J Syst Sci Syst Eng (Dec 2010) 19(4): 430-452
DOI: 10.1007/s11518-010-5151-9

ISSN: 1004-3756 (Paper) 1861-9576 (Online)
CN11-2983/N

SOME NEW SIMILARITY MEASURES FOR INTUITIONISTIC FUZZY VALUES AND THEIR APPLICATION IN GROUP DECISION MAKING*

Meimei XIA[1] **Zeshui XU**[1, 2]

[1]*School of Economics and Management, Southeast University, Nanjing, Jiangsu 211189, China*
meimxia@163.com

[2]*Institute of Sciences, PLA University of Science and Technology, Nanjing, Jiangsu 210007, China*
xu_zeshui@263.net (✉)

Abstract

We first propose a series of similarity measures for intuitionistic fuzzy values (IFVs) based on the intuitionistic fuzzy operators (Atanassov 1995). The parameters in the proposed similarity measures can control the degree of membership and the degree of non-membership of an IFV, which can reflect the decision maker's risk preference. Moreover, we can obtain some known similarity measures when some fixed values are assigned to the parameters. Furthermore, we apply the similarity measures to aggregate IFVs and develop some aggregation operators, such as the intuitionistic fuzzy dependent averaging operator and the intuitionistic fuzzy dependent geometric operator, whose prominent characteristic is that the associated weights only depend on the aggregated intuitionistic fuzzy arguments and can relieve the influence of unfair arguments on the aggregated results. Based on these aggregation operators, we develop some group decision making methods, and finally extend our results to interval-valued intuitionistic fuzzy environment.

Keywords: Intuitionistic fuzzy value, interval-valued intuitionistic fuzzy value, similarity measure, aggregation operator

1. Introduction

Since it is an important tool for determining the degree of similarity between two objects, similarity measure has attracted a lot of attention in the last decades. Intuitioistic fuzzy set (IFS), proposed by Atanossov (1986), is characterized by a membership function and a non-membership function and has been investigated by many authors (Atanassov & Gargo 1989, Gau & Buehrer 1993, Chen & Tan 1994, Bustince & Burillo 1996, Hong & Choi 2000, etc.). IFSs are more powerful in dealing with uncertainty and vagueness than traditional fuzzy sets, and more and more people are paying attention to the similarity measures for IFSs. Szmidt & Kacprzyk (2000) proposed some

* The work was supported in part by the National Science Fund for Distinguished Young Scholars of China (No.70625005), the National Natural Science Foundation of China (No.71071161), and the Program Sponsored for Scientific Innovation Research of College Graduate in Jiangsu Province (No.CX10B_059Z).

definitions of distances between IFSs, and compared them with the approach used for fuzzy sets. Li & Cheng (2002) gave a similarity formula for IFSs only based on the membership degree and the non-membership degree. Liang & Shi (2003) used some examples to show that the formula defined by Li & Cheng (2002) is not reasonable sometimes and proposed several new similarity measures for IFSs. Mitchell (2003) interpreted IFSs as ensembles of ordered fuzzy sets from the statistical point of view to modify Li & Cheng's method and applied them to pattern recognition. Huang & Yang (2004) presented a similarity measure for IFSs based on the Hausdorff distance. Grzegorzewski (2004) also suggested some methods for measuring distances between IFSs based on the Hausdorff metric. Huang & Yang (2007) proposed several reasonable measures to calculate the degree of similarity between IFSs, in which the proposed measures are induced by L_p metric. Xu (2007a) gave some similarity measures for IFSs based on distance measures. Li et al. (2007) analyzed, compared and summarized the existing similarity measures between IFSs by their counterintuitive examples in pattern recognition. Xu & Chen (2008) gave a comprehensive overview of distance and similarity measures of IFSs and defined some continuous distance and similarity measures for IFSs, in which the geometric distance model was also utilized.

From the above analysis, we can conclude that the mentioned similarity measures for IFSs are only based on the original information. However, besides the membership and non-membership information, the unknown information is an important part in IFS. Optimists anticipate desirable outcomes whereas pessimists expect unfavorable outcomes (2008). These two kinds of decision makers may hold the opposite opinions on the unknown information in IFS. Thus to do a more reasonable decision catering for different decision makers, we should get more information from the original one (Burillo & Bustince 1996, Liu & Wang 2007). To do this, some parameters can be used to control the membership degree or the non-membership degree of an intuitionistic fuzzy value (IFV) which is the basic element of an IFS. The intuitinistic fuzzy operators defined by Atanossov (1995) are very useful for dealing with such situations. In this paper, we give some new similarity measures to reflect the decision maker's risk preference. We can assign different values for the parameters in the similarity measures according to the actual situations. Moreover, when the parameters are assigned some certain values, the proposed similarity measures reduce to the ones given by Xu & Chen (2008).

Group decision making is used to obtain the best solution(s) for a problem according to the information provided by some decision makers. It is an important research field of decision science, operational research and management science. In a group decision making problem, we have a set of different alternatives to solve the problem and a set of decision makers which are usually required to provide their preference values about the alternatives. To get the group's opinions about the alternatives, aggregation techniques are required to aggregate all the individual opinions. In the last decades, a lot of aggregation operators have been developed for crisp numbers (Yager & Kacprzyk 1997, Xu &

Da 2002, Yager 2003, 2004a, 2004b, etc.), interval numbers (Yager 2004c, Yager & Xu 2006, etc.), linguistic variables (Herrera & Herrera-Viedma 1997, Xu 2004, etc.). Since more and more authors are paying attention to IFVs, many aggregation operators for IFVs have been developed, for example, the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator (Xu & Yager 2006), the intuitionistic fuzzy ordered weighted averaging (IFOWA) operator (Xu 2007b), the generalized intuitionistic fuzzy weighted averaging (GIFWA) operator (Zhao et al. 2010), and so on. Determining the weight vectors of the aggregation operators is an important issue. The approaches to determining the OWA weights have been widely studied (O'Hagan 1988, Filev & Yager 1998, Yager & Filev 1999, Yager 1993, Xu 2005, 2006). But most of these methods are based on the degrees of optimism which are given by the decision makers and have no relation with the aggregation arguments. However, in some situations, a decision maker may not specify the degree of optimism in advance due to time pressure, lack of knowledge, or the decision maker's limited expertise related with problem domain (Xu 2008). Based on the deviation measure and the normal distribution, Xu (2008) developed some dependent uncertain ordered weighted aggregation operators, such as the dependent uncertain ordered weighted averaging (DUOWA) operators and the dependent uncertain ordered weighted geometric (DUOWG) operators, etc. The prominent characteristic of these operators is that the associated weights only depend on the aggregated arguments and can relieve the influence of unfair arguments on the aggregated results. In this paper, we shall extend this idea to intuitionistic fuzzy environment.

To do that, in Section 2, we give some new similarity measures for IFVs based on the defined intuitionistic fuzzy point operators. Section 3 presents the intuitionistic fuzzy dependent averaging (IFDA) operator and the intuitionistic fuzzy dependent geometric (IFDG) operator for aggregating IFVs and applies them to group decision making. Sections 4 and 5 extend the similarity measures and decision making methods given in Sections 2 and 3 to the interval-valued intuitionistic fuzzy environment. Section 6 concludes the paper.

2. Some New Similarity Formulas for IFVs

In order to reflect the decision maker's risk preference, in this section, we shall introduce some intuitionistic fuzzy point operators with controlling parameters, based on which, some new similarity measures are further proposed.

2.1 Point Operators for IFVs

Let a set X be fixed. An IFS A on X is an object having the form (Atansssov 1986):

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\} \tag{1}$$

where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in X$ to A respectively, and for every $x \in X$:

$$0 \le \mu_A(x) + \nu_A(x) \le 1 \tag{2}$$

For each IFS A on X, let

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{3}$$

then $\pi_A(x)$ is called the degree of

indeterminacy of x to A. For computational convenience, Xu & Yager (2006) and Xu (2007b) named $\alpha_r=(\mu_{\alpha_r},\nu_{\alpha_r})$ an intuitionistic fuzzy value (IFV). Additionally, $s_{\alpha_r}=\mu_{\alpha_r}-\nu_{\alpha_r}$ and $h_{\alpha_r}=\mu_{\alpha_r}+\nu_{\alpha_r}$ are called the score (Cheb & Tan 1994) and accuracy degree (Hong & Choi 2000) of α_r, respectively.

To compare any two IFVs α_1 and α_2, Xu & Yager (2006), and Xu (2007) introduced a simple method as below:

1) If $s_{\alpha_1}<s_{\alpha_2}$, then $\alpha_1<\alpha_2$;

2) If $s_{\alpha_1}=s_{\alpha_2}$, then

a) If $h_{\alpha_1}=h_{\alpha_2}$, then $\alpha_1=\alpha_2$;

b) If $h_{\alpha_1}<h_{\alpha_2}$, then $\alpha_1<\alpha_2$.

For an IFS $A=\{<x,\mu_A(x),\nu_A(x)>|\,x\in X\}$, let $\kappa,\lambda\in[0,1]$, Atanassov (1995) gave the following operators:

1) $D_\kappa(A)=\{x,<\mu_A(x)+\kappa\pi_A(x),\ \nu_A(x)+(1-\kappa)\pi_A(x)\,|\,x\in X>\}$;

2) For $\kappa+\lambda\le 1$, $F_{\kappa,\lambda}(A)=\{x,<\mu_A(x)+\kappa\pi_A(x),\ \nu_A(x)+\lambda\pi_A(x)\,|\,x\in X\}$;

3) $G_{\kappa,\lambda}(A)=\{x,<\kappa\mu_A(x),\lambda\nu_A(x)>|\,x\in X\}$;

4) $H_{\kappa,\lambda}(A)=\{x,<\kappa\mu_A(x),\ \nu_A(x)+\lambda\pi_A(x)>|\,x\in X\}$;

5) $H^*_{\kappa,\lambda}(A)=\{x,<\kappa\mu_A(x),\ \nu_A(x)+\lambda(1-\kappa\mu_A(x)-\nu_A(x))>|\,x\in X\}$;

6) $J_{\kappa,\lambda}(A)=\{x,<\mu_A(x)+\kappa\pi_A(x),\ \lambda\nu_A(x)>|\,x\in X\}$;

7) $J^*_{\kappa,\lambda}(A)=\{x,<\mu_A(x)+\kappa(1-\mu_A(x)-\lambda\nu_A(x)),\lambda\nu_A(x)>|\,x\in X\}$;

8) For $\kappa+\lambda\le 1, P_{\kappa,\lambda}(A)=\{x,<\max(\kappa,\mu_A(x)),\min(\lambda,\nu_A(x))>|\,x\in X\}$;

9) For $\kappa+\lambda\le 1, Q_{\kappa,\lambda}(A)=\{x,<\min(\kappa,\mu_A(x)),\max(\lambda,\nu_A(x))>|\,x\in X\}$.

The parameters in the intuitionistic fuzzy operators can be assigned different values according to the decision maker's demand. Obviously, the operators can also be true for IFVs:

For an IFV $\alpha_r=(\mu_{\alpha_r},\nu_{\alpha_r})$, let $\kappa,\lambda\in[0,1]$, we define some intuitionistic fuzzy point operators: IFV $\to$ IFV as follows:

1) $D_\kappa(\alpha_r)=(\mu_{\alpha_r}+\kappa\pi_{\alpha_r},\nu_{\alpha_r}+(1-\kappa)\pi_{\alpha_r})$;

2) For $\kappa+\lambda\le 1$, $F_{\kappa,\lambda}(\alpha_r)=(\mu_{\alpha_r}+\kappa\pi_{\alpha_r},\nu_{\alpha_r}+\lambda\pi_{\alpha_r})$;

3) $G_{\kappa,\lambda}(\alpha_r)=(\kappa\mu_{\alpha_r},\lambda\nu_{\alpha_r})$;

4) $H_{\kappa,\lambda}(\alpha_r)=(\kappa\mu_{\alpha_r},\nu_{\alpha_r}+\lambda\pi_{\alpha_r})$;

5) $H^*_{\kappa,\lambda}(\alpha_r)=(\kappa\mu_{\alpha_r},\ \nu_{\alpha_r}+\lambda(1-\kappa\mu_{\alpha_r}-\nu_{\alpha_r}))$;

6) $J_{\kappa,\lambda}(\alpha_r)=(\mu_{\alpha_r}+\kappa\pi_{\alpha_r},\lambda\nu_{\alpha_r})$;

7) $J^*_{\kappa,\lambda}(\alpha_r)=(\mu_{\alpha_r}+\kappa(1-\mu_{\alpha_r}-\lambda\nu_{\alpha_r}),\lambda\nu_{\alpha_r})$;

8) For $\kappa+\lambda\le 1$, $P_{\kappa,\lambda}(\alpha_r)=(\max(\kappa,\mu_{\alpha_r}),\min(\lambda,\nu_{\alpha_r}))$;

9) For $\kappa+\lambda\le 1$, $Q_{\kappa,\lambda}(\alpha_r)=(\min(\kappa,\mu_{\alpha_r}),\max(\lambda,\nu_{\alpha_r}))$.

2.2 Some New Similarity Measures for IFVs

Based on the intuitionsitic fuzzy point operators, we can define some new similarity measures considering the decision maker's risk preference.

