Ark. Mat., **54** (2016), 455–471
DOI: 10.1007/s11512-016-0237-3

Fourier dimension of random images

Fredrik Ekström

Abstract. Given a compact set of real numbers, a random $C^{m+\alpha}$-diffeomorphism is constructed such that the image of any measure concentrated on the set and satisfying a certain condition involving a real number s, almost surely has Fourier dimension greater than or equal to $s/(m+\alpha)$. This is used to show that every Borel subset of the real numbers of Hausdorff dimension s is $C^{m+\alpha}$-equivalent to a set of Fourier dimension greater than or equal to $s/(m+\alpha)$. In particular every Borel set is diffeomorphic to a Salem set, and the Fourier dimension is not invariant under C^m-diffeomorphisms for any m.

1. Introduction

The *Fourier dimension* of a Borel probability measure on $\mathbf{R}^d$ measures the polynomial rate of decay of the Fourier transform of μ, and is defined to be the supremum of all s in $[0, d]$ such that $|\hat{\mu}(\xi)||\xi|^{s/2}$ is bounded. The Fourier dimension of a Borel set F is the supremum of the Fourier dimensions of all probability measures that give full measure to F. It can be shown that if the Fourier dimension of μ is greater than s then

$$\iint |y-x|^{-s} \, d\mu(x) \, d\mu(y) < \infty$$

(see [12, Lemma 12.12]), and it follows that the Hausdorff dimension of F is always greater than or equal to the Fourier dimension of F. If the Fourier and Hausdorff dimensions of F are equal, then F is called a *Salem set*. An example of a set that is not a Salem set is the ternary Cantor set, which has Fourier dimension 0. (More generally, a compact proper subset of $[0, 1]$ that is invariant under multiplication mod 1 by an integer greater than or equal to 2 cannot support a measure whose Fourier transform tends to 0 at infinity.)

This paper gives a construction of a random $C^{m+\alpha}$-diffeomorphism $f_\omega : \mathbf{R} \to \mathbf{R}$, given a compact subset E of $\mathbf{R}$. If λ is a probability measure on E such that the number of connected components of E^c that are included in any given interval J and have length at least x^{-1} is bounded below (up to constants) by $\log\lambda(J)+s\log x$, then the image of λ under f_ω almost surely has Fourier dimension greater than or equal to $s/(m+\alpha)$. The mentioned condition is satisfied for example if E is the attractor of a "nice" iterated function system and λ is a probability measure on E such that $\lambda(I)\le A|I|^s$ for every interval I.

Taking $E=C$ in the construction of f_ω, where C is a certain fat Cantor set, it is shown that for any Borel set F of positive s-dimensional Hausdorff measure there is a real number t such that almost surely $\dim_{\mathrm{F}} f_\omega(C\cap(F+t))\ge s/(m+\alpha)$. This is used to prove that for any Borel set F there is a $C^{m+\alpha}$-diffeomorphism f such that $\dim_{\mathrm{F}} f(F)\ge s/(m+\alpha)$, where s is the Hausdorff dimension of F (even if the s-dimensional Hausdorff measure of F is 0). In particular, every Borel subset of $\mathbf{R}$ is diffeomorphic to a Salem set, and the Fourier dimension is not C^m-invariant for any m.

A remaining question is whether there exists a Borel set F such that $\dim_{\mathrm{F}} f(F)\le\dim_{\mathrm{H}} F/(m+\alpha)$ for every $C^{m+\alpha}$-diffeomorphism f. One might also ask whether the Fourier dimension is invariant under C^∞-diffeomorphisms, since the statements that are proved here become empty when $m\to\infty$. It follows from previous work (see below) that this is not the case for subsets of $\mathbf{R}^2$, and not for subsets of $\mathbf{R}$ if non-invertible C^∞-functions are considered.

1.1. Related work

It was shown by Salem in [14] that there exist Salem subsets of $\mathbf{R}$ of any dimension between 0 and 1, using a construction of a Cantor set where the contraction ratios are chosen randomly.

The one-dimensional Brownian motion is almost surely Hölder continuous with any exponent less than 1/2, and more generally the fractional Brownian motion with Hurst index α is Hölder continuous with any exponent less than α for $\alpha\in(0,1)$. It was shown by Kahane that if E is any compact subset of $\mathbf{R}$ of Hausdorff dimension $s\le\alpha$, then the image of E under fractional Brownian motion with Hurst index α is almost surely a Salem set of dimension s/α. (See [9, Chapters 17 and 18]. In fact Kahane proved a more general statement, which allows both E and the image of E to lie in higher-dimensional Euclidean spaces.)

In [3], Bluhm gave a method for randomly perturbing a class of self-similar measures on $\mathbf{R}^d$, such that the perturbed measure almost surely has Fourier dimension equal to the similarity dimension of the original measure. For $d=1$, the uniform measures on Cantor sets with constant contraction ratio are among the measures considered by Bluhm, and if the parameters in the construction are chosen suitably then the perturbation is a bi-Lipschitz map. Thus it follows from Bluhm's result that such Cantor sets are bi-Lipschitz equivalent to Salem sets.

An explicit example of a Salem set is given by the set of α-well approximable numbers, that is, the set

$$E(\alpha)=\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{x\in[0,1]\,;\|kx\|\le k^{-(1+\alpha)}\},$$

where $\|\cdot\|$ denotes the distance to the nearest integer. By a theorem of Jarník [8] and Besicovitch [1] the set $E(\alpha)$ has Hausdorff dimension $2/(2+\alpha)$ for $\alpha>0$, and Kaufman [11] showed that there is a probability measure on $E(\alpha)$ with Fourier dimension $2/(2+\alpha)$ (see also Bluhm's paper [2]).

It follows from a result of Kaufman [10] that any C^2-curve in $\mathbf{R}^2$ with non-zero curvature has Fourier dimension 1. Since line segments in $\mathbf{R}^2$ have Fourier dimension 0 this shows that the Fourier dimension in $\mathbf{R}^2$ is not in general invariant under C^∞-diffeomorphisms.

