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Hausdorff dimension of wiggly metric spaces

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Abstract. For a compact connected set $X\subseteq\ell^\infty$, we define a quantity $\beta'(x,r)$ that measures how close X may be approximated in a ball $B(x,r)$ by a geodesic curve. We then show that there is $c>0$ so that if $\beta'(x,r)>\beta>0$ for all $x\in X$ and $r<r_0$, then $\dim X>1+c\beta^2$. This generalizes a theorem of Bishop and Jones and answers a question posed by Bishop and Tyson.

1. Introduction

1.1. Background and main results

Our starting point is a theorem of Bishop and Jones, stated below, which roughly says that a connected subset of $\mathbb{R}^2$ that is uniformly non-flat in every ball centered upon it (or in other words, is very "wiggly") must have large dimension. We measure flatness with Jones' β-numbers: if K is a subset of a Hilbert space $\mathscr{H}$, $x\in K$, and $r>0$, we define

$$\beta(x,r)=\beta_K(x,r)=\frac{1}{r}\inf_L\sup\{\operatorname{dist}(y,L):y\in K\cap B(x,r)\}, \tag{1.1}$$

where the infimum is taken over all lines $L\subseteq\mathscr{H}$.

Theorem 1.1. ([1, Theorem 1.1]) *There is a constant $c>0$ such that the following holds. Let $K\subseteq\mathbb{R}^2$ be a compact connected set and suppose that there is $r_0>0$ such that for all $r\in(0,r_0)$ and all $x\in K$, $\beta_K(x,r)>\beta_0$. Then the Hausdorff dimension*(1) *of K satisfies $\dim K\geq 1+c\beta_0^2$.*

(1) See Section 2 for the definition of Hausdorff dimension and other definitions and notation.

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There are also analogues of Theorem 1.1 for surfaces of higher topological dimension, see for example [5].

Our main theorem extends this result to the metric space setting using an alternative definition of β. Before stating our results, however, we discuss the techniques and steps involved in proving Theorem 1.1 to elucidate why the original methods do not immediately carry over, and to discuss how they must be altered for the metric space setting.

The main tool in proving Theorem 1.1 is the *analyst's traveling salesman theorem*, which we state below. First recall that for a metric space (X,d), a *maximal* ε*-net* is a maximal collection of points $X'\subseteq X$ such that $d(x,y)\geq\varepsilon$ for all $x,y\in X'$.

Theorem 1.2. ([16, Theorem 1.1]) *Let $A>1$, K be a compact subset of a Hilbert space $\mathscr{H}$, and $X_n\supseteq X_{n+1}$ be a nested sequence of maximal 2^{-n}-nets in K. For $A>1$, define*

$$\beta_A(K):=\operatorname{diam} K+\sum_{n\in\mathbb{Z}}\sum_{x\in X_n}\beta_K^2(x,A2^{-n})2^{-n}. \tag{1.2}$$

There is A_0 such that for $A>A_0$ there is $C_A>0$ (depending only on A) so that for any K, $\beta_A(K)<\infty$ implies there is a connected set Γ such that $K\subseteq\Gamma$ and

$$\mathscr{H}^1(\Gamma)\leq C_A\beta_A(K).$$

Conversely, if Γ is connected and $\mathscr{H}^1(\Gamma)<\infty$, then for any $A>1$,

$$\beta_A(\Gamma)\leq C_A\mathscr{H}^1(\Gamma). \tag{1.3}$$

At the time of [1], this was only known for the case $\mathscr{H}=\mathbb{R}^2$, due to Jones [9]. Okikiolu generalized this to $\mathbb{R}^n$ in [13] and Schul to Hilbert space in [16].

The proof of Theorem 1.1 goes roughly as follows: one constructs a *Frostman measure* μ supported on K satisfying

$$\mu(B(x,r))\leq Cr^s \tag{1.4}$$

for some $C>0$, $s=1+c\beta_0^2$, and for all $x\in K$ and $r>0$. This easily implies that the Hausdorff dimension of K is at least s (see [12, Theorem 8.8] and that section for a discussion on Frostman measures). One builds such a measure on K inductively by deciding the values $\mu(Q_n)/\mu(Q)$ for each dyadic cube Q intersecting K and for each nth generation descendant Q_n intersecting K, where n is some large number that will depend on β_0. If the number of such nth generation descendants is large enough, we can choose the ratios and hence disseminate the mass $\mu(Q)$ amongst the descendants Q_n in such a way that the ratios will be very small and (1.4) will be satisfied. To show that there are enough descendants, one looks at the skeletons of the nth generation descendants of Q and uses the second half of Theorem 1.2 coupled with the non-flatness condition in the statement of Theorem 1.1 to guarantee that the total length of this skeleton (and hence the number of cubes) will be large.

In the metric space setting, however, no such complete analogue of Theorem 1.2 exists, and it is not even clear what the appropriate analogue of a β-number should be. Note, for example, that it does not make sense to estimate the length of a metric curve Γ using the original β-number, even if we consider Γ as lying in some Banach space. A simple counterexample is if $\Gamma\subseteq L^1([0,1])$ is the image of $s\colon [0,1]\to L^1([0,1])$ defined by $t\mapsto \mathbb{1}_{[0,t]}$. This a geodesic, so in particular, it is a rectifiable curve of finite length. However, $\beta_\Gamma(x,r)$ (i.e. the width of the smallest tube containing $\Gamma\cap B(x,r)$ in L^1, rescaled by a factor r) is uniformly bounded away from zero, and in particular, $\beta_A(\Gamma)=\infty$.

In [6], Hahlomaa gives a good candidate for a β-number for a general metric space X using Menger curvature and uses it to show that if the sum in (1.2) is finite for $K=X$ (using his definition of β_X), then it can be contained in the Lipschitz image of a subset of the real line (analogous to the first half of Theorem 1.2). An example of Schul [15], however, shows that the converse of Theorem 1.2 is false in general: (1.3) with Hahlomaa's β_X does not hold with the same constant for all curves in ℓ^1. We refer to [15] for a good summary on the analyst's traveling salesman problem.

To generalize Theorem 1.1, we use a β-type quantity that differs from both Jones' and Hahlomaa's definitions. It is inspired by one defined by Bishop and Tyson in [2] that measures the deviation of a set from a geodesic in a metric space: if X is a metric space, $B_X(x,r)=\{y\in X: d(x,y)<r\}$, and $y_0,\dots,y_n\in B_X(x,r)$ is an ordered sequence, define

$$\partial(y_0,\dots,y_n)=\sum_{i=0}^{n-1} d(y_i,y_{i+1})-d(y_0,y_n)+\sup_{z\in B_X(x,r)}\min_{i=1,\dots,n} d(z,y_i) \tag{1.5}$$

and define

$$\hat\beta_X(x,r)=\inf_{\{y_i\}_{i=0}^n\subseteq B_X(x,r)}\frac{\partial(y_0,\dots,y_n)}{d(y_0,y_n)}, \tag{1.6}$$

where the infimum is over all finite ordered sequences in $B_X(x,r)$ of any length n.

In [2], Bishop and Tyson ask whether, for a compact connected metric space X, (1.6) being uniformly larger than zero is enough to guarantee that $\dim X>1$. We answer this in the affirmative.

Theorem 1.3. *There is $\theta>0$ such that the following holds: If X is a compact connected metric space and $\hat\beta_X(x,r)>\beta>0$ for all $x\in X$ and $r\in(0,r_0)$ for some $r_0>0$, then* $\dim X\geq 1+\theta\beta^4$.

Instead of $\hat{\beta}$, however, we work with a different quantity, which we define here for a general compact metric space X. First, by Kuratowski's embedding theorem, we may assume that X is a subset of ℓ^∞, whose norm we denote by $|\cdot|$. Let $B(x,r)=B_{\ell^\infty}(x,r)$ and define

$$\beta'_X(x,r)=\inf_s \frac{\ell(s)-|s(0)-s(1)|+\sup_{z\in X\cap B(x,r)}\operatorname{dist}(z,s([0,1]))}{|s(0)-s(1)|}, \tag{1.7}$$

where the infimum is over all curves $s\colon [0,1]\to B(x,r)\subseteq \ell^\infty$ and

$$\ell(s)=\sup_{\{t_i\}_{i=0}^n}\sum_{i=0}^{n-1}|s(t_i)-s(t_{i+1})|$$

is the length of s, where the supremum is over all partitions $0=t_0<t_1<...<t_n=1$. In general, if s is defined on a union of disjoint open intervals $\{I_j\}_{j=1}^\infty$, we set

$$\ell(s|_{\bigcup_{j=1}^\infty I_j})=\sum_{j=1}^\infty \ell(s|_{I_j}).$$

The case in which s is just a straight line segment through the center of the ball with length $2r$ gives the estimate $\beta'_X(x,r)\leq\frac{1}{2}$.

The quantity $\beta'(x,r)$ measures how well $X\cap B(x,r)$ may be approximated by a geodesic. To see this, note that if the $\frac{1}{2}\beta'(x,r)|s(0)-s(1)|$-neighborhood of $s([0,1])$ contains $X\cap B(x,r)$, for some $s\colon[0,1]\to\ell^\infty$, then the length of s must be at least $\left(1+\frac{1}{2}\beta'(x,r)\right)|s(0)-s(1)|$, which is $\frac{1}{2}\beta'(x,r)|s(0)-s(1)|$ more than the length of any geodesic connecting $s(0)$ and $s(1)$. The quantity $\hat{\beta}$ similarly measures how well the portion of $X\cap B(x,r)$ may be approximated by a geodesic polygonal path with vertices in X. In Figure 1, we compare the meanings of β, $\hat{\beta}$, and β'.

We will refer to the quantities $\ell(s)$ and $\partial(y_0,...,y_n)$ as the *geodesic deviation* of s and $\{y_0,...,y_n\}$ respectively. We will also say that $\hat{\beta}_X(x,r)$ and $\beta'_X(x,r)$ measure the *geodesic deviation* of X inside the ball $B(x,r)$.

Note that for the image of $t\mapsto\mathbb{1}_{[0,t]}\in L^1([0,1])$ described earlier, it is easy to check that $\hat{\beta}(x,r)=\beta'(x,r)=0$ for all $x\in X$ and $r>0$, even though $\beta_X(x,r)$ is bounded away from zero. This, of course, makes the terminology "wiggly" rather misleading in metric spaces, since there are certainly non-flat or highly "wiggly" geodesics in L^1; we use this terminology only to be consistent with the literature. Later on in Proposition 5.2, however, we will show that in a Hilbert space we have for some $C>0$,

$$\beta'(x,r)\leq\beta(x,r)\leq C\beta'(x,r)^{1/2}. \tag{1.8}$$

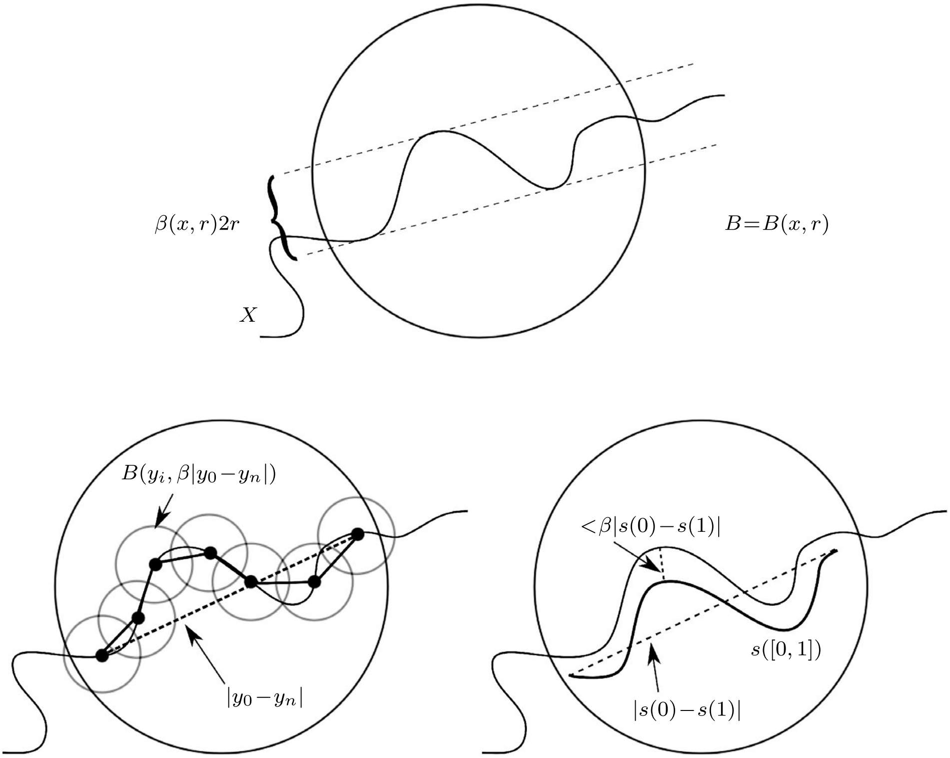

Figure 1. In each of the three figures above there is a ball $B=B(x,r)$ containing a portion of a curve X. In the first picture, $\beta(x,r)2r$ is the width of the smallest tube containing $X\cap B(x,r)$. In the second picture, we see that $\hat{\beta}(x,r)$ is such that for $\beta>\hat{\beta}(x,r)$, there are $y_0,\dots,y_n\in X$ with vertices in $X\cap B$ so that balls centered on the y_i of radius $\beta|y_0-y_n|$ cover $X\cap B$, and so that the geodesic deviation (that is, its length minus $|y_0-y_n|$) is at most $\beta|y_0-y_n|$. In the last picture, we show that if $\beta'(x,r)<\beta$, there is $s\colon[0,1]\to\ell^\infty$ whose geodesic deviation and whose distance from any point in $X\cap B$ are both at most $\beta|s(0)-s(1)|$.

