for reaching consensus are introduced and its properties discussed.

3.1 The stop conditions

The consensus reaching process is usually viewed as a set of transformations of the decision makers' diverse judgments into a group judgment with an acceptable consensus level. But along with the benefit of group consensus enhancement, there are also costs such as time, money, and energy. Therefore, as we shall see below, we need to define stop conditions for this process. Different stop conditions are adopted according to the specific decision context. Generally, a consensus reaching process should be stopped when any of the following stop conditions is fulfilled:

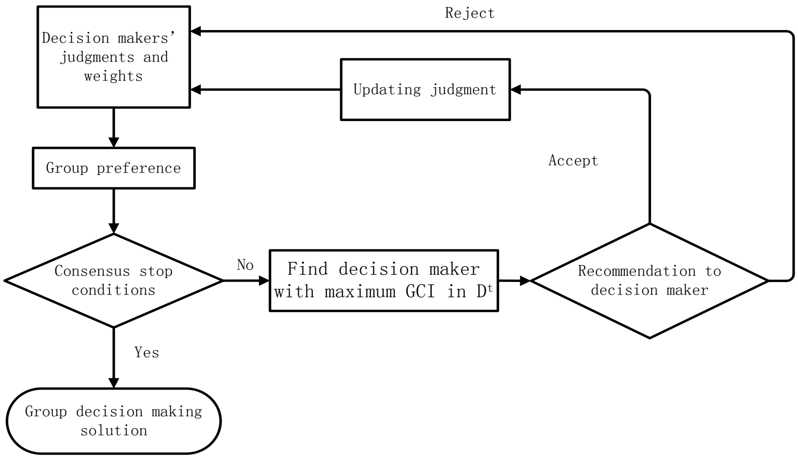

Figure 1 Diagram of the consensus reaching process

Stop condition 1 *If all members of a group with m decision makers have an acceptable consensus, we say that group consensus is achieved.*

Stop condition 2 *If all members of a group with m decision makers have rejected the recommendation of the moderator, then we stop the consensus reaching process.*

Stop condition 3 *If the predefined maximum number of iterations T, where $T \geq 1$, T, an integer, is achieved then terminate the consensus reaching process.*

3.2 Algorithm for the consensus reaching process

Input: Initial PCMs $\boldsymbol{A}_1, \boldsymbol{A}_2 \cdots, \boldsymbol{A}_m$ with acceptable consistency, weight vector of decision makers $\boldsymbol{\rho} = (\rho_1, \rho_2, \cdots, \rho_m)^{\mathrm{T}}$, the threshold value of the group consensus index ε, the maximum number of iterations T.

Output: Final PCMs $\boldsymbol{A}_1^*, \boldsymbol{A}_2^*, \cdots, \boldsymbol{A}_m^*$, solution of group decision making $\boldsymbol{G}^*$, and the number of iterations t, $0 \leq t \leq T$.

Step 1. Let $t = 0$, $\boldsymbol{A}_k^0 = \left(a_{ij(k)}^0\right)_{n \times n} = \left(a_{ij(k)}\right)_{n \times n}$, $k \in M$. Determine the weight vector $\boldsymbol{\rho} = (\rho_1, \rho_2, \cdots, \rho_m)^{\mathrm{T}}$ by using subjective judgment or by using equal weight.

Step 2. Let $\boldsymbol{G}^t = \left(g_{ij}^t\right)_{n \times n}$ be the group PCM derived from $\boldsymbol{A}_1^t, \boldsymbol{A}_2^t, \cdots, \boldsymbol{A}_m^t$, where $g_{ij}^t = \prod_{k=1}^{m} (a_{ij(k)}^t)^{\rho_k}$, $i, j \in N$.

Step 3. Calculate the group consensus index

GCI_k for $k \in M$. If any of the stop conditions is satisfied, then go to Step 5; otherwise, continue.

