On the Carleson duality

Tuomas Hytönen and Andreas Rosén

Abstract. As a tool for solving the Neumann problem for divergence-form equations, Kenig and Pipher introduced the space $\mathcal{X}$ of functions on the half-space, such that the non-tangential maximal function of their L_2 Whitney averages belongs to L_2 on the boundary. In this paper, answering questions which arose from recent studies of boundary value problems by Auscher and the second author, we find the pre-dual of $\mathcal{X}$, and characterize the pointwise multipliers from $\mathcal{X}$ to L_2 on the half-space as the well-known Carleson-type space of functions introduced by Dahlberg. We also extend these results to L_p generalizations of the space $\mathcal{X}$. Our results elaborate on the well-known duality between Carleson measures and non-tangential maximal functions.

1. Introduction

A fundamental estimate in harmonic analysis is Carleson's inequality for Carleson measures. See [3, Theorem 2] and [4, Theorem 1] for the original formulations and applications in the theory of interpolating analytic functions, or for example Stein [11, Section II.2.2] and Coifman, Meyer and Stein [5] for more recent accounts in the framework of real-variable harmonic analysis. This inequality states that for a function $f(t, x)$ and a measure $d\mu(t, x)$ in the upper half-space $\mathbf{R}^{1+n}_+ := \{(t, x); t > 0 \text{ and } x \in \mathbf{R}^n\}$, one has the estimate

$$\iint_{\mathbf{R}^{1+n}_+} |f(t, x)| \, d\mu(t, x) \lesssim \sup_Q \left(\frac{\mu(\widehat{Q})}{|Q|} \right) \int_{\mathbf{R}^n} N_* f(y) \, dy,$$

where the supremum is over all cubes Q in $\mathbf{R}^n$ and $\widehat{Q} := (0, \ell(Q)) \times Q$ is the *Carleson box*, $\ell(Q)$ and $|Q|$ being the sidelength and measure of Q. Furthermore N_* denotes the *non-tangential maximal function*

$$(N_* f)(y) := \sup_{\{(t, x); |x-y| \leq at\}} |f(t, x)|, \quad y \in \mathbf{R}^n,$$

Andreas Rosén was earlier named Andreas Axelsson.

where $a>0$ is a fixed constant determining the aperture of the cone. The exact value of a is less important, since for any $a_1, a_2>0$ the corresponding non-tangential maximal functions $N_* f$ are comparable in the $L_p(\mathbf{R}^n)$ norm for any $1\leq p\leq\infty$. See Fefferman and Stein [7, Lemma 1].

Carleson's inequality has numerous applications. Motivating for this paper is its applications to boundary value problems for elliptic partial differential equations. A recent application concerns boundary value problems for divergence-form equations

$$\operatorname{div}_{t,x} A(t,x)\nabla_{t,x} u(t,x) = 0,$$

with non-smooth coefficients $A\in L_\infty(\mathbf{R}_+^{1+n}; \mathbf{C}^{(1+n)\times(1+n)})$ with uniformly positive real part. To solve the Neumann problem with $L_2(\mathbf{R}^n)$ boundary data, Kenig and Pipher [9] introduced (a space equivalent to) the function space $\mathcal{X}$ consisting of functions $f(t,x)$, thought of as gradients of solutions $u(t,x)$, with $N_*(W_2 f)\in L_2(\mathbf{R}^n)$, where

$$(W_q f)(t,x) := |W(t,x)|^{-1/q}\|f\|_{L_q(W(t,x))}, \quad (t,x)\in\mathbf{R}_+^{1+n},$$

is the L_q *Whitney averaged function*, with

$$W(t,x) := \left\{(x,y)\in\mathbf{R}_+^{1+n}\,;\, |y-x|<c_1 t \text{ and } c_0^{-1}<\frac{s}{t}<c_0\right\}$$

being the *Whitney region* around (t,x). (Again, the precise value of the fixed constants $c_0>1$ and $c_1>0$ is less important.) The reason for replacing f by the Whitney average $W_2 f$ is that, unlike the potential $u(t,x)$, the gradient $f(t,x)=\nabla_{t,x} u(t,x)$ does not have classical interior pointwise De Giorgi–Nash–Moser bounds.

In the recent works of the second author with P. Auscher [1] and [2], the function space $\mathcal{X}$ above is fundamental. In these papers, new methods are developed to solve the Neumann (as well as the Dirichlet) problem for systems of divergence-form equations, which rely on solving certain operator-valued singular integral equations in the function space $\mathcal{X}$. Two questions arose, which motivated this paper.

• Which functions $g(t,x)$ are bounded multipliers

$$\mathcal{X}\longrightarrow L_2(\mathbf{R}_+^{1+n}; dt\,dx),$$

$$f(t,x)\longmapsto g(t,x)f(t,x)?$$

It was shown [1, Lemma 5.5], using Carleson's inequality, that g is a multiplier if the modified Carleson norm

$$(1)\qquad\qquad \sup_Q\left(\frac{1}{|Q|}\iint_{\widehat{Q}} W_\infty g(t,x)^2\,dt\,dx\right)^{1/2}$$

is finite. We show in this paper (Theorem 3.1) that this modified Carleson norm in fact is equivalent to the multiplier norm

$$\|g\|_{\mathcal{X} \to L_2(\mathbf{R}_+^{1+n}; dt\, dx)} = \sup_{f \neq 0} \left(\frac{\|gf\|_{L_2(\mathbf{R}_+^{1+n}; dt\, dx)}}{\|f\|_{\mathcal{X}}} \right).$$

The modified Carleson norm (1) has been known for some time to be fundamental in the perturbation theory for divergence-form equations. It was introduced already by Dahlberg [6]. See also Fefferman, Kenig and Pipher [8] and Kenig and Pipher [9] and [10].

• What is the dual, or pre-dual, space of $\mathcal{X}$? We show in this paper (Theorem 3.2) that $\mathcal{X}$ is the dual space of the space of functions $g(t, x)$ such that

$$\int_{\mathbf{R}^n} \left(\sup_{Q \ni z} \frac{1}{|Q|} \iint_{\widehat{Q}} W_2 g(t, x)\, dt\, dx \right)^2 dz < \infty.$$

(We here identify a function $f \in \mathcal{X}$ with the functional $g \mapsto \iint_{\mathbf{R}_+^{1+n}} fg\, dt\, dx$.) Theorem 3.2 also shows that the space $\mathcal{X}$ is not reflexive. The interest in understanding duality for the space $\mathcal{X}$ comes from the dual relation between the Dirichlet problem with $L_2(\mathbf{R}^n)$ data and the Dirichlet problem with Sobolev $H^1(\mathbf{R}^n)$ data. See [2, Theorem 1.4] and [9, Theorem 5.4].

Beyond these two results, we prove more general L_p results for the Carleson duality. On one hand, we consider not only $W_\infty g$ and $W_2 g$, but more general L_q Whitney averages. On the other hand, we measure the non-tangential maximal function and the Carleson functional in L_p norms. For example, this may have useful applications to boundary value problems with L_p data.

In Section 2, we first prove the corresponding results for a discrete vector-valued model of the Carleson duality. Then in Section 3, we prove equivalence between dyadic and non-dyadic norms, which yields the non-dyadic results.

The spaces we consider here are closely related to the tent spaces introduced by Coifman, Meyer and Stein [5], and in fact reduce to them for certain choices of the parameters. However, as a whole, the scale of spaces that we consider is new. Since the precise connection to tent spaces is somewhat technical, we postpone a more detailed commentary until Remark 3.3 below.

2. A discrete vector-valued model

In this section we study a dyadic model of the problem. We use the following notation. Let $\mathcal{D} = \bigcup_{j \in \mathbf{Z}} \mathcal{D}_j$ denote the dyadic cubes in $\mathbf{R}^n$, where

$$\mathcal{D}_j := \{2^{-j}(0, 1)^n + 2^{-j} k \, ; k \in \mathbf{Z}^n\}.$$

Let $W_Q := (\ell(Q)/2, \ell(Q)) \times Q$ denote the dyadic Whitney region, being in one-to-one correspondence with $Q \in \mathcal{D}$. Note that unlike their non-dyadic counterparts $W(t, x)$, the regions W_Q form a disjoint partition of $\mathbf{R}^{1+n}_+$ (modulo zero-sets). Define the dyadic Hardy–Littlewood maximal function

$$M_{\mathcal{D}} h(x) := \sup_{Q : x \in Q \in \mathcal{D}} \frac{1}{|Q|} \int_Q h(y)\, dy, \quad x \in \mathbf{R}^n,$$

for $h \in L_1^{\mathrm{loc}}(\mathbf{R}^n)$. Recall that $M_{\mathcal{D}}$ is bounded on $L_p(\mathbf{R}^n)$, $1 < p \le \infty$.