In [4], subsets A and B of $\mathbf{R}$ were constructed such that

$$\max(\dim_{\mathrm{F}} A, \dim_{\mathrm{F}} B)<1 \quad \text{and} \quad \dim_{\mathrm{F}} A\cup B=1,$$

and they can be taken to be included in $[1,2]$. If $f(x)=x^2$ then

$$\dim_{\mathrm{F}} f\big(\sqrt{A}\cup(-\sqrt{B})\big)=\dim_{\mathrm{F}} A\cup B=1$$

and

$$\dim_{\mathrm{F}}\big(\sqrt{A}\cup(-\sqrt{B})\big)=\max(\dim_{\mathrm{F}}\sqrt{A},\dim_{\mathrm{F}}\sqrt{B})$$

since $\sqrt{A}$ and $-\sqrt{B}$ are separated (see [5, Theorem 2]). Thus f changes the Fourier dimension of at least one of $\sqrt{A}$, $\sqrt{B}$ and $\sqrt{A}\cup(-\sqrt{B})$, showing that the Fourier dimension in $\mathbf{R}$ is not in general invariant under C^∞-functions.

1.2. Some definitions and notation

The *Fourier transform* of a probability measure μ on $\mathbf{R}^d$ is defined for $\xi\in\mathbf{R}^d$ by

$$\hat{\mu}(\xi)=\int \mathrm{e}(-\xi\cdot x)\,d\mu(x),$$

where $\mathrm{e}(y)=e^{2\pi i y}$ and $\cdot$ is the Euclidean inner product. If f and g are complex-valued functions with the same domain, then

$$f(\xi)\lesssim g(\xi)$$

means that there is a constant C such that

$$\left|f(\xi)\right| \leq C\left|g(\xi)\right|$$

for all ξ in the domain. The Fourier dimension of μ is then defined by

$$\dim_{\mathrm{F}} \mu = \sup\left\{s \in [0,d]\,;\, \hat{\mu}(\xi) \lesssim |\xi|^{-s/2}\right\},$$

and the Fourier dimension of a Borel set F is defined by

$$\dim_{\mathrm{F}} F = \sup\left\{\dim_{\mathrm{F}} \mu\,;\, \mu \text{ is a probability measure on } \mathbf{R}^d \text{ and } \mu(F)=1\right\}.$$

If μ is a measure on a measurable space X and $f\colon X\to Y$ is a measurable function, then $f\mu$ denotes the transportation of μ by f, namely, the measure on Y defined by

$$(f\mu)(A) = \mu\big(f^{-1}(A)\big).$$

The formula for change of variable in Lebesgue integrals thus reads

$$\int_Y g\, d(f\mu) = \int_X g\circ f\, d\mu,$$

where g is a real- or complex-valued function on Y.

Any increasing function $\varepsilon\colon[0,\infty)\to[0,\infty)$ such that

$$\varepsilon(0) = \lim_{t\to 0} \varepsilon(t) = 0$$

is called a *modulus of continuity*. A function $g\colon\mathbf{R}\to\mathbf{R}$ is uniformly continuous with modulus ε if

$$\left|g(y)-g(x)\right| \leq \varepsilon\big(|y-x|\big)$$

for all $x,y\in\mathbf{R}$, and a set G of functions $\mathbf{R}\to\mathbf{R}$ is *uniformly equicontinuous* with modulus ε if every $g\in G$ is uniformly continuous with modulus ε. A function is *Hölder continuous* with exponent $\alpha\in(0,1]$ if it is uniformly continuous with modulus $\varepsilon(t)=Ct^{\alpha}$ for some constant C.

2. Constructions and main results

This section contains the main constructions and the statements of the main results. Proofs are given in later sections.

Theorem 1. *Let F be a Borel subset of $\mathbf{R}$. Then there exists a $C^{m+\alpha}$-diffeomorphism $f\colon\mathbf{R}\to\mathbf{R}$ such that*

$$\dim_{\mathrm{F}} f(F) \geq \frac{\dim_{\mathrm{H}} F}{m+\alpha}.$$

The proof of Theorem 1 is based on Theorem 3 and Theorem 5 below, and is given at the end of Section 6.

2.1. The random map f_ω on a compact set E

Given a compact subset E of $\mathbf{R}$, let D be the set of bounded connected components of E^c. Choose non-negative numbers $\{\delta_\mathcal{U}\}_{\mathcal{U}\in D}$ whose sum is finite and let $\Omega=\prod_{\mathcal{U}\in D}[0,\delta_\mathcal{U}]$. For $\omega\in\Omega$, define f_ω on E by

$$f_\omega(x)=x+\sum_{\mathcal{U}\subset(-\infty,x)}\omega_\mathcal{U},$$

where the sum is over those $\mathcal{U}\in D$ that lie to the left of x (thus f_ω can be thought of as increasing the size of each hole $\mathcal{U}$ in E by an additive amount of $\omega_\mathcal{U}$). Fix some probability measure ν on $[0,1]$ such that $\lim_{|\xi|\to\infty}\hat{\nu}(\xi)=0$ and let $\nu_\mathcal{U}$ be the image of ν under the map $x\mapsto\delta_\mathcal{U}x$. Let P be the product measure on Ω that projects to $\nu_\mathcal{U}$ on the $\mathcal{U}$-coordinate.

If J is an interval and $x>0$, let

$$\psi(J,x)=\#\{\mathcal{U}\in D\,;\mathcal{U}\subset J \text{ and } \delta_\mathcal{U}\geq x^{-1}\}.$$

Theorem 2. *Let $s\in[0,1]$ and let λ be a probability measure on E. Suppose that there are constants a and $b>0$ and some x_0 such that*

$$\psi(J,x)\geq a+b\big(\log\lambda(J)+s\log x\big)$$

for every interval J and every $x\geq x_0$. Then almost surely $\dim_\mathrm{F} f_\omega\lambda\geq s$.