Let $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ and $\beta=(\beta_1,\beta_2,\ldots,\beta_n)$ be two collections of IFVs, and $\kappa,\lambda\in[0,1]$,

$$t_{a_j}=\mu_{\alpha_j}-\mu_{\beta_j},$$

$$t_{b_j}=\nu_{\alpha_j}-\nu_{\beta_j},$$

$$t_{c_j}=\pi_{\alpha_j}-\pi_{\beta_j}=1-\mu_{\alpha_j}-\nu_{\alpha_j}-1+\mu_{\beta_j}+\nu_{\beta_j}=-t_{a_j}-t_{b_j}\quad(j=1,2,\ldots,n).$$

Then some similarity measures for IFVs can be given as:

1) $S_{D_\kappa}^{\eta}(\alpha,\beta)=1-\left[\sum_{j=1}^{n}\omega_j(|(1-\kappa)t_{a_j}-\kappa t_{b_j}|^{\eta})\right]^{1/\eta}$;

2) For $\kappa+\lambda\le 1$,

$$S_{F_{\kappa,\lambda}}^{\eta}(\alpha,\beta)=1-\left[\frac{1}{2}\sum_{j=1}^{n}\omega_j(|(1-\kappa)t_{a_j}-\kappa t_{b_j}|^{\eta}+|(1-\lambda)t_{b_j}-\lambda t_{a_j}|^{\eta}+|(1-\kappa-\lambda)(t_{a_j}+t_{b_j})|^{\eta})\right]^{1/\eta};$$

3) $S_{G_{\kappa,\lambda}}^{\eta}(\alpha,\beta)=1-\left[\frac{1}{2}\sum_{j=1}^{n}\omega_j(|\kappa t_{a_j}|^{\eta}+|\lambda t_{b_j}|^{\eta}+|\kappa t_{a_j}+\lambda t_{b_j}|^{\eta})\right]^{1/\eta}$;

4) $S_{H_{\kappa,\lambda}}^{\eta}(\alpha,\beta)=1-\left[\frac{1}{2}\sum_{j=1}^{n}\omega_j(|\kappa t_{a_j}|^{\eta}+|\lambda t_{a_j}-(1-\lambda)t_{b_j}|^{\eta}+|(\kappa-\lambda)t_{a_j}+(1-\lambda)t_{b_j}|^{\eta})\right]^{1/\eta}$;

5) $S_{H_{\kappa,\lambda}^{*}}^{\eta}(\alpha,\beta)=1-\left[\frac{1}{2}\sum_{j=1}^{n}\omega_j(|\kappa t_{a_j}|^{\eta}+|\kappa\lambda t_{a_j}-(1-\lambda)t_{b_j}|^{\eta}+|(1-\lambda)(\kappa t_{a_j}+t_{b_j})|^{\eta})\right]^{1/\eta}$;

6) $S_{J_{\kappa,\lambda}}^{\eta}(\alpha,\beta)=1-\left[\frac{1}{2}\sum_{j=1}^{n}\omega_j(|(1-\kappa)t_{a_j}-\kappa t_{b_j}|^{\eta}+|\lambda t_{b_j}|^{\eta}+|(1-\kappa)t_{a_j}+(\lambda-\kappa)t_{b_j}|^{\eta})\right]^{1/\eta}$;

7) $S_{J_{\kappa,\lambda}^{*}}^{\eta}(\alpha,\beta)=1-\left[\frac{1}{2}\sum_{j=1}^{n}\omega_j(|(1-\kappa)t_{a_j}-\kappa\lambda t_{b_j}|^{\eta}+|\lambda t_{b_j}|^{\eta}+|(1-\kappa)(t_{a_j}+\lambda t_{b_j})|^{\eta})\right]^{1/\eta}$;

8) For $\kappa+\lambda\le 1$, $S_{P_{\kappa,\lambda}}^{\eta}(\alpha,\beta)=1-\Big[\frac{1}{2}\sum_{j=1}^{n}\omega_j(|\max(\kappa,\mu_{\alpha_j})-\max(\kappa,\mu_{\beta_j})|^{\eta}$

$+|\min(\lambda,\nu_{\alpha_j})-\min(\lambda,\nu_{\beta_j})|^{\eta}+|\max(\kappa,\mu_{\alpha_j})-\max(\kappa,\mu_{\beta_j})+\min(\lambda,\nu_{\alpha_j})-\min(\lambda,\nu_{\beta_j})|^{\eta})\Big]^{1/\eta}$;

9) For $\kappa+\lambda\le 1$, $S_{Q_{\kappa,\lambda}}^{\eta}(\alpha,\beta)=1-\Big[\frac{1}{2}\sum_{j=1}^{n}\omega_j(|\min(\kappa,\mu_{\alpha_j})-\min(\kappa,\mu_{\beta_j})|^{\eta}$

$+|\max(\lambda,\nu_{\alpha_j})-\max(\lambda,\nu_{\beta_j})|^{\eta}+|\min(\kappa,\mu_{\alpha_j})-\min(\kappa,\mu_{\beta_j})+\max(\lambda,\nu_{\alpha_j})-\max(\lambda,\nu_{\beta_j})|^{\eta})\Big]^{1/\eta}$;

It is easy to prove that the above similarity measures satisfy the following properties:

1) $0\le S(\alpha,\beta)\le 1$;
2) $S(\alpha,\beta)=1$, if $\alpha=\beta$;
3) $S(\alpha,\beta)=S(\beta,\alpha)$.

where $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ and $\beta=(\beta_1,\beta_2,\ldots,\beta_n)$ are two collections of IFVs and $S(\alpha,\beta)$ is the similarity degree between them.

Furthermore, if the parameters in the developed similarity formulas change, we can get some interesting special cases:

1) For $\kappa+\lambda\le 1$, $\lim_{\eta\to+\infty}S_{F_{\kappa,\lambda}}^{\eta}(\alpha,\beta)=$

$1-\max_j\{|(1-\kappa)t_{a_j}-\kappa t_{b_j}|,$

$|(1-\lambda)t_{b_j}-\lambda t_{a_j}|,|(1-\kappa-\lambda)(t_{a_j}+t_{b_j})|\}$;

2) $\lim_{\eta\to+\infty} S^{\eta}_{G_{\kappa,\lambda}}(\alpha,\beta)=$

$$1-\max_j\{|\kappa t_{a_j}|,|\lambda t_{b_j}|,|\kappa t_{a_j}+\lambda t_{b_j}|\};$$

3) $\lim_{\eta\to+\infty} S^{\eta}_{H_{\kappa,\lambda}}(\alpha,\beta)=1-\max_j\{|\kappa t_{a_j}|,$

$$|\lambda t_{a_j}-(1-\lambda)t_{b_j}|,|(\kappa-\lambda)t_{a_j}+(1-\lambda)t_{b_j}|\};$$

4) $\lim_{\eta\to+\infty} S^{\eta}_{H^*_{\kappa,\lambda}}(\alpha,\beta)=1-\max_j\{|\kappa t_{a_j}|,$

$$|\kappa\lambda t_{a_j}-(1-\lambda)t_{b_j}|,|(1-\lambda)(\kappa t_{a_j}+t_{b_j})|\};$$

5) $\lim_{\eta\to+\infty} S^{\eta}_{J_{\kappa,\lambda}}(\alpha,\beta)=$

$$1-\max_j\{|(1-\kappa)t_{a_j}-\kappa t_{b_j}|,$$

$$|\lambda t_{b_j}|,|(1-\kappa)t_{a_j}+(\lambda-\kappa)t_{b_j}|\}$$

6) $\lim_{\eta\to+\infty} S^{\eta}_{J^*_{\kappa,\lambda}}(\alpha,\beta)=$

$$1-\max_j\{|(1-\kappa)t_{a_j}-\kappa\lambda t_{b_j}|,$$

$$|\lambda t_{b_j}|,|(1-\kappa)(t_{a_j}+\lambda t_{b_j})|\};$$

7) For $\kappa+\lambda\le 1$, $\lim_{\eta\to+\infty} S^{\eta}_{P_{\kappa,\lambda}}(\alpha,\beta)=1-$

$$\max_j\{|\max(\kappa,\mu_{\alpha_j})-\max(\kappa,\mu_{\beta_j})|,$$

$$|\min(\lambda,\nu_{\alpha_j})-\min(\lambda,\nu_{\beta_j})|,|\max(\kappa,\mu_{\alpha_j})-$$

$$\max(\kappa,\mu_{\beta_j})+\min(\lambda,\nu_{\alpha_j})-\min(\lambda,\nu_{\beta_j})|\};$$

8) For $\kappa+\lambda\le 1$, $\lim_{\eta\to+\infty} S^{\eta}_{Q_{\kappa,\lambda}}(\alpha,\beta)=1-$

$$\max_j\{|\min(\kappa,\mu_{\alpha_j})-\min(\kappa,\mu_{\beta_j})|,$$

$$|\max(\lambda,\nu_{\alpha_j})-\max(\lambda,\nu_{\beta_j})|,|\min(\kappa,\mu_{\alpha_j})-$$

$$\min(\kappa,\mu_{\beta_j})+\max(\lambda,\nu_{\alpha_j})-\max(\lambda,\nu_{\beta_j})|\};$$

9) $S^{\eta}_{F_{0,0}}(\alpha,\beta)=S^{\eta}_{G_{1,1}}(\alpha,\beta)=S^{\eta}_{H_{1,0}}(\alpha,\beta)$

$$=S^{\eta}_{H^*_{1,0}}(\alpha,\beta)=S^{\eta}_{J_{0,1}}(\alpha,\beta)=S^{\eta}_{J^*_{0,1}}(\alpha,\beta)$$

$$=S^{\eta}_{P_{0,1}}(\alpha,\beta)=S^{\eta}_{Q_{1,0}}(\alpha,\beta)$$

$$=1-\left[\frac{1}{2}\sum_{j=1}^{n}\omega_j(|t_{a_j}|^{\eta}+|t_{b_j}|^{\eta}+|t_{c_j}|^{\eta})\right]^{1/\eta};$$

10) $\lim_{\eta\to+\infty} S^{\eta}_{F_{0,0}}(\alpha,\beta)=\lim_{\eta\to+\infty} S^{\eta}_{G_{1,1}}(\alpha,\beta)$

$$=\lim_{\eta\to+\infty} S^{\eta}_{H_{1,0}}(\alpha,\beta)=\lim_{\eta\to+\infty} S^{\eta}_{H^*_{1,0}}(\alpha,\beta)$$

$$=\lim_{\eta\to+\infty} S^{\eta}_{J_{0,1}}(\alpha,\beta)=\lim_{\eta\to+\infty} S^{\eta}_{J^*_{0,1}}(\alpha,\beta)$$

$$=\lim_{\eta\to+\infty} S^{\eta}_{P_{0,1}}(\alpha,\beta)=\lim_{\eta\to+\infty} S^{\eta}_{Q_{1,0}}(\alpha,\beta)$$

$$=1-\max_j\{|t_{a_j}|,|t_{b_j}|,|t_{c_j}|\}.$$

Both 9) and 10) were developed by Xu & Chen (2008).

2.3 Application to Pattern Recognition

In this section, several examples (Mitchell 2003, Liang & Shi 2003, Huang & Yang 2004, 2007, 2008a, 2008b) related to pattern recognition are given to compare the developed similarity measures and some existing ones (for computational convenience, let $\omega=(1/n, 1/n,\ldots,1/n)^{\mathrm{T}}$.

Example 1 Assume that there are three patterns denoted by IFVs in $X=\{x_1,x_2,x_3\}$:

$$A_1=\{(x_1,0.3,0.3),(x_2,0.2,0.2),(x_3,0.1,0.1)\};$$

$$A_2=\{(x_1,0.2,0.2),(x_2,0.2,0.2),(x_3,0.2,0.2)\};$$

$$A_3=\{(x_1,0.4,0.4),(x_2,0.4,0.4),(x_3,0.4,0.4)\}.$$

and a sample is given as:

$$B=\{(x_1,0.3,0.3),(x_2,0.2,0.2),(x_1,0.1,0.1)\}$$

Using the developed measures proposed in Section 2.2, we get

$$S^5_{D_{0.2}}(A_1,B)=1,\ S^5_{D_{0.2}}(A_2,B)=0.9600,$$
$$S^5_{D_{0.2}}(A_3,B)=0.8800,$$

$$S^5_{F_{0.2,0.1}}(A_1,B)=1,\ S^5_{F_{0.2,0.1}}(A_2,B)=0.9176,$$
$$S^5_{F_{0.2,0.1}}(A_3,B)=0.7527,$$

$$S^5_{G_{0.7,0.8}}(A_1,B)=1,\ S^5_{G_{0.7,0.8}}(A_2,B)=0.9118,$$
$$S^5_{G_{0.7,0.8}}(A_3,B)=0.7355,$$

$$S^5_{H_{0.8,0.3}}(A_1,B)=1,\ S^5_{H_{0.8,0.3}}(A_2,B)=0.9286,$$

$$S^5_{H_{0.8,0.3}}(A_3,B)=0.7857,$$

$$S^5_{D_{0.2}}(A_2,B)=0.8867,\ S^5_{H^*_{0.9,0.2}}(A_2,B)=0.9103,$$

$$S^5_{H^*_{0.9,0.2}}(A_3,B)=0.7310,$$

$$S^5_{J_{0.1,0.6}}(A_1,B)=1,\ S^5_{J_{0.1,0.6}}(A_2,B)=0.9176,$$

$$S^5_{J_{0.1,0.6}}(A_3,B)=0.7527,$$

$$S^5_{J^*_{0.2,0.7}}(A_1,B)=1,\ S^5_{J^*_{0.2,0.7}}(A_2,B)=0.9201,$$

$$S^5_{J^*_{0.2,0.7}}(A_3,B)=0.7603,$$

$$S^5_{J_{0.1,0.6}}(A_1,B)=0.8880,S^5_{P_{0.1,0.9}}(A_2,B)=0.8825,$$

$$S^5_{P_{0.1,0.9}}(A_3,B)=0.6475,$$

$$S^5_{Q_{0.8,0.1}}(A_1,B)=1,\ S^5_{Q_{0.8,0.1}}(A_2,B)=0.8825,$$

$$S^5_{Q_{0.8,0.1}}(A_3,B)=0.6475.$$

It's obvious that the sample B belongs to the pattern A_1 according to the maximum degree of similarity. The same results can be obtained using some existing similarity measures (Mitchell 2003, Liang & Shi 2003, Huang & Yang 2004, 2007, 2008a, 2008b).