That the two should be correlated in this setting seems natural as $\beta(x,r)$ is measuring how far X is deviating from a straight line, which are the only geodesics in Hilbert space.

In Lemma 5.1 below, we will also show that for some $C>0$,

$$\beta'(x,r)\leq\hat{\beta}(x,r)\leq C\beta'(x,r)^{1/2}$$

so that Theorem 1.3 follows from the following theorem, which is our main result.

Theorem 1.4. *There is $c_0>0$ such that the following holds: If X is a compact connected metric space and $\beta'_X(x,r)>\beta>0$ for all $x\in X$ and $r\in(0,r_0)$ for some $r_0>0$, then* $\dim X\geq 1+c_0\beta^2$.

We warn the reader, however, that the quadratic dependence on β appears in Theorems 1.4 and 1.1 for completely different reasons. In Theorem 1.1, it comes from using Theorem 1.2, or ultimately from the Pythagorean theorem, which of course does no hold in general metric spaces; in Theorem 1.4, it seems to be an artifact of the construction and can perhaps be improved.

Our approach to proving Theorem 1.4 follows the original proof of Theorem 1.1 described earlier: to show that a metric curve X has large dimension, we approximate it by a polygonal curve, estimate its length from below, and use this estimate to construct a Frostman measure, but in lieu of a traveling salesman theorem. (In fact, taking $\beta'(x,A2^{-n})$ instead of $\beta(x,A2^{-n})^2$ in Theorem 1.2 does not lead to a metric version of Theorem 1.2 for a similar reason that Hahlomaa's β-number does not work; one need only consider Schul's example [15, Section 3.3.1].)

1.2. An application to conformal dimension

The original context of Bishop and Tyson's conjecture, and the motivation for Theorem 1.4, concerned conformal dimension. Recall that a *quasisymmetric map* $f\colon X\to Y$ between two metric spaces is a map for which there is an increasing homeomorphism $\eta\colon(0,\infty)\to(0,\infty)$ such that for any distinct $x,y,z\in X$,

$$\frac{|f(x)-f(y)|}{|f(z)-f(y)|}\leq\eta\left(\frac{|x-y|}{|z-y|}\right).$$

The *conformal dimension* of a metric space X is

$$\text{C-dim}\,X=\inf_f \dim f(X),$$

where the infimum ranges over all quasisymmetric maps $f\colon X\to f(X)$. For more information about recent work on conformal dimension, see for example [11].

In [2], it is shown that the antenna set has conformal dimension one, yet every quasisymmetric image of it into any metric space has dimension strictly larger than one. The *antenna set* is a self-similar fractal lying in $\mathbb{C}$ whose similarities are

$$f_1(z)=\frac{z}{2},\quad f_2(z)=\frac{z+1}{2},\quad f_3(z)=i\alpha z+\frac{1}{2},\quad\text{and}\quad f_4(z)=-i\alpha z+\frac{1}{2}+i\alpha,$$

where $\alpha\in\left(0,\frac{1}{2}\right)$ is some fixed angle (see Figure 2).

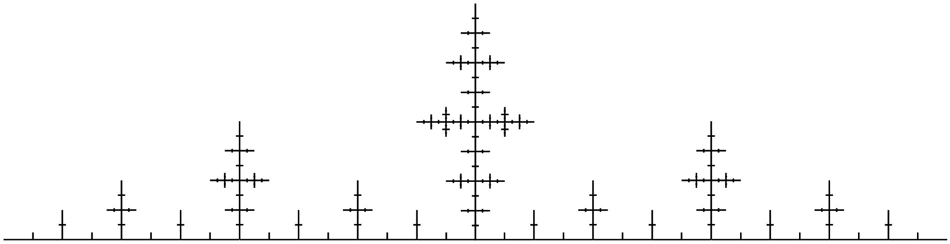

Figure 2. The antenna set with $\alpha=\frac{1}{4}$.

To show that the conformal dimension 1 is never attained under any quasisymmetric image of the antenna set, the authors show by hand that any quasisymmetric map of the antenna set naturally induces a Frostman measure of dimension larger than one. At the end of the paper, however, the authors suggested another way of showing the same result by proving an analogue of Theorem 1.1 for a β-number, which is uniformly large for the antenna set as well as any quasisymmetric image of it.

Theorem 1.4 does not just give a much longer proof of Bishop and Tyson's result, but it lends itself to more general sets lacking any self-similar structure.

Definition 1.5. Let $c>0$ and $Y=[0,e_1]\cup[0,e_2]\cup[0,e_3]\subseteq\mathbb{R}^3$, where e_j is the jth standard basis vector in $\mathbb{R}^3$, and let X be a compact connected metric space. For $x\in X$, $r>0$, we say that $B_X(x,r)$ has a *c-antenna* if there is a homeomorphism $h\colon Y\to h(Y)\subseteq B_X(x,r)$ such that the distance between $h(e_i)$ and $h([0,e_j]\cup[0,e_k])$ is at least cr for all permutations (i,j,k) of $(1,2,3)$. We say that X is *c-antenna-like* if $B_X(x,r)$ has a c-antenna for every $x\in X$ and $r<\frac{1}{2}\operatorname{diam} X$.

Clearly, the classical antenna set in $\mathbb{R}^2$ is antenna-like.

Theorem 1.6. *Let X be a compact connected metric space in ℓ^∞.*

(1) *If $B_X(x,r)$ has a c-antenna, then $\beta'(x,r)>c/7$. Hence, if X is c-antenna-like, we have* $\dim X\geq 1+c_0/49c^2$.

(2) *Any quasisymmetric image of an antenna-like set into any metric space is also antenna-like and hence has dimension strictly larger than one.*

Note that this result does not say that the conformal dimension of an antenna-like set is larger than one, only that no quasisymmetric image of it has dimension equal to one. However, see [10], where Mackay bounds the conformal dimension of a set from below using a different quantity.

1.3. Outline

In Section 2, we go over some necessary notation and tools before proceeding to the proof of Theorem 1.4 in Section 3. In Section 4, we prove Theorem 1.6, and in Section 5 we compare β', $\hat{\beta}$, and β.

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2. Preliminaries

2.1. Basic notation

Since we are only dealing with compact metric spaces, by the Kuratowski embedding theorem, we will implicitly assume that all our metric spaces are contained in ℓ^∞, whose norm we will denote by $|\cdot|$.

For $x \in \ell^\infty$ and $r > 0$, we will write

$$B(x,r) = \{y \in \ell^\infty : |x-y| < r\} \subseteq \ell^\infty.$$

If $B = B(x,r)$ and $\lambda > 0$, we write λB for $B(x, \lambda r)$. For a set $A \subseteq \ell^\infty$ and $\delta > 0$, define

$$A_\delta = \{x \in \ell^\infty : \operatorname{dist}(x,A) < \delta\} \quad \text{and} \quad \operatorname{diam} A = \sup\{|x-y| : x, y \in A\},$$

where

$$\operatorname{dist}(A,B) = \inf\{|x-y| : x \in A \text{ and } y \in B\} \quad \text{and} \quad \operatorname{dist}(x,A) = \operatorname{dist}(\{x\}, A).$$

For a set $E \subseteq \mathbb{R}$, let $|E|$ denote its Lebesgue measure. For an interval $I \subseteq \mathbb{R}$, we will write a_I and b_I for its left and right endpoints respectively. For $s > 0$, $\delta \in (0, \infty]$ and $A \subseteq \ell^\infty$, define

$$\mathscr{H}^s_\delta(A) = \inf\left\{\sum_{j=1}^\infty \operatorname{diam} A_j : A \subseteq \bigcup_{j=1}^\infty A_j, \operatorname{diam} A_j < \delta\right\} \quad \text{and} \quad \mathscr{H}^s(A) = \lim_{\delta \to 0} \mathscr{H}^1_\delta(A).$$

The *Hausdorff dimension* of a set A is $\dim A := \inf\{s : \mathscr{H}^s(A) = 0\}$.

2.2. Cubes

In this section, we construct a family of subsets of ℓ^∞, tailored to a metric space X, that have properties similar to dyadic cubes in Euclidean space. These cubes appeared in [16] (where they were alternatively called "*cores*") and are similar to the so-called *Christ–David cubes* [3], [4] in some respects, although they are not derived from them.

Fix $M>0$ and $c\in\left(0,\frac{1}{8}\right)$. Let $X_n\subseteq X$ be a nested sequence of maximal M^{-n}-nets in X. Let

$$\mathscr{B}_n=\{B(x,M^{-n}):x\in X_n\} \quad \text{and} \quad \mathscr{B}=\bigcup_{n=1}^{\infty}\mathscr{B}_n.$$

For $B=B(x,M^{-n})\in\mathscr{B}_n$, define

$$\begin{aligned} Q_B^0&=cB,\\ Q_B^j&=Q_B^{j-1}\cup\bigcup\left\{cB:B\in\bigcup_{n=m}^{\infty}\mathscr{B}_m \text{ and } cB\cap\bigcap_{n=m}^{\infty}Q_B^{j-1}\neq\varnothing\right\},\quad j=1,2,...,\\ Q_B&=\bigcup_{j=0}^{\infty}Q_B^j. \end{aligned}$$

Basically, Q_B is the union of all balls B' that may be connected to B by a chain $\{cB_j\}_{j=1}^n$ with $B_j\in\mathscr{B}$, $\operatorname{diam}B_j\leq\operatorname{diam}B$, and $cB_j\cap cB_{j+1}\neq\varnothing$ for all j.

For such a cube Q constructed from $B(x,M^{-n})$, we let $x_Q=x$ and $B_Q=B(x,cM^{-n})$.

Let

$$\Delta_n=\{Q_B:B\in\mathscr{B}_n\} \quad \text{and} \quad \Delta=\bigcup_{n=0}^{\infty}\Delta_n.$$

Note that, for $Q\in\Delta_n$, $x_Q\in X_n$.

Lemma 2.1. *If $c<\frac{1}{8}$, then for X and Δ as above, the family of cubes Δ satisfy the following properties*:

(1) *If $Q,R\in\Delta$ and $Q\cap R\neq\varnothing$, then $Q\subseteq R$ or $R\subseteq Q$;*

(2) *For $Q\in\Delta$,*

$$B_Q\subseteq Q\subseteq(1+8M^{-1})B_Q. \tag{2.1}$$

The proof is essentially in [14], but with slightly different parameters. So that the reader need not perform the needed modifications, we provide a proof here.

Proof. Part 1 follows from the definition of the cubes Q. To prove Part 2, we first claim that if $\{B_j\}_{j=0}^n$ is a chain of balls with centers x_j for which $cB_j \cap cB_{j+1} \neq \varnothing$, then for $C = 1/(1-2M^{-1})$,

$$\sum_{j=0}^{n} \operatorname{diam} cB_j \leq C \max_{j=0,\ldots,n} \operatorname{diam} cB_j. \tag{2.2}$$

We prove (2.2) by induction. Let x_j denote the center of B_j. If $n=1$, $\operatorname{diam} B_0 \leq \operatorname{diam} B_1$, and x_0 and x_1 are the centers of B_0 and B_1 respectively, then $\operatorname{diam} B_0 \leq M^{-1} \operatorname{diam} B_1$ since otherwise $B_0, B_1 \in \mathscr{B}_N$ for some N and

$$M^{-n} \leq |x_0 - x_1| \leq \tfrac{1}{2} \operatorname{diam} cB_0 + \tfrac{1}{2} \operatorname{diam} cB_1 = 2cM^{-n} < M^{-n}$$

as $c < \frac{1}{8}$, which is a contradiction. Hence,

$$\operatorname{diam} cB_0 + \operatorname{diam} cB_1 \leq (1 + 2M^{-1}) \operatorname{diam} cB_1 \leq C \operatorname{diam} cB_1.$$

Now suppose $n>1$. Let $j_0 \in \{1, ..., n\}$ and N be an integer so that

$$\operatorname{diam} B_{j_0} = \max_{j=1,\ldots,n} \operatorname{diam} B_j = 2M^{-N}. \tag{2.3}$$

Recall that all balls in $\mathscr{B}$ have radii that are powers of M^{-1}, so there exists an N so that the above happens.