2.2. Extending f_ω to $\mathbf{R}$

Let φ be an increasing C^∞-function on $\mathbf{R}$ that is 0 on $(-\infty,0]$ and 1 on $[1,\infty)$. Then

$$f_\omega(x)=x+\sum_{\mathcal{U}\in D}\omega_\mathcal{U}\varphi\left(\frac{x-\inf\mathcal{U}}{|\mathcal{U}|}\right)$$

is an extension of f_ω to all of $\mathbf{R}$. Let m be a positive integer and let $\alpha\in[0,1]$. From now on choose $\delta_\mathcal{U}=|\mathcal{U}|^m\delta(|\mathcal{U}|)$, where

$$\delta(t)=\begin{cases}\frac{1}{\max(-\log t,\log 2)} & \text{if } \alpha=0\\ t^\alpha & \text{if } \alpha\in(0,1].\end{cases}$$

Theorem 3. *The function f_ω is a $C^{m+\alpha}$-diffeomorphism for every $\omega\in\Omega$, and $\{f_\omega^{(m)}\}_{\omega\in\Omega}$ is uniformly equicontinuous with modulus $2\|\varphi^{(m+1)}\|_\infty\,\delta$.*

2.3. Consequences of Theorem 2

Theorem 4. *Let* $\Phi=\{F_1,...,F_N\}$ *be an iterated function system on* $\mathbf{R}$ *consisting of contracting* $C^{1+\beta}$*-diffeomorphisms. Take* E *to be the attractor of* Φ *and assume that the interiors of the convex hulls of* $F_i(E)$ *and* $F_j(E)$ *are disjoint whenever* $i\neq j$*. Let* λ *be a probability measure on* E *such that* $\lambda(I)\leq A|I|^s$ *for every interval* I*. Then almost surely* $\dim_{\mathrm{F}} f_\omega\lambda\geq s/(m+\alpha)$.

If $s=\dim_{\mathrm{H}} E$ then there is a probability measure λ on E such that $\lambda(I)\leq A|I|^s$ for every interval I (see [7, Theorem 5.3]), and thus $\dim_{\mathrm{F}} f_\omega E\geq \dim_{\mathrm{H}} E/(m+\alpha)$ almost surely. By using a particular set C in the construction of f_ω, similar results can be obtained for general Borel sets.

Let $\{c_k\}_{k=1}^{\infty}$ be an increasing sequence of positive numbers that converges to $1/2$ and satisfies

$$\prod_{k=1}^{\infty} 2c_k > 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{\log(1-2c_k)}{k} = 0$$

(for example $c_k=1/2-1/k^2$). Let $C_0=[0,1]$ and for $k\geq 1$ let C_k be the set obtained by removing from every connected component I of C_{k-1} the open interval that is concentric with I and has length $(1-2c_k)|I|$. (Thus C_k consists of 2^k closed intervals of length $\prod_{i=1}^{k} c_i$.) Let

$$C=\bigcap_{k=0}^{\infty} C_k\,;$$

this is a compact set of positive Lebesgue measure.

Theorem 5. *Take* $E=C$ *and let* F *be a Borel subset of* $\mathbf{R}$ *such that* $\mathcal{H}^s(F)>0$*. Then there is some* $t\in\mathbf{R}$ *such that almost surely*

$$\dim_{\mathrm{F}} f_\omega\big(C\cap(F+t)\big) \geq \frac{s}{m+\alpha}.$$

3. Proof of Theorem 2

The following lemma says that an almost sure bound for the Fourier dimension of a random measure can be proved by estimating the decay rate of $\mathbf{E}|\hat{\mu}(\xi)|^{2q}$ for large q. This idea was used in the works of Salem, Kahane and Bluhm mentioned in the introduction. The proof of Theorem 2 is also much inspired by Kahane's and Bluhm's proofs.

Lemma 6. *Let (Ω, P) be a probability space and let $\Omega \ni \omega \mapsto \mu_\omega$ be a random probability measure on $\mathbf{R}$ such that $|\operatorname{supp} \mu_\omega| < M$ almost surely, for some constant M. Suppose that*

$$\mathbf{E}\big(\big|\hat{\mu}_\omega(\xi)\big|^{2q}\big) \lesssim |\xi|^{-sq+1}$$

for $q=1,2,\ldots$. Then almost surely

$$\hat{\mu}_\omega(\xi) \lesssim |\xi|^{-s/2+\varepsilon}$$

for every $\varepsilon>0$.

Proof. The assumption on the decay of $\mathbf{E}|\hat{\mu}_\omega(\xi)|^{2q}$ implies that

$$\int \sum_{\xi\in\mathbf{Z}/M} |\xi|^{sq-3}\big|\hat{\mu}_\omega(\xi)\big|^{2q}\, dP(\omega) = \sum_{\xi\in\mathbf{Z}/M} |\xi|^{sq-3}\mathbf{E}\big|\hat{\mu}_\omega(\xi)\big|^{2q} \le \text{const.} \times \sum_{\xi\in\mathbf{Z}/M} |\xi|^{-2} < \infty.$$

Thus for a.e. ω the sum in the first expression is finite, so

$$\lim_{\substack{|\xi|\to\infty \\ \xi\in\mathbf{Z}/M}} |\xi|^{sq-3}\big|\hat{\mu}_\omega(\xi)\big|^{2q} = 0,$$

and in particular

$$\big|\hat{\mu}(\xi)\big| \lesssim |\xi|^{-s/2+3/(2q)}, \quad \xi \in \mathbf{Z}/M.$$

It follows from a lemma of Kahane [9, p. 252] that $\hat{\mu}_\omega(\xi) \lesssim |\xi|^{-s/2+3/(2q)}$ for $\xi\in\mathbf{R}$ as well. Letting $q\to\infty$ establishes the conclusion for any fixed $\varepsilon>0$, and letting $\varepsilon\to 0$ along a countable set then proves the lemma. $\square$

