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# Unicity of graded covers of the category $\mathcal{O}$ of Bernstein–Gelfand–Gelfand

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**Abstract.** We show that the standard graded cover of the well-known category $\mathcal{O}$ of Bernstein–Gelfand–Gelfand can be characterized by its compatibility with the action of the center of the enveloping algebra.

## 1. Introduction

Let $\mathfrak{g}\supset\mathfrak{b}\supset\mathfrak{h}$ be a complex semisimple Lie algebra with a choice of a Borel and a Cartan subalgebra. In [BGG] Bernstein, Gelfand and Gelfand introduced the so-called category $\mathcal{O}=\mathcal{O}(\mathfrak{g},\mathfrak{b})$ of representations of $\mathfrak{g}$. Later on Beilinson and Ginzburg [BG] argued that it is natural to study $\mathbb{Z}$-gradings of category $\mathcal{O}$, see also [BGS]. In this article we introduce the notion of a graded cover, a generalization of the notion of a $\mathbb{Z}$-grading which seemed to be more natural to us, and prove the following uniqueness theorem for graded covers of $\mathcal{O}$, to be explained in more detail in the later parts of this introduction.

**Theorem 1.1.** (Uniqueness of graded covers of $\mathcal{O}$)

(1) *Category $\mathcal{O}$ admits a graded cover compatible with the action of the center of the universal enveloping algebra of $\mathfrak{g}$;*

(2) *If two graded covers of $\mathcal{O}$ are both compatible with the action of the center of the universal enveloping algebra, they are cover-equivalent.*

*Note* 1.2. Graded covers of category $\mathcal{O}$ which are compatible with the action of the center have already been constructed in [S1] and [BGS]. The main point of this article is to show that compatibility with the action of the center already determines a graded cover of $\mathcal{O}$ up to cover-equivalence. To our knowledge this

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statement is already new in the case of $\mathbb{Z}$-gradings [BGS, Section 4.3], although to see this case treated the reader could skip most of the paper and go directly to the proof of Theorem 9.3, which mainly reduces to a careful application of the bicentralizer property as discussed in Section 7. In the body of the paper, we mainly investigate the notion of a graded cover.

*Note* 1.3. An analogous theorem holds with the same proof for the modular versions of category $\mathcal{O}$ considered in [S3] and [RSW]. It is for this case, that the generalization from gradings to graded covers is really needed. An analogous theorem also holds for the category of all Harish-Chandra modules over a complex connected reductive algebraic group, considered as a real Lie group. If we interpret our Harish-Chandra modules as bimodules over the enveloping algebra in the usual way and restrict to the subcategories of objects killed from the left by a fixed power of a given maximal ideal in the center of the enveloping algebra, the same proof in conjunction with [S2] will work. To deduce the general case, some additional arguments are needed to justify passing to the limit, which can be found in the diploma thesis of Rottmaier [R].

*Definition* 1.4. An abelian category in which each object has finite length will henceforth be called an *artinian category.*

*Note* 1.5. This terminology is different from the terminology introduced in [G, p. 356]. There a category is called artinian if it is abelian and all its objects satisfy a descending chain condition, noetherian if it is abelian and all its objects satisfy a ascending chain condition, and finite if it is abelian and all its objects are of finite length.

*Definition* 1.6. By a *graded cover* of an artinian category $\mathcal{A}$ we understand a triple $(\tilde{\mathcal{A}}, v, \varepsilon)$ consisting of an abelian category $\tilde{\mathcal{A}}$ equipped with a strict automorphism [1] "shift the grading", an exact functor $v\colon \tilde{\mathcal{A}} \to \mathcal{A}$ "forget the grading" and an isotransformation of functors $\varepsilon\colon v \stackrel{\sim}{\Longrightarrow} v[1]$, such that the following hold:

(1) For all $M, N \in \tilde{\mathcal{A}}$ the pair $(v, \varepsilon)$ induces an isomorphism on the homomorphism groups $\bigoplus_{i\in\mathbb{Z}} \tilde{\mathcal{A}}(M, N[i]) \xrightarrow{\sim} \mathcal{A}(vM, vN)$;

(2) Given $M \in \mathcal{A}$, $N \in \tilde{\mathcal{A}}$ and an epimorphism $M \twoheadrightarrow vN$ there exists $P \in \tilde{\mathcal{A}}$ and an epimorphism $vP \twoheadrightarrow M$ such that the composition $vP \to vN$ comes from a morphism $P \to N$ in $\tilde{\mathcal{A}}$.

*Remark* 1.7. The main difference to the concept of a $\mathbb{Z}$-grading in the sense of [BGS, Section 4.3] is that for our graded covers we do not ask for any kind of positivity or semisimplicity. In particular, if we start with a grading and "change all degrees to their negatives", we would always get a graded cover again, but most of the time this would not be a grading anymore. If $A$ is a left-artinian ring with a $\mathbb{Z}$-grading, for which in this article we ask no positivity condition whatsoever, then the forgetting of the grading on finitely generated graded modules $v\colon \tilde{A}\text{-Modf}^{\mathbb{Z}} \to A\text{-Modf}$ with the obvious $\varepsilon$ is always a graded cover, see Example 3.1. In Proposition 4.2 we will show that the opposed category of a graded cover is a graded cover of the opposed category. In Proposition 4.2 we discuss in more detail how our notion of a graded cover generalizes the notion of a $\mathbb{Z}$-grading as given in [BGS, Section 4.3].

*Remark* 1.8. We would like to know whether condition (2) will follow in general, when we ask it only in case $N=0$. This amounts to the condition that every object from $\mathcal{A}$ is a quotient of some object coming from $\tilde{\mathcal{A}}$. We rather expect a negative answer, but still would like to see a counterexample.

*Remark* 1.9. We will try to strictly follow a notation, where calligraphic letters $\mathcal{A}$, $\mathcal{B}, \ldots$ denote categories, roman capitals $F$, $G, \ldots$ denote functors between or objects of our categories, and little Greek letters $\tau$, $\varepsilon, \ldots$ denote transformations. The only exception is the "forgetting of grading" in all its variants, to be denoted by the small letter $v$ although it is a functor.

*Definition* 1.10. We say that a graded cover $(\widetilde{\mathcal{O}}, v, \varepsilon)$ of the BGG-category $\mathcal{O}$ is *compatible with the action of the center* $Z \subset \mathrm{U}(\mathfrak{g})$ of the enveloping algebra if the following holds: Given an object $\widetilde{M} \in \widetilde{\mathcal{O}}$ and a maximal ideal $\chi \subset Z$ such that $\chi^n(v\widetilde{M})=0$ for some $n \in \mathbb{N}$, the induced morphism

$$\begin{aligned} Z/\chi^n &\longrightarrow \mathrm{End}_{\mathfrak{g}}(v\widetilde{M}), \\ z+\chi^n &\longmapsto (z\,\cdot\,), \end{aligned}$$

is homogeneous for the grading on $\mathrm{End}_{\mathfrak{g}}(v\widetilde{M})$ induced by the pair $(v, \varepsilon)$ and the natural grading on $Z/\chi^n$ induced by the Harish-Chandra-homomorphism as explained in Note 1.11.