Note that B_{j_0-1} and B_{j_0} cannot have the same diameter (which follows from the $n=1$ case we proved earlier). Since B_{j_0} has the maximum diameter of all the B_j, we in fact know that $\operatorname{diam} B_{j_0-1} \leq M^{-1} B_{j_0}$ (again, recall that all balls have radii that are powers of M^{-1}).

Let $i_0 \leq j_0$ be the minimal integer for which $\operatorname{diam} B_{i_0} \leq M^{-1} \operatorname{diam} B_{j_0}$ (which exists by the previous discussion) and let $k_0 \geq j_0$ be the maximal integer such that $B_{k_0} \leq M^{-1} \operatorname{diam} B_{j_0}$. By the induction hypothesis

$$\sum_{j=j_0+1}^{k_0} \operatorname{diam} cB_j \leq C \max_{j_0 < j \leq k_0} \operatorname{diam} cB_j \leq CM^{-1} \operatorname{diam} cB_{j_0}$$

and

$$\sum_{j=i_0}^{j_0-1} \operatorname{diam} cB_j \leq C \max_{i_0 \leq j < j_0} \operatorname{diam} cB_j \leq CM^{-1} \operatorname{diam} cB_{j_0} \tag{2.4}$$

so that

$$\sum_{j=i_0}^{k_0} \operatorname{diam} B_j \leq (1+2CM^{-1}) \operatorname{diam} cB_{j_0} = C \operatorname{diam} cB_{j_0}. \tag{2.5}$$

Claim. $i_0=0$.

Note that if $i_0>0$, then

$$\begin{aligned} |x_{i_0-1}-x_{j_0}| &\leq \sum_{i=i_0-1}^{j_0} \operatorname{diam} cB_i \\ &\leq \operatorname{diam} cB_{i_0-1} + \operatorname{diam} cB_{j_0} + \sum_{i=i_0}^{j_0-1} 2cB_{j_0} \\ &\overset{(2.3)(2.4)}{\leq} 2 \operatorname{diam} cB_{j_0} + CM^{-1} \operatorname{diam} cB_{j_0} \\ &= (2c+cCM^{-1}) \operatorname{diam} B_{j_0} \\ &= (2c+cCM^{-1}) 2M^{-N} \\ &< M^{-N} \end{aligned}$$

for $c<\frac{1}{4}$ and $M>4$ (this makes $C<2$). Since $x_{j_0} \in X_N$ and points in X_N are M^{-N}-separated, we must have $x_{i_0-1} \notin X_N$, and hence $B_{i_0-1} \notin \mathscr{B}_N$. Thus

$$\operatorname{diam} B_{i_0-1} \leq M^{-1} \operatorname{diam} B_{j_0},$$

which contradicts the minimality of i_0, and hence $i_0=0$. We can prove similarly that $k_0=n$, and this with (2.4) proves (2.2). This in turn implies that for any $N \in \mathbb{N}$, if $Q \in \Delta_N$, then $\operatorname{diam} Q \leq C \operatorname{diam} cB_Q$. Hence

$$\begin{aligned} Q &\subseteq B(x_Q, cM^{-N} + (C-1) \operatorname{diam} cB_Q) \\ &= B\left(x_Q, c\left(1+\frac{4M^{-1}}{1-2M^{-1}}\right)M^{-N}\right) \subseteq (1+8M^{-1})B_Q. \qquad \square \end{aligned}$$

For N large enough, this means we can pick our cubes so that they do not differ much from balls. We will set $8M^{-1}=\varepsilon\beta$ for some $\varepsilon \in (0,1)$ to be determined later, so that

$$B_Q \subseteq Q \subseteq (1+\varepsilon\beta)B_Q. \tag{2.6}$$

Remark 2.2. There are a few different constructions of families of metric subsets with properties similar to dyadic cubes, see [3], [4], and [8] for example, and the references therein. Readers familiar with any of these references will see that the Schul's "cores" we have just constructed are very different from the cubes constructed in the aforementioned references. In particular, each Δ_n does not partition any metric space in the same way that dyadic cubes (half-open or otherwise) would partition Euclidean space, not even up to a set of measure zero). However, for each n we do have

$$X \subseteq \bigcup \{c^{-1}Q : Q \in \Delta_n\}, \tag{2.7}$$

and we still have the familiar intersection properties in Lemma 2.1. The reason for the ad hoc construction is the crucial "roundness" property (2.6).

Lemma 2.3. *Let $\gamma\colon [0,1]\to\ell^\infty$ be a continuous piecewise linear function whose image is a finite union of line segments, set $\Gamma=\gamma([0,1])$ and let Δ be the family of cubes from Lemma 2.1 tailored to X. Then for any $Q\in\Delta$, $\mathscr{H}^1(\partial Q)=0$ and $|\gamma^{-1}(\partial Q)|=0$.*

Proof. Note that since Γ is a finite polynomial curve, $\mu=\mathscr{H}^1|_\Gamma$ is *doubling* on Γ, meaning that there is a constant C so that $\mu(B(x,Mr))\leq C\mu(B(x,r))$ for all $x\in\Gamma$ and $r>0$. If $x\in\partial Q$ for some $Q\in\Delta$, then there is a sequence $x_n\in X_n$ such that $|x_n-x|<M^{-n}$ since the X_n are maximal M^{-n}-nets. To each x_n corresponds a ball $B_n=B(x_n,M^{-n})\in\mathscr{B}_n$. Let N be such that $Q\in\Delta_N$. As $cB_n\subseteq Q_{B_n}\in\Delta_n$, we have by Lemma 2.1 that either $cB_n\subseteq Q$ (if $Q_{B_n}\cap Q\neq\varnothing$) or $cB_n\subseteq R$ for some $R\in\Delta_N$ with $Q\cap R=\varnothing$. In either case, since cubes do not contain their boundaries (since they are open), we have that $cB_n\cap\partial Q=\varnothing$. This implies that Q is porous, and it is well known that the zero measure is the only doubling measure on such a set. More precisely, the doubling condition on μ guarantees that $\lim_{n\to\infty}\mu(\partial Q\cap B(x,M^{-n}))/\mu(B(x,M^{-n}))=1$ μ-a.e. $x\in\Gamma$ (see [7, Theorem 1.8]), but if $x\in\partial Q$ and B_n is as above, then one can show using the doubling property of μ that

$$\limsup_{n\to\infty}\frac{\mu(\partial Q\cap B(x,M^{-n}))}{\mu(B(x,M^{-n}))}\leq\limsup_{n\to\infty}\frac{\mu(B(x,M^{-n})\backslash B_n)}{\mu(B(x,M^{-n}))}<1,$$

and thus $\mu(\partial Q)=0$. The last part of the theorem follows since γ is piecewise affine. $\square$

The following lemma will be used frequently.

Lemma 2.4. *Let $I\subseteq\mathbb{R}$ be an interval, $s\colon I\to\ell^\infty$ be continuous, and $I'\subseteq I$ be a subinterval. Then*

$$\ell(s|_{I'})-|s(a_{I'})-s(b_{I'})|\leq\ell(s|_I)-|s(a_I)-s(b_I)|. \tag{2.8}$$

Proof. We may assume $\ell(s_I)<\infty$, otherwise (2.8) is trivial. We estimate

$$\begin{aligned}\ell(s|_{I'})-|s(a_{I'})-s(b_{I'})| &= \ell(s|_I)-\ell(s|_{I\setminus I'})-|s(a_{I'})-s(b_{I'})|\\ &\leq \ell(s|_I)-(|s(a_I)-s(a_{I'})|+|s(b_I)-s(b_{I'})|)-|s(a_{I'})-s(b_{I'})|\\ &\leq \ell(s|_I)-|s(a_I)-s(b_I)|. \quad\square\end{aligned}$$

3. Proof of Theorem 1.4

3.1. Setup

For this section, we fix a compact connected set X satisfying the conditions of Theorem 1.4. The main tool is the following lemma, which can be seen as a weak substitute for Theorem 1.2.

Lemma 3.1. *Let $c'<\frac{1}{8}$. We can pick M large enough (by picking $\varepsilon>0$ small enough) and pick $\beta_0,\theta>0$ such that, for any X satisfying the conditions of Theorem 1.4 for some $\beta\in(0,\beta_0)$, the following holds. If X_n is any nested sequence of M^{-n}-nets in X, then there is $n_0=n_0(\beta)$ such that for $x_0\in X_n$ with $M^{-n}<\min\{r_0,\frac{1}{2}\operatorname{diam}X\}$,*

$$\#X_{n+n_0}\cap B(x_0,c'M^{-n})\geq M^{(1+\theta\beta^2)n_0}. \tag{3.1}$$

We will prove this in Section 3.2, but first, we will show why this proves Theorem 1.4.

Proof of Theorem 1.4. Without loss of generality, we may assume $r_0>2$ by scaling X if necessary. We first consider the case when $\beta<\beta_0$. Let Δ be the family of cubes from Lemma 2.1 tailored to the metric space X with $c=c'$ and define inductively,

$$\Delta_0'=\Delta_0 \quad\text{and}\quad \Delta_{n+1}'=\{R\in\Delta_{(n+1)n_0}: R\subseteq Q \text{ for some } Q\in\Delta_n\}.$$

By Lemma 3.1, for any $Q\in\Delta_n'$, if $B_Q=B(x_Q,cM^{-N})$, then

$$\#\{R\in\Delta_{n+1}', R\subseteq Q\}\geq\#X_{N+n_0}\cap Q\geq\#X_{n_0}\cap c'B_Q\geq M^{(1+\theta\beta^2)n_0} \tag{3.2}$$

and moreover, since $c'<\frac{1}{8}$,

$$2B_Q\cap 2B_R=\varnothing \quad \text{for } Q,R\in\Delta_n. \tag{3.3}$$

Define a probability measure μ inductively by picking $Q_0\in\Delta_0'$, setting $\mu(Q_0)=1$, and for $Q\in\Delta_n'$ and $R\in\Delta_{n+1}'$, with $R\subseteq Q$, setting

$$\frac{\mu(R)}{\mu(Q)}=\frac{1}{\#\{S\in\Delta_{n+1}' : S\subseteq Q\}}\stackrel{(3.2)}{\leq} M^{-(1+\theta\beta^2)n_0}. \tag{3.4}$$

Let $x\in X$, $r\in(0,r_0/M)$. Pick n so that

$$M^{-n_0(n+1)}\leq r<M^{-n_0 n}. \tag{3.5}$$

Claim. There is at most one $y\in X_{(n-1)n_0}$ such that

$$B(y,c'M^{-(n-1)n_0})\cap B(x,r)\neq\varnothing \quad \textit{and} \quad Q=Q_{B(y,c'M^{-(n-1)n_0})}\in\Delta_{n-1}'. \tag{3.6}$$

Indeed, if there were another such $y'\in X_{(n-1)n_0}$ with $B(y',c'M^{-(n-1)n_0})\cap B(x,r)\neq\varnothing$, then

$$\begin{aligned}
M^{-(n-1)n_0} &\leq |y'-y| \\
&\leq c'M^{-(n-1)n_0}+\operatorname{dist}(B(y,c'M^{-(n-1)n_0}),B(y',c'M^{-(n-1)n_0})) \\
&\quad +c'M^{-(n-1)n_0} \\
&\leq 2c'M^{-(n-1)n_0}+\operatorname{diam} B(x,r) \\
&\leq 2c'M^{-(n-1)n_0}+2r \\
&\stackrel{(3.5)}{\leq} 2M^{-(n-1)n_0}(c'+M^{-n_0}) \\
&< 4c'M^{-(n-1)n_0} \\
&< M^{-(n-1)n_0}
\end{aligned}$$

since $c'<\frac{1}{8}$ and we can pick $\varepsilon<c'/8$ so that $M^{-n_0}\leq M^{-1}<c'$, which gives a contradiction and proves the claim.