Proof of Theorem 2. Let $\mu_\omega = f_\omega\lambda$. By Lemma 6 it suffices to show that

$$\mathbf{E}\big(\big|\hat{\mu}_\omega(\xi)\big|^{2q}\big) \lesssim |\xi|^{-sq+1} \quad \text{for } q=1,2,\ldots.$$

Now,

$$\begin{aligned}\big|\hat{\mu}_\omega(\xi)\big|^{2q} &= \left(\int \mathrm{e}\big(\xi(y-x)\big)\, d\mu_\omega^2(x,y)\right)^q \\ &= \left(\int \mathrm{e}\big(\xi\big(f_\omega(y)-f_\omega(x)\big)\big)\, d\lambda^2(x,y)\right)^q \\ &= \int \mathrm{e}\left(\xi \sum_{i=1}^{q} \big(f_\omega(y_i)-f_\omega(x_i)\big)\right) d\lambda^{2q}(\bar{x},\bar{y}),\end{aligned}$$

where $\bar{x}=(x_1,...,x_q)$ and $\bar{y}=(y_1,...,y_q)$. Let

$$d_{\bar{x},\bar{y}}=\sum_{i=1}^{q}(y_i-x_i)$$

and

$$h_{\bar{x},\bar{y}}(z)=\#\{i\,;z<y_i\}-\#\{i\,;z<x_i\}.$$

If $(\bar{x},\bar{y})\in E^{2q}$ then $h_{\bar{x},\bar{y}}$ is constant on each $\mathcal{U}$ in D, and

$$\sum_{i=1}^{q}\big(f_\omega(y_i)-f_\omega(x_i)\big)=d_{\bar{x},\bar{y}}+\sum_{\mathcal{U}\in D}h_{\bar{x},\bar{y}}(\mathcal{U})\omega_{\mathcal{U}}.$$

Thus

$$\big|\hat{\mu}_\omega(\xi)\big|^{2q}=\int \mathrm{e}(\xi d_{\bar{x},\bar{y}})\mathrm{e}\bigg(\xi\sum_{\mathcal{U}\in D}h_{\bar{x},\bar{y}}(\mathcal{U})\omega_{\mathcal{U}}\bigg)\,d\lambda^{2q}(\bar{x},\bar{y}),$$

and integrating over ω gives

$$\begin{aligned}\mathbf{E}\big|\hat{\mu}_\omega(\xi)\big|^{2q}&\le\int\bigg|\int\prod_{\mathcal{U}\in D}\mathrm{e}\big(h_{\bar{x},\bar{y}}(\mathcal{U})\xi\omega_{\mathcal{U}}\big)\,dP(\omega)\bigg|\,d\lambda^{2q}(\bar{x},\bar{y})\\&=\int\bigg|\prod_{\mathcal{U}\in D}\int\mathrm{e}\big(h_{\bar{x},\bar{y}}(\mathcal{U})\xi\omega_{\mathcal{U}}\big)\,d\nu_{\mathcal{U}}(\omega_{\mathcal{U}})\bigg|\,d\lambda^{2q}(\bar{x},\bar{y})\\&=\int\prod_{\mathcal{U}\in D}\big|\hat{\nu}\big(h_{\bar{x},\bar{y}}(\mathcal{U})\delta_{\mathcal{U}}\xi\big)\big|\,d\lambda^{2q}(\bar{x},\bar{y}).\end{aligned}$$

Let B_r be the set of $(\bar{x},\bar{y})$ such that $\lambda(J)\le r$ whenever J is an interval and $h_{\bar{x},\bar{y}}$ is non-zero everywhere on J. If $(\bar{x},\bar{y})\in B_r$ then for each x_i there must be some y_j such that the open interval between x_i and y_j has λ-measure less than or equal to r, since either $x_i=y_j$ for some j or $h_{\bar{x},\bar{y}}$ increases by 1 at x_i. Thus for each fixed $\bar{y}$ there is a set of λ-measure $2qr$ that contains all the x_i:s whenever $(\bar{x},\bar{y})\in B_r$. Hence

$$\lambda^{2q}(B_r)=\int\lambda^q\big(\{\bar{x}\,;(\bar{x},\bar{y})\in B_r\}\big)\,d\lambda^q(\bar{y})\le(2qr)^q.$$

If on the other hand $(\bar{x},\bar{y})\notin B_r$ then there is an interval J such that $\lambda(J)\ge r$ and $h_{\bar{x},\bar{y}}\neq 0$ on J. Then for any $K>0$ and for $|\xi|\ge Kx_0$,

$$\begin{aligned}\prod_{\mathcal{U}\in D}\big|\hat{\nu}\big(h_{\bar{x},\bar{y}}(\mathcal{U})\delta_{\mathcal{U}}\xi\big)\big|&\le\prod_{\substack{\mathcal{U}\in D\\ \mathcal{U}\subset J}}g\big(\delta_{\mathcal{U}}|\xi|\big)\le g(K)^{\psi(J,K^{-1}|\xi|)}\\&\le g(K)^{a+b(\log r+s\log(K^{-1}|\xi|))},\end{aligned}$$

where

$$g(x)=\sup_{|\xi|\geq x}\left|\hat{\nu}(\xi)\right|.$$

Thus for any positive r and K, and $|\xi|\geq Kx_0$,

$$\mathbf{E}\left|\hat{\mu}_\omega(\xi)\right|^{2q}\leq(2qr)^q+g(K)^{a+b(\log r+s\log(K^{-1}|\xi|))}.$$

In particular this holds if r is chosen such that

$$g(K)^{a+b(\log r+s\log(K^{-1}|\xi|))}=r^q,$$

or equivalently,

$$\log r=-s\log|\xi|\left(\frac{a}{bs\log|\xi|}+\left(1-\frac{\log K}{\log|\xi|}\right)\right)\frac{b\log g(K)}{b\log g(K)-q}.$$

For a fixed K the factor in the middle converges to 1 when $|\xi|\to\infty$, and hence

$$\mathbf{E}\left|\hat{\mu}_\omega(\xi)\right|^{2q}\lesssim|\xi|^{-sq\frac{b\log g(K)}{b\log g(K)-q}+\frac{1}{2}}$$

for any K. Taking K large enough shows that $\mathbf{E}|\hat{\mu}_\omega(\xi)|^{2q}\lesssim|\xi|^{-sq+1}$. $\square$

4. Lemma 7 and the proof of Theorem 3

The following lemma is used in the proof of Theorem 3 at the end of this section, and also in the proof of Theorem 1.