*Note* 1.11. (The natural grading on $Z/\chi^n$) Let $S=\mathrm{S}(\mathfrak{h})$ be the symmetric algebra of our Cartan subalgebra. The Weyl group $W$ acts on it in a natural way. We have the Harish-Chandra isomorphism $Z \xrightarrow{\sim} S^W$. For any maximal ideal $\lambda \subset S$ let $W_\lambda$ be its isotropy group and $Y=Y(\lambda) \subset S$ be the $W_\lambda$-invariants and put $\chi=\lambda \cap Z$ and $\mu=\lambda \cap Y$. Now general results in invariant theory [L, Exercise 3.18] tell us, that the natural maps are in fact isomorphisms

$$Z_\chi^\wedge \xrightarrow{\sim} (S_\lambda^\wedge)^{W_\lambda} \xleftarrow{\sim} Y_\mu^\wedge$$

from the completion of the invariants to the invariants of the completion, leading to isomorphisms $Z/\chi^n \xrightarrow{\sim} Y/\mu^n$. Moving the double of the obvious nonnegative $W$-invariant grading on the polynomial ring $S$, living only in even degrees, with the comorphism of the translation by $\lambda$, we obtain a nonnegative $W_\lambda$-invariant grading on $S$, which induces a nonnegative grading on $Y$ with $\mu = Y^{>0}$ being its part of positive degree. This way we get a natural nonnegative grading on $Y/\mu^n$. The induced nonnegative grading on $Z/\chi^n$ does not depend on the choice of $\lambda$. Indeed, any other choice $\lambda'$ will be conjugate under the Weyl group, so in formulas we have $\lambda' = x\lambda$ for some $x \in W$. We deduce the commutative diagram

$$\begin{tikzcd}
Z^\wedge_\chi \arrow[r, "\sim"] \arrow[d, equal] & (S^\wedge_\lambda)^{W_\lambda} \arrow[d, "x"', "\wr"] & Y(\lambda)^\wedge_\mu \arrow[l, "\sim"'] \arrow[d, "x"', "\wr"] \\
Z^\wedge_\chi \arrow[r, "\sim"] & (S^\wedge_{\lambda'})^{W_{\lambda'}} & Y(\lambda')^\wedge_{\mu'} . \arrow[l, "\sim"']
\end{tikzcd}$$

Now the action of $x$ induces an isomorphism $Y(\lambda) \xrightarrow{\sim} Y(\lambda')$ of graded rings mapping the maximal ideal $\mu$ to $\mu'$ and thus inducing a graded isomorphism $Y(\lambda)/\mu^n \xrightarrow{\sim} Y(\lambda')/(\mu')^n$ compatible with our isomorphism from $Z/\chi^n$ to both sides. Thus the induced grading on $Z/\chi^n$ does not depend on the choice of $\lambda$. We call it the *natural grading on $Z/\chi^n$*.

*Note* 1.12. The graded cover of the category $\mathcal{O}$ coming from Koszul duality is compatible with the action of the center. Our main Theorem 1.1 holds also for graded covers compatible when we take the half of our natural grading on the $Z/\chi^n$, or any integer multiple of it. The proof remains the same.

*Note* 1.13. We obtain as Corollary 9.4, that any two nonnegative $\mathbb{Z}$-gradings on the endomorphism ring of a projective generator of a block of category $\mathcal{O}$, which are compatible with the action of the center and semisimple in degree zero, coincide up to conjugation with a unit from our endomorphism ring.

*Definition* 1.14. Let $(\tilde{\mathcal{A}}, \tilde{v}, \tilde{\varepsilon})$ and $(\hat{\mathcal{A}}, \hat{v}, \hat{\varepsilon})$ be graded covers of an artinian category $\mathcal{A}$. A *cover-equivalence* is a triple $(F, \pi, \varepsilon)$, where $F\colon \tilde{\mathcal{A}} \to \hat{\mathcal{A}}$ is an additive functor and $\varepsilon\colon [1]F \stackrel{\sim}{\Rightarrow} F[1]$ and $\pi\colon \hat{v}F \stackrel{\sim}{\Rightarrow} \tilde{v}$ are isotransformations of functors such that the following diagram of isotransformations of functors commutes:

$$\begin{tikzcd}
\hat{v}[1]F \arrow[r, Rightarrow, "\varepsilon", "\sim"'] \arrow[d, Rightarrow, "\hat{\varepsilon}"', "\wr"] & \hat{v}F[1] \arrow[r, Rightarrow, "\pi", "\sim"'] & \tilde{v}[1] \arrow[d, Rightarrow, "\tilde{\varepsilon}"', "\wr"] \\
\hat{v}F \arrow[rr, Rightarrow, "\pi", "\sim"'] & & \tilde{v}.
\end{tikzcd}$$

Two covers are said to be *cover-equivalent* if there is a cover-equivalence from one to the other. We will show in Note 8.1 that this is indeed an equivalence relation.

*Note* 1.15. This notion of cover-equivalence generalizes the definition of equivalence of gradings given in [BGS, Definition 4.3.1.2]. We will show in Proposition 6.1 that given a left-artinian ring $A$ every graded cover of $A$-Modf is cover-equivalent to the graded cover given by a $\mathbb{Z}$-grading on $A$. The question when two graded covers of this type are cover-equivalent to each other is discussed in Proposition 6.2.

*Note* 1.16. We thank the referee, whose pertinent remarks helped to make the article more readable.

## 2. Construction of compatible graded covers of $\mathcal{O}$

Let us recall how a graded cover of $\mathcal{O}$ is constructed in [S1]. The construction is blockwise. Given a block $\mathcal{O}_\lambda$ one chooses an indecomposable antidominant projective $Q_\lambda \in \mathcal{O}_\lambda$ and shows that the action of the center $Z$ of the enveloping algebra leads to a surjection onto its endomorphism ring $Z \twoheadrightarrow \mathrm{End}_{\mathfrak{g}} Q_\lambda$. Then one shows that the functor $\mathbb{V}_\lambda = \mathrm{Hom}_{\mathfrak{g}}(Q_\lambda, \cdot) \colon \mathcal{O}_\lambda \to \text{Mod-}Z$ is fully faithful on projective objects and if $\chi \subset Z$ is a maximal ideal and $n$ an integer with $\chi^n Q_\lambda = 0$ and $P_\lambda$ is a projective generator of $\mathcal{O}_\lambda$, then $\mathbb{V}_\lambda P_\lambda$ admits a $\mathbb{Z}$-grading making it a $\mathbb{Z}$-graded module over $Z/\chi^n$ for the natural grading on $Z/\chi^n$ explained in Note 1.11. Any choice of such a $\mathbb{Z}$-grading on $\mathbb{V}_\lambda P_\lambda$ induces a $\mathbb{Z}$-grading first on $\mathrm{End}_Z(\mathbb{V}_\lambda P_\lambda)$ and then on $\mathrm{End}_{\mathfrak{g}}(P_\lambda)$. Our graded cover is then obtained as the composition

$$\text{Modfg}^{\mathbb{Z}}\text{-}\mathrm{End}_{\mathfrak{g}}(P_\lambda) \longrightarrow \text{Modfg-}\mathrm{End}_{\mathfrak{g}}(P_\lambda) \xrightarrow{\approx} \mathcal{O}_\lambda$$

of the forgetting of the grading with the equivalence given by tensoring with $P_\lambda$ over $\mathrm{End}_{\mathfrak{g}}(P_\lambda)$. In fact, in [S1] it is even proven that the above choices can be made in such a way that the graded ring $\mathrm{End}_{\mathfrak{g}}(P_\lambda)$ is a Koszul ring.

