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# On Möbius orthogonality for interval maps of zero entropy and orientation-preserving circle homeomorphisms

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**Abstract.** We will prove Sarnak's conjecture on Möbius disjointness for continuous interval maps of zero entropy and also for orientation-preserving circle homeomorphisms by reducing these result to a well-known theorem of Davenport from 1937.

## 1. Introduction

In [7], Sarnak discussed the disjointness conjecture concerning the Möbius function $\mu(n)$. The assertion of the conjecture is the following. Let $X$ be a compact metric space and $f$ be a continuous map of zero entropy. Then for any pair $(X, f)$ as $n \to \infty$,

$$S_n(f(x), \varphi) = \frac{1}{n} \sum_{k=1}^{n} \mu(k) \varphi(f^k(x)) = o(1), \tag{1}$$

where $x \in X$ and $\varphi \in C(X)$.

The conjecture is known to be true for several dynamical systems. For a Kronecker flow it is proved in [9] and [3], while when $(X, f)$ is a translation on a compact nilmanifold it is proved in [4]. In [2] it is also established for horocycle flows. In this paper we are going to prove the following two theorems.

**Theorem 1.1.** *Let $f: [0,1] \to [0,1]$ and $\varphi: [0,1] \to \mathbb{R}$ be continuous maps, and assume that $f$ has zero entropy. Then (1) holds.*

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**Theorem 1.2.** *Let $f\colon S^1\to S^1$ be an orientation-preserving circle homeomorphism and $\varphi\colon S^1\to\mathbb{R}$ be a continuous map. Then (1) holds.*

## 2. Some preliminary results

Before proving Theorems 1.1 and 1.2 we present some technical results which will be used later. We first recall Lemmas 5 and 6 in Davenport's paper [3].

**Lemma 2.1.** *Let $h>0$ and $N\in\mathbb{N}$. If $l,q\in\mathbb{N}$, $(l,q)=1$ and $q\le(\log N)^h$, then*

$$\sum_{\substack{n=1\\ n\equiv l \,(\mathrm{mod}\, q)}}^{N} \mu(n)=O(Ne^{-C(h)\sqrt{\log N}}).$$

**Lemma 2.2.** *In Lemma 2.1 the condition $(l,q)=1$ can be omitted.*

Combining these two lemmas we obtain the following result.

**Lemma 2.3.** *Let $h>0$ and $N\in\mathbb{N}$. Then if $l,q\in\mathbb{N}$ and $q\le(\log N)^h$, then*

$$\sum_{\substack{n=1\\ n\equiv l \,(\mathrm{mod}\, q)}}^{N} \mu(n)=O(Ne^{-C(h)\sqrt{\log N}}).$$

We will also need Theorem 1, p. 319 in [3], which can be formulated as follows.

**Theorem 2.4.** *For any given $K>0$,*

$$\sum_{n=1}^{N}\mu(n)e^{2\pi in\theta}=O(N(\log N)^{-K}),$$

*uniformly in $\theta$.*

From the above theorem, using approximation of a continuous function by trigonometric polynomials one can obtain the following corollary.

**Corollary 2.5.** *If $\varphi\in C(S^1)$ and $R_\rho$ is a rigid rotation by the irrational number $\rho$, then*

$$\sum_{n=1}^{N}\mu(n)\varphi(R_\rho^n(x))=o(N)\quad \textit{for all } x\in S^1.$$

Next we will continue with the following lemmas.

**Lemma 2.6.** *Let* $\{x_n\}_{n=1}^{\infty}$ *be an eventually periodic sequence, i.e.* $x_n=x_{n+m}$ *for some fixed number* $m\in\mathbb{N}$ *and for any* $n\geq n_0$, *then*

$$\sum_{n=1}^{N}\mu(n)x_n=o(N).$$

*Proof.* It is clear, that

$$\sum_{n=1}^{N}\mu(n)x_n=\sum_{l=0}^{m-1}x_{n_0+l}\Bigg(\sum_{\substack{1\leq n\leq N\\ n\equiv n_0+l\ (\mathrm{mod}\, m)}}\mu(n)\Bigg)+o(N).$$