Lemma 7. *Let $\{V_k\}_{k=1}^\infty$ be disjoint open intervals such that $V=\bigcup_k V_k$ is bounded and let $\{g_k\}_{k=1}^\infty$ be increasing functions $\mathbf{R}\to\mathbf{R}$ that are m times differentiable ($m\geq1$), such that $\{g_k^{(m)}\}$ is uniformly equicontinuous with modulus ε and*

$$(1)\qquad g_k(\inf V_k)=0\quad\textit{and}\quad g_k'=\ldots=g_k^{(m)}=0\ \textit{on}\ V_k^c$$

for all k. Define g:$\mathbf{R}\to\mathbf{R}$ by

$$g(x)=\sum_{k=1}^\infty g_k(x).$$

Then g is m times continuously differentiable with

$$g(\inf V)=0\quad\textit{and}\quad g'=\ldots=g^{(m)}=0\ \textit{on}\ V^c,$$

and $g^{(m)}$ is uniformly continuous with modulus 2ε.

Proof. It follows from (1) that

$$\left|g_k^{(m)}(x)\right| \le \varepsilon\big(\operatorname{dist}(x, V^c)\big) \tag{2}$$

and

$$g_k(\inf V_k + t) \le \frac{\varepsilon(t)}{m!} t^m \quad \text{for } t \ge 0 \tag{3}$$

for all k. In particular,

$$g(x) \le \sum_k \left|g_k(V_k)\right| \le \frac{\varepsilon(\sup_k |V_k|)}{m!} \sum_k |V_k|^m < \infty,$$

so that g is well-defined.

It is clear that $g^{(m)}$ exists and is continuous on V, and by (2),

$$\lim_{\substack{x \to V^c \\ x \in V}} g^{(m)}(x) = 0.$$

To prove that g is m times continuously differentiable it will be shown that $g^{(m)}$ exists and is 0 on V^c. For this it suffices to consider limits from the right, since $x \mapsto -g(-x) + |g(V)|$ has the same form as g, with $x \mapsto -g_k(-x) + |g_k(V_k)|$ instead of g_k. So take any $x \in V^c$ and any $h \ge 0$. Then

$$g(x+h) - g(h) = \sum_{V_k \subset (x, x+h)} \left|g_k(V_k)\right| + \sum_{x+h \in V_n} g_n(x+h)$$

(the second sum has one term if $x+h \in V$ and is empty otherwise). By (3),

$$\begin{aligned} \sum_{V_k \subset (x, x+h)} \left|g_k(V_k)\right| &\le \frac{\varepsilon(h)}{m!} \sum_{V_k \subset (x, x+h)} |V_k|^m \\ &\le \frac{\varepsilon(h)}{m!} \left(\sum_{V_k \subset (x, x+h)} |V_k| \right)^m \le \frac{\varepsilon(h)}{m!} h^m \end{aligned}$$

and

$$\sum_{x+h \in V_n} g(x+h) \le \frac{\varepsilon(h)}{m!} h^m,$$

and thus

$$g(x+h) - g(x) \le \frac{2\varepsilon(h)}{m!} h^m.$$

Hence g is m times differentiable at x, and

$$g'(x) = \ldots = g^{(m)}(x) = 0.$$

Finally, it will be shown that $g^{(m)}$ is uniformly continuous with modulus 2ε. If there is some n such that x and y both lie in $\bar{V}_n$ then

$$\left|g^{(m)}(y)-g^{(m)}(x)\right|=\left|g_n^{(m)}(y)-g_n^{(m)}(x)\right|\leq\varepsilon\big(|y-x|\big).$$

Otherwise the open interval between x and y intersects V^c, so

$$\begin{aligned}\left|g^{(m)}(y)-g^{(m)}(x)\right| &\leq \left|g^{(m)}(y)\right|+\left|g^{(m)}(x)\right| \\ &\leq \varepsilon\big(\operatorname{dist}(y,V^c)\big)+\varepsilon\big(\operatorname{dist}(x,V^c)\big)\leq 2\varepsilon\big(|y-x|\big). \qquad \square\end{aligned}$$

Proof of Theorem 3. Note that $\delta(t)/t$ is decreasing for $t>0$. Let

$$g_{\mathcal{U}}(x)=\omega_{\mathcal{U}}\varphi\left(\frac{x-\inf\mathcal{U}}{|\mathcal{U}|}\right).$$

Then

$$g_{\mathcal{U}}^{(k)}=\frac{\omega_{\mathcal{U}}}{|\mathcal{U}|^k}\varphi^{(k)}\left(\frac{x-\inf\mathcal{U}}{|\mathcal{U}|}\right)$$

for all k, and thus for all $x,y\in\bar{\mathcal{U}}$,

$$\begin{aligned}\left|g_{\mathcal{U}}^{(m)}(y)-g_{\mathcal{U}}^{(m)}(x)\right| &\leq \left\|\varphi^{(m+1)}\right\|_\infty\frac{\delta_{\mathcal{U}}}{|\mathcal{U}|^{m+1}}|y-x|=\left\|\varphi^{(m+1)}\right\|_\infty\frac{\delta(|\mathcal{U}|)}{|\mathcal{U}|}|y-x| \\ &\leq \left\|\varphi^{(m+1)}\right\|_\infty\frac{\delta(|y-x|)}{|y-x|}|y-x|=\left\|\varphi^{(m+1)}\right\|_\infty\delta\big(|y-x|\big).\end{aligned}$$