Our discrete vector-valued setup is as follows. We assume that to each $Q \in \mathcal{D}$, there are two associated Banach spaces $\mathcal{X}_Q$ and $\mathcal{Y}_Q$. For a sequence $f = \{f_Q\}_{Q \in \mathcal{D}}$, where $f_Q \in \mathcal{X}_Q$, we define its non-tangential maximal function

$$(N_{\mathcal{X}} f)(x) := \sup_{Q : x \in Q \in \mathcal{D}} \|f_Q\|_{\mathcal{X}_Q}, \quad x \in \mathbf{R}^n.$$

For fixed $1 \le p < \infty$, let $\mathcal{X}_p$ denote the space of all sequences f such that $\|f\|_{\mathcal{X}_p} := \|N_{\mathcal{X}} f\|_{L_p(\mathbf{R}^n)} < \infty$. For a sequence $g = \{g_Q\}_{Q \in \mathcal{D}}$, where $g_Q \in \mathcal{Y}_Q$, we define the Carleson functional

$$(C_{\mathcal{Y}} g)(x) := \sup_{Q : x \in Q \in \mathcal{D}} \frac{1}{|Q|} \sum_{\substack{R \subset Q \\ R \in \mathcal{D}}} \|g_R\|_{\mathcal{Y}_R}, \quad x \in \mathbf{R}^n.$$

For a fixed number $1 < p' \le \infty$, let $\mathcal{Y}_{p'}$ denote the space of all sequences g such that $\|g\|_{\mathcal{Y}_{p'}} := \|C_{\mathcal{Y}} g\|_{L_{p'}(\mathbf{R}^n)} < \infty$. Note that the case $p' = 1$ is not interesting, since $g = 0$ necessarily if $\|C_{\mathcal{Y}} g\|_{L_1(\mathbf{R}^n)} < \infty$.

We assume that for each $Q \in \mathcal{D}$ there is a duality $\langle \mathcal{X}_Q, \mathcal{Y}_Q \rangle$ as below, with constants C uniformly bounded with respect to Q.

Definition 2.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces. By a *duality* $\langle \mathcal{X}, \mathcal{Y} \rangle$, we mean a bilinear map $\mathcal{X} \times \mathcal{Y} \ni (f, g) \mapsto \langle f, g \rangle \in \mathbf{R}$ and a constant $0 < C < \infty$ such that

$$|\langle f, g \rangle| \le C \|f\|_{\mathcal{X}} \|g\|_{\mathcal{Y}}, \qquad f \in \mathcal{X}, \ g \in \mathcal{Y},$$

$$\|f\|_{\mathcal{X}} \le C \sup_{\|g\|_{\mathcal{Y}} = 1} \langle f, g \rangle, \quad f \in \mathcal{X},$$

$$\|g\|_{\mathcal{Y}} \le C \sup_{\|f\|_{\mathcal{X}} = 1} \langle f, g \rangle, \quad g \in \mathcal{Y}.$$

We prove the following duality result.

Theorem 2.2. *Let $\{\mathcal{X}_Q\}_{Q\in\mathcal{D}}$ and $\{\mathcal{Y}_Q\}_{Q\in\mathcal{D}}$ be pairwise dual Banach spaces as above, and let $1/p+1/p'=1$, $1\leq p<\infty$. Then there is a constant $0<C<\infty$ such that*

$$\sum_{Q\in\mathcal{D}}|\langle f_Q, g_Q\rangle| \leq C\|N_{\mathcal{X}}f\|_{L_p(\mathbf{R}^n)}\|C_{\mathcal{Y}}g\|_{L_{p'}(\mathbf{R}^n)}, \quad f_Q\in\mathcal{X}_Q,\ g_Q\in\mathcal{Y}_Q,$$

$$\|N_{\mathcal{X}}f\|_{L_p(\mathbf{R}^n)} \leq C\sup_{\|C_{\mathcal{Y}}g\|_{L_{p'}(\mathbf{R}^n)}=1}\sum_{Q\in\mathcal{D}}\langle f_Q, g_Q\rangle, \quad f_Q\in\mathcal{X}_Q,$$

$$\|C_{\mathcal{Y}}g\|_{L_{p'}(\mathbf{R}^n)} \leq C\sup_{\|N_{\mathcal{X}}f\|_{L_p(\mathbf{R}^n)}=1}\sum_{Q\in\mathcal{D}}\langle f_Q, g_Q\rangle, \quad g_Q\in\mathcal{Y}_Q.$$

The application we have in mind is the following. For functions $f(t,x)$ in $\mathbf{R}_+^{1+n}$, let $f_Q:=f|_{W_Q}\in L_q(W_Q)=:\mathcal{X}_Q$, where the Banach space has norm $\|f\|_{\mathcal{X}_Q}:=|W_Q|^{-1/q}\|f_Q\|_{L_q(W_Q)}$ so that

$$N_{L_q}f = \sup_{Q:x\in Q\in\mathcal{D}}|W_Q|^{-1/q}\|f\|_{L_q(W_Q)}.$$

For functions $g(t,x)$ in $\mathbf{R}_+^{1+n}$, let $g_Q:=g|_{W_Q}\in L_{\tilde{q}}(W_Q)=:\mathcal{Y}_Q$, where the Banach space has norm $\|g\|_{\mathcal{Y}_Q}:=|W_Q|^{1-1/\tilde{q}}\|g_Q\|_{L_{\tilde{q}}(W_Q)}$ so that

$$C_{L_{\tilde{q}}}f = \sup_{Q:x\in Q\in\mathcal{D}}\frac{1}{|Q|}\sum_{\substack{R\subset Q\\ R\in\mathcal{D}}}|W_R|^{1-1/\tilde{q}}\|g\|_{L_{\tilde{q}}(W_R)}.$$

We generalize slightly the Carleson functional and define

$$C_{L_{\tilde{q}}}^r f(x) = \sup_{Q:x\in Q\in\mathcal{D}}\left(\frac{1}{|Q|}\sum_{\substack{R\subset Q\\ R\in\mathcal{D}}}|W_R|(|W_R|^{-1/\tilde{q}}\|g\|_{L_{\tilde{q}}(W_R)})^r\right)^{1/r}$$

for $x\in\mathbf{R}^n$ and $1\leq r<\infty$.

Corollary 2.3. *Let $1/p+1/\tilde{p}=1/q+1/\tilde{q}=1/r$, with $r\leq p<\infty$, $r\leq q\leq\infty$ and $1\leq r<\infty$. Then there is a constant $0<C<\infty$ such that*

$$\|fg\|_{L_r(\mathbf{R}_+^{1+n})} \leq C\|N_{L_q}f\|_{L_p(\mathbf{R}^n)}\|C_{L_{\tilde{q}}}^r g\|_{L_{\tilde{p}}(\mathbf{R}^n)},$$

$$\|N_{L_q}f\|_{L_p(\mathbf{R}^n)} \leq C\sup_{\|C_{L_{\tilde{q}}}^r g\|_{L_{\tilde{p}}(\mathbf{R}^n)}=1}\|fg\|_{L_r(\mathbf{R}_+^{1+n})},$$

$$\|C_{L_{\tilde{q}}}^r g\|_{L_{\tilde{p}}(\mathbf{R}^n)} \leq C\sup_{\|N_{L_q}f\|_{L_p(\mathbf{R}^n)}=1}\|fg\|_{L_r(\mathbf{R}_+^{1+n})}.$$

Note that the case $p=q=r=2$ solves a dyadic version of the multiplier question for the space $\mathcal{X}$ from the introduction. In this case $\tilde{p}=\tilde{q}=\infty$. Note also that the case $p=q=2$, $r=1$, together with Theorem 2.4 below, solves a dyadic version of the dual space question for the space $\mathcal{X}$ from the introduction. In this case $\tilde{p}=\tilde{q}=2$.