## 3. Graded covers of artinian categories

*Example* 3.1. Let $A$ be a left-artinian ring. Given any $\mathbb{Z}$-grading $\tilde{}$ on $A$ let $(\tilde{A}\text{-Modf}^{\mathbb{Z}}, v, \varepsilon)$ be the category of finitely generated $\mathbb{Z}$-graded left $\tilde{A}$-modules with morphisms homogeneous of degree 0 and let $(v, \varepsilon)$ be the natural forgetting of the grading. Then $(\tilde{A}\text{-Modf}^{\mathbb{Z}}, v, \varepsilon)$ is a graded cover of $A$-Modf. To check the second condition in Definition 1.6, let $M \twoheadrightarrow N$ be a surjection of a not necessarily graded module onto a graded module. A generating system of $M$ gives a generating system of $N$. Take the nonzero homogeneous components of its elements. It is possible to choose preimages of these components in $M$ in such a way that they generate $M$. Then a suitable graded free $A$-module $P$ with its basis vectors going to these preimages will do the job.

*Example* 3.2. Let $A$ be a ring and $\tilde{}$ be a $\mathbb{Z}$-grading on $A$ and suppose $\mathfrak{m}\subset A$ is a homogeneous ideal such that $A/\mathfrak{m}^n$ is left artinian for all $n$. Then the category of all finitely generated $\mathbb{Z}$-graded left $\tilde{A}$-modules killed by some power of $\mathfrak{m}$ is in a similar way a graded cover of the category of all finitely generated left $A$-modules killed by some power of $\mathfrak{m}$.

*Remark* 3.3. Every graded cover of an artinian category is also artinian, as the length can get only bigger if we apply an exact functor which does not annihilate any object. Our forgetting of grading cannot annihilate any object, as an object with nonzero endomorphism ring by Definition 1.6 (1) is mapped to an object with nonzero endomorphism ring.

*Definition* 3.4. Given a graded cover $(\tilde{\mathcal{A}}, v, \varepsilon)$ of an artinian category $\mathcal{A}$, a *$\mathbb{Z}$-graded lift*, or for short *lift*, of an object $M\in\mathcal{A}$ is a pair $(\widetilde{M}, \varphi)$ with $\widetilde{M}\in\tilde{\mathcal{A}}$ and an isomorphism $\varphi\colon v\widetilde{M}\xrightarrow{\sim}M$ in $\mathcal{A}$.

**Lemma 3.5.** *If for an indecomposable object there exists a lift, then this lift is unique up to isomorphism and shift.*

*Proof.* Given an indecomposable object $M\in\mathcal{A}$, its endomorphism ring $\mathcal{A}(M,M)$ is a local ring. Suppose there are two lifts $(\widetilde{M}, \varphi)$ and $(\widehat{M}, \psi)$ of $M$. We can then decompose the identity morphism of $M$ into homogeneous components. Because the nonunits in $\mathcal{A}(M,M)$ form its maximal ideal, at least one of the homogeneous components has to be a unit, i.e. an isomorphism. □

**Lemma 3.6.** *Given a graded cover $(\tilde{\mathcal{A}}, v, \varepsilon)$ of an artinian category $\mathcal{A}$, a nontrivial object $0\neq M\in\tilde{\mathcal{A}}$ is never isomorphic to its shifted versions $M[i]$ for $i\neq 0$.*

*Proof.* It is enough to prove the statement for simple objects. From now on let $M\in\tilde{A}$ be simple. Suppose there is an isomorphism $M\xrightarrow{\sim}M[i]$ for some $i\neq 0$. Then the endomorphism ring of $vM\in\mathcal{A}$ is given by twisted Laurent polynomials over a skew-field, more precisely $\mathcal{A}(vM, vM)$ is of the form $K^\sigma[X, X^{-1}]$ where the skew-field $K=\tilde{\mathcal{A}}(M,M)$ is the endomorphism ring of $M$ in $\tilde{\mathcal{A}}$, $\sigma\colon K\to K$ is an automorphism of skew-fields and $cX=X\sigma(c)$ for all $c\in K$. More precisely, let $i>0$ be minimal such there exists a nonzero morphism $M\to M[i]$ and let $X$ be such a morphism. It necessarily is an isomorphism and has to be a basis of its homomorphism space as a right and also as a left $K$-module. In addition, there is no nonzero morphism $M \to M[j]$ unless $j \in \mathbb{Z}i$, and $X^n$ is a $K$-basis of the space of homomorphisms $M \to M[ni]$ from the left as well as from the right. Finally, we find an automorphism $\sigma\colon K \to K$ of skew-fields such that $cX = X\sigma(c)$ for all $c \in K$, and these data then completely determine the structure of the endomorphism ring. Obviously 0 and 1 are the only idempotents in $K^\sigma[X, X^{-1}]$, so $vM \in \mathcal{A}$ is an indecomposable object. On the other hand it has finite length by assumption. Thus, by a version of Fittings lemma, all elements in the endomorphism ring $\mathcal{A}(vM, vM)$ have to be either units or nilpotent, and this is just not the case. □

**Lemma 3.7.** *Given a graded cover $(\tilde{\mathcal{A}}, v, \varepsilon)$ of an artinian category $\mathcal{A}$, forgetting of the grading induces a bijection of sets*

$$(\operatorname{irr} \tilde{\mathcal{A}})/\mathbb{Z} \xrightarrow{\sim} \operatorname{irr} \mathcal{A}$$

*between the isomorphism classes of simple objects in the graded cover modulo shift and the isomorphism classes of simple objects in $\mathcal{A}$.*

*Proof.* First we show that each epimorphism $vM \twoheadrightarrow L$ in $\mathcal{A}$, where $M \in \tilde{\mathcal{A}}$ is simple and $L \in \mathcal{A}$ is nonzero, has to be an isomorphism. By definition of a graded cover we find an object $N \in \tilde{\mathcal{A}}$ such that $vN$ maps epimorphically onto $\ker(vM \twoheadrightarrow L)$, i.e. there is a resolution $vN \xrightarrow{\varphi} vM \twoheadrightarrow L$ of $L$ in $\mathcal{A}$ by objects having a $\mathbb{Z}$-graded lift. We can decompose $\varphi$ into homogeneous components $\varphi = \sum_{i \in \mathbb{Z}} \varphi_i$ and each summand $\varphi_i\colon N \to M[i]$ has to be either trivial or an epimorphism in $\tilde{\mathcal{A}}$. Using Lemma 3.6, the nontrivial summands give an epimorphism

$$(\varphi_i)\colon N \twoheadrightarrow \bigoplus_{\substack{i \in \mathbb{Z} \\ \varphi_i \neq 0}} M[i]$$

in $\tilde{\mathcal{A}}$. This has to stay an epimorphism when we forget the grading and postcompose with the morphism $\bigoplus_{\varphi_i \neq 0} vM \twoheadrightarrow vM$ adding up the components. So if $\varphi$ is nonzero, it is a surjection. Thus we have shown that the forgetting of the grading induces a map $(\operatorname{irr} \tilde{\mathcal{A}})/\mathbb{Z} \to \operatorname{irr} \mathcal{A}$. To see that it is surjective, take $L \in \operatorname{irr} \mathcal{A}$. By our assumptions there is an $M \in \tilde{\mathcal{A}}$ together with an epimorphism $vM \twoheadrightarrow L$; consider the set of all objects in $\tilde{\mathcal{A}}$ which map, after forgetting the grading, epimorphically onto $L$ and choose among them an object $M \in \tilde{\mathcal{A}}$ of minimal length. If $M$ is not simple, there is a nontrivial subobject $K \subset M$ with nontrivial quotient and we get a short exact sequence

$$0 \longrightarrow vK \longrightarrow vM \longrightarrow vC \longrightarrow 0.$$