Since $g_{\mathcal{U}}^{(m)}$ is constant on $\mathcal{U}^c$ the inequality

$$\left|g_{\mathcal{U}}^{(m)}(y)-g_{\mathcal{U}}^{(m)}(x)\right|\leq\left\|\varphi^{(m+1)}\right\|_\infty\delta\big(|y-x|\big)$$

then holds for all $x,y\in\mathbf{R}$, and the conclusion follows by Lemma 7 since

$$f_\omega(x)=x+\sum_{\mathcal{U}\in D}g_{\mathcal{U}}(x). \qquad \square$$

5. Proof of Theorem 4

Recall that $\Phi=\{F_1,...,F_N\}$ is an iterated function system where each F_i is a contracting $C^{1+\beta}$ diffeomorphism, that E is the attractor of Φ and that the interiors of the convex hulls of $F_i(E)$ and $F_j(E)$ are assumed to be disjoint when $i\neq j$. If $\rho=\rho_1...\rho_n$ is a finite string over $\{1,...,N\}$, let $F_\rho=F_{\rho_1}\circ...\circ F_{\rho_n}$ and $E_\rho=F_\rho(E)$. By the principle of bounded distortion there is a constant B such that

$$B^{-1}|E_\rho||x-y|\leq\left|F_\rho(x)-F_\rho(y)\right|\leq B|E_\rho||x-y|$$

for every ρ and every x, y in a compact ball that includes E (this is proved in [7, Proposition 4.2] for iterated function systems consisting of contracting C^2-diffeomorphisms, and the proof goes through with a small modification for contracting $C^{1+\beta}$-diffeomorphisms). In particular, there is some $\gamma \in (0,1)$ such that

$$\gamma|x-y| \le \bigl|F_i(x)-F_i(y)\bigr|$$

for every i and every x, y in that ball.

Lemma 8. *Let λ be a probability measure on E such that*

$$\lambda(I) \le A|I|^s$$

for every interval I. Then there are constants a and $b>0$ such that

$$\#\bigl\{\mathcal{U}\in D\,;\mathcal{U}\subset J \text{ and } |\mathcal{U}|\ge x^{-1}\bigr\} \ge a+b\bigl(\log\lambda(J)+s\log x\bigr)$$

for every interval J and every $x>0$.

Proof. For $y>0$, let

$$\eta(y) = \#\bigl\{\mathcal{U}\in D\,; |\mathcal{U}| \ge y^{-1}\bigr\},$$

and fix some $\mathcal{U}_0 \in D$. Given y, let n be the unique integer such that

$$\gamma^{n+1}|\mathcal{U}_0| < y^{-1} \le \gamma^n|\mathcal{U}_0|.$$

If $n\ge 0$ then $|F_\rho(\mathcal{U}_0)|\ge y^{-1}$ for each ρ of length n and there are N^n such ρ:s, so $\eta(y)\ge N^n \ge n\log N$. If $n<0$, it is still true that $\eta(y)\ge n\log N$. Thus

$$\eta(y) \ge n\log N \ge \left(\frac{\log y+\log|\mathcal{U}_0|}{-\log\gamma}-1\right)\log N.$$

Now assume that $s>0$, since otherwise the conclusion of the lemma holds trivially. Take an interval J with positive λ-measure and some positive x. Consider the sets of the form E_ρ that are maximal (with respect to inclusion) subject to the condition of having λ-measure less than or equal to $\lambda(J)/3$. Their union is E and the intersection of two different such sets contains at most one point. Therefore J intersects at least three of them, and must include at least one (the one in the middle). Thus there is some ρ such that $E_\rho \subset J$ and

$$|E_\rho| \ge \gamma|E_{\rho'}| \ge \gamma\left(\frac{\lambda(E_{\rho'})}{A}\right)^{1/s} \ge \gamma\left(\frac{\lambda(J)}{3A}\right)^{1/s},$$

where ρ' is the longest proper prefix of ρ. Every $\mathcal{U}\in D$ satisfies $F_\rho(\mathcal{U})\in D$ and $F_\rho(\mathcal{U})\subset \operatorname{conv} E_\rho$ and $|F_\rho(\mathcal{U})|\geq B^{-1}|E_\rho||\mathcal{U}|$, so it follows that

$$\begin{aligned}\#\{\mathcal{U}\in D;\mathcal{U}\subset J \text{ and } |\mathcal{U}|\geq x^{-1}\} &\geq \#\{\mathcal{U}\in D;\mathcal{U}\subset \operatorname{conv} E_\rho \text{ and } |\mathcal{U}|\geq x^{-1}\}\\ &\geq \eta\big(B^{-1}|E_\rho|x\big)\geq \eta\big(B^{-1}\gamma(3A)^{-1/s}\lambda(J)^{1/s}x\big)\\ &\geq \left(\frac{\log(B^{-1}\gamma(3A)^{-1/s})+\log|\mathcal{U}_0|}{-\log\gamma}-1\right)\log N\\ &\quad+\frac{\log N}{-s\log\gamma}\big(\log\lambda(J)+s\log x\big). \qquad \square\end{aligned}$$

Proof of Theorem 4. Take any $\varepsilon>0$. Using Lemma 8 at the last step, there are constants a and $b>0$ and some x_0 such that if $x\geq x_0$ then

$$\begin{aligned}\psi(J,x)&=\#\{\mathcal{U}\in D;\mathcal{U}\subset J \text{ and } |\mathcal{U}|^m\delta\big(|\mathcal{U}|\big)\geq x^{-1}\}\\ &\geq \#\{\mathcal{U}\in D;\mathcal{U}\subset J \text{ and } |\mathcal{U}|^{m+\alpha+\varepsilon}\geq x^{-1}\}\\ &= \#\{\mathcal{U}\in D;\mathcal{U}\subset J \text{ and } |\mathcal{U}|\geq x^{-1/(m+\alpha+\varepsilon)}\}\\ &\geq a+b\left(\log\lambda(J)+\frac{s}{m+\alpha+\varepsilon}\log x\right),\end{aligned}$$

and thus $\dim_{\mathrm{F}} f_\omega\lambda\geq s/(m+\alpha+\varepsilon)$ almost surely by Theorem 2. Letting $\varepsilon\to 0$ along a countable set proves the theorem. $\square$

6. Proof of Theorem 5 and Theorem 1

Lemma 9. *Let F be a Borel subset of* $\mathbf{R}$ *such that $\mathcal{H}^s(F)>0$. Then there exists some $t\in\mathbf{R}$ and a probability measure λ on $C\cap(F+t)$ such that*

$$\lambda(I)\leq A|I|^s$$

for some constant A and every interval I.