Step 4. Let D^t be the set of decision makers who have not rejected the updating recommendation from the moderator before the t th iteration. Identify the most discordant decision maker DM_h with $GCI_h^t = \max_{l \in D^t} \{GCI_l^t\}$. Then the moderator suggests to DM_h to update his/her PCM $\boldsymbol{A}_h^t$ to $\boldsymbol{A}_h^{t+1}$ by using

$$a_{ij(h)}^{t+1} = \left(a_{ij(h)}^t\right)^{\alpha^t} \left(g_{ij}^t\right)^{1-\alpha^t} \tag{7}$$

where the parameter α^t associated with judgment updating, $0 \le \alpha^t \le 1$. The value of α^t can be set according to the situation of each iteration. The smaller α^t is, the larger will be the revision of the selected decision maker's judgment.

If DM_h accepts this suggestion, we have $\boldsymbol{A}_k^{t+1} = \left(a_{ij(k)}^{t+1}\right)_{n \times n}, k \in M$, where

$$a_{ij(k)}^{t+1} = \begin{cases} \left(a_{ij(h)}^t\right)^{\alpha^t} \left(g_{ij}^t\right)^{1-\alpha^t}, & k = h \\ a_{ij(k)}^t, & k \neq h \end{cases}. \tag{8}$$

Then set $t = t + 1$ and return to Step 2.

If DM_h rejects this suggestion, we have $\boldsymbol{A}_k^{t+1} = \boldsymbol{A}_k^t$, where $k \in M$. Then set $t = t + 1$ and return to Step 2.

Step 5. Let $\boldsymbol{A}_k^* = \boldsymbol{A}_k^t$. The output solution of the group decision making is $\boldsymbol{G}^*$ and the number of iterations is t. Stop the algorithm.

3.3 Properties

3.3.1 Comfortable environment

The main advantage and function of our consensus reaching model is that we show sufficient respect to the decision makers involved. Previous studies of consensus reaching usually force decision makers to change their judgments according to the moderator's suggestion. Mathematical proofs in their studies indicate that the group can achieve consensus by using these models. However, it is important to remember that, a decision maker usually provides low quality information in a group decision environment in which he does not feel comfortable. In addition, the judgments in the group decision context are usually supposed to be made by the decision makers, not by the moderator. If the decision makers are forced to update their judgments, then this is not a suggestion or recommendation from the moderator, but a command. As shown in our algorithm, decision makers can accept or reject the suggestions from the moderator. Moreover, judgment updating suggestions from the moderator, which consist of the current group judgment and his (her) own judgment, can be seen as feedback from other decision makers in an anonymous decision environment. This way the proposed consensus reaching model provides a comfortable environment for the decision makers.

3.3.2 Acceptable Consistency

Lemma 1 *Suppose in the* t *th iteration of the proposed consensus reaching model,* $\boldsymbol{A}_1^t, \boldsymbol{A}_2^t, \cdots, \boldsymbol{A}_m^t$ *are of acceptable consistency, where* $t \in [0, T]$*, then* $\boldsymbol{A}_1^{t+1}, \boldsymbol{A}_2^{t+1}, \cdots, \boldsymbol{A}_m^{t+1}$ *are of acceptable consistency.*

Proof. After the t th iteration, we will get $\boldsymbol{A}_1^{t+1}, \boldsymbol{A}_2^{t+1}, \cdots, \boldsymbol{A}_m^{t+1}$. Here we consider three cases.

Case 1. The algorithm is stopped in the t th iteration. Thus according to Step 3 and Step 5 of the algorithm we have $\boldsymbol{A}_k^* = \boldsymbol{A}_k^{t+1} = \boldsymbol{A}_k^t$, $k \in M$.

Then $A_k^*(A_k^{t+1})$ is of acceptable consistency for all $k \in M$.

Case 2. The stop condition is not fulfilled in the t th iteration and the selected decision maker DM_h rejects the suggestion of updating judgment. Then $A_k^{t+1} = A_k^t$, $k \in M$. Thus we get $\forall k \in M$, A_k^{t+1} is of acceptable consistency.

Case 3. The stop condition is not fulfilled in the t th iteration and the selected decision maker DM_h accepts the suggestion of updating judgment. Then A_h^{t+1} is a weighted combination of A_k^t and G^t, where both A_k^t and G^t are of acceptable consistency. Therefore as we mentioned in Section 2, A_h^{t+1} is also of acceptable consistency. For $k \neq h$, $k \in M$, we have $A_k^{t+1} = A_k^t$. Thus $\forall k \in M$, A_k^{t+1} is of acceptable consistency.