Then the restriction of $vM \twoheadrightarrow L$ to $vK$ must be trivial, otherwise it contradicts the minimal length assumption on $M$; therefore $vC$ has to map epimorphically onto $L$ again contradicting the minimal length assumption on $M$. We conclude that $M \in \tilde{A}$ was already simple. The proof of injectivity is left to the reader. $\square$

**Proposition 3.8.** *Projective objects do lift.*

*Remark* 3.9. Using the stability of graded covers by passing to the opposed categories of Proposition 4.2, we easily deduce that injective objects do lift as well.

*Proof.* It is enough to prove the statement for indecomposable projective objects. Let $P$ be one of those. It is known that $P$ admits a unique simple quotient, which in turn by Lemma 3.7 admits a graded lift, so that we can write

$$P \longrightarrow\!\!\!\!\!\rightarrow vL$$

with $L \in \tilde{A}$ being a simple object. By our definition of a graded cover we can find an epimorphism $vM \twoheadrightarrow P$ such that the composition $vM \twoheadrightarrow P \twoheadrightarrow vL$ comes from a morphism $M \to L$. Assume now in addition, that $M$ has minimal length for such a situation. If we can show that $vM$ is indecomposable we are done, because $P$ is projective and thus the morphism $vM \twoheadrightarrow P$ splits. Suppose $vM \cong A \oplus B$. Then one summand, say $A$, has to map epimorphically onto $L$. If $B$ is not zero, then $B$ also has a simple quotient $\pi\colon B \twoheadrightarrow vE$ and we get an epimorphism $\psi\colon vM \twoheadrightarrow vL \oplus vE$. We can decompose the composition

$$\lambda = \mathrm{pr}_2 \circ \psi\colon vM \longrightarrow vL \oplus vE \longrightarrow\!\!\!\!\!\rightarrow vE$$

into homogeneous components $\sum_{i\in\mathbb{Z}} \lambda_i$. If there was a nonzero $\lambda_i\colon M \to E[i]$ with $E[i] \ncong L$, then $\ker \lambda_i$ would also surject onto $L$ and $v(\ker \lambda_i)$ would surject onto $vL$ and thus onto $P$, contradicting our assumption of minimal length. So we may assume our epimorphism $\psi$ is obtained by forgetting the grading from an epimorphism $(\tilde{\lambda}, \widetilde{\varphi})\colon M \to L \oplus L$. But then again $v(\ker \tilde{\lambda})$ will still surject onto $vL$, contradicting our assumption of minimal length. Thus $vM$ is indecomposable and the split epimorphism $vM \twoheadrightarrow P$ has to be an isomorphism. $\square$

**Corollary 3.10.** *Let $(\tilde{\mathcal{A}}, v, \varepsilon)$ be a graded cover of an artinian category $\mathcal{A}$ and suppose $\mathcal{A}$ has enough projective objects. Then forgetting the grading induces a bijection of sets*

$$(\mathrm{inProj}\, \tilde{\mathcal{A}})/\mathbb{Z} \xrightarrow{\sim} \mathrm{inProj}\, \mathcal{A}$$

*between the isomorphism classes of indecomposable projective objects in the graded cover modulo shift and the isomorphism classes of indecomposable projective objects in $\mathcal{A}$.*

*Proof.* It is well known that if an artinian category $\mathcal{A}$ has enough projective objects, taking the projective cover gives a bijection between the set of isomorphism classes of simple objects in $\mathcal{A}$ and the set of isomorphism classes of indecomposable projective objects in $\mathcal{A}$. So both sides are in bijection to the corresponding sets of isomorphism classes of simple objects. Since each projective object admits a lift by Proposition 3.8, and since such a lift clearly is again projective, the statement follows from the existence and unicity statement about lifts of simple objects of Lemma 3.7. $\square$

**Corollary 3.11.** *Let $A$ be a left-artinian ring with a $\mathbb{Z}$-grading.*

(1) *There exists a complete system of primitive pairwise orthogonal idempotents in $A$ such that all its elements are homogeneous*;

(2) *If $(1_x)_{x\in I}$ is such a system, $(\widetilde{M},\varphi)$ a graded lift of $A$ considered as a left $A$-module, and $\tilde{A}$ is the lift of $A$ given by the $\mathbb{Z}$-grading, then there exists a map $n\colon I\to\mathbb{Z}$ along with an isomorphism of graded left $\tilde{A}$-modules*

$$\widetilde{M} \xrightarrow{\sim} \bigoplus_{x\in I} \tilde{A}1_x[n(x)].$$

*Proof.* It is easy to see that every homomorphism $\tilde{A}\to\tilde{A}[i]$ of graded left $A$-Modules is the multiplication from the right with an element of $A$ homogeneous of degree $i$. There is a direct sum decomposition $\tilde{A}\cong\bigoplus_{x\in I} P_x$ into indecomposable objects in $\tilde{A}$-Modf$^{\mathbb{Z}}$, and its summands are projective. The corresponding idempotent endomorphisms of $\tilde{A}$ are right multiplications with some idempotents $1_x\in A$, homogeneous of degree zero. Forgetting the grading on the $P_x$ we get indecomposable projective $A$-modules by Corollary 3.10, and thus our family $(1_x)_{x\in I}$ is a full set of primitive orthogonal idempotents in $A$. For the second statement let $\widetilde{M}=\bigoplus_{y\in J} Q_y$ be a direct sum decomposition into indecomposable objects in $\tilde{A}$-Modf$^{\mathbb{Z}}$. Again its summands are projective, so by Corollary 3.10 they stay indecomposable when we forget the grading. Thus there is a bijection $\sigma\colon I\xrightarrow{\sim}J$ with $vP_x\cong vQ_{\sigma(x)}$ and by the uniqueness of lifts in Lemma 3.5 we find that $P_x[n(x)]\cong Q_{\sigma(x)}$ for suitable $n(x)\in\mathbb{Z}$. $\square$

## 4. Alternative definition of graded covers

*Note* 4.1. The following proposition establishes the relation to the concept of a $\mathbb{Z}$-grading as introduced in [BGS]. It also ensures our concept of graded cover to be stable upon passing to the opposed categories. Apart from that, this section is not relevant for the rest of this article.