Proof. Replacing $|f|^r$ and $|g|^r$ by f and g, we see that it suffices to consider the case $r=1$. In this case, the result follows from Theorem 2.2. $\square$

Proof of Theorem 2.2. (i) For completeness, we start with the well-known proof of the $\sum_Q |\langle f_Q, g_Q \rangle|$ estimate. It suffices to estimate $\sum_Q \|f_Q\| \|g_Q\|$. Note that

$$\sum_{R \subset Q} \|g_R\| \le |Q| \inf_{x \in Q} C_{\mathcal{Y}} g(x) \le \int_Q C_{\mathcal{Y}} g \, dx$$

for any $Q \in \mathcal{D}$. Select, for given $k \in \mathbf{Z}$, the maximal dyadic cubes $\mathcal{D}^k \subset \mathcal{D}$ such that $\|f_Q\| > 2^k$. Then $\bigcup_{Q \in \mathcal{D}^k} Q = \{x \in \mathbf{R}^n ; N_{\mathcal{X}} f(x) > 2^k\}$, and the cubes in $\mathcal{D}^k$ are disjoint. We get

$$\sum_{Q:\|f_Q\|>2^k} \|g_Q\| \le \sum_{Q \in \mathcal{D}^k} \sum_{R \subset Q} \|g_R\| \le \sum_{Q \in \mathcal{D}^k} \int_Q C_{\mathcal{Y}} g \, dx = \int_{x:N_{\mathcal{X}} f(x)>2^k} C_{\mathcal{Y}} g \, dx,$$

and hence

$$\sum_{Q \in \mathcal{D}} \|f_Q\| \|g_Q\| \approx \sum_{Q \in \mathcal{D}} \sum_{k:2^k < \|f_Q\|} 2^k \|g_Q\|$$

$$= \sum_{k \in \mathbf{Z}} 2^k \sum_{Q:\|f_Q\|>2^k} \|g_Q\|$$

$$\le \sum_{k \in \mathbf{Z}} 2^k \int_{x:N_{\mathcal{X}} f(x)>2^k} C_{\mathcal{Y}} g \, dx$$

$$= \int_{\mathbf{R}^n} \sum_{k:2^k < N_{\mathcal{X}} f(x)} 2^k C_{\mathcal{Y}} g \, dx$$

$$\approx \int_{\mathbf{R}^n} N_{\mathcal{X}} f C_{\mathcal{Y}} g \, dx$$

$$\le \|N_{\mathcal{X}} f\|_p \|C_{\mathcal{Y}} g\|_{p'}.$$

(ii) Next we prove the estimate of $\|C_{\mathcal{Y}} g\|_{p'}$. Consider first the case $p'=\infty$. Pick $Q \in \mathcal{D}$ such that

$$\frac{1}{|Q|} \sum_{R \subset Q} \|g_R\| \ge \tfrac{1}{2} \|C_{\mathcal{Y}} g\|_\infty.$$

Then construct $f=\{f_R\}_{R\in\mathcal{D}}$, choosing $f_R\in\mathcal{X}_R$ so that $\|f_R\|=1/|Q|$ and $\|g_R\|/|Q|\approx$ $\langle f_R, g_R \rangle$ if $R\subset Q$, and $f_R:=0$ if $R\not\subset Q$. It follows that $\|C_{\mathcal{Y}}g\|_\infty\approx\sum_R\langle f_R, g_R\rangle$ and $\|N_{\mathcal{X}}f\|_1=1$ since $N_{\mathcal{X}}f=1/|Q|$ on Q and $N_{\mathcal{X}}f=0$ off Q.

Next consider the case $1<p'<\infty$. Select, for given $k\in\mathbf{Z}$, the maximal dyadic cubes $\mathcal{D}^k\subset\mathcal{D}$ such that

$$\frac{1}{|Q|}\sum_{R\subset Q}\|g_R\|>2^k.$$

Then $\{x\in\mathbf{R}^n; C_{\mathcal{Y}}g(x)>2^k\}=\bigcup_{Q\in\mathcal{D}^k}Q$, and the cubes in $\mathcal{D}^k$ are disjoint. We obtain

$$|\{x\,;C_{\mathcal{Y}}g(x)>2^k\}|=\sum_{Q\in\mathcal{D}^k}|Q|\leq 2^{-k}\sum_{Q\in\mathcal{D}^k}\sum_{R\subset Q}\|g_R\|.$$

Now let

$$\hat{f}_R:=\frac{1}{|R|}\int_R C_{\mathcal{Y}}g\,dx.$$

Note that $\hat{f}_R$ does not depend on k, and that $\hat{f}_R>2^k$ for $R\subset Q\in\mathcal{D}^k$. We get $|\{x; C_{\mathcal{Y}}g(x)>2^k\}|\leq 2^{-k}\sum_{R:\hat{f}_R>2^k}\|g_R\|$ and

$$\|C_{\mathcal{Y}}g\|_{p'}^{p'}\approx\sum_{k\in\mathbf{Z}}2^{p'k}|\{x\,;C_{\mathcal{Y}}g(x)>2^k\}|$$

$$\leq\sum_{R\in\mathcal{D}}\sum_{k:2^k<\hat{f}_R}2^{(p'-1)k}\|g_R\|\approx\sum_{R\in\mathcal{D}}(\hat{f}_R)^{p'-1}\|g_R\|.$$

Now construct $f=\{f_R\}_{R\in\mathcal{D}}$, choosing $f_R\in\mathcal{X}_R$ such that

$$\|f_R\|=(\hat{f}_R)^{p'-1}\quad\text{and}\quad(\hat{f}_R)^{p'-1}\|g_R\|\approx\langle f_R, g_R\rangle.$$

We get

$$N_{\mathcal{X}}f(x)=\sup_{Q\ni x}(\hat{f}_Q)^{p'-1}=(M_{\mathcal{D}}(C_{\mathcal{Y}}g)(x))^{p'-1}.$$

Since $p(p'-1)=p'$, this gives

$$\|N_{\mathcal{X}}f\|_p^p=\|M_{\mathcal{D}}(C_{\mathcal{Y}}g)\|_{p'}^{p'}\lesssim\|C_{\mathcal{Y}}g\|_{p'}^{p'},$$

and we conclude that

$$\sum_Q\langle f_Q, g_Q\rangle\gtrsim\|C_{\mathcal{Y}}g\|_{p'}^{p'}\gtrsim\|C_{\mathcal{Y}}g\|_{p'}\|N_{\mathcal{X}}f\|_p.$$

(iii) Next we prove the estimate of $\|N_{\mathcal{X}}f\|_p$. Consider first the case $1<p<\infty$. Select, for given $k\in\mathbf{Z}$, the maximal dyadic cubes $\mathcal{D}^k\subset\mathcal{D}$ such that $\|f_Q\|>2^k$.

Then $\{x \in \mathbf{R}^n ; N_\mathcal{X} f(x) > 2^k\} = \bigcup_{Q \in \mathcal{D}^k} Q$, and the cubes in $\mathcal{D}^k$ are disjoint. Write $k_Q := \max_{Q \in \mathcal{D}^k} k \leq \log_2 \|f_Q\|$. We obtain

$$\|N_\mathcal{X} f\|_p^p \approx \sum_{k \in \mathbf{Z}} 2^{kp} |\{x ; N_\mathcal{X} f(x) > 2^k\}| = \sum_{Q \in \mathcal{D}} |Q| \sum_{k : Q \in \mathcal{D}^k} 2^{kp}$$

$$\approx \sum_{Q \in \mathcal{D}} |Q| 2^{k_Q p} = \sum_{Q \in \mathcal{D}} 2^{k_Q} |Q| 2^{k_Q(p-1)} \approx \sum_{Q \in \mathcal{D}} \|f_Q\| \left(|Q| \sum_{k : Q \in \mathcal{D}^k} 2^{k(p-1)} \right).$$

Write $\hat{g}_Q := |Q| \sum_{k : Q \in \mathcal{D}^k} 2^{k(p-1)}$ and construct $g = \{g_Q\}_{Q \in \mathcal{D}}$, choosing $g_Q \in \mathcal{Y}_Q$ such that $\|g_Q\| = \hat{g}_Q$ and $\|f_Q\| \|g_Q\| \approx \langle f_Q, g_Q \rangle$. Then

$$\frac{1}{|Q|} \sum_{R \subset Q} \|g_R\| \lesssim \sum_{k \in \mathbf{Z}} 2^{k(p-1)} \frac{1}{|Q|} \sum_{\substack{R \subset Q \\ R \in \mathcal{D}^k}} |R|$$

$$= \sum_{k \in \mathbf{Z}} 2^{k(p-1)} \frac{1}{|Q|} |\{x ; N_\mathcal{X} f(x) > 2^k\} \cap Q|$$

$$\approx \frac{1}{|Q|} \int_Q (N_\mathcal{X} f)^{p-1} \, dx$$

$$\leq \inf_Q M_\mathcal{D}((N_\mathcal{X} f)^{p-1}),$$

and therefore $\|C_\mathcal{Y} g\|_{p'}^{p'} \lesssim \|(N_\mathcal{X} f)^{p-1}\|_{p'}^{p'} = \|N_\mathcal{X} f\|_p^p$, since $p'(p-1) = p$. We conclude that

$$\sum_{Q \in \mathcal{D}} \langle f_Q, g_Q \rangle \gtrsim \|N_\mathcal{X} f\|_p^p \gtrsim \|C_\mathcal{Y} g\|_{p'} \|N_\mathcal{X} f\|_p.$$