Summarizing all three cases, we have that $\forall k \in M$, if A_k^t is of acceptable consistency, then A_k^{t+1} is of acceptable consistency.

Theorem 1 *In the proposed consensus reaching model, suppose all* m *initial PCMs* $A_1, A_2, \cdots, A_m$ *are of acceptable consistency, then the final PCMs* $A_1^*, A_2^*, \cdots, A_m^*$ *are of acceptable consistency.*

Proof. First, we get that $A_k^0 = A_k$ is of acceptable consistency, $k \in M$. Then using Lemma 1, we can complete the proof of Theorem 1.

From Theorem 1, we know that in the proposed consensus reaching model, if the input PCMs are of acceptable consistency, the output PCMs are also of acceptable consistency. When we use this consensus reaching model, we do not need to be concerned about the inconsistency issues. All we need to do is to make sure that every decision maker provides a PCM with acceptable consistency, which can also be seen as a quality check of judgments.

3.3.3 Effectiveness

In the t th iteration of the proposed consensus reaching model, suppose the selected decision maker DM_h accepts the suggestion of updating judgment, then this judgment updating is the same as the updating proposed by Wu and Xu (2012b) who proved that $GCI_h^{t+1} < GCI_h^t$. Therefore, in our consensus reaching model, the updating suggestion from the moderator also improves the consensus level of the selected decision maker.

4. Numerical Example

In order to show the detailed implementation process and validity of the proposed consensus reaching model, we consider the following group decision making problem first presented by Dong et al. (2010) and also used by Wu and Xu (2012b). At first, we evaluate the effectiveness of the proposed method using their data set. Then we make comparisons between the proposed approach and previous methods.

4.1 The results of the proposed consensus reaching method

Suppose we have four alternatives X_1, X_2, X_3 and X_4 to be ranked and five decision makers DM_1, DM_2, DM_3, DM_4, and DM_5 with PCMs $A_k = \left(a_{ij(k)}\right)$, $k = 1,2,$ $3,4,5$, where

$$A_1 = \begin{pmatrix} 1 & 4 & 6 & 7 \\ 1/4 & 1 & 3 & 4 \\ 1/6 & 1/3 & 1 & 2 \\ 1/7 & 1/4 & 1/2 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 5 & 7 & 9 \\ 1/5 & 1 & 4 & 6 \\ 1/7 & 1/4 & 1 & 2 \\ 1/9 & 1/6 & 1/2 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & 3 & 5 & 8 \\ 1/3 & 1 & 4 & 5 \\ 1/5 & 1/4 & 1 & 2 \\ 1/8 & 1/5 & 1/2 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 4 & 5 & 6 \\ 1/4 & 1 & 3 & 3 \\ 1/5 & 1/3 & 1 & 2 \\ 1/6 & 1/3 & 1/2 & 1 \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 1 & 1/2 & 1 & 2 \\ 2 & 1 & 2 & 3 \\ 1 & 1/2 & 1 & 4 \\ 1/2 & 1/3 & 1/4 & 1 \end{pmatrix}.$$

Then we compute the consistency ratios of A_k and get: $CR_{A_1} = 0.0383$, $CR_{A_2} = 0.0678$, $CR_{A_3} = 0.0339$, $CR_{A_4} = 0.0471$, $CR_{A_5} = 0.0363$, which shows that the given PCMs are of acceptable consistency. Let $\boldsymbol{\rho} = (0.1, 0.3, 0.1, 0.2, 0.3)^{\mathrm{T}}$ be the weight vector of the decision makers. Set the threshold value of the group consensus index at $\varepsilon = 1.01$ and the maximum number of iterations $T = 10$. Then we show how to apply the proposed consensus reaching model to improve the group consensus level. Let $t = 0$, $A_k^0 = A_k$, $k = 1,2,3,4,5$. By using Eq.(1), we can get the group PCM:

$$G^0 = \begin{pmatrix} 1 & 1.9786 & 2.6547 & 3.3879 \\ 0.5054 & 1 & 2.2894 & 2.8536 \\ 0.3767 & 0.4368 & 1 & 1.7818 \\ 0.2952 & 0.3504 & 0.5612 & 1 \end{pmatrix}.$$

Then we can calculate GCI_k^0 using Eq.(6), $k = 1,2,3,4,5$. These values are listed in Table 2. All decision makers do not reach the consensus threshold value, where DM_5 is the maximum one. The consensus reaching process should be continued until it fulfills the stop conditions. The group consensus index vector, the choice of decision makers and α^t at each iteration are listed in Table 2.