**Proposition 4.2.** *Let $\mathcal{A}$ be an artinian category. A triple $(\tilde{\mathcal{A}}, v, \varepsilon)$ consisting of an abelian category $\tilde{\mathcal{A}}$ equipped with a strict automorphism* [1], *an exact functor $v\colon \tilde{\mathcal{A}} \to \mathcal{A}$ and an isotransformation of functors $\varepsilon\colon v \stackrel{\sim}{\Longrightarrow} v[1]$ is a graded cover of $\mathcal{A}$ if and only if the following hold*:

(1) *For all $M, N \in \tilde{\mathcal{A}}$ and all $j$, the pair $(v, \varepsilon)$ induces isomorphisms of extension spaces $\bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}^j_{\tilde{\mathcal{A}}}(M, N[i]) \xrightarrow{\sim} \operatorname{Ext}^j_{\mathcal{A}}(vM, vN)$;*

(2) *Every irreducible object in $\mathcal{A}$ admits a graded lift.*

*Note* 4.3. Our proof of the backwards implication only uses condition (2) along with condition (1) for $j=0,1$. So it is also equivalent to ask only these seemingly weaker properties.

*Note* 4.4. The definition of a $\mathbb{Z}$-grading in [BGS] consists in asking conditions (1) and (2) and in addition asking that there exists a $\mathbb{Z}$-valued function on the set of irreducibles in $\tilde{\mathcal{A}}$ called weight such that every short exact sequence starting and ending with an irreducible splits unless the weight of the end is smaller than the weight of the starting point.

*Proof.* Let us first show that every graded cover has these two properties. For the second property, this follows from Lemma 3.7. For the first property, we use the description of extensions as homomorphism spaces in the derived category

$$\operatorname{Ext}^j_{\mathcal{A}}(vM, vN) = \varinjlim_{Q} \operatorname{Hot}^j_{\mathcal{A}}(Q, vN),$$

where $Q$ runs over the system of all resolutions $Q \to vM$ of $vM$ and $\operatorname{Hot}^j_{\mathcal{A}}$ denotes homomorphisms of homological degree $j$ in the homotopy category. Our condition (2) on a graded cover ensures that if we take all resolutions $P \to M$ in $\tilde{\mathcal{A}}$, the resolutions $vP \to vM$ will be cofinal in the system of all resolutions of $vM$ and thus give the same limit. To explain this, consider the diagram

$$\begin{tikzcd}
Q_i \arrow[r] & \dots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, two heads] & vM \\
vP_i \arrow[u] \arrow[r] & \dots \arrow[r] & vP_1 \arrow[u] \arrow[r] & vP_0 \arrow[u] \arrow[r, two heads] & vM. \arrow[u, equal]
\end{tikzcd}$$

By condition (2) on a graded cover, there exists a choice for a rightmost square as indicated, with the lower horizontal row coming from a morphism in $\tilde{\mathcal{A}}$. Now suppose inductively that we already have constructed a diagram as above up to degree $i$ with the lower horizontal row exact and coming from $\tilde{\mathcal{A}}$. Then we construct the next step as to be explained in the diagram

$$\begin{tikzcd} Q_{i+1} \arrow[r, equal] & Q_{i+1} \arrow[r, two heads, "1"] & \ker_Q \arrow[r, hook, "1"] & Q_i \arrow[r] & Q_{i-1} \\ vP_{i+1} \arrow[u, "3"'] \arrow[r, two heads, "3"] & A_{i+1} \arrow[u, "2"'] \arrow[r, two heads, "2"] & v\ker_P \arrow[u, "1"'] \arrow[r, hook, "1"] & vP_i \arrow[u] \arrow[r] & vP_{i-1}. \arrow[u] \end{tikzcd}$$

In the first step, we take kernels of the last horizontal arrows constructed and profit from the exactness of the $Q$-complex. In the second step, we form the pullback. In the third step, we again use condition (2). The claim follows. Now let us show in the other direction that our two properties ensure both conditions of the definition of a graded cover Definition 1.6. The first condition is obvious. To show the second condition, we may use pull-backs and induction on the length of the kernel to restrict to the case when $K{:=}\ker(M \twoheadrightarrow vN)$ is simple. Indeed we can otherwise find a simple subobject $E \subset K$ and put $\overline{K}{:=}K/E$ and $\overline{M}{:=}M/E$ and consider the surjection $\overline{M} \twoheadrightarrow vN$. By induction we know it can be prolonged by a surjection $v\overline{P} \twoheadrightarrow \overline{M}$ such that the composition $v\overline{P} \twoheadrightarrow \overline{M} \twoheadrightarrow vN$ comes by forgetting the grading from a morphism $\overline{P} \to N$. Now pulling back this surjection $v\overline{P} \twoheadrightarrow \overline{M}$ we get a surjection $Q \twoheadrightarrow M$ and a short exact sequence $E \hookrightarrow Q \twoheadrightarrow v\overline{P}$. Here the kernel is simple, so if we assume the case of a simple kernel known, we find a surjection $vP \twoheadrightarrow Q$ such that the composition $vP \twoheadrightarrow Q \twoheadrightarrow v\overline{P}$ comes from a morphism $P \to \overline{P}$. Putting all this together, we thus can restrict our attention to the case of a simple kernel $L$. Then by Lemma 3.7 this kernel admits a graded lift, so we arrive at a short exact sequence

$$vL \hookrightarrow M \twoheadrightarrow vN.$$

Now use the isomorphism $\bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}^1_{\tilde{\mathcal{A}}}(N, L[i]) \xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{A}}(vN, vL)$ to write the class $e$ of the above extension as a finite sum of homogeneous components $e{=}\sum_{i=a}^{b} e_i$. We then get a commutative diagram

$$\begin{tikzcd} \bigoplus_{i=a}^{b} L[i] \arrow[r, hook] & \bigoplus_{i=a}^{b} E_i \arrow[r, two heads] & \bigoplus_{i=a}^{b} N \\ \bigoplus_{i=a}^{b} L[i] \arrow[u, equal] \arrow[r, hook] & P \arrow[u] \arrow[r, two heads] & N \arrow[u, "\Delta"'] \end{tikzcd}$$

with a right pullback square and by forgetting the grading another commutative diagram

$$\begin{tikzcd}
\bigoplus_{i=a}^{b} vL \arrow[r, hook] \arrow[d, two heads, "\Sigma"] & vP \arrow[r, two heads] \arrow[d, two heads] & vN \arrow[d, equal] \\
vL \arrow[r, hook] & M \arrow[r, two heads] & vN
\end{tikzcd}$$

with a left pushout square. This finishes the proof. $\square$

## 5. Lifting functors to $\mathbb{Z}$-graded covers

*Definition* 5.1. A $\mathbb{Z}$-*category* is a category $\tilde{\mathcal{A}}$ together with a strict autoequivalence $[1]$ which we call "shift of grading". A $\mathbb{Z}$-*Functor* between $\mathbb{Z}$-categories is a pair $(F, \varepsilon)$ consisting of a functor $F$ between the underlying categories and an isotransformation $\varepsilon\colon F[1] \stackrel{\sim}{\Longrightarrow} [1]F$.