Example 2 Assume that there are two patterns denoted by IFVs in $X=\{x_1,x_2,x_3\}$:

$$A_1=\{(x_1,0.2,0.2),(x_2,0.2,0.2),(x_3,0.2,0.2)\};$$

$$A_2=\{(x_1,0.4,0.4),(x_2,0.4,0.4),(x_3,0.4,0.4)\}.$$

and a sample is given as:

$$B=\{(x_1,0.3,0.3),(x_2,0.3,0.3),(x_1,0.1,0.3)\}$$

Then, using the developed measures proposed in Section 2.2, we can obtain

$$S^5_{D_{0.2}}(A_1,B)=0.9267,\ S^5_{D_{0.2}}(A_2,B)=0.8867,$$

$$S^5_{F_{0.2,0.1}}(A_1,B)=0.8842,\ S^5_{F_{0.2,0.1}}(A_2,B)=0.8319,$$

$$S^5_{G_{0.7,0.8}}(A_1,B)=0.8866,S^5_{G_{0.7,0.8}}(A_2,B)=0.8245,$$

$$S^5_{H_{0.8,0.3}}(A_1,B)=0.8978,S^5_{H_{0.8,0.3}}(A_2,B)=0.8516,$$

$$S^5_{H^*_{0.9,0.2}}(A_1,B)=0.8789,S^5_{H^*_{0.9,0.2}}(A_2,B)=0.8156,$$

$$S^5_{J_{0.1,0.6}}(A_1,B)=0.8880,\ S^5_{J_{0.1,0.6}}(A_2,B)=0.8189,$$

$$S^5_{J^*_{0.2,0.7}}(A_1,B)=0.8917,\ S^5_{J^*_{0.2,0.7}}(A_2,B)=0.8301,$$

$$S^5_{P_{0.1,0.9}}(A_1,B)=0.8492,\ S^5_{P_{0.1,0.9}}(A_2,B)=0.7614,$$

$$S^5_{Q_{0.8,0.1}}(A_1,B)=0.8492,\ S^5_{Q_{0.8,0.1}}(A_2,B)=0.7614.$$

Obviously, the sample B belongs to the pattern A_1 by using the maximum degree of similarity. The same results can be got by some existing similarity measures (Mitchell 2003, Liang & Shi 2003, Huang & Yang 2004, 2007, 2008a, 2008b).

Example 3 Assume that there are three patterns denoted with IFVs in $X=\{x_1,x_2,x_3\}$:

$$A_1=\{(x_1,0.1,0.1),(x_2,0.5,0.1),(x_3,0.1,0.9)\};$$

$$A_2=\{(x_1,0.5,0.5),(x_2,0.7,0.3),(x_3,0.0,0.8)\};$$

$$A_3=\{(x_1,0.7,0.2),(x_2,0.1,0.8),(x_3,0.4,0.4)\}.$$

and a sample is given as:

$$B=\{(x_1,0.4,0.4),(x_2,0.6,0.2),(x_1,0.0,0.8)\}$$

Then using the developed measures proposed in Section 2.2, we can obtain

$$S^5_{D_{0.2}}(A_1,B)=0.9000,\ S^5_{D_{0.2}}(A_2,B)=0.9600,$$

$$S^5_{D_{0.2}}(A_3,B)=0.6000,$$

$$S^5_{F_{0.2,0.1}}(A_1,B)=0.7939,\ S^5_{F_{0.2,0.1}}(A_2,B)=0.9176,$$

$$S^5_{F_{0.2,0.1}}(A_3,B)=0.5954,$$

$$S^5_{G_{0.7,0.8}}(A_1,B)=0.7796,\ S^5_{G_{0.7,0.8}}(A_2,B)=0.9118,$$

$$S^5_{G_{0.7,0.8}}(A_3,B)=0.6907,$$

$$S^5_{H_{0.8,0.3}}(A_1,B)=0.8214,\ S^5_{H_{0.8,0.3}}(A_2,B)=0.9286,$$

$$S^5_{H_{0.8,0.3}}(A_3,B)=0.6280,$$

$$S^5_{H^*_{0.9,0.2}}(A_1,B)=0.7759,\ S^5_{H^*_{0.9,0.2}}(A_2,B)=0.9103,$$

$$S^5_{H^*_{0.9,0.2}}(A_3,B)=0.6170,$$

$$S^5_{J_{0.1,0.6}}(A_1,B)=0.7939,\ S^5_{J_{0.1,0.6}}(A_2,B)=0.9176,$$

$$S^5_{J_{0.1,0.6}}(A_3,B)=0.6436,$$

$S^5_{J^*_{0.2,0.7}}(A_1,B)=0.8002$, $S^5_{J^*_{0.2,0.7}}(A_2,B)=0.9201$,

$S^5_{J^*_{0.2,0.7}}(A_3,B)=0.6554$,

$S^5_{P_{0.1,0.9}}(A_1,B)=0.7317$, $S^5_{P_{0.1,0.9}}(A_2,B)=0.8825$,

$S^5_{P_{0.1,0.9}}(A_3,B)=0.6033$,

$S^5_{Q_{0.8,0.1}}(A_1,B)=0.7317$, $S^5_{Q_{0.8,0.1}}(A_2,B)=0.8825$,

$S^5_{Q_{0.8,0.1}}(A_3,B)=0.6033$.

It seems that the sample B belongs to the pattern A_2 which is also the result obtained by using some existing similarity measures (Mitchell 2003, Liang & Shi 2003, Huang & Yang 2004, 2007, 2008a, 2008b).

From the above analysis, we can conclude that the proposed similarity measures have some advantages:

1) They consider all the membership degree, the non-membership degree and the indeterminacy degree of an IFV while some existing ones only consider the membership degree and the non-membership degree of an IFV (Mitchell 2003, Liang & Shi 2003, Huang & Yang 2004, 2007, 2008b). Three-parameter formulas are not only formally correct but also consistent with the essence of IFVs. Szmidt & Kacprzyk (2006) discussed why we should take into account all three functions when calculating distances between IFSs.

2) They can get more information from the original one, while the existing ones don't have this property (Mitchell 2003, Liang & Shi 2003, Huang & Yang 2004, 2007, 2008a, 2008b, Xu & Chen 2008). If we make a good decision, we should consider all possible situations, such as pessimistic or optimistic or natural ones. For example, we can find that $S^\eta_{H_{\kappa,\lambda}}(\alpha,\beta)$, $S^\eta_{H^*_{\kappa,\lambda}}(\alpha,\beta)$ and $S^\eta_{P_{\kappa,\lambda}}(\alpha,\beta)$ can reflect the fact that optimistic decision makers estimate the decision situation positively for increasing the membership degrees of IFVs, while $S^\eta_{J_{\kappa,\lambda}}(\alpha,\beta)$, $S^\eta_{J^*_{\kappa,\lambda}}(\alpha,\beta)$, $S^\eta_{Q_{\kappa,\lambda}}(\alpha,\beta)$ can fulfill that pessimistic decision makers estimate the decision situation negatively for decreasing the membership degrees of IFVs. Chen (2010) discussed this issue in details.

3) They have many parameters providing the decision maker more choices while some existing ones only have one form (Huang and Yang 2004, 2008b). Every single one is different, and each of us views the problem from his/her particular and limited frame of reference. Thus flexibility is a very important issue in decision making, and the proposed similarity measures can be suitable for different decision makers and different situations as the parameters change.

4) They are the extended forms of some existing ones (Xu & Chen 2008).

The proposed similarity measures also have some disadvantages, for example, when the indeterminacy degree of an IFV is very small, some controlling parameters κ,λ in the similarity measures $S^\eta_{D_\kappa}, S^\eta_{F_{\kappa,\lambda}}$ are invalid. In fact, the values of the parameters can reflect the decision makers' risk preferences, so the choices of them are very subjective. The decision makers can assign them different values according to the actual situations.

3. Methods for Intuitionistic Fuzzy Group Decision Making

In this section, we apply the developed similarity measures to group decision making by giving some dependent intuitionistic fuzzy aggregation operators in which the associated weighting vectors depend on the aggregation arguments.

3.1 Dependent Aggregation Operators for IFVs

Let V be the set of all IFVs, and α_1 and $\alpha_2 \in V$, then the following operational laws are valid (Xu & Yager 2006, Xu 2007b):

1) $\alpha_1 \oplus \alpha_2 = (\mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1}\mu_{\alpha_2}, \nu_{\alpha_1}\nu_{\alpha_2})$;

2) $\alpha_1 \otimes \alpha_2 = (\mu_{\alpha_1}\mu_{\alpha_2}, \nu_{\alpha_1} + \nu_{\alpha_2} - \nu_{\alpha_1}\nu_{\alpha_2})$;

3) $\lambda\alpha_1 = (1-(1-\mu_{\alpha_1})^{\lambda}, \nu_{\alpha_1}^{\lambda}),\ \lambda > 0$;

4) $\alpha_1^{\lambda} = (\mu_{\alpha_1}^{\lambda}, 1-(1-\nu_{\alpha_1})^{\lambda}),\ \lambda > 0$.

Based on the operational laws of IFVs, some aggregation operators (Xu & Yager 2006, Xu 2007b) for IFVs can be given as:

Definition 1 An intuitionistic fuzzy weighted averaging (IFWA) operator of dimension n is a mapping IFOWA: $V^n \to V$, which has the following form:

$$\text{IFWA}_w(\alpha_1,\alpha_2,\ldots,\alpha_n) = \mathop{\oplus}_{j=1}^{n}(w_j\alpha_j) \tag{4}$$

where $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weight vector of $(\alpha_1,\alpha_2,\ldots,\alpha_m)$ with $w_j \geq 0$, $j = 1,2,\ldots,n$, and $\sum_{j=1}^{n} w_j = 1$.

Definition 2 An intuitionistic fuzzy weighted geometric (IFWG) operator of dimension n is a mapping IFWG: $V^n \to V$, which has the following form:

$$\text{IFWG}_w(\alpha_1,\alpha_2,\ldots,\alpha_n) = \mathop{\otimes}_{j=1}^{n}\alpha_j^{w_j} \tag{5}$$

where $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weight vector of $(\alpha_1,\alpha_2,\ldots,\alpha_m)$ with $w_j \geq 0$, $j = 1,2,\ldots,n$, and $\sum_{j=1}^{n} w_j = 1$.

Motivated by Xu (2008), we develop some intuitionistic fuzzy dependent operators:

Definition 3 Let $(\alpha_1,\alpha_2,...,\alpha_n)$ be a collection of IFVs. An intuitionistic fuzzy dependent averaging (IFDA) operator of dimension n is a mapping IFDA: $V^n \to V$, which has the following form:

$$\begin{aligned}\text{IFDA}(\alpha_1,\alpha_2,\ldots,\alpha_n) &= \mathop{\oplus}_{i=1}^{n}\left(\frac{\alpha_i S(\alpha_i,\bar{\alpha})}{\sum_{j=1}^{n} S(\alpha_j,\bar{\alpha})}\right) \\ &= \left(1-\prod_{i=1}^{n}(1-\mu_{\alpha_i})^{S(\alpha_i,\bar{\alpha})/\sum_{j=1}^{n}S(\alpha_j,\bar{\alpha})},\ \prod_{i=1}^{n}\nu_{\alpha_i}^{S(\alpha_i,\bar{\alpha})/\sum_{j=1}^{n}S(\alpha_j,\bar{\alpha})}\right)\end{aligned} \tag{6}$$

where $\bar{\alpha} = (\oplus_{j=1}^{n}\alpha_j)/n$, and $S(\alpha_j,\bar{\alpha})$ is the similarity measure of α_j and $\bar{\alpha}$.

We can easily prove that the IFDA operator is commutative, monotonic and bounded, which can be presented as:

1) If $(\alpha_{\sigma(1)},\alpha_{\sigma(2)},\ldots,\alpha_{\sigma(n)})$ is any permutation of $(\alpha_1,\alpha_2,\ldots,\alpha_n)$, then

$$\text{IFDA}(\alpha_1,\alpha_2,\cdots,\alpha_n) = \text{IFDA}(\alpha_{\sigma(1)},\alpha_{\sigma(2)},\ldots,\alpha_{\sigma(n)}) \tag{7}$$

2) Let $\alpha^- = \left(\min_j(\mu_{\alpha_j}), \max_j(\nu_{\alpha_j})\right)$ and $\alpha^+ = \left(\max_j(\mu_{\alpha_j}), \min_j(\nu_{\alpha_j})\right)$, then

$$\alpha^- \leq \text{IFDA}(\alpha_1,\alpha_2,\ldots,\alpha_n) \leq \alpha^+ \tag{8}$$

3) If all the arguments are the same, then

$$\text{IFDA}(\alpha,\alpha,\ldots,\alpha) = \alpha \tag{9}$$

Definition 4 Let $(\alpha_1,\alpha_2,...,\alpha_n)$ be a collection of IFVs. An intuitionistic fuzzy dependent geometric (IFDG) operator of dimension n is a mapping IFDG: $V^n \to V$, which has the following form:

$$\begin{aligned}\text{IFDG}(\alpha_1,\alpha_2,\ldots,\alpha_n) &= \overset{n}{\underset{i=1}{\otimes}} \alpha_i^{S(\alpha_i,\hat{\alpha})/\sum_{j=1}^{n} S(\alpha_j,\hat{\alpha})} \\ &= \left(\prod_{i=1}^{n} \mu_{\alpha_i}^{S(\alpha_i,\hat{\alpha})/\sum_{j=1}^{n} S(\alpha_j,\hat{\alpha})}, \right. \\ &\qquad \left. 1-\prod_{i=1}^{n} (1-\nu_{\alpha_i})^{S(\alpha_i,\hat{\alpha})/\sum_{j=1}^{n} S(\alpha_j,\hat{\alpha})} \right) \end{aligned} \tag{10}$$

where $\hat{\alpha} = (\otimes_{j=1}^{n} \alpha_j)/n$, and $S_f(\alpha_j, \hat{\alpha})$ is the similarity measure f of α_j and $\hat{\alpha}$.

Similar to the IFDA operator (6), the IFDG operator (10) is also commutative, monotonic and bounded.

By comparing the developed aggregation operators with the existing ones, we can find that the weight vectors of these operators only depend on the aggregation arguments, and any permutation of the same collection of IFVs can get the same aggregation result, which avoids some objective influence produced while the weight vectors of some aggregation operators should be given by the decision makers or by using some methods having little relation to the aggregation arguments.