Table 2 Group consensus index, the choice of selected decision maker and α^t

Round (t)	Group consensus index ($\boldsymbol{GCI}$)	Selected decision maker	Choice	α^t
0	$\boldsymbol{GCI}^0$ =(1.0502,1.1144,1.0382,1.0040,**1.3366**)$^{\mathrm{T}}$	DM_5	Accept	0.5
1	$\boldsymbol{GCI}^1$ =(1.0198,1.0634,1.0150,1.0185,**1.1435**)$^{\mathrm{T}}$	DM_5	Accept	0.5
2	$\boldsymbol{GCI}^2$ =(1.0083,1.0391,1.0079,1.0126,**1.0587**)$^{\mathrm{T}}$	DM_5	Accept	0.6
3	$\boldsymbol{GCI}^3$ =(1.0047,**1.0291**,1.0066,1.0119,*1.0301*)$^{\mathrm{T}}$	DM_5	Reject	0.7
4	$\boldsymbol{GCI}^4$ =(1.0047,**1.0291**,1.0066,1.0119,*1.0301*)$^{\mathrm{T}}$	DM_2	Accept	0.7
5	$\boldsymbol{GCI}^5$ =(1.0058,**1.0181**,1.0076,1.0108,*1.0254*)$^{\mathrm{T}}$	DM_2	Accept	0.7
6	$\boldsymbol{GCI}^6$ =(1.0070,**1.0113**,1.0088,1.0102,*1.0220*)$^{\mathrm{T}}$	DM_2	Accept	0.8
7	$\boldsymbol{GCI}^7$ =(1.0078,1.0083,1.0095,**1.0100**,*1.0203*)$^{\mathrm{T}}$	DM_4	Accept	0.9
8(Stop)	$\boldsymbol{GCI}^8$ =(1.0080,1.0083,1.0093,1.0085,*1.0201*)$^{\mathrm{T}}$	DM_2		

Note: Bold numbers shows this expert is the one with maximum GCI and do not reject the updating suggestion. Italic numbers shows this expert has rejected the updating suggestion.

4.2 Comparison with the existing methods

In Dong et al. (2010) and Wu and Xu (2012b), the final priorities of the alternatives are

$$\boldsymbol{w} = (0.5416, 0.2597, 0.1299, 0.0688)^{\mathrm{T}}$$

and

$$\boldsymbol{w} = (0.5437, 0.2722, 0.1172, 0.0669)^{\mathrm{T}}$$

respectively.

The ranks of alternatives in this model is the same as Dong et al. (2010) and Wu and Xu (2012b)'s model. However, the priority vector in our model is different. This minor difference in the priorities result based on the modified PCMs, in our opinion, is due to the different adjustment mechanisms in the consensus reaching process. In our model, the suggestion from the moderator is just a suggestion, not a command. Thus the decision is still made by decision makers. In step 3 of our method, decision maker DM_3 rejected the updating suggestion from the moderator, which cannot happen in their models. Thus DM_3 insists on his minority opinion as before. The approaches of Dong, et al. (2010) and Wu and Xu (2012b) are more aggressive to force the decision makers to change their mind in the updating process, resulting in a decision which is made by the moderator rather than the decision makers.

5. Conclusions

In this paper, we have proposed a consensus reaching model for a group based on judgment updating. We use the compatibility index to measure the consensus level of a group. A consensus reaching process is proposed to help the group reach a predefined consensus level without force the decision makers to change their opinion. The suggestions from the moderator should only be considered as a decision aid, which the decision makers use as a reference to update their opinion. The main advantages of this model are: (1) it allows the decision makers involved to accept or reject the recommendation from the moderator, which makes decision makers feel comfortable in the group decision process; (2) the final group pairwise comparison matrix will be of acceptable consistency if the individual pairwise comparison matrices are of acceptable consistency; (3) the suggestion from the moderator is effective.