If $a_l=\min_n\{n:n\equiv n_0+l\ (\mathrm{mod}\ m)$ and $1\leq n\leq N\}$, then

$$\sum_{\substack{1\leq n\leq N\\ n\equiv n_0+l\ (\mathrm{mod}\, m)}}\mu(n)=\sum_{\substack{1\leq n\leq N\\ n\equiv a_l\ (\mathrm{mod}\, m)}}\mu(n).$$

According to Lemma 2.1, for large enough integers $N$ ($m\leq(\log N)^h$, and $h>0$),

$$\frac{1}{N}\Bigg|\sum_{\substack{1\leq n\leq N\\ n\equiv a_l\ (\mathrm{mod}\, m)}}\mu(n)\Bigg|=\left|\frac{O(Ne^{-C(h)\sqrt{\log N}})}{Ne^{-C(h)\sqrt{\log N}}}e^{-C(h)\sqrt{\log N}}\right|\leq\left|\frac{C_0(h)}{e^{C(h)\sqrt{\log N}}}\right|\to 0,$$

as $N\to\infty$. $\square$

**Lemma 2.7.** *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence of real numbers such that* $|x_n|\leq 1$ *for all* $n\in\mathbb{N}$ *and assume that there are* $n_0$ *and* $k$ *such that if* $n,m\geq n_0$, $x_n\neq 0$, $x_m\neq 0$ *and* $n\neq m$, *then* $|n-m|\geq k$. *Then we have*

$$\varlimsup_{N\to\infty}\frac{1}{N}\left|\sum_{n=1}^{N}x_n\right|\leq\frac{1}{k}.$$

*Proof.* First we observe that

$$\varlimsup_{N\to\infty}\frac{1}{N}\left|\sum_{n=1}^{N}x_n\right|\leq\varlimsup_{N\to\infty}\frac{1}{N}\left|\sum_{n=1}^{n_0}x_n\right|+\varlimsup_{N\to\infty}\frac{1}{N}\left|\sum_{n=n_0+1}^{N}x_n\right|=\varlimsup_{N\to\infty}\frac{1}{N}\left|\sum_{n=n_0+1}^{N}x_n\right|.$$

Now, as the number of non-zero elements from $\{x_n\}_{n=1}^{\infty}$ of indices between $n_0+1$ and $N$ cannot be greater then $(N-n_0)/k$, we get

$$\varlimsup_{N\to\infty}\frac{1}{N}\left|\sum_{n=n_0+1}^{N}x_n\right|\leq\varlimsup_{N\to\infty}\left|\frac{N-n_0}{kN}\right|=\varlimsup_{N\to\infty}\left|\frac{1}{k}-\frac{n_0}{kN}\right|=\frac{1}{k}.\quad\square$$

## 3. Proof of Theorem 1.1

We now prove the first main result of our paper.

Recall that the $\omega$-limit set of a point $x$ is the set defined as

$$\omega(x, f) = \bigcap_{n=0}^{\infty} \overline{\{f^k(x) : k \geq n\}}.$$

We divide the proof into three different cases.

*Case* 1. *$x$ is eventually periodic.* In this case the result immediately follows from Lemma 2.6.

*Case* 2. *$\omega(x, f)$ is a finite set.* According to Lemma 5.4.3 in [6], if $\omega(x, f)$, is a finite set, then it is an orbit of some periodic point, i.e.

$$\omega(x, f) = \{x_1, x_2, ..., x_s\}, \tag{2}$$

where

$$f^k(x_1) = x_{k+1} \text{ for } k < s \quad \text{and} \quad f^s(x_1) = x_1.$$

Now let $\varepsilon > 0$ be a real number such that the intervals

$$(x_1 - \varepsilon, x_1 + \varepsilon), \quad (x_2 - \varepsilon, x_2 + \varepsilon), \quad ..., \quad (x_s - \varepsilon, x_s + \varepsilon)$$

are disjoint. For every $l \in \mathbb{N}$ define

$$A_{x_1}^l = \{n \in \mathbb{N} \colon f^n(x) \in (x_1 - \varepsilon, x_1 + \varepsilon) \text{ and } n \geq l\}.$$