3.2 Decision Making Methods Based on the Dependent Operators for IFVs

In this section, we apply the developed similarity formulas and the aggregation operators for IFVs to group decision making, which can be described as follows:

There are h decision makers to express their information about m alternatives under n criteria by using the IFVs $\alpha_{ij}^{(k)} = (\mu_{ij}^{(k)}, \nu_{ij}^{(k)})$ constructing the decision matrix $D^{(k)} = (\alpha_{ij}^{(k)})_{m\times n}$ $(k = 1,2,\ldots,h)$. Let $d = (d_1, d_2,\ldots,d_h)$ be the set of decision makers, $A = (A_1, A_2,\ldots, A_m)$ the set of alternatives, $G = (G_1, G_2,\ldots,G_n)$ the set of criteria, $w = (w_1, w_2,\ldots, w_n)^{\mathrm{T}}$ the weight vector of the criteria. In order to get the optimal alternative, the following steps are suggested:

(Method I)

Step 1 Utilize the IFWA operator:

$$z_i^{(k)} = \overset{n}{\underset{j=1}{\oplus}} (w_j \alpha_{ij}^{(k)}) \tag{11}$$

or the IFWG operator:

$$z_i^{(k)} = \overset{n}{\underset{j=1}{\otimes}} (\alpha_{ij}^{(k)})^{w_j} \tag{12}$$

to aggregate the IFVs for the alternative $A_i^{(k)}$ given by k th decision maker.

Step 2 Utilize the IFDA operator:

$$\begin{aligned} z_i &= \text{IFDA}(z_i^{(1)}, z_i^{(2)},\cdots, z_i^{(h)}) \\ &= \overset{h}{\underset{k=1}{\oplus}} \left(\frac{z_i^{(k)} S(z_i^{(k)}, \bar{z}_i)}{\sum_{k=1}^{h} S(z_i^{(k)}, \bar{z}_i)} \right) \\ &= \left(1-\prod_{k=1}^{h} (1-\mu_{z_i^{(k)}})^{S(z_i^{(k)},\bar{z}_i)/\sum_{k=1}^{h} S(z_i^{(k)},\bar{z}_i)}, \right. \\ &\qquad \left. \prod_{k=1}^{h} \nu_{z_i^{(k)}}^{S(z_i^{(k)},\bar{z}_i)/\sum_{k=1}^{h} S(z_i^{(k)},\bar{z}_i)} \right) \end{aligned} \tag{13}$$

or the IFDG operator:

$$\begin{aligned} z_i &= \text{IFDG}(z_i^{(1)}, z_i^{(2)},\cdots, z_i^{(h)}) \\ &= \overset{h}{\underset{k=1}{\otimes}} (z_i^{(k)})^{S(z_i^{(k)},\bar{z}_i)/\sum_{k=1}^{h} S(z_i^{(k)},\bar{z}_i)} \\ &= \left(\prod_{k=1}^{h} \mu_{z_i^{(k)}}^{S(z_i^{(k)},\hat{z}_i)/\sum_{k=1}^{h} S(z_i^{(k)},\hat{z}_i)}, \right. \\ &\qquad \left. 1-\prod_{k=1}^{h} (1-\nu_{z_i^{(k)}})^{S(z_i^{(k)},\bar{z}_i)/\sum_{k=1}^{h} S(z_i^{(k)},\hat{z}_i)} \right) \end{aligned} \tag{14}$$

to get the collective values for the alternative

A_i, where $\overline{z}_i = (\oplus_{k=1}^{h} z_i^{(k)})/h$ and $\hat{z}_i = (\otimes_{k=1}^{h} z_i^{(k)})/h$.

Step 3 Rank the alternatives by z_i $(i = 1, 2, \cdots, m)$ using the comparing laws of IFVs.

To illustrate Method Ⅰ, in what follows, a practical group decision making problem involving the prioritization of a set of information technology improvement projects (Ngwenyama & Bryson 1999) is given:

Example 4 The information management steering committee of Midwest American Manufacturing Corp. must prioritize for development and implementation a set of six information technology improvement projects A_j $(j = 1, 2, \cdots, 6)$, which have been proposed by area managers. The committee is concerned that the projects are prioritized from highest to lowest potential contribution to the firm's strategic goal of gaining competitive advantage in the industry. In assessing the potential contribution of each project, three factors are considered, G_1: productivity, G_2: differentiation, and G_3: management, whose weight vector is $\omega = (0.35, 0.35, 0.3)^{\mathrm{T}}$. The productivity factor assesses the potential of a proposed project to increase the effectiveness and efficiency of the firm's manufacturing and service operations. The differentiation factor assesses the potential of a proposed project to fundamentally differentiate the firm's products and services from its competitors, and to make them more desirable to its customers. The management factor assesses the potential of a proposed project to assist management in improving their planning, controlling and decision-making activities. The following is the list of proposed information systems projects:

1) A_1: quality management information.
2) A_2: inventory control.
3) A_3: customer order tracking.
4) A_4: materials purchasing management.
5) A_5: fleet management.
6) A_6: design change management.

Suppose that there are four decision makers d_k $(k = 1, 2, 3, 4)$. They provide their preferences with IFVs over the projects A_j $(j = 1, 2, \cdots, 6)$ with respect to the factors G_i $(i = 1, 2, 3)$, which are listed respectively in Tables 1 − 4.

To get the optimal alternative, the following steps are given:

Step 1 Utilize the IFWA operator (11) to aggregate the IFVs for the alternative $A_j^{(k)}$ given by k th decision maker.

$z_1^{(1)} = (0.4798, 0.1569)$, $z_2^{(1)} = (0.4059, 0.1569)$,

$z_3^{(1)} = (0.4104, 0.1872)$, $z_4^{(1)} = (0.3778, 0.1808)$,

$z_5^{(1)} = (0.4371, 0.2656)$, $z_6^{(1)} = (0.3480, 0.2000)$,

Table 1 Intuitionistic fuzzy decision matrix provided by d_1

	A_1	A_2	A_3	A_4	A_5	A_6
G_1	(0.3, 0.2)	(0.5, 0.1)	(0.4,0.3)	80.3,0.1)	(0.4, 0.3)	(0.5, 0.4)
G_2	(0.6,0.1)	(0.3, 0.2)	(0.5, 0.2)	(0.5,0.3)	(0.5,0.3)	(0.2, 0.1)
G_3	(0.5,0.2)	(0.4,0.2)	(0.3, 0.1)	(0.3,0.2)	(0.4,0.2)	(0.3,0.2)

Table 2 Intuitionistic fuzzy decision matrix provided by d_2

	A_1	A_2	A_3	A_4	A_5	A_6
G_1	(0.5,0.3)	(0.3, 0.1)	(0.3,0.4)	(0.5, 0.3)	(0.5, 0.3)	(0.5, 0.3)
G_2	(0.2,0.1)	(0.5, 0.3)	(0.4,0.3)	(0.6,0.3)	(0.3, 0.2)	(0.4,0.3)
G_3	(0.3,0.3)	(0.4,0.2)	(0.3, 0.1)	(0.5,0.2)	(0.3,0.2)	(0.2, 0.1)

Table 3 Intuitionistic fuzzy decision matrix provided by d_3

	A_1	A_2	A_3	A_4	A_5	A_6
G_1	(0.4,0.2)	(0.4,0.1)	(0.2,0.2)	(0.5, 0.4)	(0.6, 0.3)	(0.4,0.2)
G_2	(0.5,0.1)	(0.6,0.3)	(0.3,0.1)	(0.6,0.2)	(0.5,0.2)	(0.3,0.1)
G_3	(0.5,0.3)	(0.5,0.2)	(0.5,0.3)	(0.3,0.1)	(0.6,0.2)	(0.5,0.1)

Table 4 Intuitionistic fuzzy decision matrix provided by d_4

	A_1	A_2	A_3	A_4	A_5	A_6
G_1	(0.3,0.1)	(0.5,0.2)	(0.6,0.1)	(0.3, 0.2)	(0.4, 0.3)	(0.3,0.1)
G_2	(0.5,0.4)	(0.4,0.3)	(0.4,0.2)	(0.5,0.3)	(0.3,0.1)	(0.5,0.2)
G_3	(0.4,0.3)	(0.7,0.1)	(0.2,0.1)	(0.3,0.2)	(0.2,0.2)	(0.4,0.3)

$z_1^{(2)} = (0.3480, 0.2042)$, $z_2^{(2)} = (0.4059, 0.1808)$,
$z_3^{(2)} = (0.3368, 0.2386)$, $z_4^{(2)} = (0.5376, 0.2656)$,
$z_5^{(2)} = (0.3778, 0.2305)$, $z_6^{(2)} = (0.3864, 0.2158)$,
$z_1^{(3)} = (0.4671, 0.1772)$, $z_2^{(3)} = (0.5071, 0.1808)$,
$z_3^{(3)} = (0.3369, 0.1772)$, $z_4^{(3)} = (0.4884, 0.2071)$,
$z_5^{(3)} = (0.5675, 0.2305)$, $z_6^{(3)} = (0.4004, 0.1275)$,
$z_1^{(4)} = (0.4059, 0.2259)$, $z_2^{(4)} = (0.5428, 0.1872)$,
$z_3^{(4)} = (0.4325, 0.1275)$, $z_4^{(4)} = (0.3778, 0.2305)$,
$z_5^{(4)} = (0.3097, 0.1808)$, $z_6^{(4)} = (0.4059, 0.1772)$.

Step 2 Utilize the IFDA operator (13) to calculate the collective values z_j for the alternative A_j (here, let $S = S^1_{F_{0.4,0.1}}$):

$z_1 = (0.4279, 0.1892)$, $z_2 = (0.4691, 0.1760)$,
$z_3 = (0.3807, 0.1782)$, $z_4 = (0.4497, 0.2186)$,
$z_5 = (0.4309, 0.2253)$, $z_6 = (0.3856, 0.1769)$.

Step 3 Get the priority of the alternatives by comparing the score and accuracy degree of z_j $(j = 1, 2, \cdots, 6)$.

$$A_2 \succ A_1 \succ A_4 \succ A_6 \succ A_5 \succ A_3$$

It is pointed that the parameters S_f, η, κ and λ can be given different values according to different actual situations.

4. Some New Similarity Formulas for IVIFVs

From the literatures review, we can see that a lot of similarity measures for IFVs have been developed recently, but little has been done about the interval-valued intuitionistic fuzzy values (IVIFVs). In this section, we extend the results in Section 2 and get some new conclusions about the similarity measures of IVIFVs.

4.1 Some Point Operators for IVIFVs

Atanassov & Gargov (1989) generalized IFS and defined an interval-valued intuitionistic fuzzy set (IVIFS) as $\tilde{A} = \{< x_i, \mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)$

$>|x_i \in X\}$, in which the membership degree $\mu_{\tilde{A}}(x_i) \subset [0,1]$ and the non-membership degree $\nu_{\tilde{A}}(x_i) \subset [0,1]$ are intervals, which satisfy $\sup \tilde{\mu}_{\tilde{A}}(x_i) + \sup \tilde{\nu}_{\tilde{A}}(x_i) \le 1$, for every $x_i \in X$. $\pi_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x)$ is called the degree of indeterminacy of x to A. Xu & Chen (2007) called the pair $(\mu_{\tilde{\alpha}_r}(x_i), \nu_{\tilde{\alpha}_r}(x_i))$ an interval-valued intuitionistic fuzzy value (IVIFV), and for convenience, denoted an IVIFV by $\tilde{\alpha}_r = ([\mu^-_{\tilde{\alpha}_r}, \mu^+_{\tilde{\alpha}_r}], [\nu^-_{\tilde{\alpha}_r}, \nu^+_{\tilde{\alpha}_r}])$, where $[\mu^-_{\tilde{\alpha}_r}, \mu^+_{\tilde{\alpha}_r}] \subset [0,1]$, $[\nu^-_{\tilde{\alpha}_r}, \nu^+_{\tilde{\alpha}_r}] \subset [0,1]$, $\mu^+_{\tilde{\alpha}_r} + \nu^+_{\tilde{\alpha}_r} \le 1$, and $\pi_{\alpha_r} = [\pi^-_{\alpha_r}, \pi^+_{\alpha_r}] = [1 - \mu^+_{\alpha_r} - \nu^+_{\alpha_r}, 1 - \mu^-_{\alpha_r} - \nu^-_{\alpha_r}]$.

Similar to the comparison method of IFVs, Xu & Chen (2007) introduced a method for comparing any two IVIFVs $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ as follows:

Let $s_{\tilde{\alpha}_i} = (\mu^-_{\tilde{\alpha}_i} - \mu^-_{\tilde{\alpha}_i} + \mu^+_{\tilde{\alpha}_i} - \nu^+_{\tilde{\alpha}_i})/2 \quad (i = 1,2)$ be the scores of $\tilde{\alpha}_i \, (i=1,2)$ respectively, and $h_{\tilde{\alpha}_i} = (\mu^-_{\tilde{\alpha}_i} + \nu^-_{\tilde{\alpha}_i} + \mu^+_{\tilde{\alpha}_i} + \nu^+_{\tilde{\alpha}_i})/2 \quad (i=1,2)$ be the accuracy degrees of $\tilde{\alpha}_i \, (i=1,2)$, respectively, then

1) If $s_{\tilde{\alpha}_1} > s_{\tilde{\alpha}_2}$, then $\tilde{\alpha}_1$ is larger than $\tilde{\alpha}_2$, denoted by $\tilde{\alpha}_1 > \tilde{\alpha}_2$;

2) If $s_{\tilde{\alpha}_1} = s_{\tilde{\alpha}_2}$, then

a) If $h_{\tilde{\alpha}_1} = h_{\tilde{\alpha}_2}$, then there is no difference between $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, denoted by $\tilde{\alpha}_1 \sim \tilde{\alpha}_2$;

b) If $h_{\tilde{\alpha}_1} > h_{\tilde{\alpha}_2}$, then $\tilde{\alpha}_1$ is larger than $\tilde{\alpha}_2$, denoted by $\tilde{\alpha}_1 > \tilde{\alpha}_2$.