Now, assuming we have $y\in X_{(n-1)n_0}$ satisfying (3.6),

$$\begin{aligned}
B(x,r)\subseteq B(y,c'M^{-(n-1)n_0}+2r) &\stackrel{(3.5)}{\subseteq} B(y,c'M^{-(n-1)n_0}+2M^{-nn_0}) \\
&\subseteq B(y,2c'M^{-(n-1)n_0})=2B_Q
\end{aligned}$$

for M large enough (that is, for $2M^{-1}<c'$, which is possible by picking $\varepsilon<c'/16$). If $Q\notin\Delta'_{n-1}$, then (3.3) implies $2B_Q\cap 2B_R=\varnothing$ for all $R\in\Delta'_{n-1}$, and so

$$\mu(B(x,r))\leq\mu(2B_Q)=0.$$

Otherwise, if $Q\in\Delta'_{n-1}$, then $Q\subseteq Q_0$, so that

$$\mu(B(x,r))\leq\mu(2B_Q)\stackrel{(3.3)}{=}\mu(Q)\stackrel{(3.4)}{=}M^{-(1+\theta\beta^2)n_0(n-1)}\mu(Q_0)\stackrel{(3.5)}{\leq}M^{2(1+\theta\beta^2)}r^{-(1+\theta\beta^2)}.$$

Thus μ is a $(1+\theta\beta^2)$-Frostman measure supported on X, which implies that $\dim X\geq 1+\theta\beta^2$ (cf. [12, Theorem 8.8]).

Now we consider the case when $\beta\geq\beta_0$. Trivially, $\beta'(x,r)\geq\beta\geq\beta_0$ for all $x\in X$ and $r<r_0$, and our previous work gives $\dim X\geq 1+\theta t^2$ for all $t<\beta_0$. Hence $\dim X\geq 1+\theta\beta_0^2$. Since $\beta'\leq\frac{1}{2}$, we must have $\beta,\beta_0\leq\frac{1}{2}$, and so

$$\dim X\geq 1+\theta\beta_0^2\geq 1+4\theta\beta_0^2\beta^2$$

and the theorem follows with $c_0=4\theta\beta_0^2$. $\square$

To show Lemma 3.1, we will approximate X by a tree containing a sufficiently dense net in X and estimate its length from below. The following lemma relates the length of this tree to the number of net points in X.

Lemma 3.2. *Let X_{n_0} be a maximal M^{-n_0}-net for a connected metric space X, where n_0 is so that $4M^{-n_0}<\frac{1}{4}\operatorname{diam} X$. Then we may embed X into ℓ^∞ so that there is a connected union of finitely many line segments $\Gamma_{n_0}\subseteq\ell^\infty$ containing X_{n_0} such that for any $x\in X_{n_0}$ and $r\in\left(4M^{-n_0},\frac{1}{4}\operatorname{diam} X\right)$,*

$$\mathscr{H}^1\left(\Gamma_{n_0}\cap B\left(x,\frac{r}{2}\right)\right)\leq 8M^{-n_0}\#(X_{n_0}\cap B(x,r)). \tag{3.7}$$

Proof. Embed X isometrically into $\ell^\infty(\mathbb{N})$ so that for any $x\in X$, the first $\#X_{n_0}$ coordinates are all zero. Construct a sequence of trees T_j as follows. Enumerate the elements of $X_{n_0}=\{x_1,\dots,x_{\#X_{n_0}}\}$. For two points x and y, let

$$A_{xy,i}=\{tx+(1-t)y+\max\{t,1-t\}|x-y|e_i:t\in[0,1]\},$$

where e_i is the ith standard basis vector in $\ell^\infty(\mathbb{N})$ (i.e. it is equal to 1 in the ith coordinate and zero in every other coordinate).

Now construct a sequence of trees T_j in $\ell^\infty(\mathbb{N})$ inductively by setting $T_0=\{x_0\}$ and T_{j+1} equal to T_j united with $S_{j+1}:=A_{x_{j+1}x'_{j+1},j+1}$, where $x'_{j+1}\in\{x_1,\dots,x_j\}$ and $x_{j+1}\in X_{n_0}\backslash\{x_1,\dots,x_j\}$ are such that

$$|x_{j+1}-x'_{j+1}|=\operatorname{dist}(X_{n_0}\backslash\{x_1,\dots,x_j\},\{x_1,\dots,x_j\}).$$

Since X is connected, $|x_{j+1}-x'_{j+1}|\leq 2M^{-n_0}$, so that

$$\mathscr{H}^1(S_j)=\mathscr{H}^1(A_{x_jx'_j,j})\leq 2|x_j-x'_j|\leq 4\cdot 2M^{-n_0}=8M^{-n_0}.$$

Then $\Gamma_{n_0}:=T_{\#X_{n_0}}$ is a tree contained in $\ell^\infty(\mathbb{N})$ containing X_{n_0} (the reason we made the arcs S_j reach into an alternate dimension is to guarantee that the branches of the tree do not intersect except at the points X_{n_0}).

To prove (3.7), note that since $r/2>2M^{-n_0}$ and $x_j\in S_j\subseteq B(x_j,2M^{-n_0})$, we have

$$\begin{aligned}\mathscr{H}^1\left(\Gamma_{n_0}\cap B\left(x,\frac{r}{2}\right)\right) &\leq \sum_{S_j\cap B(x,r/2)\neq\varnothing}\mathscr{H}^1(S_j)\\ &\leq \sum_{x_j\in B(x,r/2+2M^{-n_0})} 8M^{-n_0}\leq 8\#(X_{n_0}\cap B(x,r)). \qquad \square\end{aligned}$$

3.2. Proof of Lemma 3.1

We now dedicate ourselves to the proof of Lemma 3.1. Again, let X be a connected metric space satisfying the conditions of Theorem 1.4. Without loss of generality, $n=0$, so that $\operatorname{diam} X>2$. Embed X into ℓ^∞ as in Lemma 3.2. Fix $n_0\in\mathbb{N}$. Let Γ_{n_0} be the tree from Lemma 3.2 containing the M^{-n_0}-net $X_{n_0}\subseteq X$.

Since Γ_{n_0} is a tree of finite length that is a union of finitely many line segments, it is not hard to show that there is a piecewise-linear arc-length-parametrized path $\gamma\colon [0,2\mathscr{H}^1(\Gamma_{n_0})]\to\Gamma_{n_0}$ that traverses almost every point in Γ_{n_0} at most twice (except at the discrete set of points X_{n_0}). The proof is similar to that of its graph-theoretic analogue.

Let Δ be the family of cubes from Lemma 2.1 tailored to Γ_{n_0} and fix $Q_0\in\Delta_0$. We will adjust the value of $c>0$ in Lemma 2.1 and the value $\varepsilon>0$ in the definition of M as we go along the proof. Note that $\operatorname{diam} X>2$ implies $\operatorname{diam}\Gamma_{n_0}>1>(1+\varepsilon\beta)c$ if $c<\frac{1}{8}$, and so $\Gamma_{n_0}\not\subseteq Q_0$. For $Q,R\in\Delta$, write $R^1=Q$ if R is a maximal cube in Δ properly contained in Q. For $n\geq 0$ and $Q\in\Delta$, define $\mathscr{L}_0(Q)=\{Q\}$ and

$$\mathscr{L}_1(Q)=\{R\in\Delta: R^1=Q\},\quad \mathscr{L}_n(Q)=\bigcup_{R\in\mathscr{L}_{n-1}(Q)}\mathscr{L}_1(R),$$

$$\widetilde{\mathscr{L}}_n(Q)=\mathscr{L}_n(Q)\cap\bigcup_{j=0}^{n_0-1}\Delta_j,\qquad \widetilde{\mathscr{L}}(Q)=\bigcup_{n=0}^{\infty}\widetilde{\mathscr{L}}_n(Q),$$

$$\widetilde{\mathscr{L}}_n=\widetilde{\mathscr{L}}_n(Q_0),\qquad\qquad \widetilde{\mathscr{L}}=\widetilde{\mathscr{L}}(Q_0).$$

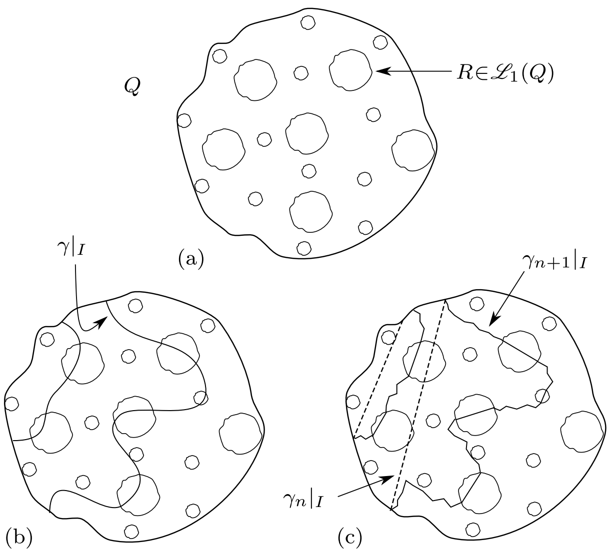

Figure 3. In (a), we have a typical cube $Q\in\Delta_n$ and some of its children in $\mathscr{L}_1(Q)$. Note that their sizes can be radically different. In (b) are the components $\gamma|_{\gamma^{-1}(Q)}$, where in this case $\gamma^{-1}(Q)$ consists of two intervals, and we have pointed at a particular component $\gamma|_I$ for some $I\in\lambda(Q)$. In (c), the *dashed lines* represent the components of $\gamma_n|_{\gamma^{-1}(Q)}$, which is affine in the cubes in Δ_n and hence is affine in Q, and the *solid* piecewise-affine curves represent the components of $\gamma_{n+1}|_{\gamma^{-1}(Q)}$, which are affine in the children of Q (since they are in Δ_{n+1}).

For $Q\in\Delta$ let

$$\lambda(Q)=\{[a,b]:(a,b) \text{ is a connected component of } \gamma^{-1}(Q)\},$$

and for $n\leq n_0$ define γ_n to be the continuous function such that for all $Q\in\mathscr{L}_n(Q_0)$ and $[a,b]\in\lambda(Q)$,

$$\gamma_n|_{[a,b]}(at+(1-t)b)=t\gamma(a)+(1-t)\gamma(b) \quad \text{for } t\in[0,1],$$

that is, γ_n is linear in all cubes in Δ_n and agrees with γ on the boundaries of the cubes (see Figure 3).

Lemma 3.1 will follow from the following two lemmas.

Lemma 3.3. *There is $K\in(0,1)$ and $\beta_0>0$ (independent of n_0 above) such that if $\beta\in(0,\beta_0)$, $n<n_0$, and $Q\in\widetilde{\mathscr{L}}_n$, either*

$$\sum_{I\in\lambda(Q)}(\ell(\gamma_{n+1}|_I)-\ell(\gamma_n|_I))\geq\frac{\varepsilon\beta}{4}\operatorname{diam} Q \tag{3.8}$$

or $Q \in \Delta_{\mathrm{Bad}}$, *where*

$$\Delta_{\mathrm{Bad}} = \{R \in \widetilde{\mathscr{L}} : \mathscr{H}^1_\infty(\Gamma_{n_0} \cap R) \geq (1+K\beta) \operatorname{diam} R\}. \tag{3.9}$$

Lemma 3.4. *With* Δ_{Bad} *defined as above, we have*

$$\sum_{Q \in \Delta_{\mathrm{Bad}}} \beta \operatorname{diam} Q \leq \frac{2}{K} \mathscr{H}^1(\Gamma_{n_0}). \tag{3.10}$$

We will prove these lemmas in Sections 3.3 and 3.4 respectively, but first let us finish the proof of Lemma 3.1.