(iii′) We finally prove the estimate of $\|N_\mathcal{X} f\|_1$, i.e. the case $p = 1$. Let $\mathcal{D}^0$ be the 2^n dyadic cubes with sidelength 2^M and one corner at the origin, where M is chosen large enough, using the monotone convergence theorem, so that $\|N_\mathcal{X} \tilde{f}\|_1 \geq \frac{1}{2} \|N_\mathcal{X} f\|_1$, where $\tilde{f}_Q := f_Q$ if $Q \subset Q_0$ for some $Q_0 \in \mathcal{D}^0$, and $\tilde{f}_Q := 0$ otherwise. Assuming the estimate proved for $\tilde{f}$, we have

$$\|N_\mathcal{X} \tilde{f}\|_1 \lesssim \frac{\sum_Q \langle \tilde{f}_Q, g_Q \rangle}{\|C_\mathcal{Y} g\|_\infty},$$

where we may assume $g_Q = 0$ unless $Q \subset Q_0$ for some $Q_0 \in \mathcal{D}^0$. This yields

$$\|N_\mathcal{X} f\|_1 \leq 2 \|N_\mathcal{X} \tilde{f}\|_1 \lesssim \sum_Q \frac{\langle \tilde{f}_Q, g_Q \rangle}{\|C_\mathcal{Y} g\|_\infty} \lesssim \sum_Q \frac{\langle f_Q, g_Q \rangle}{\|C_\mathcal{Y} g\|_\infty}.$$

Thus, replacing f by $\tilde{f}$, we may assume that $f_Q = 0$ unless $Q \subset Q_0$ for some $Q_0 \in \mathcal{D}^0$.

Given f contained by $\mathcal{D}^0$ as above, we define recursively sets of disjoint dyadic cubes $\mathcal{D}^j \subset \mathcal{D}$, $j=1,2,3,...$, as follows. Having constructed $\mathcal{D}^j$, let $Q \in \mathcal{D}^j$. Define $\mathcal{D}_Q^{j+1}$ to be the set of maximal dyadic cubes $R \in \mathcal{D}$ such that $R \subset Q$ and $\|f_R\|>2\|f_Q\|$. Then let $\mathcal{D}^{j+1}:=\bigcup_{Q \in \mathcal{D}^j} \mathcal{D}_Q^{j+1}$. Furthermore, let $\mathcal{D}^f:=\bigcup_{j=1}^{\infty} \mathcal{D}^j$ and

$$E(Q):=Q\setminus \bigcup_{R \in \mathcal{D}_Q^{j+1}} R, \quad Q \in \mathcal{D}^j.$$

From the above construction, if $x \in Q_k \subset Q_{k-1} \subset ... \subset Q_0$, where $Q_j \in \mathcal{D}^j$, then $\|f_{Q_k}\|>2^{k-1}\|f_{Q_1}\|$, $k=2,3,...$, where $\|f_{Q_1}\|>0$. Hence, if $N_{\mathcal{X}}f(x)<\infty$, then there is a minimal $Q \ni x$ with $Q \in \mathcal{D}^f$. For this Q, we have $x \in E(Q)$ and $N_{\mathcal{X}}f(x) \leq 2\|f_Q\|$. Thus

(2) $$N_{\mathcal{X}}f \leq 2 \sum_{Q \in \mathcal{D}^f} \|f_Q\| 1_{E(Q)} \quad \text{a.e.}$$

so that $\|N_{\mathcal{X}}f\|_1 \leq 2\sum_{Q \in \mathcal{D}^f} \|f_Q\| |Q|$. Conversely, if $x \in Q_k \subset Q_{k-1} \subset ... \subset Q_0$, where $Q_j \in \mathcal{D}^j$, are all the selected dyadic cubes containing x, then $N_{\mathcal{X}}f(x) \geq \|f_{Q_k}\| \geq 2\|f_{Q_{k-1}}\| \geq ... \geq 2^k\|f_{Q_0}\|$. Thus

$$\sum_{Q \in \mathcal{D}^f} \|f_Q\| |Q| = \int_{\mathbf{R}^n} \sum_{\substack{Q \in \mathcal{D}^f \\ Q \ni x}} \|f_Q\| \, dx \leq \int_{\mathbf{R}^n} N_{\mathcal{X}}f \sum_{j=0}^{\infty} 2^{-j} \, dx \leq 2\|N_{\mathcal{X}}f\|_1.$$

Now let $c \in (0,1)$ be a constant, to be chosen below, and define

$$\mathcal{D}_1^f := \{Q \in \mathcal{D}^f \, ; |E(Q)| > c|Q|\} \quad \text{and} \quad \mathcal{D}_2^f := \mathcal{D}^f \setminus \mathcal{D}_1^f.$$

From (2) we have

$$\|N_{\mathcal{X}}f\|_1 \leq 2 \sum_{Q \in \mathcal{D}_1^f} \|f_Q\| |Q| + 2c \sum_{Q \in \mathcal{D}_2^f} \|f_Q\| |Q| \leq 2 \sum_{Q \in \mathcal{D}_1^f} \|f_Q\| |Q| + 4c\|N_{\mathcal{X}}f\|_1.$$

Choose $c=\frac{1}{8}$ to obtain $\|N_{\mathcal{X}}f\|_1 \leq 4 \sum_{Q \in \mathcal{D}_1^f} \|f_Q\| |Q|$. Construct $g=\{g_Q\}_{Q \in \mathcal{D}}$, choosing $g_Q \in \mathcal{Y}_Q$ such that $\|g_Q\|=|Q|$ and $\langle f_Q, g_Q \rangle \approx \|f_Q\| |Q|$ if $Q \in \mathcal{D}_1^f$, and $g_Q:=0$ otherwise. Then $\|N_{\mathcal{X}}f\|_1 \lesssim \sum_{Q \in \mathcal{D}_1^f} \langle f_Q, g_Q \rangle$. To estimate

$$\frac{1}{|Q|} \sum_{R \subset Q} \|g_R\| = \frac{1}{|Q|} \sum_{\substack{R \subset Q \\ R \in \mathcal{D}_1^f}} |R|,$$

note that if $R \in \mathcal{D}_1^f \cap \mathcal{D}^j$, then $\sum_{R' \in \mathcal{D}_R^{j+1}} |R'| \leq \frac{7}{8}|R|$. Thus

$$\frac{1}{|Q|} \sum_{\substack{R \subset Q \\ R \in \mathcal{D}_1^f}} |R| \leq \frac{1}{|Q|} \sum_{j=0}^{\infty} (\tfrac{7}{8})^j |Q| = 8.$$

Thus $\|C_{\mathcal{Y}} g\|_\infty \leq 8$. This completes the proof of the theorem. $\quad\square$

Consider now a duality $\langle \mathcal{X}, \mathcal{Y} \rangle$ between two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ as in Definition 2.1. We define the linear map $L \colon \mathcal{X} \to \mathcal{Y}^*$ sending $f \in \mathcal{X}$ to the linear functional

$$\Lambda_f \colon \mathcal{Y} \longrightarrow \mathbf{R},$$
$$g \longmapsto \langle f, g \rangle.$$

The estimate $|\langle f, g \rangle| \leq C \|f\|_{\mathcal{X}} \|g\|_{\mathcal{Y}}$ shows that $\|L\|_{\mathcal{X} \to \mathcal{Y}^*} \leq C$, whereas it follows from the estimate $\|f\|_{\mathcal{X}} \leq C \sup_{\|g\|_{\mathcal{Y}}=1} \langle f, g \rangle$ that L is injective with closed range $L(\mathcal{X}) \subset \mathcal{Y}^*$. Thus the duality gives a topological, but not in general isometric, identification, through L, of $\mathcal{X}$ with the closed subspace $L(\mathcal{X})$ of $\mathcal{Y}^*$. The estimate $\|g\|_{\mathcal{Y}} \leq C \sup_{\|f\|_{\mathcal{X}}=1} \langle f, g \rangle$ furthermore shows that this subspace is "large" in the sense that its pre-annihilator is

$$^\perp L(\mathcal{X}) := \{ g \in \mathcal{Y} \,;\, \Lambda g = 0 \text{ for all } \Lambda \in L(\mathcal{X}) \} = \{0\}.$$

In general we may have that $L(\mathcal{X}) \subsetneq \mathcal{Y}^*$, but if $\mathcal{Y}$ is reflexive, then necessarily $L(\mathcal{X}) = \mathcal{Y}^*$. Below we identify $\mathcal{X}$ and $L(\mathcal{X})$, and thus write $\mathcal{X} = \mathcal{Y}^*$ if $L(\mathcal{X}) = \mathcal{Y}^*$. We also note that the above also holds with the roles of $\mathcal{X}$ and $\mathcal{Y}$ interchanged, giving an identification of $\mathcal{Y}$ with a large closed subspace of $\mathcal{X}^*$.