For an IVIFV $\tilde{\alpha}_r = ([\mu^-_{\tilde{\alpha}_r}, \mu^+_{\tilde{\alpha}_r}], [\nu^-_{\tilde{\alpha}_r}, \nu^+_{\tilde{\alpha}_r}])$, let $\kappa, \lambda \in [0,1]$, motivated by Atanossov (1995), we define some IVIFV operators: IVIF $\to$ IVIFV as follows:

1) $D_\kappa(\tilde{\alpha}_r) = ([\mu^-_{\tilde{\alpha}_r} + \kappa\pi^-_{\tilde{\alpha}_r}, \mu^+_{\tilde{\alpha}_r} + \kappa\pi^-_{\tilde{\alpha}_r}], [\nu^-_{\tilde{\alpha}_r} + (1-\kappa)\pi^-_{\tilde{\alpha}_r}, \nu^+_{\tilde{\alpha}_r} + (1-\kappa)\pi^-_{\tilde{\alpha}_r}])$;

2) For $\kappa + \lambda \le 1$, $F_{\kappa,\lambda}(\tilde{\alpha}_r) = ([\mu^-_{\tilde{\alpha}_r} + \kappa\pi^-_{\tilde{\alpha}_r}, \mu^+_{\tilde{\alpha}_r} + \kappa\pi^-_{\tilde{\alpha}_r}], [\nu^-_{\tilde{\alpha}_r} + \lambda\pi^-_{\tilde{\alpha}_r}, \nu^+_{\tilde{\alpha}_r} + \lambda\pi^-_{\tilde{\alpha}_r}])$;

3) $G_{\kappa,\lambda}(\tilde{\alpha}_r) = ([\kappa\mu^-_{\tilde{\alpha}_r}, \kappa\mu^+_{\tilde{\alpha}_r}], [\lambda\nu^-_{\tilde{\alpha}_r}, \lambda\nu^+_{\tilde{\alpha}_r}])$;

4) $H_{\kappa,\lambda}(\tilde{\alpha}_r) = ([\kappa\mu^-_{\tilde{\alpha}_r}, \kappa\mu^+_{\tilde{\alpha}_r}], [\nu^-_{\tilde{\alpha}_r} + \lambda\pi^-_{\tilde{\alpha}_r}, \nu^+_{\tilde{\alpha}_r} + \lambda\pi^-_{\tilde{\alpha}_r}])$;

5) $H^*_{\kappa,\lambda}(\tilde{\alpha}_r) = ([\kappa\mu^-_{\tilde{\alpha}_r}, \kappa\mu^+_{\tilde{\alpha}_r}], [\nu^-_{\tilde{\alpha}_r} + \lambda(1 - \kappa\mu^+_{\tilde{\alpha}_r} - \nu^+_{\tilde{\alpha}_r}), \nu^+_{\tilde{\alpha}_r} + \lambda(1 - \kappa\mu^+_{\tilde{\alpha}_r} - \nu^+_{\tilde{\alpha}_r})])$;

6) $J_{\kappa,\lambda}(\tilde{\alpha}_r) = ([\mu^-_{\tilde{\alpha}_r} + \kappa\pi^-_{\tilde{\alpha}_r}, \mu^+_{\tilde{\alpha}_r} + \kappa\pi^-_{\tilde{\alpha}_r}], [\lambda\nu^-_{\tilde{\alpha}_r}, \lambda\nu^+_{\tilde{\alpha}_r}])$;

7) $J^*_{\kappa,\lambda}(\tilde{\alpha}_r) = ([\mu^-_{\tilde{\alpha}_r} + \kappa(1 - \mu^+_{\tilde{\alpha}_r} - \lambda\nu^+_{\tilde{\alpha}_r}), \mu^+_{\tilde{\alpha}_r} + \kappa(1 - \mu^+_{\tilde{\alpha}_r} - \lambda\nu^+_{\tilde{\alpha}_r})], [\lambda\nu^-_{\tilde{\alpha}_r}, \lambda\nu^+_{\tilde{\alpha}_r}])$;

8) For $\kappa + \lambda \le 1$, $P_{\kappa,\lambda}(\tilde{\alpha}_r) = ([\max(\kappa, \mu^-_{\tilde{\alpha}_r}), \max(\kappa, \mu^+_{\tilde{\alpha}_r})], [\min(\lambda, \nu^-_{\tilde{\alpha}_r}), \min(\lambda, \nu^+_{\tilde{\alpha}_r})])$;

9) For $\kappa + \lambda \le 1$, $Q_{\kappa,\lambda}(\tilde{\alpha}_r) = ([\min(\kappa, \mu^-_{\tilde{\alpha}_r}), \min(\kappa, \mu^+_{\tilde{\alpha}_r})], [\max(\lambda, \nu^-_{\tilde{\alpha}_r}), \max(\lambda, \nu^+_{\tilde{\alpha}_r})])$.

4.2 Some New Similarity Formulas for IVIFVs

Let $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n)$ and $\tilde{\beta} = (\tilde{\beta}_1, \tilde{\beta}_2, \cdots, \tilde{\beta}_n)$ be two collections of IVIFVs, and let $t^-_{a_j} = \mu^-_{\alpha_j} - \mu^-_{\beta_j}$, $t^+_{a_j} = \mu^+_{\alpha_j} - \mu^+_{\beta_j}$, $t^-_{b_j} = \nu^-_{\alpha_j} - \nu^-_{\beta_j}$, $t^+_{b_j} = \nu^+_{\alpha_j} - \nu^+_{\beta_j}$, $t^-_{c_j} = \pi^-_{\alpha_j} - \pi^-_{\beta_j} = 1 - \mu^+_{\alpha_j} - \nu^+_{\alpha_j} - 1 + \mu^+_{\beta_j} + \nu^+_{\beta_j} = -t^+_{a_j} - t^+_{b_j}$, $t^+_{c_j} = \pi^+_{\alpha_j} - \pi^+_{\beta_j} = 1 - \mu^-_{\alpha_j} - \nu^-_{\alpha_j} - 1 + \mu^-_{\beta_j} + \nu^-_{\beta_j} = -t^-_{a_j} - t^-_{b_j}$ $(j = 1,2,\cdots,n)$, $\kappa, \lambda \in [0,1]$, then a series of similarity measures for $\tilde{\alpha}$ and $\tilde{\beta}$ IVIFVs can be developed:

1) $S^{\eta}_{D_\kappa}(\tilde{\alpha},\tilde{\beta}) = 1-\left[\frac{1}{4}\sum_{j=1}^{n}\omega_j(|t^-_{a_j}-\kappa(t^+_{a_j}+t^+_{b_j})|^\eta+|t^-_{b_j}-(1-\kappa)(t^+_{a_j}+t^+_{b_j})|^\eta+2|(1-\kappa)t^+_{a_j}-\kappa t^+_{b_j}|^\eta + |t^-_{a_j}+t^-_{b_j}-t^+_{a_j}-t^+_{b_j}|^\eta)\right]^{1/\eta}$;

2) For $\kappa+\lambda\le 1$,

$$S^{\eta}_{F_{\kappa,\lambda}}(\tilde{\alpha},\tilde{\beta}) = 1-\left[\frac{1}{4}\sum_{j=1}^{n}\omega_j(|t^-_{a_j}-\kappa(t^+_{a_j}+t^+_{b_j})|^\eta+|(1-\kappa)t^+_{a_j}-\kappa t^+_{b_j}|^\eta+|t^-_{b_j}-\lambda(t^+_{a_j}+t^+_{b_j})|^\eta + |\lambda t^+_{a_j}-(1-\lambda)t^+_{b_j}|^\eta+|t^-_{a_j}+t^-_{b_j}-(\kappa+\lambda)(t^+_{a_j}+t^+_{b_j})|^\eta+|(1-\kappa-\lambda)(t^+_{a_j}+t^+_{b_j})|^\eta)\right]^{1/\eta};$$

3) $S^{\eta}_{G_{\kappa,\lambda}}(\tilde{\alpha},\tilde{\beta}) = 1-\left[\frac{1}{4}\sum_{j=1}^{n}\omega_j(|\kappa t^-_{a_j}|^\eta+|\kappa t^+_{a_j}|^\eta + |\lambda t^-_{b_j}|^\eta+|\lambda t^+_{b_j}|^\eta+|\kappa t^-_{a_j}+\lambda t^-_{b_j}|^\eta+|\kappa t^+_{a_j}+\lambda t^+_{b_j}|^\eta)\right]^{1/\eta}$;

4) $S^{\eta}_{H_{\kappa,\lambda}}(\tilde{\alpha},\tilde{\beta}) = 1-\left[\frac{1}{4}\sum_{j=1}^{n}\omega_j(|\kappa t^-_{a_j}|^\eta+|\kappa t^+_{a_j}|^\eta+|t^-_{b_j}-\lambda(t^+_{a_j}+t^+_{b_j})|^\eta+|\lambda t^+_{a_j}-(1-\lambda)t^+_{b_j}|^\eta + |\kappa t^-_{a_j}+t^-_{b_j}-\lambda(t^+_{a_j}+t^+_{b_j})|^\eta+|(\kappa-\lambda)t^+_{a_j}+(1-\lambda)t^+_{b_j}|^\eta)\right]^{1/\eta}$

5) $S^{\eta}_{H^*_{\kappa,\lambda}}(\tilde{\alpha},\tilde{\beta}) = 1-\left[\frac{1}{4}\sum_{j=1}^{n}\omega_j(|\kappa t^-_{a_j}|^\eta+|\kappa t^+_{a_j}|^\eta+|t^-_{b_j}-\lambda(\kappa t^+_{a_j}+t^+_{b_j})|^\eta+|\kappa\lambda t^+_{a_j}-(1-\lambda)t^+_{b_j}|^\eta + |\kappa t^-_{a_j}+t^-_{b_j}-\lambda(\kappa t^+_{a_j}+t^+_{b_j})|^\eta+|(1-\lambda)(\kappa t^+_{a_j}+t^+_{b_j})|^\eta)\right]^{1/\eta}$;

6) $S^{\eta}_{J_{\kappa,\lambda}}(\tilde{\alpha},\tilde{\beta}) = 1-\left[\frac{1}{4}\sum_{j=1}^{n}\omega_j(|t^-_{a_j}-\kappa(t^+_{a_j}+t^+_{b_j})|^\eta+|(1-\kappa)t^+_{a_j}-\kappa t^+_{b_j}|^\eta+|\lambda t^-_{b_j}|^\eta+|\lambda t^+_{b_j}|^\eta + |t^-_{a_j}+\lambda t^-_{b_j}-\kappa(t^+_{a_j}+t^+_{b_j})|^\eta+|(1-\kappa)t^+_{a_j}+(\lambda-\kappa)t^+_{b_j}|^\eta)\right]^{1/\eta}$

7) $S^{\eta}_{J^*_{\kappa,\lambda}}(\tilde{\alpha},\tilde{\beta}) = 1-\left[\frac{1}{4}\sum_{j=1}^{n}\omega_j(|t^-_{a_j}-\kappa(t^+_{a_j}+\lambda t^+_{b_j})|^\eta+|(1-\kappa)t^+_{a_j}-\kappa\lambda t^+_{b_j}|^\eta+|\lambda t^-_{b_j}|^\eta+|\lambda t^+_{b_j}|^\eta + |t^-_{a_j}+\lambda t^-_{b_j}-\kappa(t^+_{a_j}+\lambda t^+_{b_j})|^\eta+|(1-\kappa)(t^+_{a_j}+\lambda t^+_{b_j})|^\eta)\right]^{1/\eta}$;

8) For $\kappa+\lambda\le 1$, $S^{\eta}_{P_{\kappa,\lambda}}(\tilde{\alpha},\tilde{\beta}) = 1-\left[\frac{1}{4}\sum_{j=1}^{n}\omega_j(|\max(\kappa,\mu^-_{\tilde{\alpha}_j})-\max(\kappa,\mu^-_{\tilde{\beta}_j})|^\eta + |\max(\kappa,\mu^+_{\tilde{\alpha}_j})-\max(\kappa,\mu^+_{\tilde{\beta}_j})|^\eta+|\min(\lambda,\nu^+_{\tilde{\alpha}_j})-\min(\lambda,\nu^+_{\tilde{\beta}_j})|^\eta+|\max(\kappa,\mu^+_{\tilde{\alpha}_j})-\max(\kappa,\mu^+_{\tilde{\beta}_j}) + \min(\lambda,\nu^+_{\tilde{\alpha}_j})-\min(\lambda,\nu^+_{\tilde{\beta}_j})|^\eta+|\max(\kappa,\mu^-_{\tilde{\alpha}_j})-\max(\kappa,\mu^-_{\tilde{\beta}_j})+\min(\lambda,\nu^-_{\tilde{\alpha}_j})-\min(\lambda,\nu^-_{\tilde{\beta}_j})|^\eta)\right]^{1/\eta}$;

9) For $\kappa+\lambda \le 1$, $S^{\eta}_{Q_{\kappa,\lambda}}(\tilde{\alpha},\tilde{\beta}) = 1-\Big[\frac{1}{4}\sum_{j=1}^{n}\omega_j (|\min(\kappa,\mu^-_{\tilde{\alpha}_j})-\min(\kappa,\mu^-_{\tilde{\beta}_j})|^{\eta}$

$+|\min(\kappa,\mu^+_{\tilde{\alpha}_j})-\min(\kappa,\mu^+_{\tilde{\beta}_j})|^{\eta} + |\max(\lambda,\nu^+_{\tilde{\alpha}_j})-\max(\lambda,\nu^+_{\tilde{\beta}_j})|^{\eta} + |\min(\kappa,\mu^+_{\tilde{\alpha}_j})-\min(\kappa,\mu^+_{\tilde{\beta}_j})$

$+\max(\lambda,\nu^+_{\tilde{\alpha}_j})-\max(\lambda,\nu^+_{\tilde{\beta}_j})|^{\eta} + |\min(\kappa,\mu^-_{\tilde{\alpha}_j})-\min(\kappa,\mu^-_{\tilde{\beta}_j}) + \max(\lambda,\nu^-_{\tilde{\alpha}_j})-\max(\lambda,\nu^-_{\tilde{\beta}_j})|^{\eta})\Big]^{1/\eta}$;

Similar to Section 2, these similarity measures satisfy the following properties:

1) $0 \le S(\tilde{\alpha},\tilde{\beta}) \le 1$; 2) $S(\tilde{\alpha},\tilde{\beta}) = 0$, if $\tilde{\alpha}=\tilde{\beta}$; 3) $S(\tilde{\alpha},\tilde{\beta}) = S(\tilde{\beta},\tilde{\alpha})$.

where $\tilde{\alpha}=(\tilde{\alpha}_1,\tilde{\alpha}_2,\dots,\tilde{\alpha}_n)$ and $\tilde{\beta}=(\tilde{\beta}_1,\tilde{\beta}_2,\dots,\tilde{\beta}_n)$ are two collections of IVIFVs, and $S(\tilde{\alpha},\tilde{\beta})$ is the similarity measure between $\tilde{\alpha}$ and $\tilde{\beta}$.