Then it is clear, that

$$\lim_{\substack{n \to \infty \\ n \in A_{x_1}^l}} f^n(x) = x_1.$$

Hence

$$\lim_{\substack{n \to \infty \\ n \in A_{x_1}^l + r}} f^n(x) = x_{r+1}$$

for $r < s$, where $A_{x_1}^l + r = \{n + r \colon n \in A_{x_1}^l\}$. Since $x_1$ is a periodic point, we get

$$\lim_{\substack{n \to \infty \\ n \in A_{x_1}^l + s}} f^n(x) = x_1.$$

So, if $l$ is sufficiently large, then

$$A_{x_1}^l + s \subset A_{x_1}^l.$$

From the above mentioned properties, it follows, that for a sufficiently large number $l_0\in\mathbb{N}$ for any $n\in A_{x_1}^{l_0}$, and $m\in\mathbb{N}_0$, we will have

$$f^{n+m}(x)\in(x_{r+1}-\varepsilon, x_{r+1}+\varepsilon),\quad \text{where } 0\le r<s \text{ and } m\equiv r \pmod{s}.$$

Therefore we get, that if

$$n_0=\min_n\{n\in A_{x_1}^{l_0}\},$$

then

$$|f^{n_0+n}(x)-f^n(x_1)|\le\varepsilon\quad \text{for any } n\in\mathbb{N}.$$

If we define $x'$ as the $n_0$-th preimage of $x_1$ on its $\omega$-limit set (2), i.e. $f^{n_0}(x')=x_1$, then

$$|f^{n+n_0}(x)-f^{n+n_0}(x')|\le\varepsilon\quad \text{for } n\in\mathbb{N}.$$

Fix $\varepsilon'>0$. Then for a given continuous function $\varphi$ we use uniform continuity to find $\varepsilon>0$ and $n_0=n_0(\varepsilon)$ such that

$$|\varphi(f^n(x))-\varphi(f^n(x'))|\le\varepsilon'\quad \text{for } n\ge n_0. \tag{3}$$

From (3) we get

$$\overline{\lim_{n\to\infty}}\,|[S_n(f(x),\varphi)-S_n(f(x'),\varphi)]|\le\varepsilon_0. \tag{4}$$

But the sequence

$$x_n=\varphi(f^n(x'))$$

is periodic. Therefore from Lemma 2.6 we get that

$$\left|\frac{1}{n}\sum_{k=1}^{n}\mu(k)\varphi(f^k(x'))\right|\to 0,\quad \text{when } n\to 0. \tag{5}$$

Then from (4) and (5) together we get (1).

*Case* 3. *$\omega(x,f)$ is an infinite set.* We now need a result essentially due to Smítal [8], but also implicitly contained in several papers of Sharkovskii and stated without proof in a paper of Blokh [1]. For our purposes we need a somewhat extended version as formulated in the notes of Ruette [6]. According to Proposition 5.4.5 of [6] for any interval map of zero entropy and $x\in[0,1]$ such that $\omega(x,f)$ is infinite, for all $k\ge 0$, one can find a sequence of closed intervals $\{L_k^i\}_{k\ge 0, 0\le i<2^k}$ such that

(i) $f(L_k^i)=L_k^{(i+1) \pmod{2^k}}$ for $0\le i<2^k$;
(ii) the intervals $\{L_k^i\}_{0\le i<2^k}$ have pairwise disjoint interiors;
(iii) $\omega(x,f)\subset\bigcup_{i=0}^{2^k-1}L_k^i$.

Let us fix $k\in\mathbb{N}$. Since $\varphi$ is a continuous function there exists a function $\varphi_0$ of the form

$$\varphi_0(x)=\sum_{r=1}^{m} d_r\psi_{(a_r,b_r)}(x),$$

where

$$\psi_{(a,b)}(x)=\begin{cases}1, & x\in(a,b),\\ \frac{1}{2}, & x=a \text{ or } x=b,\\ 0, & \text{otherwise,}\end{cases} \tag{6}$$

such that

$$|\varphi(x)-\varphi_0(x)|<\varepsilon, \quad \text{for } x\in(0,1), \tag{7}$$

the intervals $\{(a_r,b_r)\}_{r=1}^{m}$ are disjoint and the points $a_r$, $b_r$, $r=1,...,m$, are different from the endpoints of the intervals $L_k^i$, $i=0,...,2^k-1$.