The following result describes when the duality in Theorem 2.2 gives the full dual spaces.

Theorem 2.4. *With the above notation, consider the duality* $\langle \mathcal{X}_p, \mathcal{Y}_{p'} \rangle$,

$$(f, g) \longmapsto \sum_{Q \in \mathcal{D}} \langle f_Q, g_Q \rangle$$

from Theorem 2.2. *We have* $\mathcal{Y}_{p'} \subsetneq \mathcal{X}_p^*$ *for any* $1 \leq p < \infty$, *as well as* $\mathcal{X}_1 \subsetneq \mathcal{Y}_\infty^*$.
If furthermore the duality $\langle \mathcal{X}_Q, \mathcal{Y}_Q \rangle$ *is such that* $\mathcal{X}_Q = \mathcal{Y}_Q^*$ *for all* $Q \in \mathcal{D}$, *and if* $1 < p < \infty$, *then* $\mathcal{X}_p = \mathcal{Y}_{p'}^*$.

Proof. (i) We first prove that $\mathcal{X}_1 \subsetneq \mathcal{Y}_\infty^*$. Let $Q_1 \supsetneq Q_2 \supsetneq Q_3 \supsetneq ...$ be dyadic cubes. Define the functionals $\Lambda_j g := \langle f_{Q_j}, g_{Q_j} \rangle$ on $\mathcal{Y}_\infty$, where we have chosen $f_{Q_j} \in \mathcal{X}_{Q_j}$ such that $\|f_{Q_j}\| = 1/|Q_j|$. It is clear that $\|\Lambda_j\|_{\mathcal{Y}_\infty^*} \approx 1$. Consider the sequence space $\ell_\infty(\mathbf{Z}_+)$ and use Hahn–Banach's theorem to construct $\lim \in \ell_\infty(\mathbf{Z}_+)^*$ such that

$$\lim(\{x_n\}_{n=1}^\infty) = \lim_{n\to\infty} x_n$$

for all convergent sequences $\{x_n\}_{n=1}^\infty$. Set $\Lambda g := \lim(\{\Lambda_j g\}_{j=1}^\infty)$. It is straightforward to verify that $\Lambda \in \mathcal{Y}_\infty^* \setminus \mathcal{X}_1$.

(ii) We next prove that $\mathcal{Y}_{p'} \subsetneq \mathcal{X}_p^*$ for $1 \le p < \infty$. Fix some cube $Q_0 \in \mathcal{D}$ with $\ell(Q) = 1$. Define functionals

$$\Lambda_j f := \sum_{\substack{R \subset Q_0 \\ \ell(R) = 2^{-j}}} \langle f_R, g_R \rangle$$

on $\mathcal{X}_p$, where $g_R \in \mathcal{Y}_R$ is chosen such that $\|g_R\| = |R|$. Then

$$|\Lambda_j f| \lesssim \sum_{\substack{R \subset Q_0 \\ \ell(R) = 2^{-j}}} \|f_R\| \, |R| \le \int_{Q_0} N_{\mathcal{X}} f \, dx \le \|N_{\mathcal{X}} f\|_p.$$

Define $\Lambda f := \lim(\{\Lambda_j f\}_{j=1}^\infty)$. It is straightforward to verify that $\Lambda \in \mathcal{X}_p^* \setminus \mathcal{Y}_{p'}$.

(iii) Finally we assume that $\mathcal{X}_Q = \mathcal{Y}_Q^*$ and $1 < p < \infty$, and aim to show that $\mathcal{X}_p = \mathcal{Y}_{p'}^*$. Let $\Lambda \in \mathcal{Y}_{p'}^*$, and let $Q \in \mathcal{D}$. Pick $f_Q \in \mathcal{X}_Q = \mathcal{Y}_Q^*$ such that $\langle f_Q, g_Q \rangle = \Lambda(\{g_Q \delta_{QR}\}_{R \in \mathcal{D}})$ for all $g_Q \in \mathcal{Y}_Q$, where $\delta_{QR} = 1$ if $R = Q$ and $\delta_{QR} = 0$ otherwise. Let $f := \{f_Q\}_{Q \in \mathcal{D}}$. Then

$$(3) \qquad\qquad \Lambda g = \sum_{Q \in \mathcal{D}} \langle f_Q, g_Q \rangle$$

holds whenever $g_Q \ne 0$ only for finitely many Q. From the monotone convergence theorem it follows that $\|N_{\mathcal{X}} f\|_p \lesssim \|\Lambda\|_{\mathcal{Y}_{p'}^*}$, so that $f \in \mathcal{X}_p$. We now use Lemma 2.5 below to deduce that (3) holds for all $g \in \mathcal{Y}_{p'}$ by continuity. $\quad\square$

Lemma 2.5. *Assume that $1 < p' < \infty$. Then the subspace consisting of sequences $g = \{g_Q\}_{Q \in \mathcal{D}}$ with $g_Q \ne 0$ for finitely many $Q \in \mathcal{D}$, is dense in $\mathcal{Y}_{p'}$.*

Proof. (i) Let $g \in \mathcal{Y}_{p'}$ and let $\varepsilon > 0$. Let $Q_1, ..., Q_{2^n}$ be the dyadic cubes with one corner at the origin and sidelength 2^M. Choose M large enough so that $\int_{\mathbf{R}_n \setminus Q_0} |C_{\mathcal{Y}} g|^{p'} dx \le \varepsilon^{p'}$, where $Q_0 := Q_1 \cup ... \cup Q_{2^n}$. Set

$$g_Q^1 := \begin{cases} g_Q, & Q \not\subset Q_0, \\ 0, & Q \subset Q_0. \end{cases}$$

Let Q'_j be a sibling of Q_j, $1 \leq j \leq 2^n$. Since $g^1_Q = 0$ for $Q \subset Q_j$, it is clear that we have $\sup_{Q_j} C_{\mathcal{Y}} g \leq \inf_{Q'_j} C_{\mathcal{Y}} g$. Therefore

$$\|C_{\mathcal{Y}} g^1\|^{p'}_{p'} = \int_{\mathbf{R}^n \setminus Q_0} |C_{\mathcal{Y}} g^1|^{p'} \, dx + \sum_{j=1}^{2^n} \int_{Q_j} |C_{\mathcal{Y}} g^1|^{p'} \, dx$$

$$\leq \int_{\mathbf{R}^n \setminus Q_0} |C_{\mathcal{Y}} g|^{p'} \, dx + \sum_{j=1}^{2^n} \int_{Q'_j} |C_{\mathcal{Y}} g|^{p'} \, dx \lesssim \varepsilon^{p'},$$

as $C_{\mathcal{Y}} g^1 \leq C_{\mathcal{Y}} g$.

(ii) Next we consider small cubes inside Q_0. Define

$$C_j h(x) := \sup_{\substack{Q \ni x \\ \ell(Q) \leq 2^{-j}}} \frac{1}{|Q|} \sum_{R \subset Q} \|h_R\|, \quad h \in \mathcal{Y}_{p'}.$$

Then $C_j g(x) \to 0$, as $j \to \infty$, for almost all x, by Lemma 2.6 below. Since $C_j g \leq C_{\mathcal{Y}} g \in L_{p'}(\mathbf{R}^n)$, it follows by dominated convergence that we can choose $j < \infty$ such that $\|C_j g\|_{p'} \leq \varepsilon$. Next choose $\delta > 0$ such that $\sum_{R:R \subset Q_0, \ell(R) \leq \delta} \|g_R\| \leq \varepsilon 2^{-nj} |Q_0|^{-1/p'}$. Set

$$g^2_Q := \begin{cases} g_Q, & \text{if } Q \subset Q_0 \text{ and } \ell(Q) \leq \delta, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$C_{\mathcal{Y}} g^2(x) = \max\left(C_j g^2(x), \sup_{\substack{Q \ni x \\ \ell(Q) > 2^{-j}}} \frac{1}{|Q|} \sum_{R \subset Q} \|g^2_R\| \right)$$

$$\leq \max(C_j g^2(x), \min(2^{nj}, d(x, Q_0)^{-n}) \, \varepsilon 2^{-nj} |Q_0|^{-1/p'})$$

$$\leq \max(C_j g^2(x), \varepsilon |Q_0|^{-1/p'} \min(1, d(x, Q_0)^{-n})),$$

where $d(x, Q_0) := \inf_{y \in Q_0} |x - y|$. This shows that $\|C_{\mathcal{Y}} g^2\|_{p'} \lesssim \varepsilon$, since

$$\|\min(1, d(x, Q_0)^{-n})\|_{p'} \lesssim |Q_0|^{1/p'}.$$

It follows that $g - g^1 - g^2$ is finitely non-zero, with $\|g^1 + g^2\|_{\mathcal{Y}_{p'}} \lesssim \varepsilon$. $\quad\square$

Lemma 2.6. *Let $Q_0 \in \mathcal{D}$ and assume that $\sum_{Q \subset Q_0} a_Q < \infty$, where $0 \leq a_Q < \infty$ for $Q \subset Q_0$. Then*

$$\frac{1}{|Q|} \sum_{R \subset Q} a_R \to 0, \quad \text{as } Q \ni x, \ \ell(Q) \to 0,$$

for almost all $x \in Q_0$.