For $Q \in \widetilde{\mathscr{L}}$, let $n(Q)$ be such that $Q \in \mathscr{L}_n$ and define

$$d(Q) = \sum_{I \in \lambda(Q)} (\ell(\gamma_{n(Q)+1}|_I) - \ell(\gamma_{n(Q)}|_I)).$$

By telescoping sums and Lemma 2.3, we have

$$\begin{aligned}
\sum_{Q \in \widetilde{\mathscr{L}}} d(Q) &= \sum_{n=0}^{n_0-1} \sum_{Q \in \widetilde{\mathscr{L}}_n} \sum_{I \in \lambda(Q)} (\ell(\gamma_{n+1}|_I) - \ell(\gamma_n|_I)) \\
&= \sum_{n=0}^{n_0-1} (\ell(\gamma_{n+1}|_{\gamma^{-1}(Q_0)}) - \ell(\gamma_n|_{\gamma^{-1}(Q_0)})) \\
&\leq \ell(\gamma|_{\gamma^{-1}(Q_0)}) \\
&= 2\mathscr{H}^1(\Gamma_{n_0} \cap Q_0). \qquad (3.11)
\end{aligned}$$

Note that $\operatorname{diam}(\Gamma_{n_0} \cap Q_0) \geq 1$ since $Q_0 \in \Delta_0$, $\operatorname{diam} \Gamma_{n_0} > 1$, and Γ_{n_0} is connected. This and Lemmas 3.3 and 3.4 imply that

$$\begin{aligned}
\frac{10}{K\varepsilon} \mathscr{H}^1(\Gamma_{n_0} \cap Q_0) &\geq \frac{2}{K\varepsilon} \mathscr{H}^1(\Gamma_{n_0} \cap Q_0) + \frac{8}{\varepsilon} \mathscr{H}^1(\Gamma_{n_0} \cap Q_0) \\
&\overset{(3.9)}{\underset{(3.11)}{\geq}} \sum_{Q \in \Delta_{\mathrm{Bad}}} \beta \operatorname{diam} Q + \frac{4}{\varepsilon} \sum_{Q \in \widetilde{\mathscr{L}} \setminus \Delta_{\mathrm{Bad}}} d(Q) \\
&\overset{(3.8)}{\geq} \sum_{Q \in \Delta_{\mathrm{Bad}}} \beta \operatorname{diam} Q + \sum_{Q \in \widetilde{\mathscr{L}} \setminus \Delta_{\mathrm{Bad}}} \beta \operatorname{diam} Q \\
&= \sum_{Q \in \widetilde{\mathscr{L}}} \beta \operatorname{diam} Q
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{n_0-1} \sum_{Q\in\Delta_n} \beta \operatorname{diam} Q \\
&\geq \sum_{n=0}^{n_0-1} \sum_{Q\in\Delta_n} \beta \operatorname{diam} B_Q \\
&= \sum_{n=0}^{n_0-1} c \sum_{Q\in\Delta_n} \beta \operatorname{diam} \frac{1}{c} B_Q \\
&\stackrel{(2.7)}{\geq} c n_0 \beta \operatorname{diam}(\Gamma_{n_0} \cap Q_0) \\
&\geq c n_0 \beta
\end{aligned}
$$

so that

$$\frac{K c n_0 \beta \varepsilon}{10} \leq \mathscr{H}^1(\Gamma_{n_0} \cap Q_0).$$

By Lemma 3.2, and since B_{Q_0} has radius c,

$$
\begin{aligned}
\mathscr{H}^1(\Gamma_{n_0} \cap Q_0) &\leq \mathscr{H}^1(\Gamma_{n_0} \cap (1+\varepsilon\beta) B_{Q_0}) \leq \mathscr{H}^1(\Gamma_{n_0} \cap B(x, 2c)) \\
&\leq 8\#(X_{n_0} \cap B(x, 4c)) M^{-n_0}.
\end{aligned}
$$

Combining these two estimates we have for $c<c'/4$ that

$$\delta n_0 M^{n_0} \beta \leq \#(X_{n_0} \cap B(x_0, c')), \quad \text{where } \delta = \frac{K c \varepsilon}{80}.$$

Pick $n_0 = \lceil 8/\delta\beta^2\varepsilon \rceil$. Since $8/\varepsilon\beta = M$, we get

$$
\begin{aligned}
\#(X_{n_0} \cap B(x_0, c')) &\geq \delta n_0 M^{n_0} \beta = n_0 \left(\frac{\delta\varepsilon\beta^2}{8}\right) M^{n_0} \frac{8}{\varepsilon\beta} \geq M^{n_0+1} \\
&= M^{n_0(1+1/n_0)} \geq M^{n_0(1+1/(8/\delta\beta^2-1))} \geq M^{n_0(1+\delta\beta^2/16)}
\end{aligned}
$$

since $8/\delta\beta^2 \geq 2$, and this proves Lemma 3.1 with $\theta = \delta/16$.

Remark 3.5. By inspecting the proof of Lemma 3.3 below, one can solve for explicit values of ε, c, β_0, and K. In particular, one can choose $\varepsilon < \frac{1}{12288}$, $K < \frac{1}{4096}$, $c < \frac{1}{64}$, and $\beta_0 = \frac{1}{356}$, so that the supremum of permissible values of θ is at least 2^{-41}, and is by no means tight.

In the next two subsections, we prove Lemmas 3.3 and 3.4.

3.3. Proof of Lemma 3.3

Fix Q as in the statement of the lemma. For any $I\in\lambda(Q)$,

$$\begin{aligned}\ell(\gamma_{n+1}|_I)-\ell(\gamma_n|_I) &\geq \ell(\gamma_{n+1}|_I)-|\gamma_n(a_I)-\gamma_n(b_I)|\\ &= \ell(\gamma_{n+1}|_I)-|\gamma_{n+1}(a_I)-\gamma_{n+1}(b_I)|\geq 0.\end{aligned}$$

Hence, to prove the lemma, it suffices to show that either $Q\in\Delta_{\text{Bad}}$ or there is an interval $I\in\lambda(Q)$ for which

$$\ell(\gamma_{n+1}|_I)-\ell(\gamma_n|_I)\geq \frac{\varepsilon\beta}{4}\operatorname{diam} Q.$$

Fix N so that $Q\in\Delta_N$. Let $\widetilde{Q}\in\Delta_{N+1}$ be such that

$$x_Q\in\widetilde{Q}\subset\widetilde{Q}^1=Q$$

and pick $I\in\lambda(Q)$ such that $\gamma_{n+1}(I)\cap\widetilde{Q}\neq\varnothing$. Note that $\gamma_n|_I\subseteq Q$ is a segment with the same endpoints as $\gamma_{n+1}|_I$. Hence

$$\begin{aligned}\ell(\gamma_n|_I) &= \mathscr{H}^1(\gamma_n(I))=\operatorname{diam}\gamma_n(I)=|\gamma_n(a_I)-\gamma_n(b_I)|\\ &= |\gamma_{n+1}(a_I)-\gamma_{n+1}(b_I)|\leq \operatorname{diam} Q. \end{aligned}\tag{3.12}$$

Before proceeding, we will give a rough idea of how the proof will go. We will consider a few cases, which are illustrated in Figure 4.

In the first case, we assume the diameter of $\gamma_n(I)$ is small with respect to Q; since $\gamma_{n+1}|_I$ has the same endpoints as $\gamma_n|_I$ and intersects the center cube $\widetilde{Q}$, there must be a large difference in length between $\gamma_{n+1}(I)$ and $\gamma_n(I)$ as the former must enter Q, hit $\widetilde{Q}$, and then exit Q, and so (3.8) will hold. For the next two cases, we assume that $\gamma_n(I)$ has large diameter. The case 2a assumes that $\gamma_{n+1}(I)$ contributes more length than $\gamma_n(I)$, again implying (3.8) trivially. (It is possible to combine this case with (1), but we found this split to be somewhat convenient.) In the final case 2b we assume that the difference in length between $\gamma_{n+1}(I)$ and $\gamma_n(I)$ is small. Since $\beta_X(B_Q)>\beta$, we can show that this implies the existence of $z\in X$ far away from $\gamma_{n+1}(I)$ (as $\gamma_{n+1}|_I$ has small geodesic deviation, so it cannot approximate all of X in B_Q). Since Γ_{n_0} approximates X, we can find a large curve $\rho\subseteq\Gamma_{n_0}$ entering B_Q, approaching z, and then leaving B_Q. The presence of both $\gamma(I)$ and ρ inside Q implies that the total length of $\Gamma_{n_0}\cap Q$ must be large, which means that $Q\in\Delta_{\text{Bad}}$.

Now we proceed with the actual proof.

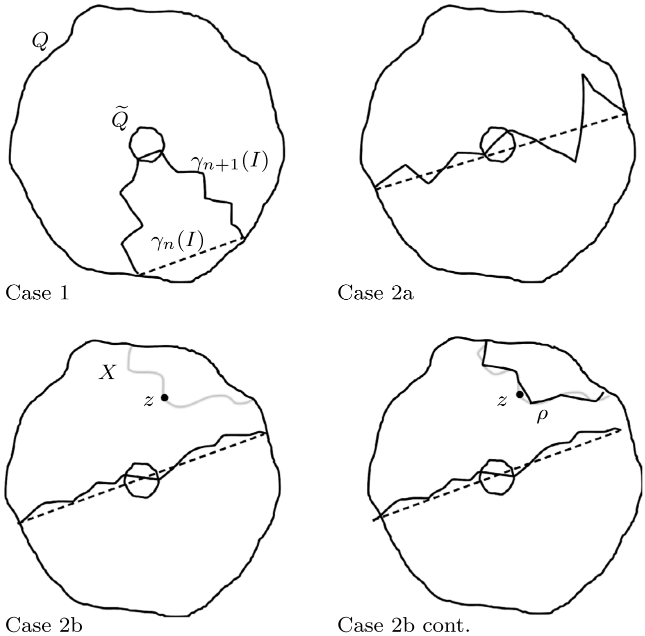

Figure 4. Illustrations of cases 1, 2a, and 2b.

Case 1. *Suppose* $\ell(\gamma_n(I)) < \frac{1}{4}\operatorname{diam} Q$. Since $\gamma_{n+1}|_I$ is a path entering Q, hitting $\widetilde{Q}$, and then leaving Q, we can estimate

$$\begin{aligned}
\ell(\gamma_{n+1}|_I) &\geq 2\operatorname{dist}(\widetilde{Q}, Q^c) \\
&\overset{(2.6)}{\geq} 2\operatorname{dist}((1+\varepsilon\beta)B_{\widetilde{Q}}, B_Q) \\
&= 2(cM^{-N} - (1+\varepsilon\beta)cM^{-N-1}) \\
&= 2cM^{-N}(1-(1+\varepsilon\beta)M^{-1}) \\
&\geq \operatorname{diam} B_Q\left(1 - \frac{\varepsilon\beta}{8} - \frac{\varepsilon^2\beta^2}{8}\right) \\
&> (1-\varepsilon\beta)\operatorname{diam} B_Q \\
&\overset{(2.6)}{\geq} \frac{1-\varepsilon\beta}{1+\varepsilon\beta}\operatorname{diam} Q
\end{aligned}$$

$$= \left(\frac{1+\varepsilon\beta}{1+\varepsilon\beta} - \frac{2\varepsilon\beta}{1+\varepsilon\beta}\right) \operatorname{diam} Q$$

$$\geq (1-2\varepsilon\beta) \operatorname{diam} Q. \tag{3.13}$$

Thus,

$$\ell(\gamma_{n+1}|_I) - \ell(\gamma_n|_I) \overset{(3.13)}{\geq} (1-2\varepsilon\beta) \operatorname{diam} Q - \tfrac{1}{4} \operatorname{diam} Q \geq \tfrac{1}{8} \operatorname{diam} Q$$

if $\varepsilon < \frac{1}{16}$, which implies the lemma in this case.

Case 2. *Suppose*

$$\ell(\gamma_n|_I) \geq \tfrac{1}{4} \operatorname{diam} Q. \tag{3.14}$$

We again split into two cases.

Case 2a. *Suppose*

$$\ell(\gamma_{n+1}|_I) \geq (1+\varepsilon\beta)\ell(\gamma_n|_I).$$

Then

$$\ell(\gamma_{n+1}|_I) - \ell(\gamma_n|_I) \geq \varepsilon\beta\ell(\gamma_n|_I) \overset{(3.14)}{\geq} \frac{\varepsilon\beta}{4} \operatorname{diam} Q.$$

Case 2b. *Now suppose*

$$\ell(\gamma_{n+1}|_I) < (1+\varepsilon\beta)\ell(\gamma_n(I)). \tag{3.15}$$

Note that in this case, we have a better lower bound on $\ell(\gamma_n|_I)$, namely,

$$\ell(\gamma_n|_I) \overset{(3.15)}{\geq} \frac{\ell(\gamma_{n+1}|_I)}{1+\varepsilon\beta} \overset{(3.13)}{\geq} \frac{1-2\varepsilon\beta}{1+\varepsilon\beta} \operatorname{diam} Q \geq (1-3\varepsilon\beta) \operatorname{diam} Q. \tag{3.16}$$

Let $C \in (0,1)$ (we will pick its value later).