Proof. Since $\mathcal{H}^s(F)>0$, there is a probability measure μ on F such that

$$\mu(I)\leq A_0|I|^s$$

for some constant A_0 and every Borel set I (see [6, Proposition 4.11 and Corollary 4.12], and also [13, Theorem 48]). Then, using that Lebesgue measure is invariant under translation,

$$0<\mu(\mathbf{R})\mathcal{L}(C)=\iint \chi_C(t)\,dt\,d\mu(x)=\iint \chi_C(x+t)\,dt\,d\mu(x)$$
$$=\iint \chi_C(x+t)\,d\mu(x)\,dt=\int \mu_t(C)\,dt,$$

where μ_t denotes the translation of μ by t. It follows that there is some t such that $\mu_t(C)>0$. For that t, let

$$\lambda=\frac{\mu_t|_C}{\mu_t(C)}.$$

Then λ is a probability measure on $C\cap(F+t)$ and

$$\lambda(I)\le\frac{A_0}{\mu_t(C)}|I|^s$$

for every interval I. □

Lemma 10. *Take $E=C$ and let λ be a probability measure on C such that*

$$\lambda(I)\le A|I|^s$$

for every interval I. Then there are constants a and $b>0$ and a function θ, such that $\lim_{x\to\infty}\theta(x)=1$ and

$$\#\{\mathcal{U}\in D\,;\mathcal{U}\subset J \text{ and } |\mathcal{U}|\ge x^{-1}\}\ge a+b\big(\log\lambda(J)+s\theta(x)\log x\big)$$

for every interval J and every $x\ge(1-2c_1)^{-1}$.

Proof. Fix J and x and let n be the unique integer such that

$$(1-2c_{n+2})\prod_{i=1}^{n+1}c_i<x^{-1}\le(1-2c_{n+1})\prod_{i=1}^{n}c_i.$$

For any two elements of D that have the same size there is a larger element of D that lies between them (with respect to the order of $\mathbf{R}$), so there is a unique largest $\mathcal{U}\in D$ that intersects J. Then there is a connected component J' of $J\setminus\mathcal{U}$ such that $\lambda(J')\ge\lambda(J)/2$, and thus

$$|J'|\ge\left(\frac{\lambda(J)}{2A}\right)^{1/s}.$$

Let

$$k=\left\lceil\frac{\log(2A)-\log\lambda(J)}{s\log 2}\right\rceil.$$

Then

$$\prod_{i=1}^{k} c_i \le 2^{-k} \le |J'|,$$

so J' intersects a connected component of $[0,1]\setminus C_k$. That connected component is smaller than $\mathcal{U}$, and therefore $\mathcal{U}\subset[0,1]\setminus C_k$ as well. It follows that J' includes one of the connected components of C_k, and hence there are at least 2^{n-k} elements of D that are included in J and have size $(1-2c_{n+1})\prod_{i=1}^{n} c_i$. Thus

$$\begin{aligned}\#\{\mathcal{U}\in D\,;\mathcal{U}\subset J \text{ and } |\mathcal{U}|\ge x^{-1}\} \ge 2^{n-k} \ge (n-k)\log 2 \\ \ge -\frac{s\log 2+\log(2A)}{s}+\frac{\log\lambda(J)}{s} \\ -\frac{n\log 2\log x}{\log(1-2c_{n+2})+\sum_{i=1}^{n+1}\log c_i},\end{aligned}$$

which has the desired form with

$$a=-\frac{s\log 2+\log(2A)}{s}, \quad b=\frac{1}{s}$$

and

$$\theta(x)=\frac{-n\log 2}{\log(1-2c_{n+2})+\sum_{i=1}^{n+1}\log c_i}.$$

This proves the lemma since $n\to\infty$ when $x\to\infty$ and $c_i\to 1/2$ when $i\to\infty$. □

Proof of Theorem 5. By Lemma 9 there is some $t\in\mathbf{R}$ and a probability measure λ on $C\cap(F+t)$ such that

$$\lambda(I)\le A|I|^s$$

for every interval I. Take any $\varepsilon>0$. Since λ is a probability measure on C, it follows from Lemma 10 that there exist constants a and $b>0$ and some x_0 such that if $x\ge x_0$ then

$$\begin{aligned}\psi(J,x) &= \#\{\mathcal{U}\in D\,;\mathcal{U}\subset J \text{ and } |\mathcal{U}|^m\delta(|\mathcal{U}|)\ge x^{-1}\} \\ &\ge \#\{\mathcal{U}\in D\,;\mathcal{U}\subset J \text{ and } |\mathcal{U}|^{m+\alpha+\varepsilon}\ge x^{-1}\} \\ &= \#\{\mathcal{U}\in D\,;\mathcal{U}\subset J \text{ and } |\mathcal{U}|\ge x^{-1/(m+\alpha+\varepsilon)}\} \\ &\ge a+b\Bigl(\log\lambda(J)+\Bigl(\frac{s}{m+\alpha+\varepsilon}-\varepsilon\Bigr)\log x\Bigr).\end{aligned}$$

Thus $\dim_{\mathrm{F}} f_\omega\lambda\ge s/(m+\alpha+\varepsilon)-\varepsilon$ almost surely by Theorem 2, and letting $\varepsilon\to 0$ along a countable set shows that $\dim_{\mathrm{F}} f_\omega\lambda\ge s/(m+\alpha)$ almost surely. □

Proof of Theorem 1. Let $s=\dim_{\mathrm{H}} F$. If $s=0$ then the statement is trivial, so assume that $s>0$ and let $\{d_k\}_{k=1}^{\infty}$ be an increasing sequence of positive numbers that converges to s. Then there are disjoint open intervals $\{J_k\}_{k=1}^{\infty}$ such that $\mathcal{H}^{d_k}(F\cap J_k)>0$ for all k. They can be constructed recursively along with open intervals $\{I_k\}_{k=1}^{\infty}$ as follows.