The numerical example shows that the proposed consensus reaching model effectively improves the consensus level and treats the decision makers with respect. The proposed models can be extended to other types of preference relations and adopting different aggregation schemes.

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References

[1] Aczél, J. & Alsina, C., (1987). Synthesizing judgements: a functional equations approach. Mathematical Modelling, 9(3-5): 311-320

[2] Aczél, J. & Roberts, F. S., (1989). On the possible merging functions. Mathematical Social Sciences, 17(3): 205-243

[3] Aczél, J. & Saaty, T. L., (1983). Procedures for synthesizing ratio judgments. Journal of Mathematical Psychology, 27(1): 93-102

[4] Aguarón, J. & Moreno-Jiménez, J. M. a., (2003). The geometric consistency index:

Approximated thresholds. European Journal of Operational Research, 147(1): 137-145

[5] Aull-Hyde, R., Erdogan, S. & Duke, J. M., (2006). An experiment on the consistency of aggregated comparison matrices in AHP. European Journal of Operational Research, 171(1): 290-295

[6] Brock, H. W., (1980). The problem of "Utility Weights" in group preference aggregation. Operations Research, 28(1): 176-187

[7] Bryson, N., (1996). Group decision-making and the analytic hierarchy process: exploring the consensus - relevant information content. Computers & Operations Research, 23(1): 27-35

[8] Chen, P. S., Chu, P. & Lin, M., (2002). On Vargas's proof of consistency test for 3x3 comparison matrices in AHP. Journal of the Operations Research Society of Japan, 45(3): 233-242

[9] Chiclana, F., Herrera-Viedma, E., Herrera, F. & Alonso, S., (2007). Some induced ordered weighted averaging operators and their use for solving group decision-making problems based on fuzzy preference relations. European Journal of Operational Research, 182(1): 383-399

[10] Chiclana, F., Mata, F., Martinez, L., Herrera-Viedma, E. & Alonso, S., (2008). Integration of a consistency control module within a consensus model. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 16(supp01): 35-53

[11] Dong, Y., Xu, Y. & Yu, S., (2009). Computing the numerical scale of the linguistic term set for the 2-tuple fuzzy linguistic representation model. IEEE Transactions on Fuzzy Systems, 17(6): 1366-1378

[12] Dong, Y., Zhang, G., Hong, W.-C. & Xu, Y., (2010). Consensus models for AHP group decision making under row geometric mean prioritization method. Decision Support Systems, 49(3): 281-289

[13] Forman, E. & Peniwati, K., (1998). Aggregating individual judgments and priorities with the analytic hierarchy process. European Journal of Operational Research, 108(1): 165-169

[14] Forman, E. H. & Gass, S. I., (2001). The Analytic Hierarchy Process - An Exposition. Operations Research, 49(4): 469-486

[15] Fu, C. & Yang, S., (2011). An attribute weight based feedback model for multiple attributive group decision analysis problems with group consensus requirements in evidential reasoning context. European Journal of Operational Research, 212(1): 179-189

[16] Grošelj, P. & Zadnik Stirn, L., (2012). Acceptable consistency of aggregated comparison matrices in analytic hierarchy process. European Journal of Operational Research, 223(2): 417-420

[17] Herrera-Viedma, E., Herrera, F. & Chiclana, F., (2002). A consensus model for multiperson decision making with different preference structures. IEEE Transactions on Systems, Man and Cybernetics, Part A: Systems and Humans, 32(3): 394-402

[18] Herrera-Viedma, E., Martinez, L., Mata, F. & Chiclana, F., (2005). A consensus support system model for group decision-making problems with multigranular linguistic

preference relations. IEEE Transactions on Fuzzy Systems, 13(5): 644-658

[19] Herrera, F., Herrera-Viedma, E. & Verdegay, J. L., (1996). A model of consensus in group decision making under linguistic assessments. Fuzzy Sets and Systems, 78(1): 73-87