Lemma 3.6. *Assuming the conditions in case 2b, let $I' \subseteq I$ be the smallest interval with*

$$\gamma_{n+1}(a_{I'}), \gamma_{n+1}(b_{I'}) \in \partial((1-C\beta)B_Q) \quad \textit{and} \quad \gamma_{n+1}(I') \cap \widetilde{Q} \neq \varnothing.$$

Then

$$\ell(\gamma_{n+1}|_{I'}) - |\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})| \leq 2\varepsilon\beta|\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})|. \tag{3.17}$$

Proof. Since γ_{n+1} enters $(1-C\beta)B_Q$, hits $\widetilde{Q}$, and then leaves $(1+C\beta)B_Q$, we have

$$
\begin{aligned}
\ell(\gamma_{n+1}|_{I'}) &\geq 2\operatorname{dist}(\widetilde{Q},(1-C\beta)B_Q^c)\\
&\overset{(2.6)}{\geq} 2\operatorname{dist}((1+\varepsilon\beta)B_{\widetilde{Q}},(1-C\beta)B_Q^c)\\
&= 2((1-C\beta)cM^{-N}-(1+\varepsilon\beta)cM^{-N-1})\\
&= 2cM^{-N}(1-C\beta-(1+\varepsilon\beta)M^{-1})\\
&> (1-C\beta-2M^{-1})\operatorname{diam} B_Q\\
&= \left(1-C\beta-\frac{\varepsilon\beta}{4}\right)\operatorname{diam} B_Q\\
&\overset{(2.6)}{\geq} \frac{1-C\beta-\frac{1}{4}\varepsilon\beta}{1+\varepsilon\beta}\operatorname{diam} Q\\
&= \left(\frac{1+\varepsilon\beta}{1+\varepsilon\beta}-\frac{C\beta+\frac{5}{4}\varepsilon\beta}{1+\varepsilon\beta}\right)\operatorname{diam} Q\\
&> (1-C\beta-2\varepsilon\beta)\operatorname{diam} Q. \qquad (3.18)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&|\gamma_{n+1}(a_I)-\gamma_{n+1}(b_I)|-|\gamma_{n+1}(a_{I'})-\gamma_{n+1}(b_{I'})|\\
&\qquad\leq |\gamma_{n+1}(a_I)-\gamma_{n+1}(a_{I'})|+|\gamma_{n+1}(b_I)-\gamma_{n+1}(b_{I'})|\\
&\qquad\leq \ell(\gamma_{n+1}|_{I\setminus I'})\\
&\qquad= \ell(\gamma_{n+1}|_I)-\ell(\gamma_{n+1}|_{I'})\\
&\qquad\overset{(3.15),(3.18)}{\leq} (1+\varepsilon\beta)\ell(\gamma_n(I))-(1-C\beta-2\varepsilon\beta)\operatorname{diam} Q\\
&\qquad\overset{(3.13)}{\leq} (1+\varepsilon\beta)\operatorname{diam} Q-(1-C\beta-2\varepsilon\beta)\operatorname{diam} Q\\
&\qquad= (3\varepsilon\beta+C\beta)\operatorname{diam} Q\\
&\qquad\overset{(3.12),(3.14)}{\leq} 4(3\varepsilon\beta+C\beta)|\gamma_{n+1}(a_I)-\gamma_{n+1}(b_I)|. \qquad (3.19)
\end{aligned}
$$

Thus,

$$
|\gamma_{n+1}(a_I)-\gamma_{n+1}(b_I)| \leq \frac{|\gamma_{n+1}(a_{I'})-\gamma_{n+1}(b_{I'})|}{1-4(3\varepsilon\beta+C\beta)} \leq 2|\gamma_{n+1}(a_{I'})-\gamma_{n+1}(b_{I'})| \qquad (3.20)
$$

if we pick $\varepsilon < \frac{1}{24}$ and $\beta < \frac{1}{8}$ (recall that $C \in (0,1)$). By Lemma 2.4,

$$\begin{aligned}\ell(\gamma_{n+1}|_{I'}) - |\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})| &\overset{(2.8)}{\leq} \ell(\gamma_{n+1}|_{I}) - |\gamma_{n+1}(a_{I}) - \gamma_{n+1}(b_{I})| \\ &\overset{(3.15)}{<} \varepsilon\beta|\gamma_{n+1}(a_I) - \gamma_{n+1}(b_I)| \\ &\overset{(3.20)}{\leq} 2\varepsilon\beta|\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})|,\end{aligned}$$

which proves (3.17). $\square$

By the main assumption in Theorem 1.4, and because we are assuming $n=0$ so that $M^{-n}=1<r_0$,

$$\begin{aligned}\beta &< \beta'_X(x_Q, (1-C\beta)cM^{-N}) \\ &\leq \frac{\ell(\gamma_{n+1}|_{I'}) - |\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})| + \sup_{z\in(1-C\beta)B_Q\cap X} \operatorname{dist}(z, \gamma_{n+1}(I'))}{|\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})|} \\ &\overset{(3.17)}{\leq} \frac{2\varepsilon\beta|\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})| + \sup_{z\in(1-C\beta)B_Q\cap X} \operatorname{dist}(z, \gamma_{n+1}(I'))}{|\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})|} \\ &= 2\varepsilon\beta + \frac{\sup_{z\in(1-C\beta)B_Q\cap X} \operatorname{dist}(z, \gamma_{n+1}(I'))}{|\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})|}\end{aligned}$$

so there is $z \in X\cap(1-C\beta)B_Q$ with

$$\begin{aligned}\operatorname{dist}(z, \gamma_{n+1}(I')) &\geq (\beta - 2\varepsilon\beta)|\gamma_{n+1}(a_{I'}) - \gamma_{n+1}(b_{I'})| \\ &\overset{(3.20)}{\geq} \frac{\beta-2\varepsilon\beta}{2}|\gamma_{n+1}(a_I) - \gamma_{n+1}(b_I)| \\ &\overset{(3.14)}{\geq} \frac{\beta-2\varepsilon\beta}{8}\operatorname{diam} Q \\ &\geq \frac{\beta}{16}\operatorname{diam} Q \end{aligned} \tag{3.21}$$

if $\varepsilon < \frac{1}{4}$.

Since $\gamma_{n+1}([0,1])$ hits every cube in $\mathscr{L}_1(Q)$, which all have diameter at most $2(1+\varepsilon\beta)cM^{-N-1}$ by (2.6) (recall that N was chosen so that $Q\in\Delta_N$),

$$\Gamma_{n_0}\cap Q \subseteq \gamma_{n+1}([0,1])_{2(1+\varepsilon\beta)cM^{-N-1}} \subseteq \gamma_{n+1}([0,1])_{4cM^{-N-1}}.$$

Note that since $Q\in\widetilde{\mathscr{L}}_n$, we have $N<n_0$. Since $X_{n_0}\subseteq\Gamma_{n_0}\cap X$ and $N<n_0$,

$$\begin{aligned}X\cap(1-C\beta)B_Q &\subseteq X\cap Q \\ &\subseteq (\Gamma_{n_0}\cap Q)_{2M^{-n_0}}\end{aligned}$$

$$
\begin{aligned}
&\subseteq \gamma_{n+1}([0,1])_{4cM^{-N-1}+2M^{-n_0}}\\
&\subseteq \gamma_{n+1}([0,1])_{4cM^{-N-1}+2M^{-N-1}}\\
&= \gamma_{n+1}([0,1])_{(2+1/c)M^{-1}2cM^{-N}}\\
&= \gamma_{n+1}([0,1])_{(2+1/c)M^{-1}\operatorname{diam} B_Q}\\
&\subseteq \gamma_{n+1}([0,1])_{(2/Mc)\operatorname{diam} B_Q}
\end{aligned}
$$

since $c<\frac{1}{8}$. Since $z\in X\cap(1-C\beta)B_Q$, there is $t\in[0,1]$ such that

$$
|\gamma_{n+1}(t)-z|<\frac{2}{c}M^{-1}\operatorname{diam} B_Q=\frac{\varepsilon\beta}{4c}\operatorname{diam} Q \tag{3.22}
$$

and so

$$
\begin{aligned}
\operatorname{dist}(\gamma_{n+1}(t),\gamma_{n+1}(I'))&\geq \operatorname{dist}(z,\gamma_{n+1}(I'))-|\gamma_{n+1}(t)-z|\\
&\overset{(3.21)(3.22)}{\geq}\left(\frac{\beta}{16}-\frac{\varepsilon\beta}{4c}\right)\operatorname{diam} Q\\
&\geq \frac{\beta}{32}\operatorname{diam} Q \qquad (3.23)
\end{aligned}
$$

for $\varepsilon<c/8$. Also, since $z\in(1-C\beta)B_Q$, we know that

$$
\begin{aligned}
B_Q&\supseteq B\left(z,\frac{C\beta}{2}\operatorname{diam} B_Q\right)\overset{(2.6)}{\supseteq} B\left(z,\frac{C\beta}{2(1+\varepsilon\beta)}\operatorname{diam} Q\right)\supseteq B\left(z,\frac{C\beta}{4}\operatorname{diam} Q\right)\\
&\overset{(3.22)}{\supseteq} B\left(\gamma_{n+1}(t),\left(\frac{C\beta}{4}-\frac{\varepsilon\beta}{4c}\right)\operatorname{diam} Q\right)\supseteq B\left(\gamma_{n+1}(t),\frac{C\beta}{8}\operatorname{diam} Q\right) \qquad (3.24)
\end{aligned}
$$

for $\varepsilon<Cc/2$. In particular, $t\in\gamma_{n+1}^{-1}(B_Q)$. Note that

$$
\begin{aligned}
\operatorname{dist}(\gamma_{n+1}(t),\gamma_{n+1}(I))&\geq \operatorname{dist}(\gamma_{n+1}(t),\gamma_{n+1}(I'))\\
&\qquad -\max\{\operatorname{diam}\gamma([a_I,a_I']),\operatorname{diam}\gamma([b_I',b_I])\}\\
&\geq \operatorname{dist}(\gamma_{n+1}(t),\gamma_{n+1}(I'))-\ell(\gamma|_{I/I'})\\
&\overset{(3.19)(3.23)}{\geq}\left(\frac{\beta}{32}-(3\varepsilon\beta+C\beta)\right)\operatorname{diam} Q\\
&\geq \frac{\beta}{64}\operatorname{diam} Q
\end{aligned}
$$

for $\varepsilon<\frac{1}{384}$ and $C<\frac{1}{128}$. Thus, since of course $C/8<\frac{1}{128}$, we have

$$
B\left(\gamma_{n+1}(t),\frac{C\beta}{8}\operatorname{diam} Q\right)\subseteq Q\backslash\gamma_{n+1}(I)_{(\beta/128)\operatorname{diam} Q}.
$$

In particular, $\gamma_{n+1}(t)\in Q$, and so by construction, $t\in[a,b]$ for some $[a,b]\in\lambda(Q)$, where $\gamma_{n+1}(a)$ and $\gamma_{n+1}(b)$ are both in Γ_{n_0}. In particular, $\gamma_{n+1}((a,b))$ is a line segment in a cube $R\in\widetilde{\mathscr{L}}_1(Q)$. If $\zeta:=\gamma_{n+1}(a)\in\Gamma_{n_0}$, then

$$\begin{aligned}|\zeta-\gamma_{n+1}(t)| &\leq \operatorname{diam} R \stackrel{(2.6)}{\leq} (1+\varepsilon\beta)\operatorname{diam} B_R = 2(1+\varepsilon\beta)cM^{-N-1}\\ &\leq (1+\varepsilon\beta)M^{-1}\operatorname{diam} Q = (1+\varepsilon\beta)\frac{\varepsilon\beta}{8}\operatorname{diam} Q \leq \frac{\varepsilon\beta}{4}\operatorname{diam} Q \leq \frac{C\beta}{16}\operatorname{diam} Q \end{aligned} \tag{3.25}$$

for $\varepsilon<\frac{C}{4}$, and so

$$B\left(\zeta,\frac{C\beta}{16}\operatorname{diam} Q\right)\subseteq B\left(\gamma_{n+1}(t),\frac{C\beta}{8}\operatorname{diam} Q\right)\subseteq \frac{Q}{\gamma_{n+1}(I)_{(\beta/128)\operatorname{diam} Q}}. \tag{3.26}$$

Thus, since Γ_{n_0} is connected and $\operatorname{diam}\Gamma_{n_0}>\operatorname{diam} Q_0>(C\beta/16)\operatorname{diam} Q$, we know there is a curve

$$\rho\subseteq\Gamma_{n_0}\cap B\left(\zeta,\frac{C\beta}{16}\operatorname{diam} Q\right)$$

connecting ζ to $B(\zeta,(C\beta/16)\operatorname{diam} Q)^c$, and therefore the curve has diameter at least $\frac{1}{16}C\beta\operatorname{diam} Q$. Hence,

$$\mathscr{H}^1_\infty(\rho)\geq\operatorname{diam}\rho\geq\frac{C\beta}{16}\operatorname{diam} Q.$$

Moreover,

$$\mathscr{H}^1_\infty(\gamma(I))\geq\operatorname{diam}\gamma(I)\geq|\gamma(a_I)-\gamma(b_I)|\stackrel{(3.12)}{=}|\gamma_n(a_I)-\gamma_n(b_I)|\stackrel{(3.16)}{\geq}(1-3\varepsilon\beta)\operatorname{diam} Q.$$

Thus, since any cube in $\mathscr{L}^1(Q)$ intersecting ρ has diameter at most $(\varepsilon\beta/4)\operatorname{diam} Q<\beta/128$ by (3.25), they are disjoint from those intersecting $\gamma(I)$ by (3.26) if we choose $\varepsilon<\frac{1}{128}$ (as if they intersect $\gamma(I)$, they also intersect $\gamma_{n+1}(I)$ by the definition of γ_{n+1}). Hence, we have

$$\mathscr{H}^1_\infty(Q)\geq\frac{C\beta}{16}\operatorname{diam} Q+(1-3\varepsilon\beta)\operatorname{diam} Q\geq\left(1+\frac{C\beta}{32}\right)\operatorname{diam} Q$$

for $\varepsilon<C/96$. Picking $K=C/32$ gives that $Q\in\Delta_{\text{Bad}}$, which finishes the proof of Lemma 3.3.