Proof. We argue by contradiction. Assume there exists $\delta>0$ such that

$$E:=\left\{x\in Q_0\,;\limsup_{\substack{Q\ni x\\ \ell(Q)\to0}}\frac{1}{|Q|}\sum_{R\subset Q}a_R>\delta\right\}$$

has positive measure. Let $A:=\sum_{Q\subset Q_0}a_Q<\infty$. Choose $j<\infty$ such that we have $\sum_{Q\subset Q_0,\ell(Q)>2^{-j}}a_Q>A-\delta|E|/2$. Select the maximal cubes $Q_k\subset Q_0$, $k=1,2,...$, such that $\ell(Q_k)\leq2^{-j}$ and $\sum_{R\subset Q_k}a_R>\delta|Q_k|$. We get that $E\subset\bigcup_{k=1}^\infty Q_k$, where the cubes Q_k are disjoint. This gives

$$A=\sum_{Q\subset Q_0}a_Q=\sum_{\substack{Q\subset Q_0\\ \ell(Q)>2^{-j}}}a_Q+\sum_{\substack{Q\subset Q_0\\ \ell(Q)\leq2^{-j}}}a_Q$$

$$\geq\left(A-\frac{\delta|E|}{2}\right)+\sum_{k=1}^\infty\delta|Q_k|\geq A+\frac{\delta|E|}{2},$$

which is a contradiction. The conclusion follows. $\square$

3. The non-dyadic results

In this section, we derive the corresponding non-dyadic results on the Carleson duality from the dyadic results in Section 2. We use the following notation. For fixed constants $c_0>1,c_1>0$ and $a>0$, we use Whitney regions $W(t,x)$, L_q Whitney averages W_qf of functions $f\in L_q^{\mathrm{loc}}(\mathbf{R}_+^{1+n})$, and non-tangential maximal functions N_*f, as in the introduction. Also define the Carleson functionals

$$C^rg(z):=\sup_{Q\ni z}\left(\frac{1}{|Q|}\iint_{\widehat{Q}}|g(t,x)|^r\,dt\,dx\right)^{1/r},\quad z\in\mathbf{R}^n,$$

for $1\leq r<\infty$, and the Hardy–Littlewood maximal function

$$Mh(z):=\sup_{Q\ni z}\frac{1}{|Q|}\int_Q h(y)\,dy,\quad z\in\mathbf{R}^n,$$

for $h\in L_1^{\mathrm{loc}}(\mathbf{R}^n)$. Here the suprema are over all (non-dyadic) axis-parallel cubes in $\mathbf{R}^n$ containing z. We write $C^1g=Cg$ when $r=1$.

We aim to prove the following non-dyadic version of Corollary 2.3.

Theorem 3.1. *Let* $1/p+1/\tilde{p}=1/q+1/\tilde{q}=1/r$, *with* $r\leq p<\infty$, $r\leq q\leq\infty$ *and* $1\leq r<\infty$. *Then there is a constant* $0<C<\infty$ *such that*

$$\|fg\|_{L_r(\mathbf{R}_+^{1+n})}\leq C\|N_*(W_qf)\|_{L_p(\mathbf{R}^n)}\|C^r(W_{\tilde{q}}g)\|_{L_{\tilde{p}}(\mathbf{R}^n)},$$

$$\|N_*(W_q f)\|_{L_p(\mathbf{R}^n)} \le C \sup_{\|C^r(W_{\tilde{q}} g)\|_{L_{\tilde{p}}(\mathbf{R}^n)} = 1} \|fg\|_{L_r(\mathbf{R}_+^{1+n})},$$

$$\|C^r(W_{\tilde{q}} g)\|_{L_{\tilde{p}}(\mathbf{R}^n)} \le C \sup_{\|N_*(W_q f)\|_{L_p(\mathbf{R}^n)} = 1} \|fg\|_{L_r(\mathbf{R}_+^{1+n})}.$$

For $r=1$, this means that there is a duality

$$(f,g) \longmapsto \iint_{\mathbf{R}_+^{1+n}} fg \, dt \, dx$$

between the Banach spaces $N_{p,q}$ and $C_{p',q'}$, defined by the norms

$$\|f\|_{N_{p,q}} := \|N_*(W_q f)\|_{L_p(\mathbf{R}^n)} \quad \text{and} \quad \|g\|_{C_{p',q'}} := \|C^1(W_{q'} g)\|_{L_{p'}(\mathbf{R}^n)},$$

with $1/p+1/p'=1$, $1/q+1/q'=1$, $1 \le p < \infty$ and $1 \le q \le \infty$. We also prove the following non-dyadic version of Theorem 2.4.

Theorem 3.2. *With the above notation, consider the duality $\langle N_{p,q}, C_{p',q'} \rangle$. We have, for $1 \le q \le \infty$, $C_{p',q'} \subsetneq N_{p,q}^*$ for any $1 \le p < \infty$, as well as $N_{1,q} \subsetneq C_{\infty,q'}^*$.*
If $1 < q \le \infty$ and $1 < p < \infty$, then $N_{p,q} = C_{p',q'}^*$.

Remark 3.3. (Relation to the Coifman–Meyer–Stein tent spaces.) It is immediate that for $\tilde{q}=r$, we have the pointwise equivalence $C^r(W_{\tilde{q}} g) = C^r(W_r g) \approx C^r g$. For $r=2$, this is the functional denoted simply by C by Coifman, Meyer and Stein [5]. They show [5, Theorem 3] that there is further the L^p equivalence

$$\|C^2(g)\|_{L_p(\mathbf{R}^n)} \approx \|A^2(g)\|_{L_p(\mathbf{R}^n)} =: \|g\|_{T_{p,2}}, \quad p \in (2, \infty),$$

where

$$A^2(g) := \left(\iint_{|y-x|<t} |g(t,y)|^2 \frac{dt \, dy}{t^n} \right)^{1/2}$$

is the area integral and $T_{p,2}$ is the tent space. Observe also that $N_*(W_\infty g)$ is pointwise dominated by the non-tangential maximal function of g with a different aperture, and hence

$$\|N_*(W_\infty g)\|_{L_p(\mathbf{R}^n)} \approx \|N_* g\|_{L_p(\mathbf{R}^n)}.$$

In view of the previous observations, taking $\tilde{q}=r=2$ (and then $q=\infty$) in Theorem 3.1, it gives the following characterization of pointwise multipliers from the tent space $T_{\tilde{p},2}$ to $L_2(\mathbf{R}^{1+n})$, where $1/p+1/\tilde{p}=\frac{1}{2}$ and $\tilde{p}>2$,

$$\|fg\|_{L_2(\mathbf{R}_+^{1+n})} \le C \|N_* f\|_{L_p(\mathbf{R}^n)} \|g\|_{T_{\tilde{p},2}},$$

$$\|N_*f\|_{L_p(\mathbf{R}^n)} \leq C \sup_{\|g\|_{T_{\tilde{p},2}}=1} \|fg\|_{L_2(\mathbf{R}^{1+n}_+)},$$

$$\|g\|_{T_{\tilde{p},2}} \leq C \sup_{\|N_*f\|_{L_p(\mathbf{R}^n)}=1} \|fg\|_{L_2(\mathbf{R}^{1+n}_+)}.$$

On the other hand, Theorem 3.1 does not contain the known duality results for these tent space, since duality in Theorem 3.1 corresponds to $r=1$, and for this exponent the spaces appearing in the statement are outside the scale of classical tent spaces as introduced by Coifman, Meyer and Stein.