[20] Herrera, F. & Martínez, L., (2000). A 2-tuple fuzzy linguistic representation model for computing with words. IEEE Transactions on Fuzzy Systems, 8(6): 746-752

[21] Hsi-Mei, H. & Chen-Tung, C., (1996). Aggregation of fuzzy opinions under group decision making. Fuzzy Sets and Systems, 79(3): 279-285

[22] Kacprzyk, J., Fedrizzi, M. & Nurmi, H., (1992). Group decision making and consensus under fuzzy preferences and fuzzy majority. Fuzzy Sets and Systems, 49(1): 21-31

[23] Keeney, R. L., (1976). A group preference axiomatization with cardinal utility. Management Science, 23(2): 140-145

[24] Keeney, R. L. & Kirkwood, C. W., (1975). Group decision making using cardinal social welfare functions. Management Science, 22(4): 430-437

[25] Lin, R., Lin, J. S. J., Chang, J., Tang, D., Chao, H. & Julian, P. C., (2008). Note on group consistency in analytic hierarchy process. European Journal of Operational Research, 190(3): 672-678

[26] Ramanathan, R. & Ganesh, L. S., (1994). Group preference aggregation methods employed in AHP: an evaluation and an intrinsic process for deriving members' weightages. European Journal of Operational Research, 79(2): 249-265

[27] Saaty, T. L., (1977). A scaling method for priorities in hierarchical structures. Journal of Mathematical Psychology, 15(3): 234-281

[28] Saaty, T. L. (1980). The Analytic Hierarchy Process: Planning, priority setting, resource allocaton. McGraw-Hill, NewYork

[29] Saaty, T. L., (1990). How to make a decision: the analytic hierarchy process. European Journal of Operational Research, 48(1): 9-26

[30] Saaty, T. L. (1994a). Fundamentals of decision making and priority theory with the Analytic Hierarchy Process. RWS Publications, Pittsburgh

[31] Saaty, T. L., (1994b). A ratio scale metric and the compatibility of ratio scales: the possibility of arrow's impossibility theorem. Applied Mathematics Letters, 7(6): 45-49

[32] Saaty, T. L. & Peniwati, K. (2012). Group decision making: drawing out and reconciling differences. RWS Publications, Pittsburgh

[33] Subramanian, N. & Ramanathan, R., (2012). A review of applications of Analytic Hierarchy Process in operations management. International Journal of Production Economics, 138(2): 215-241

[34] Tan, C., (2011). A multi-criteria interval-valued intuitionistic fuzzy group decision making with Choquet integral-based TOPSIS. Expert Systems with Applications, 38(4): 3023-3033

[35] Tanino, T., (1984). Fuzzy preference orderings in group decision making. Fuzzy Sets and Systems, 12(2): 117-131

[36] Vaidya, O. S. & Kumar, S., (2006). Analytic Hierarchy Process: an overview of applications. European Journal of

Operational Research, 169(1): 1-29

[37] Wang, R.-C. & Chuu, S.-J., (2004). Group decision-making using a fuzzy linguistic approach for evaluating the flexibility in a manufacturing system. European Journal of Operational Research, 154(3): 563-572

[38] Wu, Z. & Xu, J., (2012a). A concise consensus support model for group decision making with reciprocal preference relations based on deviation measures. Fuzzy Sets and Systems, 206(0): 58-73

[39] Wu, Z. & Xu, J., (2012b). A consistency and consensus based decision support model for group decision making with multiplicative preference relations. Decision Support Systems, 52(3): 757-767

[40] Xu, Y., Li, K. W. & Wang, H., (2013). Distance-based consensus models for fuzzy and multiplicative preference relations. Information Sciences, 253(0): 56-73

[41] Xu, Z., (2000). On consistency of the weighted geometric mean complex judgment matrix in AHP. European Journal of Operational Research, 126(3): 683-687

[42] Xu, Z., (2009). An automatic approach to reaching consensus in multiple attribute group decision making. Computers & Industrial Engineering, 56(4): 1369-1374

[43] Yu, P. L., (1973). A class of solutions for group decision problems. Management Science, 19(8): 936-946

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