3.4. Geometric martingales and the proof of Lemma 3.4

For $Q\in\Delta$, let $k(Q)$ be the number of cubes in Δ_{Bad} that properly contain Q, and set

$$\begin{aligned}\Delta_{\mathrm{Bad},j} &= \{Q\in\Delta_{\mathrm{Bad}} : k(Q)=j\},\\ \mathrm{Bad}_j(Q) &= \{R\subseteq Q : k(R)=k(Q)+j\},\\ G(Q) &= \frac{\Gamma_{n_0}\cap Q}{\bigcup_{R\in\mathrm{Bad}_1(Q)} R}.\end{aligned}$$

We will soon define, for each $Q\in\Delta_{\mathrm{bad}}$, a non-negative weight function $w_Q\colon \Gamma_{n_0}\to[0,\infty)$ $\mathscr{H}^1|_{\Gamma_{n_0}}$-a.e. in a martingale fashion by defining it as a limit of a sequence w_Q^j. Each w_Q^j will be constant on various subsets of Γ_{n_0} that partition Γ_0. We will actually decide the value of w_Q^j on an element A of the partition, say, by declaring the value of

$$w_Q^j(A) := \int_{\Gamma_{n_0}\cap A} w_Q^j \, d\mathscr{H}^1.$$

Then we will define w_Q^{j+1} to be constant on sets in a partition subordinate to the previous partition so that, on sets A in the jth partition, $w_Q^{j+1}(A)=w_Q^j(A)$, and so forth. We do this in such a way that we disseminate the mass of the weight function w_Q so that w_Q is supported in Q, has integral $\operatorname{diam} Q$, and so that $w_Q(x)\le 1/(1+K\beta)^{k(x)-k(Q)}$, where $k(x)$ is the total number of bad cubes containing x. By geometric series, this will mean that $\sum_{Q\in\Delta_{\mathrm{Bad}}} w_Q \mathbb{1}_Q$ is a bounded function, so that its total integral is at most a constant times $\mathscr{H}^1(\Gamma_0)$. However, the integral of each of these functions w_Q is $\operatorname{diam} Q$, and so the integral is also equal to $\sum_{Q\in\Delta_{\mathrm{Bad}}} \operatorname{diam} Q$, which gives us (3.9). This method appears in [16]. Now we proceed with the proof.

First set

$$w_Q^0(Q) = \operatorname{diam} Q \quad \text{and} \quad w_Q^0|_{Q^c} \equiv 0, \tag{3.27}$$

and construct w_Q^{j+1} from w_Q^j as follows:

1. If $R\in\mathrm{Bad}_j(Q)$ for some j and $S\in\mathrm{Bad}_1(R)$, set w_Q^{j+1} to be constant in S so that

$$w_Q^{j+1}(S) = w_Q^j(R)\frac{\operatorname{diam} S}{\sum_{T\in\mathrm{Bad}_1(R)} \operatorname{diam} T + \mathscr{H}^1(G(R))}. \tag{3.28}$$

2. Set w_Q^{j+1} to be constant in $G(R)$ so that

$$w_Q^{j+1}(G(R))=w_Q^j(R)-\sum_{S\in \mathrm{Bad}_1(R)} w_Q^{j+1}(S). \tag{3.29}$$

3. For points x not in any $R\in\mathrm{Bad}_j(Q)$, set $w_Q^{j+1}(x)=w_Q^j(x)$.
Like a martingale, we have by our construction that, if $R\in\mathrm{Bad}_j(Q)$, then $w_Q^i(R)=w_Q^j(R)$ for all $i\geq j$, and in particular, $w_Q^j(Q)=\operatorname{diam} Q$ for all $j\geq 0$.

We will need the inequality

$$\sum_{T\in\mathrm{Bad}_1(R)} \operatorname{diam} T+\mathscr{H}^1(G(R))\geq \mathscr{H}^1_\infty(R\cap\Gamma_{n_0})\geq (1+K\beta)\operatorname{diam} R. \tag{3.30}$$

The first inequality comes from the fact that if $\delta>0$ and A_i is a cover of $G(R)$ by sets so that $\sum_{i=1}^\infty \operatorname{diam} A_i<\mathscr{H}^1(G(R))+\delta$, then $\{A_i\}_i\cup\mathrm{Bad}_1(R)$ is a cover of R (up to a set of $\mathscr{H}^1$-measure zero by Lemma 2.3), and so

$$\sum_{T\in\mathrm{Bad}_1(R)} \operatorname{diam} T+\mathscr{H}^1(G(R))+\delta>\sum_{i=1}^\infty \operatorname{diam} A_i+\sum_{T\in\mathrm{Bad}_1(R)} \operatorname{diam} T\geq \mathscr{H}^1_\infty(R\cap\Gamma_{n_0}),$$

which gives the first inequality in (3.30) by taking $\delta\to 0$. The last inequality in (3.30) is from the definition of Δ_{Bad}.

For $S\in\mathrm{Bad}_1(R)$ and $R\in\mathrm{Bad}_j(Q)$, by induction we have

$$\begin{aligned}\frac{w_Q^{j+1}(S)}{\operatorname{diam} S}&\overset{(3.28)}{=}\frac{w_Q^j(R)}{\sum_{T\in\mathrm{Bad}_1(R)}\operatorname{diam} T+\mathscr{H}^1(G(R))}\overset{(3.30)}{\leq}\frac{w_Q^j(R)}{\operatorname{diam} R}\frac{1}{1+K\beta}\\ &\leq\frac{w_Q^0(Q)}{\operatorname{diam} Q}\frac{1}{(1+K\beta)^{j+1}}\overset{(3.27)}{=}\frac{1}{(1+K\beta)^{j+1}}.\end{aligned} \tag{3.31}$$

Hence, since w_Q^{j+1} is constant in S, for $x\in S\cap\Gamma_{n_0}$,

$$\begin{aligned}w_Q^{j+1}(x)&\overset{(3.28)}{=}w_Q^j(R)\frac{\operatorname{diam} S}{\sum_{T\in\mathrm{Bad}_1(R)}\operatorname{diam} T+\mathscr{H}^1(G(R))}\frac{1}{\mathscr{H}^1(S\cap\Gamma_{n_0})}\\ &\overset{(3.30)}{\leq}\frac{w_Q^j(R)}{\sum_{T\in\mathrm{Bad}_1(R)}\operatorname{diam} T+\mathscr{H}^1(G(R))}\frac{1}{1+K\beta}\\ &\overset{(3.30)}{\leq}\frac{w_Q^j(R)}{\operatorname{diam} R}\frac{1}{(1+K\beta)^2}\end{aligned}$$

$$\overset{(3.31)}{\leq} \frac{w_Q^0(Q)}{\operatorname{diam} Q} \frac{1}{(1+K\beta)^{j+2}}$$

$$= \frac{1}{(1+K\beta)^{j+2}}. \tag{3.32}$$

Moreover, if $x \in G(R)$,

$$\begin{aligned}
w_Q^{j+1}(x) &= \frac{w_Q^{j+1}(G(R))}{\mathscr{H}^1(G(R))} \\
&\overset{(3.29)}{=} \frac{w_Q^j(R) - \sum_{S\in \mathrm{Bad}_1(R)} w_Q^{j+1}(S)}{\mathscr{H}^1(G(R))} \\
&\overset{(3.28)}{=} \frac{w_Q^j(R)}{\mathscr{H}^1(G(R))} \left(1 - \sum_{S\in \mathrm{Bad}_1(R)} \frac{\operatorname{diam} S}{\sum_{T\in \mathrm{Bad}_1(R)} \operatorname{diam} T + \mathscr{H}^1(G(R))}\right) \\
&= \frac{w_Q^j(R)}{\mathscr{H}^1(G(R))} \frac{\mathscr{H}^1(G(R))}{\sum_{T\in \mathrm{Bad}_1(R)} \operatorname{diam} T + \mathscr{H}^1(G(R))} \\
&= \frac{w_Q^j(R)}{\sum_{T\in \mathrm{Bad}_1(R)} \operatorname{diam} T + \mathscr{H}^1(G(R))} \\
&\overset{(3.30)}{<} \frac{w_Q^j(R)}{\operatorname{diam} R} \frac{1}{1+K\beta} \\
&\overset{(3.31)}{\leq} \frac{1}{(1+K\beta)^{j+1}}. \qquad (3.33)
\end{aligned}$$

Since $\Delta_{\mathrm{Bad}} \subseteq \bigcup_{j=0}^{n_0} \Delta_j$, and $\mathscr{H}^1\left(\bigcup_{Q\in\Delta} \partial Q\right) = 0$, almost every point $x \in Q_0 \cap \Gamma_{n_0}$ is contained in at most finitely many cubes in Δ_{Bad}, and hence the value of $w_Q^{j+1}(x)$ changes only finitely many times in j. Thus the limit $w_Q = \lim_{j\to\infty} w_Q^j$ is well defined almost everywhere. For $x \in Q \cap \Gamma_{n_0}$, set $k(x) = k(R)$, where $R \subseteq Q$ is the smallest cube in Δ_{Bad} containing x. Then (3.32) and (3.33) imply that

$$w_Q(x) \leq \frac{1}{(1+K\beta)^{k(x)-k(Q)}},$$

and so

$$\sum_{x\in Q\in\Delta_{\mathrm{Bad}}} w_Q(x) \leq \sum_{j=0}^{k(x)} \frac{1}{(1+K\beta)^j} \leq \sum_{j=0}^{\infty} \frac{1}{(1+K\beta)^j} = \frac{1+K\beta}{K\beta} \leq \frac{2}{K\beta}$$

since $K\beta<1$. Hence,

$$\sum_{Q\in\Delta_{\mathrm{Bad}}}\operatorname{diam} Q=\sum_{Q\in\Delta_{\mathrm{Bad}}}\int_Q w_Q(x)\,d\mathscr{H}^1(x)=\int_{\Gamma_{n_0}}\Bigg(\sum_{x\in Q\in\Delta_{\mathrm{Bad}}} w_Q(x)\Bigg)\,d\mathscr{H}^1(x)$$
$$\leq\frac{2}{K\beta}\mathscr{H}^1(\Gamma_{n_0}),$$

which finishes the proof of Lemma 3.4.

4. Antenna-like sets

This section is devoted to the proof of Theorem 1.6.

It is easy to verify using the definitions that being antenna-like is a quasisymmetric invariant quantitatively, so by Theorem 1.4, it suffices to verify that, if X is c-antenna-like, then any ball $B(x,r)$ with $x\in X$ and $0<r<\frac{1}{2}\operatorname{diam} X$ has $\beta'(x,r)>c/7$.

Fix such a ball, so there is a homeomorphism $h\colon \bigcup_{i=1}^3[0,e_i]\to X\cap B(x,r)$ so that

$$\operatorname{dist}(h(e_i),h([0,e_j]\cup[0,e_k]))\geq cr \tag{4.1}$$

for all permutations (i,j,k) of $(1,2,3)$ (see Figure 5).

Let $s\colon[0,1]\to B(x,r)$ satisfy

$$\ell(s|_{[0,1]})-|s(0)-s(1)|+\sup_{z\in X\cap B(x,r)}\operatorname{dist}(z,s([0,1]))<2\beta'(x,r)|s_0-s_1|=:\beta.$$

Set $x_i=h(e_i)$ for $i=1,2,3$ and let

$$t_1=\inf s^{-1}\Bigg(\bigcup_{i=1}^3 B(x_i,\beta)\Bigg).$$

This always exists since $X\cap B(x,r)\subseteq s([0,1])_\beta$. Without loss of generality, assume that $s(t_1)\in B(x_1,\beta)$. Similarly, let

$$t_2=\inf s^{-1}\Bigg(\bigcup_{i=2}^3 B(x_i,\beta)\Bigg) \tag{4.2}$$

and again, without loss of generality, assume that $s(t_2)\in B(x_2,\beta)$.