Note that the norm of our space $N_{p,\infty}$ is the same as the norm of the tent space $T_{p,\infty}$ of Coifman, Meyer and Stein; however, our space consists of all measurable functions for which this norm is finite, whereas the definition of $T_{p,\infty}$ involves an additional continuity condition, including certain continuity on the boundary of $\mathbf{R}^{1+n}_+$. This explains the seeming contradiction between our duality result that $C_{\infty,1} \subsetneq N^*_{1,\infty}$ and the well-known result of Coifman–Meyer–Stein that the dual of $T_{1,\infty}$ coincides with the space of Carleson measures. In fact, our example which shows the strict containment of the two spaces is precisely based on problems that occur when approaching the boundary.

We prove Theorems 3.1 and 3.2 by showing equivalence of the corresponding dyadic and non-dyadic norms. For this, we require the following two lemmata.

Lemma 3.4. *Let* $0\leq u\in L_1^{\mathrm{loc}}(\mathbf{R}^{1+n}_+)$. *Assume that* $W\subset\bigcup_{j=1}^N W_j\subset\mathbf{R}^{1+n}_+$, *where* $|W_j|\leq C|W|$ *for* $j=1,...,N$. *Then for some* $1\leq j\leq N$, *we have*

$$\frac{1}{|W_j|}\iint_{W_j} u\,dt\,dx \geq \frac{1}{CN}\left(\frac{1}{|W|}\iint_W u\,dt\,dx\right).$$

Proof. The conclusion follows directly from the inequalities

$$\iint_W u\,dt\,dx \leq \sum_{j=1}^N \iint_{W_j} u\,dt\,dx \leq N\max_j \iint_{W_j} u\,dt\,dx. \quad \square$$

The following lemma uses the estimation technique from [7, Lemma 1].

Lemma 3.5. *Consider two functions* $f,g\colon \mathbf{R}^n\to[0,\infty)$. *Assume that there are constants* $0<c_1,c_2<\infty$ *such that* $f(z)>\lambda$ *implies* $g>c_1\lambda$ *on some set* $B\subset\mathbf{R}^n$ *with* $0<\sup\{|y-z|;y\in B\}^n\leq c_2|B|$. *Then there is a constant* $0<c_3<\infty$ *such that*

$$\|f\|_{L_p(\mathbf{R}^n)} \leq c_3\|g\|_{L_p(\mathbf{R}^n)},$$

for any $1\leq p\leq\infty$.

Proof. Let $\lambda>0$. Let $E_\lambda:=\{y;g(y)>c_1\lambda\}$ and consider the indicator (characteristic) function 1_{E_λ}. Let $z\in\mathbf{R}^n$ be such that $f(z)>\lambda$. Then, by hypothesis, there exists a set $B\subset E_\lambda$ and the hypothesis implies that

$$M(1_{E_\lambda})(z)\gtrsim\frac{|B|}{\sup\{|y-z|\,;\,y\in B\}^n}\geq\frac{1}{c_2}>0.$$

By the weak L_1-boundedness of M, we have

$$|\{z\,;\,f(z)>\lambda\}|\leq|\{z\,;\,M(1_{E_\lambda})(z)\gtrsim 1\}|\lesssim\|1_{E_\lambda}\|_1=|E_\lambda|.$$

This proves the estimate for $p=\infty$. For $1\leq p<\infty$, we estimate

$$\int_{\mathbf{R}^n}|f(x)|^p\,dx=\int_0^\infty|\{z\,;\,f(z)>\lambda\}|p\lambda^{p-1}\,d\lambda$$
$$\lesssim\int_0^\infty|\{z\,;\,g(z)>c_1\lambda\}|p\lambda^{p-1}\,d\lambda\approx\int_{\mathbf{R}^n}|g(x)|^p\,dx.\quad\square$$

In order to compare the Banach spaces $N_{p,q}$ and $C_{p',q'}$ with their dyadic counterparts, we make the following definitions. With notation as in Section 2, denote by $N_{p,q}^{\mathcal{D}}$ the space $\mathcal{X}_p$ with $\mathcal{X}_Q=L_q(W_Q)$, so that

$$\|f\|_{N_{p,q}^{\mathcal{D}}}=\|N_{L_q}(f)\|_{L_p(\mathbf{R}^n)}.$$

Similarly denote by $C_{p',q'}^{\mathcal{D}}$ the space $\mathcal{Y}_{p'}$ with $\mathcal{Y}_Q=L_{q'}(W_Q)$, so that

$$\|g\|_{C_{p',q'}^{\mathcal{D}}}=\|C_{L_{q'}}(g)\|_{L_{p'}(\mathbf{R}^n)}.$$

In what follows, we shall identify functions $f\in L_1^{\mathrm{loc}}(\mathbf{R}_+^{1+n})$ and sequences $\{f_Q\}_{Q\in\mathcal{D}}$, where $f_Q\in L_1(W_Q)$, in the natural way, i.e. given f we set $f_Q:=f|_{W_Q}$, and given $\{f_Q\}_{Q\in\mathcal{D}}$ we set $f:=f_Q$ on W_Q.

Proposition 3.6. *Let $1\leq p<\infty$ and $1\leq q\leq\infty$. Under the above identification, the spaces $N_{p,q}$ and $N_{p,q}^{\mathcal{D}}$ are equal, with equivalent norms*

$$\|N_*(W_q f)\|_{L_p(\mathbf{R}^n)}\approx\|N_{L_q}(f)\|_{L_p(\mathbf{R}^n)}.$$

In particular, up to equivalence of norms, the left-hand side is independent of the exact choice of $a\geq 0$, $c_0>1$ and $c_1>0$, and the right-hand side is independent of the exact choice of dyadic system.

Note that this shows that we here in fact can choose $a=0$, i.e. the vertical maximal function, for $N_*(W_q f)$. This is because we already have some non-tangential control in $W_q f$.

Proof. (i) To prove the estimate $\|N_*(W_q f)\|_{L_p(\mathbf{R}^n)} \gtrsim \|N_{L_q}(f)\|_{L_p(\mathbf{R}^n)}$, we use Lemma 3.5. Assume $N_{L_q} f(z) > \lambda$. Then there is a cube $Q \in \mathcal{D}$ such that $z \in Q$ and

$$|W_Q|^{-1/q} \|f\|_{L_q(W_Q)} \geq \lambda.$$

Consider (non-dyadic) cubes $W \subset \mathbf{R}_+^{1+n}$ with $\operatorname{diam} W = c_2 \operatorname{dist}(W, \mathbf{R}^n)$. We fix $c_2 > 0$ small enough, depending on c_0 and c_1, so that

$$W \subset \bigcap_{(s,y) \in W} W(s,y).$$

It is clear that there is an integer $N < \infty$ such that W_Q is the union of at most N such cubes W, uniformly for all Q. Lemma 3.4 shows that one of these cubes W, say W_0, has $|W_0|^{-1/q} \|f\|_{L_q(W_0)} \gtrsim \lambda$. It follows that $|W(t,x)|^{-1/q} \|f\|_{L_q(W(t,x))} \gtrsim \lambda$ for $(t,x) \in W_0$, and therefore $N_*(W_q f) \gtrsim \lambda$ on the projection $B \subset \mathbf{R}^n$ of $W_0 \subset \mathbf{R}_+^{1+n}$, and the stated estimate follows from Lemma 3.5.

(ii) Conversely, to prove the estimate $\|N_*(W_q f)\|_{L_p(\mathbf{R}^n)} \lesssim \|N_{L_q}(f)\|_{L_p(\mathbf{R}^n)}$, we again apply Lemma 3.5. Assume $N_*(W_q f)(z) > \lambda$. Then

$$|W(t,x)|^{-1/q} \|f\|_{L_q(W(t,x))} \geq \lambda$$

for some (t,x) such that $|x-z| \leq at$. We see that there is an integer $N < \infty$ such that $W(t,x)$ is contained in the union of at most N dyadic Whitney regions W_Q, with N independent of (t,x). Thus by Lemma 3.4, for some $c>0$, $|W_Q|^{-1/q} \|f\|_{L_q(W_Q)} \geq c\lambda$ for one of these Q. Since $N_{L_q}(f) > c\lambda$ on Q and $\operatorname{dist}(z,Q) \lesssim t \approx \ell(Q)$, Lemma 3.5 completes the proof. $\square$

Proposition 3.7. *Let $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Under the above identification, the spaces $C_{p',q'}$ and $C_{p',q'}^{\mathcal{D}}$ are equal, with equivalent norms*

$$\|C(W_{q'} g)\|_{L_{p'}(\mathbf{R}^n)} \approx \|C_{L_{q'}} g\|_{L_{p'}(\mathbf{R}^n)}.$$

In particular, up to equivalence of norms, the left-hand side is independent of the exact choice of $c_0 > 1$ and $c_1 > 0$, and the right-hand side is independent of the exact choice of dyadic system.