This is a slight modification of the standard approximation of continuous function by step functions and follows from the uniform continuity of $\varphi$.

First we note that, if for any $x\in[0,1]$ the iterates of $x$ visit any of the endpoints 0 and 1 more then once, then $x$ is eventually periodic and (1) follows from Lemma 2.6. Therefore, we can suppose, that the orbit of $x$ does not contain any of the points 0 and 1. Then, from (7) it follows that it is enough to prove (1) for functions of the form (6), i.e. when

$$\varphi(x)=\psi_{(a,b)}(x),$$

where $0\le a\le b\le 1$. In the following we write $I=(a,b)$.

Since $\omega(x,f)$ is infinite and from property (iii), the interior of one of the intervals $\{L_k^i\}_{i=0}^{2^k-1}$ will contain infinitely many points from $\omega(x,f)$. Let say it is the interval $L_k^i$. So

$$\omega(x,f)\cap\operatorname{int}(L_k^i)\neq\varnothing.$$

Therefore there exists $n_0\in\mathbb{N}$ such that

$$f^{n_0}(x)\in\operatorname{int}(L_k^i).$$

But in this case, from property (i),

$$f^{n_0+m}(x)\in L_k^{(m+i)\,(\mathrm{mod}\,2^k)} \quad \text{for } m\in\mathbb{N}. \tag{8}$$

Then

$$\sum_{n\le N}\mu(n)\psi_{(a,b)}(f^n(x))=\sum_{n<n_0}\mu(n)\psi_{(a,b)}(f^n(x))+\sum_{n=n_0}^{N}\mu(n)\psi_{(a,b)}(f^n(x))$$

$$=o(N)+\sum_{s=0}^{2^k-1}\sum_{\substack{n_0\le n\le N\\ f^n(x)\in L_k^s}}\mu(n)\psi_{(a,b)}(f^n(x)).$$

Now consider the sum

$$A_N^s = \sum_{\substack{n_0 \le n \le N \\ f^n(x) \in L_k^s}} \mu(n)\psi_{(a,b)}(f^n(x)).$$

For $s=0,\ldots,2^k-1$ define the sequences

$$x_n^s = \begin{cases} 0, & n<n_0, \\ \psi_{(a,b)}(f^n(x))\chi_{L_k^s}(f^n(x)), & n\ge n_0, \end{cases} \tag{9}$$

where $\chi_{L_k^s}(x)$ is the characteristic function of the interval $L_k^s$. One can see that, if $L_k^s$ does not contain any of the endpoints of $I$, then the sequence $\{x_n^s\}_{n=1}^{\infty}$ is eventually periodic (according to (8) and (ii)). Therefore from Lemma 2.6 we will get

$$A_N^s = o(N). \tag{10}$$

If $L_k^s$ contains any of the endpoint of $I$, then we note that, if $x_n^s \neq 0$ and $x_m^s \neq 0$, where $n, m \ge n_0$, $n \neq m$, then $|n-m| \ge 2^k$ (again from (8) and (ii)). Therefore according to Lemma 2.7,

$$\varlimsup_{N\to\infty} \left| \frac{A_N^s}{N} \right| \le \frac{1}{2^k}. \tag{11}$$

Since the intervals $\{L_k^i\}_{i=0}^{2^k-1}$ are disjoint, at most two of them can contain an endpoint of $I$, and for all those intervals we will have the estimate (11). Therefore from (10) and (11),

$$\varlimsup_{N\to\infty} \frac{1}{N} \left| \sum_{n\le N} \mu(n)\psi_{(a,b)}(f^n(x)) \right| \le \varlimsup_{N\to\infty} \frac{1}{N} \sum_{s=0}^{2^k-1} |A_N^s| \le \frac{2}{2^k} = \frac{1}{2^{k-1}}.$$

As $k$ is an arbitrary integer, we get (1).

## 4. Proof of Theorem 1.2

Let us now proceed to the second part of the paper, the proof of Theorem 1.2. It will be based on a semi-conjugacy to the known case of a circle rotation. We first state and prove a result, which will be an important step of the proof in the case when the rotation number is irrational.