Obviously, the followings are true:

1) $\lim_{\eta\to+\infty} S^{\eta}_{D_\kappa}(\tilde{\alpha},\tilde{\beta}) = 1-\max_j\{|t^-_{a_j}-\kappa(t^+_{a_j}+t^+_{b_j})|, |t^-_{b_j}-(1-\kappa)(t^+_{a_j}+t^+_{b_j})|, |(1-\kappa)t^+_{a_j}-\kappa t^+_{b_j}|,$

$|t^-_{a_j}+t^-_{b_j}-t^+_{a_j}-t^+_{b_j}|\}$;

2) For $\kappa+\lambda\le 1$, $\lim_{\eta\to+\infty} S^{\eta}_{F_{\kappa,\lambda}}(\tilde{\alpha},\tilde{\beta}) = 1-\max_j\{|t^-_{a_j}-\kappa(t^+_{a_j}+t^+_{b_j})|, |(1-\kappa)t^+_{a_j}-\kappa t^+_{b_j}|,$

$|t^-_{b_j}-\lambda(t^+_{a_j}+t^+_{b_j})|, |\lambda t^+_{a_j}-(1-\lambda)t^+_{b_j}|^{\eta}, |t^-_{a_j}+t^-_{b_j}-(\kappa+\lambda)(t^+_{a_j}+t^+_{b_j})|^{\eta}, |(1-\kappa-\lambda)(t^+_{a_j}+t^+_{b_j})|^{\eta}\}$;

3) $\lim_{\eta\to+\infty} S^{\eta}_{G_{\kappa,\lambda}}(\tilde{\alpha},\tilde{\beta}) = 1-\max_j\{|\kappa t^-_{a_j}|, |\kappa t^+_{a_j}|, |\lambda t^-_{b_j}|, |\lambda t^+_{b_j}|, |\kappa t^-_{a_j}+\lambda t^-_{b_j}|, |\kappa t^+_{a_j}+\lambda t^+_{b_j}|^{\eta}\}$;

4) $\lim_{\eta\to+\infty} S^{\eta}_{H_{\kappa,\lambda}}(\tilde{\alpha},\tilde{\beta}) = 1-\max_j\{|\kappa t^-_{a_j}|, |\kappa t^+_{a_j}|, |t^-_{b_j}-\lambda(t^+_{a_j}+t^+_{b_j}),| \lambda t^+_{a_j}-(1-\lambda)t^+_{b_j}|,$

$|\kappa t^-_{a_j}+t^-_{b_j}-\lambda(t^+_{a_j}+t^+_{b_j})|, |(\kappa-\lambda)t^+_{a_j}+(1-\lambda)t^+_{b_j}|\}$;

5) $\lim_{\eta\to+\infty} S^{\eta}_{H^*_{\kappa,\lambda}}(\tilde{\alpha},\tilde{\beta}) = 1-\max_j\{|\kappa t^-_{a_j}|, |\kappa t^+_{a_j}|, |t^-_{b_j}-\lambda(\kappa t^+_{a_j}+t^+_{b_j})|, |\kappa\lambda t^+_{a_j}-(1-\lambda)t^+_{b_j}|,$

$|\kappa t^-_{a_j}+t^-_{b_j}-\lambda(\kappa t^+_{a_j}+t^+_{b_j})|, |(1-\lambda)(\kappa t^+_{a_j}+t^+_{b_j})|\}$;

6) $\lim_{\eta\to+\infty} S^{\eta}_{J_{\kappa,\lambda}}(\tilde{\alpha},\tilde{\beta}) = 1-\max_j\{|t^-_{a_j}-\kappa(t^+_{a_j}+t^+_{b_j})|, |(1-\kappa)t^+_{a_j}-\kappa t^+_{b_j}|, |\lambda t^-_{b_j}|, |\lambda t^+_{b_j}|,$

$|t^-_{a_j}+\lambda t^-_{b_j}-\kappa(t^+_{a_j}+t^+_{b_j})|, |(1-\kappa)t^+_{a_j}+(\lambda-\kappa)t^+_{b_j}|\}$;

7) $\lim_{\eta\to+\infty} S^{\eta}_{J^*_{\kappa,\lambda}}(\tilde{\alpha},\tilde{\beta}) = 1-\max_j\{|t^-_{a_j}-\kappa(t^+_{a_j}+\lambda t^+_{b_j})|, |(1-\kappa)t^+_{a_j}-\kappa\lambda t^+_{b_j}|, |\lambda t^-_{b_j}|, |\lambda t^+_{b_j}|,$

$|t^-_{a_j}+\lambda t^-_{b_j}-\kappa(t^+_{a_j}+\lambda t^+_{b_j})|, |(1-\kappa)(t^+_{a_j}+\lambda t^+_{b_j})|\}$;

8) For $\kappa+\lambda\le 1$, $\lim_{\eta\to+\infty} S^{\eta}_{P_{k,\lambda}}(\tilde{\alpha},\tilde{\beta}) = 1-\max_j\{|\max(\kappa,\mu^-_{\tilde{\alpha}_j})-\max(\kappa,\mu^-_{\tilde{\beta}_j})|,$

$|\max(\kappa,\mu^+_{\tilde{\alpha}_j})-\max(\kappa,\mu^+_{\tilde{\beta}_j})|, |\min(\lambda,\nu^-_{\tilde{\alpha}_j})-\min(\lambda,\nu^-_{\tilde{\beta}_j})|, |\min(\lambda,\nu^+_{\tilde{\alpha}_j})-\min(\lambda,\nu^+_{\tilde{\beta}_j})|,$

$|\max(\kappa,\mu^+_{\tilde{\alpha}_j})-\max(\kappa,\mu^+_{\tilde{\beta}_j}) + \min(\lambda,\nu^+_{\tilde{\alpha}_j})-\min(\lambda,\nu^+_{\tilde{\beta}_j})|, |\max(\kappa,\mu^-_{\tilde{\alpha}_j})-\max(\kappa,\mu^-_{\tilde{\beta}_j})$

$+\min(\lambda,\nu_{\tilde{\alpha}_j}^-)-\min(\lambda,\nu_{\tilde{\beta}_j}^-)|\}$;

9) For $\kappa+\lambda\le 1$, $\lim_{\eta\to+\infty} Q_{P_{k,\lambda}}^{\eta}(\tilde{\alpha},\tilde{\beta})=1-\max_j\{|\min(\kappa,\mu_{\tilde{\alpha}_j}^-)-\min(\kappa,\mu_{\tilde{\beta}_j}^-)|$,

$|\min(\kappa,\mu_{\tilde{\alpha}_j}^+)-\min(\kappa,\mu_{\tilde{\beta}_j}^+)|$, $|\max(\lambda,\nu_{\tilde{\alpha}_j}^-)-\max(\lambda,\nu_{\tilde{\beta}_j}^-)|$, $|\max(\lambda,\nu_{\tilde{\alpha}_j}^+)-\max(\lambda,\nu_{\tilde{\beta}_j}^+)|$,

$|\min(\kappa,\mu_{\tilde{\alpha}_j}^+)-\min(\kappa,\mu_{\tilde{\beta}_j}^+)+\max(\lambda,\nu_{\tilde{\alpha}_j}^+)-\max(\lambda,\nu_{\tilde{\beta}_j}^+)|$, $|\min(\kappa,\mu_{\tilde{\alpha}_j}^-)-\min(\kappa,\mu_{\tilde{\beta}_j}^-)$

$+\max(\lambda,\nu_{\tilde{\alpha}_j}^-)-\max(\lambda,\nu_{\tilde{\beta}_j}^-)|\}$;

10) $S_{F_{0,0}}^{\eta}(\tilde{\alpha},\tilde{\beta})=S_{G_{1,1}}^{\eta}(\tilde{\alpha},\tilde{\beta})=S_{H_{1,0}}^{\eta}(\tilde{\alpha},\tilde{\beta})=S_{H_{1,0}^*}^{\eta}(\tilde{\alpha},\tilde{\beta})=S_{J_{0,1}}^{\eta}(\tilde{\alpha},\tilde{\beta})=S_{J_{0,1}^*}^{\eta}(\tilde{\alpha},\tilde{\beta})$

$$=S_{P_{0,1}}^{\eta}(\tilde{\alpha},\tilde{\beta})=S_{Q_{1,0}}^{\eta}(\tilde{\alpha},\tilde{\beta})=1-\left[\frac{1}{4}\sum_{j=1}^{n}\omega_j(|t_{a_j}^-|^{\eta}+|t_{a_j}^+|^{\eta}+|t_{b_j}^-|^{\eta}+|t_{b_j}^+|^{\eta}+|t_{c_j}^-|^{\eta}+|t_{c_j}^+|^{\eta})\right]^{1/\eta};$$

11) $\lim_{\eta\to+\infty} S_{F_{0,0}}^{\eta}(\tilde{\alpha},\tilde{\beta})=\lim_{\eta\to+\infty} S_{G_{1,1}}^{\eta}(\tilde{\alpha},\tilde{\beta})=\lim_{\eta\to+\infty} S_{H_{1,0}}^{\eta}(\tilde{\alpha},\tilde{\beta})=\lim_{\eta\to+\infty} S_{H_{1,0}^*}^{\eta}(\tilde{\alpha},\tilde{\beta})$

$=\lim_{\eta\to+\infty} S_{J_{0,1}}^{\eta}(\tilde{\alpha},\tilde{\beta})=\lim_{\eta\to+\infty} S_{J_{0,1}^*}^{\eta}(\tilde{\alpha},\tilde{\beta})=\lim_{\eta\to+\infty} S_{P_{0,1}}^{\eta}(\tilde{\alpha},\tilde{\beta})=\lim_{\eta\to+\infty} S_{Q_{1,0}}^{\eta}(\tilde{\alpha},\tilde{\beta})$

$=1-\max_j\{|t_{a_j}^-|,|t_{a_j}^+|,|t_{b_j}^-|,|t_{b_j}^+|,|t_{c_j}^-|,|t_{c_j}^+|\}$.

We can find that both 10) and 11) were given by Xu (2007c).

5. Methods for Interval-Valued Intuitionistic Fuzzy Group Decision Making

In this section, we shall extend our results in Section 3 to interval-valued intuitiontistic fuzzy environment.

5.1 Dependent Aggregation Operators for IVIFVs

For any two IVIFVs $\tilde{\alpha}_1=([\mu_{\tilde{\alpha}_1}^-,\mu_{\tilde{\alpha}_1}^+],[\nu_{\tilde{\alpha}_1}^-,\nu_{\tilde{\alpha}_1}^+])$ and $\tilde{\alpha}_2=([\mu_{\tilde{\alpha}_2}^-,\mu_{\tilde{\alpha}_2}^+],[\nu_{\tilde{\alpha}_2}^-,\nu_{\tilde{\alpha}_2}^+])$, the following operations can be given (Xu & Chen 2007):

1) $\tilde{\alpha}_1\oplus\tilde{\alpha}_2=([\mu_{\tilde{\alpha}_1}^-+\mu_{\tilde{\alpha}_2}^--\mu_{\tilde{\alpha}_1}^-\mu_{\tilde{\alpha}_2}^-,\ \mu_{\tilde{\alpha}_1}^++\mu_{\tilde{\alpha}_2}^+-\mu_{\tilde{\alpha}_1}^+\mu_{\tilde{\alpha}_2}^+],[\nu_{\tilde{\alpha}_1}^-\nu_{\tilde{\alpha}_2}^-,\nu_{\tilde{\alpha}_1}^+\nu_{\tilde{\alpha}_2}^+])$;

2) $\tilde{\alpha}_1\otimes\tilde{\alpha}_2=([\mu_{\tilde{\alpha}_1}^-\mu_{\tilde{\alpha}_2}^-,\mu_{\tilde{\alpha}_1}^+\mu_{\tilde{\alpha}_2}^+],\ [\nu_{\tilde{\alpha}_1}^-+\nu_{\tilde{\alpha}_2}^--\nu_{\tilde{\alpha}_1}^-\nu_{\tilde{\alpha}_2}^-,\nu_{\tilde{\alpha}_1}^++\nu_{\tilde{\alpha}_2}^+-\nu_{\tilde{\alpha}_1}^+\nu_{\tilde{\alpha}_2}^+])$;

3) $\lambda\tilde{\alpha}_1=([1-(1-\mu_{\tilde{\alpha}_1}^-)^{\lambda},1-(1-\mu_{\tilde{\alpha}_1}^+)^{\lambda}],\ [(\nu_{\tilde{\alpha}_1}^-)^{\lambda},(\nu_{\tilde{\alpha}_1}^+)^{\lambda}])$, $\lambda>0$;

4) $\tilde{\alpha}_1^{\lambda}=([(\mu_{\tilde{\alpha}_1}^-)^{\lambda},(\mu_{\tilde{\alpha}_1}^+)^{\lambda}],[1-(1-\nu_{\tilde{\alpha}_1}^-)^{\lambda},\ 1-(1-\nu_{\tilde{\alpha}_1}^+)^{\lambda}])$, $\lambda>0$.

Let $\tilde{V}$ be the set of all IVIFVs, based on the operational laws of IVIFVs, the following aggregation operators for IVIFVs can be given (Xu 2007d, Xu & Chen 2007):

Definition 5 An interval-valued intuitionistic fuzzy weighted averaging (IVIFWA) operator of dimension n is a mapping IVIFWA: $\tilde{V}^n\to\tilde{V}$, which has the following form:

$$\text{IVIFWA}_w(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)=\mathop{\oplus}_{j=1}^{n}(w_j\tilde{\alpha}_j)\qquad(15)$$

where $w=(w_1,w_2,\ldots,w_n)^{\mathrm{T}}$ is the weight vector of $(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)$ with $w_j\ge 0$, $j=1,2,\ldots,n$, and $\sum_{j=1}^{n}w_j=1$.