Note that $h([0,e_1]\cup[0,e_3])$ is a path connecting x_1 to x_3, where the latter point is not contained in $s([t_1,t_2])_\beta$ by our choices of t_1 and t_2, although the former point is; otherwise, there would be $t\in[t_1,t_2]$ such that $s(t)\in B(x_3,\beta)$, contradicting the minimality of t_2. Since $h([0,e_1]\cup[0,e_3])$ is connected and $s([t_1,t_2])_\beta$ contains x_1 but

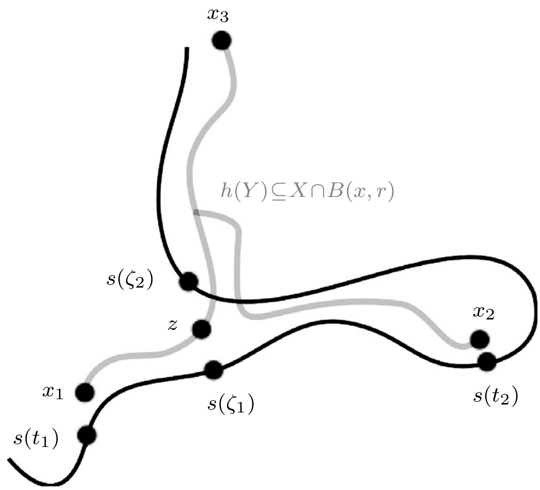

Figure 5.

not x_3, we can pick a point $z\in h([0,e_1]\cup[0,e_3])$ so that $\operatorname{dist}(z,s([t_1,t_2]))=\beta$. Pick $\zeta_1\in[t_1,t_2]$ and $\zeta_2\in(t_2,1]$ so that

$$|s(\zeta_1)-z|=\operatorname{dist}(z,s([t_1,t_2]))=\beta \quad\text{and}\quad |s(\zeta_2)-z|<\beta. \tag{4.3}$$

Then by Lemma 2.4,

$$\begin{aligned}
2\beta'(x,r)|s_0-s_1| &> \ell(s|_{[0,1]})-|s(0)-s(1)|\\
&\geq \ell(s|_{[\zeta_1,\zeta_2]})-|s(\zeta_1)-s(\zeta_2)|\\
&\geq \ell(s|_{[\zeta_1,t_2]})+\ell(s|_{[t_2,\zeta_2]})-|s(\zeta_1)-z|-|z-s(\zeta_2)|\\
&\overset{(4.3)}{>} |s(\zeta_1)-s(t_2)|+|s(t_2)-s(\zeta_2)|-\beta-\beta\\
&\geq |z-x_2|-|s(\zeta_1)-z|-|x_2-s(t_2)|\\
&\qquad +|x_2-z|-|s(t_2)-x|-|s(\zeta_2)-z|-2\beta\\
&\overset{\substack{(4.1)\\(4.3)}}{\geq} cr-\beta-\beta+cr-\beta-\beta-2\beta\\
&= 2cr-6\beta\\
&\geq c|s(0)-s(1)|-12\beta(x,r)|s(0)-s(1)|,
\end{aligned}$$

which yields $\beta'(x,r)\geq c/7$ and completes the proof of Theorem 1.6.

5. Comparison of the β-numbers

For quantities A and B, we will write $A\lesssim B$ if there is a universal constant C so that $A\leq CB$, and $A\sim B$ if $A\lesssim B\lesssim A$.

Lemma 5.1. *Let $X\subseteq\ell^\infty$ be a compact connected set. Also let $x\in X$ and $0<r<\frac{1}{2}\operatorname{diam} X$. Then*

$$\beta'(x,r)\leq\hat{\beta}(x,r)\lesssim\beta'(x,r)^{1/2}. \tag{5.1}$$

Proof. The first inequality follows trivially from the definitions, since each sequence $y_0,...,y_n\in X$ induces a finite polygonal Lipschitz path s in ℓ^∞ for which

$$\ell(s)-|s(0)-s(1)|=\sum_{i=0}^{n-1}|y_i-y_{i+1}|-|y_0-y_n|.$$

For the opposite inequality, let $s\colon[0,1]\to\ell^\infty$ be such that

$$\frac{\ell(s)-|s(0)-s(1)|+\sup_{z\in B(x,r)\cap X}\operatorname{dist}(z,s([0,1]))}{|s(0)-s(1)|}\leq 2\beta'(x,r)=:\beta. \tag{5.2}$$

Let

$$A=s^{-1}(s([0,1])_{2\beta|s_0-s_1|}),$$

which is a relatively open subset of $[0,1]$. Let $a=\inf A$ and define $a=t_0<t_1<...<t_n\leq 1$ inductively by setting

$$t_{i+1}=\inf\{t\in A\cap(t_i,b]:\operatorname{dist}(s(t),s([t_0,t_i]))>\beta^{1/2}|s(0)-s(1)|\}.$$

We claim that

$$n\sim\beta^{-1/2}|s(0)-s(1)|. \tag{5.3}$$

To see this, note that since $|s(t_i)-s(t_{i+1})|\geq\beta^{1/2}|s(0)-s(1)|$, the sets

$$B\big(s(t_i),\tfrac{1}{2}\beta^{1/2}|s(0)-s(1)|\big)$$

are disjoint, so that

$$\tfrac{1}{2}n\beta^{1/2}|s(0)-s(1)|\leq\ell(s)\leq(1+\beta)|s(0)-s(1)|\leq 2|s(0)-s(1)|,$$

which gives $n\leq 4\beta^{-1/2}$. On the other hand, the balls $B(s(t_i),2\beta^{1/2}|s(0)-s(1)|)$ cover $s([0,1])$, and so

$$|s(0)-s(1)| \leq \ell(s) \leq \sum_{i=0}^{n} \operatorname{diam} B(s(t_i), 2\beta^{1/2}|s(0)-s(1)|)$$
$$\leq (n+1)4\beta^{1/2}|s(0)-s(1)| \leq 8n\beta^{1/2}|s(0)-s(1)|,$$

which gives $n \geq 1/8\beta$, and this proves (5.3).

By the definition of A, there are

$$y_i \in \overline{B(s(t_i), 2\beta|s(0)-s(1)|)}.$$

Then

$$\sum_{i=0}^{n-1} |y_i - y_{i+1}| - |y_0 - y_1| \leq \sum_{i=0}^{n-1} |s(t_i) - s(t_{i+1})| + 4n\beta|s(0)-s(1)| - |s(t_0) - s(t_n)|$$
$$\stackrel{(5.3)}{\leq} \ell(s|_{[t_0,t_n]}) - |s(t_0) - s(t_n)| + C\beta^{1/2}|s(0)-s(1)|$$
$$\stackrel{(5.2)}{\leq} \beta|s_0 - s_1| + C\beta^{1/2}|s(0)-s(1)|$$
$$\lesssim \beta^{1/2}|s(0)-s(1)|.$$

Claim. $|s(0)-s(1)| \lesssim |s(t_0)-s(t_n)|$.

Since X is connected and $r < \frac{1}{2} \operatorname{diam} X$, there is a path connecting x to $B(x,r)^c$, which naturally must be of diameter at least r. Hence

$$|s(0)-s(1)| \leq 2r \leq 2(\ell(s|_{[t_0,t_n]}) - 4\beta|s_0 - s_1|) \leq 2|s(t_0)-s(t_n)| + C\beta^{1/2}|s(0)-s(1)|,$$

which, if $\beta^{1/2}$ is small enough, implies

$$|s(0)-s(1)| \leq 4|s(t_0)-s(t_n)| = 4|y_0 - y_n|$$

so that the above estimates imply that

$$\sum_{i=0}^{n-1} |y_i - y_{i+1}| - |y_0 - y_n| \lesssim \beta^{1/2}|s(0)-s(1)| \leq 4\beta^{1/2}|y_0 - y_n|. \tag{5.4}$$

Moreover,

$$X \cap B(x,r) \subseteq s([0,1])_{\beta|s(0)-s(1)|}$$
$$\subseteq \bigcup_{i=0}^{n} B(s(t_i), (2\beta^{1/2}+\beta)|s(0)-s(1)|)$$

$$\subseteq \bigcup_{i=0}^{n} B(y_i, (2\beta^{1/2}+\beta+2\beta)|s(0)-s(1)|)$$
$$\subseteq \bigcup_{i=0}^{n} B(y_i, 5\beta^{1/2}|s(0)-s(1)|)$$
$$\subseteq \bigcup_{i=0}^{n} B(y_i, 20\beta^{1/2}|y_0-y_n|). \tag{5.5}$$

Thus (5.4) and (5.5) imply that $\hat{\beta}(x,r)\leq 20\beta^{1/2}=20\sqrt{2}\beta'(x,r)^{1/2}$. $\square$

Proposition 5.2. *For a compact connected subset X of some Hilbert space,*

$$\beta''(x,r)\leq \beta(x,r)\lesssim \beta''(x,r) \quad \text{for } x\in\Gamma \text{ and } r<\tfrac{1}{2}\operatorname{diam} X,$$

where

$$\beta''(x,r)=\inf_s\left(\left(\frac{\ell(s)-|s(0)-s(1)|}{|s(0)-s(1)|}\right)^{1/2}+\frac{\sup_{z\in B(x,r)\cap X}\operatorname{dist}(z,s([0,1]))}{|s(0)-s(1)|}\right).$$

In particular,

$$\beta'(x,r)\leq \beta(x,r)\lesssim \beta'(x,r)^{1/2}. \tag{5.6}$$

Note that (5.6) is tight in the sense that if $X\subseteq\mathbb{C}$, $0\in X$, and $B(0,1)\cap\Gamma=[-1,1]\cup[0,i\varepsilon]$, then by Theorem 1.6 and (5.6), for all $\varepsilon>0$,

$$\beta(0,1)\leq\varepsilon\leq 7\beta'(0,1)\leq 7\beta(0,1)\leq 7\varepsilon.$$

However, if $X\cap B(x,r)=[-1,0]\cup[0,e^{i\varepsilon}]$, then for all $\varepsilon>0$, again by (5.6) (and estimating $\beta''(0,1)$ by letting s be the path traversing the segments $[-1,0]\cup[0,e^{i\varepsilon}]$),

$$\beta(0,1)^2\sim\varepsilon^2\gtrsim\beta'(0,1)\gtrsim\beta(0,1)^2.$$

Proof. For the first inequality, simply let $s\colon [0,1]\to\mathscr{H}$ be the line segment spanning $L\cap B(x,r)$, where L is some line passing through $B(x,r/2)$. Then $\ell(s)=\mathscr{H}^1(L\cap B(x,r))\geq r$ and hence

$$\beta''(x,r)\leq\frac{\sup_{z\in B(x,r)\cap X}\operatorname{dist}(z,s([0,1]))}{|s(0)-s(1)|}\leq\frac{\sup_{z\in B(x,r)\cap X}\operatorname{dist}(z,L)}{r}.$$

Since $x\in X$, the range of admissible lines in the infimum in (1.1) can be taken to be lines intersecting $B(x,r/2)$. Using this fact and infimizing the above inequality over all such lines proves the first inequality in (5.6).

For the opposite inequality, let s satisfy

$$\left(\frac{\ell(s)-|s(0)-s(1)|}{|s(0)-s(1)|}\right)^{1/2}+\frac{\sup_{z\in B(x,r)\cap X}\operatorname{dist}(z,s([0,1]))}{|s(0)-s(1)|}\le 2\beta''(B(x,r))=:\beta.$$

Let

$$\beta(s):=\sup_{t\in[0,1]}\operatorname{dist}(s(t),[s(0),s(1)]).$$

Then by the Pythagorean theorem, there is $c>0$ so that

$$(1+c\beta(s)^2)|s(0)-s(1)|\le \ell(s)\le (1+\beta^2)|s(0)-s(1)|$$

and thus $\beta(s)\le c^{-1}\beta$. Hence, if L is the line passing through $s(0)$ and $s(1)$,

$$\begin{aligned}\beta(x,r)&\le \sup_{z\in B(x,r)\cap X}\operatorname{dist}(z,L)\le \sup_{z\in B(x,r)\cap X}\operatorname{dist}(z,[s(0),s(1)])\\ &\le \beta(s)+\sup_{z\in B(x,r)\cap X}\operatorname{dist}(z,s([0,1]))\le c^{-1}\beta+\beta\lesssim\beta. \qquad \square\end{aligned}$$

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