*Definition* 5.2. Let $\mathcal{A}$ and $\mathcal{B}$ be artinian categories, $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor and let $(\tilde{\mathcal{A}}, v, \varepsilon)$ and $(\widetilde{\mathcal{B}}, w, \eta)$ be graded covers of $\mathcal{A}$ and $\mathcal{B}$ respectively. A $\mathbb{Z}$-*graded lift of* $F$ is a triple $(\widetilde{F}, \pi, \varepsilon)$, where $\widetilde{F}\colon \tilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ is an additive $\mathbb{Z}$-functor and $\pi\colon w\widetilde{F} \stackrel{\sim}{\Longrightarrow} Fv$ and $\varepsilon\colon [1]\widetilde{F} \stackrel{\sim}{\Longrightarrow} \widetilde{F}[1]$ are isotransformations of functors, such that the following diagram of isotransformations of functors commutes:

$$\begin{tikzcd}
w[1]\widetilde{F} \arrow[r, Rightarrow, "\varepsilon", "\sim"'] \arrow[d, Rightarrow, "\eta"', "\wr"] & w\widetilde{F}[1] \arrow[r, Rightarrow, "\pi", "\sim"'] & Fv[1] \arrow[d, Rightarrow, "\varepsilon"', "\wr"] \\
w\widetilde{F} \arrow[rr, Rightarrow, "\pi", "\sim"'] & & Fv.
\end{tikzcd}$$

*Note* 5.3. By definition, a cover-equivalence as defined in Definition 1.14 between two graded covers of a given artinian category is a graded lift of the identity functor in the sense of Definition 5.2.

*Note* 5.4. Take two $\mathbb{Z}$-graded left-artinian rings $\tilde{A}$ and $\widetilde{B}$ and in addition a $B$-$A$-bimodule $X$ of finite length as left $B$-module. Then obviously the functor $F = X \otimes_A\colon A\text{-Modf} \to B\text{-Modf}$ admits a graded lift $\widetilde{F}\colon \tilde{A}\text{-Modf}^{\mathbb{Z}} \to \widetilde{B}\text{-Modf}^{\mathbb{Z}}$ if and only if $X$ admits a $\mathbb{Z}$-grading making it into a graded $\widetilde{B}$-$\tilde{A}$-bimodule $\widetilde{X}$.

## 6. Comparing graded covers of module categories

**Proposition 6.1.** *Let $A$ be a left artinian ring and $(\tilde{\mathcal{A}}, v, \varepsilon)$ be a graded cover of $A$-Modf. Then there exists a $\mathbb{Z}$-grading $\tilde{}$ on $A$ such that $\tilde{A}$-Modf$^{\mathbb{Z}}$ is cover-equivalent to $(\tilde{\mathcal{A}}, v, \varepsilon)$.*

*Proof.* By Proposition 3.8 there exists a lift $(\widetilde{M}, \varphi)$ of $A$ in $\tilde{\mathcal{A}}$. By assumption we obtain isomorphisms

$$\bigoplus_i \tilde{\mathcal{A}}(\widetilde{M}, \widetilde{M}[i]) \xrightarrow{\sim} \mathrm{End}_A(A) \xleftarrow{\sim} A^{\mathrm{opp}}.$$

Here the left map comes from forgetting the grading and the right map from the action by right multiplication. We leave it to the reader to check that this grading on $A$ will do the job. □

**Proposition 6.2.** *Let $A$ be a left-artinian ring and let $\tilde{}$ and $\hat{}$ be two $\mathbb{Z}$-gradings on $A$. Then the following statements are equivalent:*

(1) *The $\mathbb{Z}$-graded covers $\tilde{A}$-Modf$^{\mathbb{Z}}$ and $\hat{A}$-Modf$^{\mathbb{Z}}$ of $A$-Modf are cover-equivalent;*

(2) *There exists a $\mathbb{Z}$-grading on the abelian group $A$ making it a graded $\hat{A}$-$\tilde{A}$-bimodule $\hat{\tilde{A}}$;*

(3) *For each complete system of primitive pairwise orthogonal idempotents $(1_x)_{x\in I}$ in $A$, homogeneous for the grading $\hat{A}$, there exist a unit $u\in A^{\times}$ and a function $n\colon I\to\mathbb{Z}$ such that the homogeneous elements of $\tilde{A}$ in degree $i$ are given by*

$$\tilde{A}_i = \bigoplus_{x,y\in I} u 1_x \hat{A}_{n(x)-n(y)+i} 1_y u^{-1};$$

(4) *There exist a complete system of primitive pairwise orthogonal idempotents $(1_x)_{x\in I}$ in $A$, homogeneous for the grading $\hat{A}$, a unit $u\in A^{\times}$ and a function $n\colon I\to\mathbb{Z}$ such that the homogeneous elements of $\tilde{A}$ in degree $i$ are given by*

$$\tilde{A}_i = \bigoplus_{x,y\in I} u 1_x \hat{A}_{n(x)-n(y)+i} 1_y u^{-1}.$$

*Proof.* (1) ⇔ (2) follows from Note 5.4. Next we prove (2) ⇒ (3). Our graded bimodule $\hat{\tilde{A}}$ with $\mathrm{id}\colon \hat{v}\hat{\tilde{A}} \xrightarrow{\sim} A$ is a $\mathbb{Z}$-graded lift in $\hat{A}$-Modf$^{\mathbb{Z}}$ of the left $A$-module $A$. By Corollary 3.11, for each complete system of pairwise orthogonal idempotents $1_x \in A$, homogeneous for $\hat{A}$, there exist integers $n(x)$ along with an isomorphism

$$\psi\colon \hat{\tilde{A}} \xrightarrow{\sim} \bigoplus_{x\in I} \hat{A} 1_x [n(x)]$$

of graded left $\hat{A}$-modules. Here both sides, when considered as ungraded left $A$-modules, admit obvious natural isomorphisms to the left $A$-module $A$. In these terms $\psi$ has to correspond to the right multiplication by a unit $u\in A^{\times}$. Now certainly $h\mapsto\psi h\psi^{-1}$ is an isomorphism between the endomorphism rings of these graded left $\hat{A}$-modules and with $a\mapsto u^{-1}au$ in the lower horizontal row we get a commutative diagram

$$\begin{tikzcd}
\mathrm{End}_A(\tilde{\hat{A}}) \arrow[r, "\sim"] & \mathrm{End}_A(\bigoplus_{x\in I}\hat{A}1_x[n(x)]) \\
A \arrow[u, "\wr"'] \arrow[r, "\sim"] & A. \arrow[u, "\wr"']
\end{tikzcd}$$

Here $\mathrm{End}_A$ means endomorphism rings of ungraded modules, but with the grading coming from the grading on our modules, and the vertical arrows are meant to map $a\in A$ to the multiplication by $a$ from the right, modulo the obvious natural isomorphisms mentioned above. In particular, the vertical maps are not compatible but rather "anticompatible" with the multiplication. Nevertheless, the lower horizontal row has to be homogeneous for the gradings induced by the vertical isomorphisms from the upper horizontal row and from that we deduce that

$$u^{-1}\tilde{A}_i u = \bigoplus_{x,y} 1_x \hat{A}_{n(x)-n(y)+i} 1_y.$$

To prove (3) $\Rightarrow$ (4), just recall that by Example 3.11 we can always find a complete system of primitive pairwise orthogonal idempotents $(1_x)_{x\in I}$ in $A$, which are homogeneous for the grading $\hat{A}$. To finally check (4) $\Rightarrow$ (2), just equip $A$ with the grading $\tilde{A}$ for which the right multiplication by $u$ as a map $(\,\cdot\,u)\colon \tilde{\hat{A}} \xrightarrow{\sim} \bigoplus_x \hat{A}1_x[-n(x)]$ is homogeneous of degree zero. $\square$