Definition 6 An interval-valued intuitionistic fuzzy weighted geometric (IVIFWG) operator of dimension n is a mapping IVIFWG: $\tilde{V}^n \to \tilde{V}$, which has the following form:

$$\text{IVIFWG}_w(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n) = \overset{n}{\underset{j=1}{\otimes}} \tilde{\alpha}_j^{w_j} \tag{16}$$

where $w=(w_1,w_2,\ldots,w_n)^{\mathrm{T}}$ is the weight vector of $(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)$ with $w_j \geq 0$, $j=1,2,\ldots,n$, and $\sum_{j=1}^n w_j = 1$.

In what follows, we develop some dependent operators for IVIFVs:

Definition 7 Let $(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)$ be a collection of IVIFVs. An interval-valued intuitionistic fuzzy dependent averaging (IVIFDA) operator of dimension n is a mapping IVIFDA: $\tilde{V}^n \to \tilde{V}$, which has the following form:

$$\begin{aligned}
&\text{IVIFDA}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n) \\
&= \overset{n}{\underset{i=1}{\oplus}} \left(\frac{\tilde{\alpha}_i S(\tilde{\alpha}_i,\dot{\alpha})}{\sum_{j=1}^n S(\tilde{\alpha}_j,\dot{\alpha})} \right) \\
&= \left(\left[1-\prod_{i=1}^n (1-\mu_{\alpha_i}^-)^{S(\tilde{\alpha}_i,\dot{\alpha})/\sum_{j=1}^n S(\tilde{\alpha}_j,\dot{\alpha})}, \right.\right. \\
&\qquad \left. 1-\prod_{i=1}^n (1-\mu_{\alpha_i}^+)^{S(\tilde{\alpha}_i,\dot{\alpha})/\sum_{j=1}^n S(\tilde{\alpha}_j,\dot{\alpha})} \right], \\
&\qquad \left[\prod_{i=1}^n (\nu_{\alpha_i}^-)^{S(\tilde{\alpha}_i,\dot{\alpha})/\sum_{j=1}^n S(\tilde{\alpha}_j,\dot{\alpha})} \right. \\
&\qquad \left.\left. \prod_{i=1}^n (\nu_{\alpha_i}^+)^{S(\tilde{\alpha}_i,\dot{\alpha})/\sum_{j=1}^n S(\tilde{\alpha}_j,\dot{\alpha})} \right] \right)
\end{aligned} \tag{17}$$

where $\dot{\alpha} = (\oplus_{j=1}^n \tilde{\alpha}_j)/n$, and $S(\tilde{\alpha}_j,\dot{\alpha})$ is the similarity measure f of $\tilde{\alpha}_j$ and $\dot{\alpha}$.

Definition 8 Let $(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)$ be a collection of IVIFVs. An interval-valued intuitionistic fuzzy dependent geometric (IVIFDG) operator of dimension n is a mapping IVIFDG: $\tilde{V}^n \to \tilde{V}$, which has the following form:

$$\begin{aligned}
&\text{IVIFDG}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n) \\
&= \overset{n}{\underset{i=1}{\otimes}} \tilde{\alpha}_i^{S(\tilde{\alpha}_i,\ddot{\alpha})/\sum_{j=1}^n S(\tilde{\alpha}_j,\ddot{\alpha})} \\
&= \left(\left[\prod_{i=1}^n (\mu_{\alpha_i}^-)^{S(\tilde{\alpha}_i,\ddot{\alpha})/\sum_{j=1}^n S(\tilde{\alpha}_j,\ddot{\alpha})}, \right.\right. \\
&\qquad \left. \prod_{i=1}^n (\mu_{\alpha_i}^+)^{S(\tilde{\alpha}_i,\ddot{\alpha})/\sum_{j=1}^n S(\tilde{\alpha}_j,\ddot{\alpha})} \right], \\
&\qquad \left[1-\prod_{i=1}^n (1-\nu_{\alpha_i}^-)^{S(\tilde{\alpha}_i,\ddot{\alpha})/\sum_{j=1}^n S(\tilde{\alpha}_j,\ddot{\alpha})}, \right. \\
&\qquad \left.\left. 1-\prod_{i=1}^n (1-\nu_{\alpha_i}^+)^{S(\tilde{\alpha}_i,\ddot{\alpha})/\sum_{j=1}^n S(\tilde{\alpha}_j,\ddot{\alpha})} \right] \right)
\end{aligned} \tag{18}$$

where $\ddot{\alpha} = (\otimes_{j=1}^n \tilde{\alpha}_j)/n$, and $S(\tilde{\alpha}_j,\ddot{\alpha})$ is the similarity measure of $\tilde{\alpha}_j$ and $\ddot{\alpha}$.

It is easy to prove that both the IVIFDA operator (17) and the IVIFDG operator (18) are commutative, monotonic and bounded.

5.2 Decision Making Methods Based on the Dependent Operators for IVIFVs

For the decision making problem in Section 3.2, if the k th decision maker expresses his/her information about m alternatives under n criteria using the IVIFVs $\tilde{\alpha}_{ij}^{(k)} (i=1,2,\ldots,m;$ $j=1,2,\ldots,n;\ k=1,2,\ldots,h)$ to construct the decision matrix $D^{(k)}=(\tilde{\alpha}_{ij}^{(k)})_{m\times n}$ $(k=1,2,\ldots,h)$, then the following steps are given to get the optimal alternative.

(Method II)

Step 1 Utilize the IVIFDA operator:

$$\tilde{z}_i^{(k)} = \overset{n}{\underset{j=1}{\oplus}} (w_j \tilde{\alpha}_{ij}^{(k)}) \tag{19}$$

or the IVIFDG operator:

$$\tilde{z}_i^{(k)} = \underset{j=1}{\overset{n}{\otimes}} (\tilde{\alpha}_{ij}^{(k)})^{w_j} \tag{20}$$

to aggregate the IFVs for the alternative $A_i^{(k)}$ given by k th decision maker.

Step 2 Utilize the IVIFDA operator:

$$\tilde{z}_i = \text{IVIFDA}(\tilde{z}_i^{(1)}, \tilde{z}_i^{(2)}, \cdots, \tilde{z}_i^{(h)}) = \underset{k=1}{\overset{h}{\oplus}} \left(\frac{\tilde{z}_i^{(k)} S(\tilde{z}_i^{(k)}, \dot{\tilde{z}}_i)}{\sum_{k=1}^{h} S(\tilde{z}_i^{(k)}, \dot{\tilde{z}}_i)} \right)$$

$$= \left(\left[1 - \prod_{k=1}^{h} (1 - \mu^{-}_{\tilde{z}_i^{(k)}})^{S(\tilde{z}_i^{(k)}, \dot{\tilde{z}}_i) / \sum_{k=1}^{h} S(\tilde{z}_i^{(k)}, \dot{\tilde{z}}_i)}, 1 - \prod_{k=1}^{h} (1 - \mu^{-}_{\tilde{z}_i^{(k)}})^{S(\tilde{z}_i^{(k)}, \dot{\tilde{z}}_i) / \sum_{k=1}^{h} S(\tilde{z}_i^{(k)}, \dot{\tilde{z}}_i)} \right], \left[\prod_{k=1}^{h} (\nu^{-}_{\tilde{z}_i^{(k)}})^{S(\tilde{z}_i^{(k)}, \dot{\tilde{z}}_i) / \sum_{k=1}^{h} S(\tilde{z}_i^{(k)}, \dot{\tilde{z}}_i)}, \prod_{k=1}^{h} (\nu^{+}_{\tilde{z}_i^{(k)}})^{S(\tilde{z}_i^{(k)}, \dot{\tilde{z}}_i) / \sum_{k=1}^{h} S(\tilde{z}_i^{(k)}, \dot{\tilde{z}}_i)} \right] \right) \tag{21}$$

or the IVIFDG operator:

$$\tilde{z}_i = \text{IVIFDG}(\tilde{z}_i^{(1)}, \tilde{z}_i^{(2)}, \cdots, \tilde{z}_i^{(h)}) = \underset{k=1}{\overset{h}{\otimes}} (\tilde{z}_i^{(k)})^{S(\tilde{z}_i^{(k)}, \ddot{\tilde{z}}_i) / \sum_{k=1}^{h} S(\tilde{z}_i^{(k)}, \ddot{\tilde{z}}_i)}$$

$$= \left(\left[\prod_{k=1}^{h} (\mu^{-}_{\tilde{z}_i^{(k)}})^{S(\tilde{z}_i^{(k)}, \ddot{\tilde{z}}_i) / \sum_{k=1}^{h} S(\tilde{z}_i^{(k)}, \ddot{\tilde{z}}_i)}, \prod_{k=1}^{h} (\mu^{+}_{\tilde{z}_i^{(k)}})^{S(\tilde{z}_i^{(k)}, \ddot{\tilde{z}}_i) / \sum_{k=1}^{h} S(\tilde{z}_i^{(k)}, \ddot{\tilde{z}}_i)} \right], \left[1 - \prod_{k=1}^{h} (1 - \nu^{-}_{\tilde{z}_i^{(k)}})^{S(\tilde{z}_i^{(k)}, \ddot{\tilde{z}}_i) / \sum_{k=1}^{h} S(\tilde{z}_i^{(k)}, \ddot{\tilde{z}}_i)}, 1 - \prod_{k=1}^{h} (1 - \nu^{+}_{\tilde{z}_i^{(k)}})^{S(\tilde{z}_i^{(k)}, \ddot{\tilde{z}}_i) / \sum_{k=1}^{h} S(\tilde{z}_i^{(k)}, \ddot{\tilde{z}}_i)} \right] \right) \tag{22}$$

to calculate the collective values for the alternative A_i, where $\dot{\tilde{z}}_i = (\oplus_{k=1}^{h} z_i^{(k)}) / h$, and $\ddot{\tilde{z}}_i = (\otimes_{k=1}^{h} z_i^{(k)}) / h$.

Step 3 Rank the alternatives by comparing the scores and accuracy degrees of $\tilde{z}_i$ $(i = 1, 2, \cdots, m)$.

In Example 4, if the decision makers d_k $(k = 1, 2, 3, 4)$ provide their preferences with IVIFVs over the projects A_j $(j = 1, 2, \ldots, 6)$ with respect to the factors G_i $(i = 1, 2, 3)$ (see Tables 5 − 8), then we can use Method Ⅱ to get the priority of the projects.

Table 5 Interval-valued intuitionistic fuzzy decision matrix provided by d_1

	A_1	A_2	A_3	A_4	A_5	A_6
G_1	([0.2,0.3], [0.1,0.2])	([0.3,0.5], [0.2,0.3])	([0.3,0.5], [0.2,0.3])	([0.2,0.3], [0.2,0.3])	([0.2,0.4], [0.3,0.4])	([0.3,0.5], [0.1,0.3])
G_2	([0.5,0.6], [0.1,0.2])	([0.2,0.3], [0.1,0.2])	([0.4,0.5], [0.2,0.4])	([0.3,0.5], [0.1,0.3])	([0.3,0.4], [0.2,0.3])	([0.1,0.2], [0.1,0.2])
G_3	([0.4,0.5], [0.2,0.3])	([0.3,0.4], [0.1,0.2])	([0.1,0.2], [0.2,0.4])	([0.3,0.4], [0.1,0.2])	([0.2,0.3], [0.1,0.2])	([0.3,0.4], [0.2,0.3])

Table 6 Interval-valued intuitionistic fuzzy decision matrix provided by d_2

	A_1	A_2	A_3	A_4	A_5	A_6
G_1	([0.4,0.5], [0.2,0.3])	([0.2,0.3], [0.1,0.2])	([0.3,0.4], [0.2,0.3])	([0.4,0.5], [0.1,0.3])	([0.2,0.5], [0.1,0.2])	([0.4,0.5], [0.2,0.3])
G_2	([0.1,0.2], [0.1,0.2])	([0.5,0.6], [0.2,0.3])	([0.3,0.5], [0.1,0.2])	([0.5,0.6], [0.1,0.2])	([0.1,0.3], [0.2,0.4])	([0.3,0.4], [0.2,0.4])
G_3	([0.3,0.4], [0.3,0.5])	([0.2,0.4], [0.1,0.3])	([0.2,0.3], [0.1,0.2])	([0.6,0.7], [0.2,0.3])	([0.3,0.4], [0.1,0.3])	([0.2,0.3], [0.1,0.2])

Table 7 Interval-valued intuitionistic fuzzy decision matrix provided by d_3

	A_1	A_2	A_3	A_4	A_5	A_6
G_1	([0.3,0.4], [0.1,0.2])	([0.2,0.4], [0.1,0.2])	([0.2,0.3] , [0.3,0.4])	([0.4,0.5], [0.3,0.4])	([0.4,0.6], [0.1,0.2])	([0.4,0.5], 0.2,0.3])
G_2	([0.3,0.5], [0.1,0.3])	([0.5,0.7], [0.1,0.2])	([0.3,0.4], [0.2,0.3])	([0.6,0.7], [0.1,0.2])	([0.5,0.6], [0.2,0.4])	([0.2,0.3], [0.1,0.2])
G_3	([0.4,0.5], [0.2,0.3])	([0.4,0.6], [0.2,0.4])	([0.4,0.5], [0.1,0.3])	([0.3,0.4], [0.2,0.3])	([0.4,0.5], [0.3,0.4])	([0.3,0.5], [0.2,0.3])

Table 8 Interval-valued intuitionistic fuzzy decision matrix provided by d_4

	A_1	A_2	A_3	A_4	A_5	A_6
G_1	([0.2,0.3], [0.1,0.2])	([0.5,0.6], [0.1,0.2])	([0.4,0.5] , [0.1,0.3])	([0.3,0.4], [0.2,0.3])	([0.4,0.5], [0.0,0.3])	([0.3,0.5], [0.0,0.2])
G_2	([0.4,0.5], [0.3,0.4])	([0.4,0.6], [0.2,0.3])	([0.3,0.5], [0.1,0.2])	([0.4,0.5], [0.3,0.4])	([0.3,0.4], [0.1,0.2])	([0.5,0.6], [0.1,0.2])
G_3	([0.3,0.4], [0.2,0.4])	([0.6,0.7], [0.1,0.2])	([0.2,0.3], [0.1,0.2])	([0.2,0.5], 0.1,0.3])	([0.2,0.3], [0.2,0.3])	([0.3,0.4], [0.3,0.4])

Step 1 Utilize the IVIFWG operator (16) to aggregate the IVIFVs for the alternative $A_j^{(k)}$ given by k th decision maker.