**Lemma 4.1.** *Let $J \subseteq S^1$ be an interval which can be either closed, open or half open. Then for a rigid rotation $R_\rho$ with irrational rotation number $\rho$, we have*

$$\sum_{n=1}^{N} \mu(n)\chi_J(R_\rho^n(x)) = o(N) \tag{12}$$

*for all $x \in S^1$.*

*Proof.* First note that the endpoints of $J$ do not play any role in the limit (12), as all the iterates of $x$ are distinct. So let us suppose that $J$ is a closed interval. If $J = \varnothing$, then there is nothing to prove. If $J$ is the entire $S^1$, then (12) follows from Corollary 2.5. If the interior of $S^1 \setminus J$ is not empty, then for sufficiently small $\varepsilon$, one can find an $\varepsilon$-neighbourhood $J_0$ of $J$ containing $J$, that is if the endpoints of $J$ are the points $c$ and $d$, then we can find an interval $J_0 = [c-\varepsilon, d+\varepsilon]$ the interior of which will contain the closure of $J$. So we will be able to extend the function $\chi_J$ from $J$ onto $J_0$ to get a new function $\varphi_{J_0}(x)$ which will be continuous. We can define $\varphi_{J_0}(x)$ as follows

$$\varphi_{J_0}(x) = \begin{cases} 1, & \text{for } x \in (c,d), \\ 0, & x \notin (c-\varepsilon, d+\varepsilon), \\ & \text{linearly and continuously otherwise.} \end{cases} \tag{13}$$

As mentioned the function $\varphi_{J_0}(x)$ will be continuous. It follows, that

$$\begin{aligned} \overline{\lim_{n\to\infty}} \left| \frac{1}{n}\sum_{k=1}^{n} \mu(k)\chi_J(R_\rho^k(x)) - \frac{1}{n}\sum_{k=1}^{n} \mu(k)\varphi_{J_0}(R_\rho^k(x)) \right| & \\ = \overline{\lim_{n\to\infty}} \frac{1}{n} \left| \sum_{k=1}^{n} \mu(k)(\chi_J(R_\rho^k(x)) - \varphi_{J_0}(R_\rho^k(x))) \right| & \\ \le \overline{\lim_{n\to\infty}} \frac{2}{n} \sum_{k=1}^{n} \chi_{(c-\varepsilon,c)}(R_\rho^k(x)) + \overline{\lim_{n\to\infty}} \frac{2}{n} \sum_{k=1}^{n} \chi_{(d,d+\varepsilon)}(R_\rho^k(x)) & \\ = 4\varepsilon. & \qquad (14) \end{aligned}$$

To compute the limits in the last expression we used the equidistribution property of the sequence $\{n\rho\}_{n=1}^{\infty}$ on $S^1$. Now since $\varphi_{J_0}$ is a continuous function, according to Corollary 2.5,

$$\lim_{n\to\infty} \frac{1}{n} \left| \sum_{k=1}^{n} \mu(k)\varphi_{J_0}(R_\rho^k(x)) \right| = 0. \tag{15}$$

Hence from (14),

$$\lim_{n\to\infty}\frac{1}{n}\left|\sum_{k=1}^{n}\mu(k)\chi_J(R_\rho^k(x))\right|\le 4\varepsilon. \tag{16}$$

Since $\varepsilon$ is arbitrary we get (12). □

*Proof of Theorem* 1.2. Let us first suppose, that the rotation number of $f$ is rational and $\rho(f)=p/q$. According to Proposition 11.2.2, p. 394 in [5], for any $x\in S^1$ the compositions of $f^q$ are convergent, that is

$$\lim_{n\to\infty} f^{nq}(x)=y$$

for some $y\in S^1$. We conclude that

$$\lim_{n\to\infty} f^{nq+s}(x)=\lim_{n\to\infty} f^{nq}(f^s(x))=y'.$$

Therefore, if we write

$$\frac{1}{n}\sum_{k=1}^{n}\mu(k)\varphi(f^k(x))=\frac{1}{n}\sum_{l=1}^{q}\sum_{\substack{1\le k\le n\\ k\equiv l\ (\mathrm{mod}\ q)}}\mu(k)\varphi(f^k(x)),$$