## 7. Gradings and bicentralizing modules

*Definition* 7.1. Let $A$ be a ring. An $A$-module $Q$ is called *bicentralizing* if the obvious map is an isomorphism

$$A \xrightarrow{\sim} \mathrm{End}_{\mathrm{End}_A Q}\, Q.$$

*Note* 7.2. For the artinian rings $A$ describing blocks of category $\mathcal{O}$, the modules $Q$ corresponding to the antidominant projective objects are bicentralizing. Indeed, they are known to be injective and the struktursatz [S1] tells us in this case, that the functor

$$\mathrm{Hom}_A(\,\cdot\,,Q)\colon A\text{-Modf}^{\mathrm{opp}} \longrightarrow (\mathrm{End}_A Q)\text{-Modf}$$

is fully faithful on projective modules. On the other hand it maps $A$ to $Q$ and thus induces an isomorphism $(\mathrm{End}_A A)^{\mathrm{opp}} \xrightarrow{\sim} \mathrm{End}_{\mathrm{End}_A Q}\, Q$. We learned this from [KSX].

*Definition* 7.3. Let $A$ be a ring and $Q \in A$-Mod be a bicentralizing $A$-module. Then we call a $\mathbb{Z}$-grading on $A$ and a $\mathbb{Z}$-grading on $\mathrm{End}_A Q$ *compatible* if there exists a $\mathbb{Z}$-grading on $Q$ such that $Q$ becomes a graded module for both of them.

**Proposition 7.4.** (Compatibility implies cover-equivalence) *Let $A$ be a left-artinian ring and $Q$ be a bicentralizing $A$-module. Then any two $\mathbb{Z}$-gradings on $A$, which are compatible with the same $\mathbb{Z}$-grading on* $\mathrm{End}_A Q$, *give rise to cover-equivalent covers of* $A$-Modf.

*Proof.* Let $\tilde{A}$ and $\hat{A}$ be our two $\mathbb{Z}$-gradings on $A$. By assumption, there exist $\mathbb{Z}$-gradings $\widetilde{Q}$ and $\widehat{Q}$ on $Q$ compatible with the given grading on $\mathrm{End}_A Q$ and compatible with the gradings $\tilde{A}$ and $\hat{A}$ of $A$ respectively. But then let us consider the isomorphism

$$A \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{End}_A Q}(\widetilde{Q}, \widehat{Q})$$

given by the left action of $A$ on $Q$ and denote by $\hat{\tilde{A}}$ the group $A$ with the $\mathbb{Z}$-grading coming from the right-hand side of this isomorphism by transport of structure. Then $\hat{\tilde{A}}$ with its obvious left and right action is a $\mathbb{Z}$-graded $\hat{A}$-$\tilde{A}$-bimodule and the proof is finished by Proposition 6.2. □

**Proposition 7.5.** (Compatibility criterion) *Let $A$ be a left-artinian ring and let $Q$ be a projective finite length bicentralizing left $A$-module with commutative endomorphism ring. Then the compatibility of a grading on $A$ with a grading on* $\mathrm{End}_A Q$ *is equivalent to the homogeneity of the composition*

$$\mathrm{End}_A Q \longrightarrow \mathrm{End}_{\mathrm{End}_A Q}\, Q \xleftarrow{\sim} A.$$

*Proof.* If the gradings are compatible, clearly this composition is homogeneous. If on the other hand the composition is homogeneous, then any grading on $Q$ making it a graded $A$-module will show the compatibility. However such a grading always exists by Proposition 3.8. □

## 8. Cover equivalence is an equivalence relation

*Note* 8.1. Clearly cover-equivalence of graded covers is a reflexive relation. We are now going to show that it is also symmetric and transitive, so it is indeed an equivalence relation on the set of graded covers of a fixed artinian category. This allows us to speak of cover-equivalent graded covers without having to specify from which of the two there exists a cover-equivalence to the other.

**Lemma 8.2.** *Any graded lift $\widetilde{F}$ of an equivalence $F$ of artinian categories is again an equivalence of categories.*

*Proof.* Since a direct sum of morphisms of abelian groups is an isomorphism if and only if the individual morphisms are isomorphisms, a graded lift of a fully faithful additive functor is clearly fully faithful itself. We just have to show that if $F$ is an equivalence of categories, then $\widetilde{F}$ is essentially surjective. So take an object $\widetilde{B}\in\widetilde{\mathcal{B}}$. By assumption there is an object $A\in\mathcal{A}$ with an isomorphism $FA\xrightarrow{\sim}w\widetilde{B}$. By the definition of graded cover and Proposition 4.2, there are $\widetilde{X},\widetilde{Y}\in\tilde{\mathcal{A}}$ with an epimorphism and a monomorphism $v\widetilde{X}\twoheadrightarrow A\hookrightarrow v\widetilde{Y}$. Applying $F$ we find an epimorphism and a monomorphism $w\widetilde{F}\widetilde{X}\twoheadrightarrow w\widetilde{B}\hookrightarrow w\widetilde{F}\widetilde{Y}$. Now if $\lambda_i\colon \widetilde{F}\widetilde{X}[i]\to\widetilde{B}$, for $i$ running through a finite set $I\subset\mathbb{Z}$ of degrees, are the homogeneous components of the first map, then they together define the left epimorphism of a sequence

$$\bigoplus_{i\in I}\widetilde{F}\widetilde{X}[i]\longrightarrow\!\!\!\!\!\rightarrow \widetilde{B}\hookrightarrow\bigoplus_{j\in J}\widetilde{F}\widetilde{Y}[j]$$

in $\widetilde{\mathcal{B}}$. The left monomorphism is constructed dually. But the composition in this sequence has to come from a morphism in $\tilde{\mathcal{A}}$, and the image of this morphism is the looked-for object of $\tilde{\mathcal{A}}$ essentially going to $\widetilde{B}$ under our functor $\widetilde{F}$. $\square$

*Note* 8.3. (Symmetry of cover-equivalence) In particular, given a cover-equivalence $(F,\pi)$ of graded covers the functor $F$ is always an equivalence of categories. Given a quasiinverse $(G,\eta)$ with $\eta\colon \mathrm{Id}\overset{\sim}{\Longrightarrow}FG$ being an isotransformation, from $\pi\colon \hat{v}F\overset{\sim}{\Longrightarrow}\tilde{v}$ we get as the composition $\hat{v}\overset{\sim}{\Longrightarrow}\hat{v}FG\overset{\sim}{\Longrightarrow}\tilde{v}G$, or more precisely $(\pi G)(\hat{v}\eta)$, an isotransformation $\tau\colon \hat{v}\overset{\sim}{\Longrightarrow}\tilde{v}G$. Similarly from $\varepsilon\colon [1]F\overset{\sim}{\Longrightarrow}F[1]$ we get a unique $\varepsilon\colon [1]G\overset{\sim}{\Longrightarrow}G[1]$ such that the composition

$$[1]\overset{\sim}{\Longrightarrow}[1]FG\overset{\sim}{\Longrightarrow}F[1]G\overset{\sim}{\Longrightarrow}FG[1]\overset{\sim}{\Longrightarrow}[1]$$

with our adjointness $\eta$ at both ends and the old and the newly to be defined $\varepsilon$ in the middle is the identity transformation. Then one may check that $(G,\tau,\varepsilon)$ is also a cover-equivalence.