$\tilde{z}_1^{(1)} = ([0.3393, 0.4457], [0.1312, 0.2314])$,
$\tilde{z}_2^{(1)} = ([0.2603, 0.3911], [0.1363, 0.2365])$,
$\tilde{z}_3^{(1)} = ([0.2386, 0.3798], [0.2000, 0.3667])$,
$\tilde{z}_4^{(1)} = ([0.2603, 0.3911], [0.1363, 0.2714])$,
$\tilde{z}_5^{(1)} = ([0.2305, 0.3669], [0.2091, 0.2044])$,
$\tilde{z}_6^{(1)} = ([0.2042, 0.3393], [0.1312, 0.2665])$,
$\tilde{z}_1^{(2)} = ([0.2259, 0.3393], [0.1991, 0.3369])$,
$\tilde{z}_2^{(2)} = ([0.2756, 0.4168], [0.1363, 0.2665])$,
$\tilde{z}_3^{(2)} = ([0.2656, 0.3967], [0.1363, 0.2365])$,
$\tilde{z}_4^{(2)} = ([0.4884, 0.5896], [0.1312, 0.2665])$,
$\tilde{z}_5^{(2)} = ([0.1772, 0.3911], [0.1363, 0.3050])$,
$\tilde{z}_6^{(2)} = ([0.2938, 0.3967], [0.1712, 0.3097])$,
$\tilde{z}_1^{(3)} = ([0.3270, 0.4624], [0.1312, 0.2665])$,
$\tilde{z}_2^{(3)} = ([0.3393, 0.5495], [0.1312, 0.2661])$,
$\tilde{z}_3^{(3)} = ([0.2838, 0.3867], [0.2091, 0.3368])$,
$\tilde{z}_4^{(3)} = ([0.4229, 0.5261], [0.2044, 0.3050])$,
$\tilde{z}_5^{(3)} = ([0.4325, 0.5681], [0.1991, 0.3364])$,
$\tilde{z}_6^{(3)} = ([0.2879, 0.4181], [0.1663, 0.2665])$,
$\tilde{z}_1^{(4)} = ([0.2879, 0.3911], [0.2044, 0.3364])$,
$\tilde{z}_2^{(4)} = ([0.4884, 0.6284], [0.1363, 0.2365])$,
$\tilde{z}_3^{(4)} = ([0.2938, 0.4290], [0.1000, 0.2365])$,
$\tilde{z}_4^{(4)} = ([0.2938, 0.4624], [0.2091, 0.3368])$,
$\tilde{z}_5^{(4)} = ([0.2938, 0.3967], [0.0986, 0.2665])$,
$\tilde{z}_6^{(4)} = ([0.3587, 0.4984], [0.1340, 0.2661])$.

Step 2 Utilize the IVIFDG operator (20) to calculate the collective values $\tilde{z}_j\,(j = 1, 2, \ldots, 6)$ for the alternatives $A_j\,(j = 1, 2, \ldots, 6)$ (here, let $S = S^1_{H^*_{0.8,0.1}}$):

$\tilde{z}_1 = ([0.2916, 0.4068], [0.1672, 0.2943])$,
$\tilde{z}_2 = ([0.3282, 0.4856], [0.1350, 0.2520])$,
$\tilde{z}_3 = ([0.2696, 0.3975], [0.1627, 0.2966])$,
$\tilde{z}_4 = ([0.3542, 0.4867], [0.1719, 0.2962])$,
$\tilde{z}_5 = ([0.2656, 0.4212], [0.1605, 0.2786])$,
$\tilde{z}_6 = ([0.2690, 0.4254], [0.1623, 0.2808])$.

Step 3 Get the priority of the alternatives by comparing the scores and accuracy degrees of $\tilde{z}_j\ (j = 1, 2, \cdots, 6)$.

$$A_2 \succ A_4 \succ A_6 \succ A_5 \succ A_1 \succ A_3$$

Additionally, we can also choose other similarity measures and give different values for the parameters κ, λ, η which may derive different decision results.

6. Conclusions

In this paper, some simialrity measures with parameters for IFVs have been studied based on the intuitionistic fuzzy operators defined by Atanossov (1995). These similarity measures can consider the three parts of each IFV and can provide the decision makers more choices. Actually, the parameters in these similarity measures can control the membership degree or the non-membership degree of an IFV, thus the decision makers can chose the most suitable similarity measure according to their demands. We have also developed some intuitionistic fuzzy dependent aggregation operators for IFVs using the given similarity measures. The prominent characteristic of these aggregation operators is that the associated weights only depend on the aggregated IFVs and can relieve the influence of unfair IFVs on the aggregated results. Furthermore, we have extended our results for IFVs to interval-valued environment and get some useful conclusions. In the future study, we shall consider to investigate how to give the values of the parameters of the developed similarity measures more reasonably.

Acknowledgments

The authors are very grateful to the anonymous referees for their insightful and constructive comments and suggestions that have led to an improved version of this paper.

References

[1] Atanassov, K. (1986). Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20: 87-96

[2] Atanassov, K. & Gargov, G. (1989). Interval-valued intuitionistic fuzzy sets. Fuzzy Sets and Systems, 31: 343-349

[3] Atanassov, K. (1995). Remark on the intuitionistic fuzzy sets - Ⅲ. Fuzzy Sets and Systems, 75: 401-402

[4] Bustince, H. & Burillo, P. (1996). Vague sets are intuitionistic fuzzy sets. Fuzzy Sets and Systems, 79: 403-405

[5] Burillo, P. & Bustince, H. (1996). Construction theorems of intuitionistic fuzzy sets. Fuzzy Sets and Systems, 84: 271-281

[6] Chen, S.M. & Tan, J.M. (1994). Handling multicriteria fuzzy decision making problems based on vague set theory. Fuzzy Sets and Systems, 67: 163-172

[7] Chen, T.Y. (2010). An outcome-oriented approach to multicriteria decision analysis with intuitionistic fuzzy optimistic / pessimistic operators. Expert Systems with Applications, 37: 7762-7774

[8] Filev, D.P. & Yager, R.R. (1998). On the issue of obtaining OWA operator weights. Fuzzy Sets and Systems, 94: 157-169

[9] Fischer, R. & Chalmers, A. (2008). Is optimism universal? A meta-analytical investigation of optimism levels across 22 nations. Personality and Individual Differences, 45: 378-382

[10] Gau, W.L. & Buehrer, D.L. (1993). Vague sets. IEEE Transactions on Systems, Man, and Cybernetics, 23: 610-614

[11] Grzegorzewski, P. (2004). Distances between intuitionistic fuzzy sets and/or interval-valued fuzzy sets based on the Hausdorff metric. Fuzzy Sets and Systems, 148: 319-328

[12] Herrera, F. & Herrera-Viedma, E. (1997).

Aggregation operators for linguistic weighted information. IEEE Transactions on Systems, Man, and Cybernetics, 27: 646-656

[13] Hong, D.H. & Choi, C.H. (2000). Multicriteria fuzzy decision-making problems based on vague set theory. Fuzzy Sets and Systems, 114: 103-113

[14] Hung, W.L. & Yang, M.S. (2004). Similarity measures of intuitionistic fuzzy sets based on Hausdorff distance. Pattern Recognition Letters, 25: 1603-1611

[15] Hung, W.L. & Yang, M.S. (2007). Similarity measures of intuitionistic fuzzy sets based on L_p metric. International Journal of Approximate Reasoning, 46: 120-136

[16] Hung, W.L. & Yang, M.S. (2008a). On the *J*-divergence of intuitionistic fuzzy sets with its application to pattern recognition. Information Sciences, 178: 1641-1650

[17] Hung, W.L. & Yang, M.S. (2008b). On similarity measures between intuitionistic fuzzy sets. International Journal of Intelligent Systems, 23: 364-383

[18] Li, D.F. & Cheng, C.T. (2002). New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions. Pattern Recognition Letters, 23: 221-225

[19] Liang, Z.Z. & Shi, P.F. (2003). Similarity measures on intuitionistic fuzzy sets. Pattern Recognition Letters, 24: 2687- 2693

[20] Li, Y.H., Olson, D.L. & Qin, Z. (2007). Similarity measures between intuitionistic fuzzy (vague) sets: a comparative analysis. Pattern Recognition Letters, 28: 278-285

[21] Liu, H.W. & Wang, G.J. (2007). Multi-criteria decision making methods based on intuitionistic fuzzy sets. European Journal of Operational Research, 197: 220-233

[22] Mitchell, H.B. (2003). On the Dengfeng-Chuntian similarity measure and its application to pattern recognition. Pattern Recognition Letters, 24: 3101-3104

[23] Ngwenyama, O. & Bryson, N. (1999). Eliciting and mapping qualitative preferences to numeric rankings in group decision making. European Journal of Operational Research, 116: 487-497

[24] O'Hagan, M. (1998). Aggregating template rule antecedents in real-time expert systems with fuzzy set logic. In: Proceedings of the 22nd Annnual IEEE Asilomar Conference on Signals, Systems and Computers, 681-689, Pacific Grove, CA

[25] Szmidt, E. & Kacprzyk, J. (2000). Distances between intuitionistic fuzzy sets. Fuzzy Sets and Systems, 114: 505-518

[26] Szmidt, E. & Kacprzyk, J. (2006). Distances between intuitionistic fuzzy sets: straightforward approaches may not work. In: 3rd Int. IEEE Conf. Intelligent Systems, 716-721

[27] Xu, Z.S. & Da, Q.L. (2002). The ordered weighted geometric averaging operators. International Journal of Intelligent Systems, 17: 709-716

[28] Xu, Z.S. (2004). Uncertain linguistic aggregation operators based approach to multiple attribute group decision making under uncertain linguistic environment. Information Sciences, 168: 171-184

[29] Xu, Z.S. (2005). An overview of methods for determining OWA weights. International Journal of Intelligent Systems, 20: 843-865

[30] Xu, Z.S. & Yager, R.R. (2006). Some geometric aggregation operators based on

intuitionistic fuzzy sets. International Journal of General Systems, 35: 417-433

[31] Xu, Z.S. (2006). Dependent OWA operators. Lecture Notes in Artificial Intelligence, 3885: 172-178

[32] Xu, Z.S. (2007a). Some similarity measures of intuitionistic fuzzy sets and their applications to multiple attribute decision making. Fuzzy Optimization and Decision Making, 6: 109-121

[33] Xu, Z.S. (2007b). Intuitionistic fuzzy aggregation operators. IEEE Transactions on Fuzzy Systems, 15: 1179-1187

[34] Xu, Z.S. (2007c). On similarity measures of interval-valued intuitionistic fuzzy set and their application to pattern recognitions. Journal of Southeast University (English Edition), 23: 139-143

[35] Xu, Z.S. (2007d). Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision making. Control and Decision, 22: 215-219

[36] Xu, Z.S. & Chen, J. (2007). On geometric aggregation over interval-valued intuitionistic fuzzy information. In: The 4th International Conference on Fuzzy Systems and Knowledge Discovery (FSKD'07), 2; 466-471, Haikou, China, August 24-27, 2007

[37] Xu, Z.S. & Chen, J. (2008). An overview of distance and similarity measures of intuitionistic fuzzy sets. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 16: 529-555

[38] Xu, Z.S. (2008). Dependent uncertain ordered weighted aggregation operators. Information Fusion, 9: 310-316

[39] Yager, R.R. (2003). Induced aggregation operators. Fuzzy Sets and Systems, 137: 59-69

[40] Yager, R.R. (2004a). Generalized OWA aggregation operators. Fuzzy Optimization and Decision Making, 3: 93-107

[41] Yager, R.R. & Kacprzyk, J. (1997). The Ordered Weighted Averaging Operators. Theory and Applications, Kluwer: Norwell, MA

[42] Yager, R.R. (2004b). Choquet aggregation using order inducing variables. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 12: 69-88

[43] Yager, R.R. (2004c). OWA aggregation over a continuous interval argument with applications to decision making. IEEE Transactions on Systems, Man, and Cybernetics, 34: 1952-1963

[44] Yager, R.R. & Xu, Z.S. (2006). The continuous ordered weighted geometric operator and its application to decision making. Fuzzy Sets and Systems, 157: 1393-1402

[45] Yager, R.R. & Filev, D.P. (1999). Induced ordered weighted averaging operators. IEEE Transactions on Systems, Man, and Cybernetics, 29: 141-150

[46] Yager, R.R. (1993). Families of OWA operators. Fuzzy Sets and Systems, 59: 125-148

[47] Zhao, H., Xu, Z.S., Ni, M.F. & Liu, S.S. (2010). Generalized aggregation operators for intuitionistic fuzzy sets. International Journal of Intelligent Systems, 25: 1-30

Meimei Xia received the B.E. degree in information management and information system from College of Operations Research and Management, Qufu Normal University,

Rizhao, China, in 2009. She is currently working toward the Ph.D. degree at School of Economics and Management, Southeast University. Her research interests include aggregation operators and group decisoin making.

Zeshui Xu received the Ph.D. degree in management science and engineering from Southeast University, Nanjing, China, in 2003. From April 2003 to May 2005, he was a Postdoctoral Researcher with the School of Economics and Management, Southeast University. From October 2005 to December 2007, he was a Postdoctoral Researcher with the School of Economics and Management, Tsinghua University, Beijing, China. He is an Adjunct Professor with the School of Economics and Management, Southeast University. He is a Chair Professor with the Institute of Sciences, PLA University of Science and Technology, Nanjing. He is a member of the Editorial Boards of Information: An International Journal, the International Journal of Applied Management Science, the International Journal of Data Analysis Techniques and Strategies, and the System Engineering - Theory and Practice and Fuzzy Systems and Mathematics. He has authored three books and contributed more than 250 journal articles to professional journals. His current research interests include information fusion, group decision making, computing with words, and aggregation operators.