then the limit

$$\lim_{\substack{k\to\infty\\ k\equiv l\ (\mathrm{mod}\ q)}}\varphi(f^k(x))$$

will exist as a real number. Hence for the sequence

$$x_k=\begin{cases}\varphi(f^k(x)), & k\equiv l\ (\mathrm{mod}\ q),\\ 0, & \text{otherwise},\end{cases}$$

there exists a sequence $\{x'_k\}_{k=1}^{\infty}$ which is eventually periodic and

$$|x_k-x'_k|\to 0$$

as $k\to\infty$. Therefore,

$$\lim_{n\to\infty}\frac{1}{n}\left|\sum_{k=1}^{n}\mu(k)\varphi(f^k(x))-\sum_{k=1}^{n}\mu(k)x'_k\right|=0. \tag{17}$$

But from Lemma 2.6,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mu(k)x'_k=0.$$

From the last expression and (17) we obtain (1).

Now let $\rho(f)$ be an irrational number.

We again start by approximating the function $\varphi$ with the function $\varphi_0$ of the form

$$\varphi_0(x)=\sum_{r=1}^{m} d_k\psi_{I_k}(x),$$

where $\{I_k\}_{k=1}^m$ are open, disjoint intervals in $S^1$, the functions $\{\psi_{I_k}\}_{k=1}^m$ are defined as in (6) (to avoid problems at the endpoints of the intervals we can suppose that $m>1$), and finally $\varphi_0$ satisfies

$$|\varphi(x)-\varphi_0(x)|<\varepsilon \quad \text{for } x\in S^1. \tag{18}$$

So from (18) it follows that it is enough to prove (1) for functions of the form (6). Since the rotation number of $f$ is an irrational number, it does not have periodic points, so we can ignore the values of the functions $\psi_{I_k}$ at the endpoints of $I_k$ and define it to be 0 there. So it will be enough to prove (1) for characteristic functions of open intervals, i.e. when

$$\varphi(x)=\chi_{(a,b)}(x), \quad \text{where } a,b\in[0,1).$$

According to Theorem 11.2.7, p. 397 in [5], there is a semi-conjugacy between $f$ and $R_\rho$, i.e. there exists an orientation-preserving, continuous map $\pi\colon S^1\to S^1$ such that

$$\pi(f(x))=R_\rho(\pi(x)).$$

In general $\pi$ is not invertible, but one can see that the restriction of $\pi$ to the orbit of any $x\in S^1$ is an invertible map. Indeed,

$$\pi(f^k(x))=\pi(f(f^{k-1}(x)))=R_\rho(\pi(f^{k-1}(x)))=\ldots=R_\rho(\pi^k(f(x))).$$

So

$$\pi(f^k(x))=\pi(f^m(x)) \quad \text{if and only if} \quad R_\rho^k(\pi(x))=R_\rho^m(\pi(x)).$$

But $R_\rho^k(\pi(x))=R_\rho^m(\pi(x))$ if and only if $k=m$, as $\rho$ is an irrational number. Therefore, for any interval $I\subset S^1$ the following is true

$$\{n\in\mathbb{N}_0 : f^n(x)\in I\}=\{n\in\mathbb{N}_0 : R_\rho^n(x)\in\pi(I)\}.$$

Equivalently

$$\chi_I(f^n(x))=\chi_{\pi(I)}(R_\rho^n(x)) \quad \text{for all } n\in\mathbb{N}_0.$$

From the last expression we get

$$\sum_{n\le N}\mu(n)\chi_{(a,b)}(f^n(x))=\sum_{n\le N}\mu(n)\chi_{\pi((a,b))}(R_\rho^n(x)).$$

Since $\pi$ is a continuous map, it is clear that $\pi((a,b))$ is also an interval in $S^1$, which can be either a single point, or half-open, closed or open. Therefore, according to Lemma 4.1,

$$\sum_{n\leq N} \mu(n)\chi_{\pi((a,b))}(R_\rho^n(x)) = o(N).$$

From this we get (1). $\square$

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