*Note* 8.4. (Transitivity of cover-equivalence) Let finally be given artinian categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, additive functors $F\colon\mathcal{A}\to\mathcal{B}$ and $G\colon\mathcal{B}\to\mathcal{C}$, graded covers $(\tilde{\mathcal{A}},v,\varepsilon)$, $(\widetilde{\mathcal{B}},w,\eta)$, $(\tilde{\mathcal{C}},u,\theta)$ of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and graded lifts $(\widetilde{F},\pi,\varepsilon)$ of $F$ and $(\widetilde{G},\tau,\varepsilon)$ of $G$. Then $(\widetilde{G}\widetilde{F},(G\pi)(\tau\widetilde{F}),\varepsilon)$ is a lift of $GF$, where we leave the definition of the last $\varepsilon$ to the reader. In particular the relation of cover-equivalence is transitive.

## 9. Proof of the main theorem

*Note* 9.1. (Pushforward of graded covers) Let $(\tilde{\mathcal{A}}, \tilde{v}, \tilde{\varepsilon})$ be a graded cover of an artinian category $\mathcal{A}$ and let $E\colon \mathcal{A} \xrightarrow{\approx} \mathcal{B}$ be an equivalence of categories. Then obviously $(\tilde{\mathcal{A}}, E\tilde{v}, E(\tilde{\varepsilon}))$ is a graded cover of $\mathcal{B}$. We call it the "pushforward" of our graded cover of $\mathcal{A}$ along $E$. Obviously two graded covers of $\mathcal{A}$ are cover-equivalent if and only if their pushforwards are cover-equivalent as graded covers of $\mathcal{B}$.

*Note* 9.2. Recall the block decomposition of $\mathcal{O}$. It is enough to prove Theorem 1.1 for each block $\mathcal{O}_\lambda$ of $\mathcal{O}$.

**Theorem 9.3.** (Uniqueness of graded covers of $\mathcal{O}$) *If two graded covers $(\widetilde{\mathcal{O}}_\lambda, \tilde{v}, \tilde{\varepsilon})$ and $(\widehat{\mathcal{O}}_\lambda, \hat{v}, \hat{\varepsilon})$ of a block $\mathcal{O}_\lambda$ of category $\mathcal{O}$ are both compatible with the action of the center, they are cover-equivalent.*

*Proof.* We find a left artinian ring $A$ along with an equivalence

$$E\colon \mathcal{O}_\lambda \xrightarrow{\sim} A\text{-Modf}.$$

It is sufficient to show that the pushforwards of our covers along $E$ are cover-equivalent. Both these pushforwards are cover-equivalent to covers corresponding to $\mathbb{Z}$-gradings $\hat{A}$ and $\tilde{A}$ on $A$, which are compatible with the action of the center in the sense that the maps $Z/\chi^n \to \hat{A}$ and $Z/\chi^n \to \tilde{A}$ are homogeneous for the corresponding central character $\chi \in \operatorname{Max} Z$ and $n$ so big that our maps are well defined. But now the antidominant projective objects of our block corresponds to a bicentralizing $A$-module $Q$ by Note 7.2, and the action of the center induces a surjection $Z/\chi^n \twoheadrightarrow \operatorname{End}_A Q$ by the endomorphismensatz of [S1], and by the compatibility criterion in Proposition 7.5 both our $\mathbb{Z}$-gradings on $A$ are compatible with the same $\mathbb{Z}$-grading on $\operatorname{End}_A Q$. Then however the corresponding covers are cover-equivalent by Proposition 7.4. □

**Corollary 9.4.** *Any two nonnegative $\mathbb{Z}$-gradings on the endomorphism ring of a projective generator of a block of category $\mathcal{O}$, which are both compatible with the action of the center and semisimple in degree zero, coincide up to conjugation with a unit from our endomorphism ring.*

*Proof.* By [BGS], on the endomorphism ring $A$ of our projective generator there exists a $\mathbb{Z}$-grading with all the above properties and with the additional property, that the equivalence relation $\sim$ on the set of indecomposable homogeneous idempotents $e, f, \ldots$ generated by asking $((eAf)_1 \neq 0 \neq (fAe)_1 \Rightarrow e \sim f)$ has only one equivalence class. Now by Proposition 6.2 part (4) any $\mathbb{Z}$-grading giving rise to a cover-equivalent graded cover is related to the given one by the conjugation with a unit and a $\mathbb{Z}$-valued function on a complete system of homogeneous pairwise orthogonal idempotents. If the new grading is nonnegative and semisimple in degree zero too, then the said function has to be constant on equivalence classes and thus in our case has to be constant. This in turn means that the two gradings are conjugate to one another by a unit. $\square$

## References


[BGG] Bernstein, J. N., Gelfand, I. M. and Gelfand, S. I., A certain category of $\mathfrak{g}$-modules, *Funktsional. Anal. i Prilozhen* **10**:2 (1976), 1–8 (Russian). English transl.: *Funct. Anal. Appl.* **10** (1976), 87–92.

[BG] Beilinson, A. A. and Ginsburg, V., Mixed categories, Ext-duality and representations (results and conjectures), *Preprint*, 1986.

[BGS] Beilinson, A. A., Ginsburg, V. and Soergel, W., Koszul duality patterns in representation theory, *J. Appl. Math. Stoch. Anal.* **9** (1996), 473–527.

[G] Gabriel, P., Des catégories abéliennes, *Bull. Soc. Math. France* **90** (1962), 323–448.

[KSX] König, S., Slungård, I. H. and Xi, C., Double centralizer properties, dominant dimension, and tilting modules, *J. Algebra* **240** (2001), 393–412.

[L] Liu, Q., *Algebraic Geometry and Arithmetic Curves*, Oxford Graduate Texts in Mathematics, Oxford, 2009.

[R] Rottmaier, M., *Diploma thesis*, University of Freiburg, Freiburg, 2012.

[RSW] Riche, S., Soergel, W. and Williamson, G., Modular Koszul duality, *Compos. Math.* **150** (2014), 273–332.

[S1] Soergel, W., Kategorie $\mathscr{O}$, Perverse Garben und Moduln über den Koinvarianten zur Weylgruppe, *J. Amer. Math. Soc.* **3** (1990), 421–445.

[S2] Soergel, W., The combinatorics of Harish-Chandra bimodules, *J. Reine Angew. Math.* **429** (1992), 49–74.

[S3] Soergel, W., On the relation between intersection cohomology and representation theory in positive characteristic, *J. Pure Appl. Algebra* **152** (2000), 311–335.



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