Let $I_1=\mathbf{R}$. Assuming that I_k has been defined and that $\dim_{\mathrm{H}} F\cap I_k=s$, there is a compact subset F_k of $F\cap I_k$ such that $0<\mathcal{H}^{d_k}(F_k)<\infty$ (see [13, Theorem 48]). Let x_k be a point such that

$$\mathcal{H}^{d_k}\big(F_k\cap(-\infty,x_k]\big)=\mathcal{H}^{d_k}\big(F_k\cap[x_k,\infty)\big).$$

Then $I_k\setminus\{x_k\}$ is a disjoint union of two open intervals. Choose I_{k+1} to be one of these intervals so that $\dim_{\mathrm{H}}(F\cap I_{k+1})=s$, and let J_k be the other interval. Note that $\mathcal{H}^{d_k}(F\cap J_k)\geq\mathcal{H}^{d_k}(F_k)/2>0$.

Take $E=C$ in the construction of f_ω. By Theorem 5, there is for each k some $t_k\in\mathbf{R}$ and $\omega_k\in\Omega$ such that

$$\dim_{\mathrm{F}} f_{\omega_k}\big(C\cap(F\cap J_k+t_k)\big)\geq\frac{d_k}{m+\alpha}.$$

Let $a_k=\inf J_k\cap(C-t_k)$ and $b_k=\sup J_k\cap(C-t_k)$. Define g_k on $[a_k,b_k]$ by

$$g_k(x)=\big(f_{\omega_k}(x+t_k)-x\big)-\big(f_{\omega_k}(a_k+t_k)-a_k\big),$$

and set g_k to 0 on $(-\infty,a_k)$ and to $g_k(b_k)$ on (b_k,∞). By Theorem 3 the function f_{ω_k} is m times differentiable and $f_{\omega_k}^{(m)}$ is uniformly continuous with modulus $2\|\varphi^{(m+1)}\|_\infty\delta$, and it is clear from the way that f_ω was extended to $\mathbf{R}$ (see Section 2.2) that

$$f'_{\omega_k}=1 \quad\text{and}\quad f''_{\omega_k}=\ldots=f_{\omega_k}^{(m)}=0 \text{ on } C.$$

It follows that g_k is m times differentiable and that $g_k^{(m)}$ is uniformly continuous with modulus $2\|\varphi^{(m+1)}\|_\infty\delta$. Thus by Lemma 7,

$$g(x)=\sum_{k=1}^{\infty}g_k(x)$$

is m times differentiable and $g^{(m)}(x)$ is uniformly continuous with modulus $4\|\varphi^{(m+1)}\|_\infty\delta$. Let

$$f(x)=x+g(x).$$

For $x\in(a_n,b_n)$

$$f(x)=\text{const.}+f_{\omega_n}(x+t_n),$$

and thus

$$\dim_{\mathrm{F}} f(F)\geq\sup_n \dim_{\mathrm{F}} f\big(F\cap(a_n,b_n)\big)\geq\sup_n\frac{d_n}{m+\alpha}=\frac{s}{m+\alpha}. \qquad\square$$

References

1. Besicovitch, A., Sets of fractional dimension (IV): on rational approximation to real numbers, *J. Lond. Math. Soc.* (2) **9** (1934), 126–131.
2. Bluhm, C., On a theorem of Kaufman: Cantor-type construction of linear fractal Salem sets, *Ark. Mat.* **36** (1998), 307–316.
3. Bluhm, C., Fourier asymptotics of statistically self-similar measures, *J. Fourier Anal. Appl.* **5** (1999), 355–362.
4. Ekström, F., The Fourier dimension is not finitely stable, *Real Anal. Exchange* **40** (2015), 397–402.
5. Ekström, F., Persson, T. and Schmeling, J., On the Fourier dimension and a modification, *J. Fractal Geom.* **2** (2015), 309–337.
6. Falconer, K., *Fractal Geometry—Mathematical Foundations and Applications*, Wiley, New York, 1990.
7. Falconer, K., *Techniques in Fractal Geometry*, Wiley, New York, 1997.
8. Jarník, V., Diophantischen Approximationen und Hausdorffsches Mass, *Mat. Sb.* **36** (1929), 371–382.
9. Kahane, J.-P., *Some Random Series of Functions*, 2nd ed., Cambridge University Press, Cambridge, 1985.
10. Kaufman, R., Random measures on planar curves, *Ark. Mat.* **14** (1976), 245–250.
11. Kaufman, R., On the theorem of Jarník and Besicovitch, *Acta Arith.* **39** (1981), 265–267.
12. Mattila, P., *Geometry of Sets and Measures in Euclidean Spaces—Fractals and Rectifiability*, Cambridge University Press, Cambridge, 1995.
13. Rogers, C. A., *Hausdorff Measures*, Cambridge University Press, Cambridge, 1970.
14. Salem, R., On singular monotonic functions whose spectrum has a given Hausdorff dimension, *Ark. Mat.* **1** (1951), 353–365.

Fredrik Ekström
Centre for Mathematical Sciences
Lund University
Box 118, SE-221 00 Lund
Sweden
fredrike@maths.lth.se

Received February 7, 2016
in revised form June 14, 2016
published online July 22, 2016