Proof. It is straightforward to check that the estimates below go through for $q'=\infty$ by properly interpreting the integrals.

(i) To prove the estimate

$$\|C(W_{q'}g)\|_{L_{p'}(\mathbf{R}^n)} \gtrsim \|C_{L_{q'}}g\|_{L_{p'}(\mathbf{R}^n)},$$

assume that $C_{L_{q'}}g(z)>\lambda$. Then there is a cube $Q\in\mathcal{D}$ such that $z\in Q$ and

$$\frac{1}{|Q|}\sum_{R\subset Q}|W_R|^{1-1/q'}\|g\|_{L_{q'}(W_R)} > \lambda.$$

We claim that there is a constant $c>0$ such that

$$\frac{1}{|W_R|}\iint_{W_R}\left(\frac{1}{|W(t,x)|}\iint_{W(t,x)}|g|^{q'}\,ds\,dy\right)^{1/q'}\,dt\,dx$$

$$\geq c\left(\frac{1}{|W_R|}\iint_{W_R}|g|^{q'}\,dt\,dx\right)^{1/q'}.$$

Given this estimate, it follows that

$$c\lambda < \frac{1}{|Q|}\sum_{R\subset Q}\iint_{W_R}W_{q'}g\,dt\,dx = \frac{1}{|Q|}\iint_{\widehat{Q}}W_{q'}g\,dt\,dx \leq C(W_{q'}g)(z)\,dt\,dx,$$

and hence $cC_{L_{q'}}g(z)\leq C(W_{q'}g)(z)$, even pointwise, from which the inequality in $L_{p'}(\mathbf{R}^n)$ follows.

To prove the claimed reverse Hölder estimates, consider (non-dyadic) cubes $W\subset\mathbf{R}_+^{1+n}$ with diam $W=c_2\operatorname{dist}(W,\mathbf{R}^n)$. We fix $c_2>0$ small enough, depending on c_0 and c_1, so that

$$W \subset \bigcap_{(s,y)\in W}W(s,y).$$

It is clear that there is an integer $N<\infty$ such that W_R is the union of at most N such cubes W, uniformly for all R. Lemma 3.4 shows that one of these cubes W, say W_0, has

$$\frac{1}{|W_0|}\iint_{W_0}|g|^{q'}\,dt\,dx \gtrsim \frac{1}{|W_R|}\iint_{W_R}|g|^{q'}\,dt\,dx.$$

We obtain

$$\frac{1}{|W_R|}\iint_{W_R}\left(\frac{1}{|W(t,x)|}\iint_{W(t,x)}|g|^{q'}\,ds\,dy\right)^{1/q'}\,dt\,dx$$

$$\gtrsim \frac{1}{|W_0|}\iint_{W_0}\left(\frac{1}{|W(t,x)|}\iint_{W(t,x)}|g|^{q'}\,ds\,dy\right)^{1/q'}\,dt\,dx$$

$$\gtrsim \frac{1}{|W_0|} \iint_{W_0} \left(\frac{1}{|W_0|} \iint_{W_0} |g|^{q'} \, ds \, dy \right)^{1/q'} dt \, dx$$

$$= \left(\frac{1}{|W_0|} \iint_{W_0} |g|^{q'} \, dt \, dx \right)^{1/q'}$$

$$\gtrsim \left(\frac{1}{|W_R|} \iint_{W_R} |g|^{q'} \, dt \, dx \right)^{1/q'}.$$

(ii) Conversely, to prove the estimate $\|C(W_{q'}g)\|_{L_{p'}(\mathbf{R}^n)} \lesssim \|C_{L_{q'}}g\|_{L_{p'}(\mathbf{R}^n)}$, assume that $C(W_{q'}g)(z) > \lambda$. Then there is a cube Q such that $z \in Q$ and

$$\frac{1}{|Q|} \iint_{\widehat{Q}} W_{q'}g \, dt \, dx > \lambda.$$

There is an integer $N < \infty$ such that $\bigcup_{(t,x) \in \widehat{Q}} W(t,x) \subset \bigcup_{j=1}^N \widehat{Q}_j =: U$ for some dyadic cubes $Q_j \in \mathcal{D}$ with $\ell(Q) \le \ell(Q_j) \le N\ell(Q)$, $1 \le j \le N$. Note that we can choose N independent of Q. Let $h := |g| 1_U$ and $\widetilde{W}_R := \bigcup_{(t,x) \in W_R} W(t,x)$, and note that there are finitely many $S \in \mathcal{D}$ such that W_S intersect $\widetilde{W}_R$ (all with $\ell(S) \approx \ell(R)$), uniformly in R. Then

$$\lambda|Q| < \iint_{\widehat{Q}} \left(\frac{1}{|W(t,x)|} \iint_{W(t,x)} h^{q'} \, ds \, dy \right)^{1/q'} dt \, dx$$

$$= \sum_{R \in \mathcal{D}} \iint_{W_R} \left(\frac{1}{|W(t,x)|} \iint_{W(t,x)} h^{q'} \, ds \, dy \right)^{1/q'} dt \, dx$$

$$\le \sum_{R \in \mathcal{D}} |W_R|^{1-1/q'} \left(\iint_{W_R} \left(\frac{1}{|W(t,x)|} \iint_{W(t,x)} h^{q'} \, ds \, dy \right) dt \, dx \right)^{1/q'}$$

$$\lesssim \sum_{R \in \mathcal{D}} |W_R|^{1-1/q'} \left(\iint_{\widetilde{W}_R} h^{q'} \, dt \, dx \right)^{1/q'}$$

$$\lesssim \sum_{R \in \mathcal{D}} \sum_{\substack{S \in \mathcal{D} \\ W_S \cap \widetilde{W}_R \ne \varnothing}} |W_S|^{1-1/q'} \left(\iint_{W_S} h^{q'} \, dt \, dx \right)^{1/q'}$$

$$= \sum_{S \in \mathcal{D}} |W_S|^{1-1/q'} \left(\iint_{W_S} h^{q'} \, dt \, dx \right)^{1/q'} \sum_{\substack{R \in \mathcal{D} \\ \widetilde{W}_R \cap W_S \ne \varnothing}} 1$$

$$\lesssim \sum_{\substack{S \in \mathcal{D} \\ W_S \subset U}} |W_S|^{1-1/q'} \|g\|_{L_{q'}(W_S)}$$

$$\leq \sum_{j=1}^{N} |Q_j| \inf_{Q_j} C_{L_{q'}} g.$$

Thus there is $c>0$ and $1 \leq j \leq N$ such that $C_{L_{q'}} g > c\lambda$ on Q_j. Lemma 3.5 applies since we may assume that $\mathrm{dist}(z, Q_j) \lesssim \ell(Q) \leq \ell(Q_j)$. $\square$

Proof of Theorem 3.1. The result follows from Corollary 2.3 and Propositions 3.6 and 3.7. Note that by replacing $|f|^r$ and $|g|^r$ by f and g, it suffices to consider the case $r=1$. $\square$

Proof of Theorem 3.2. The result follows from Theorem 2.4 and Propositions 3.6 and 3.7. $\square$

References

1. AUSCHER, P. and AXELSSON [ROSÉN], A., Weighted maximal regularity estimates and solvability of non-smooth elliptic systems I, *Invent. Math.* **184** (2011), 47–115.
2. AUSCHER, P. and ROSÉN, A., Weighted maximal regularity estimates and solvability of non-smooth elliptic systems II, to appear in *Anal. PDE*.
3. CARLESON, L., An interpolation problem for bounded analytic functions, *Amer. J. Math.* **80** (1958), 921–930.
4. CARLESON, L., Interpolations by bounded analytic functions and the corona problem, *Ann. of Math.* **76** (1962), 547–559.
5. COIFMAN, R. R., MEYER, Y. and STEIN, E. M., Some new function spaces and their applications to harmonic analysis, *J. Funct. Anal.* **62** (1985), 304–335.
6. DAHLBERG, B., On the absolute continuity of elliptic measures, *Amer. J. Math.* **108** (1986), 1119–1138.
7. FEFFERMAN, C. and STEIN, E. M., H^p spaces of several variables, *Acta Math.* **129** (1972), 137–193.
8. FEFFERMAN, R. A., KENIG, C. E. and PIPHER, J., The theory of weights and the Dirichlet problem for elliptic equations, *Ann. of Math.* **134** (1991), 65–124.
9. KENIG, C. and PIPHER, J., The Neumann problem for elliptic equations with non-smooth coefficients, *Invent. Math.* **113** (1993), 447–509.

10. KENIG, C. and PIPHER, J., The Neumann problem for elliptic equations with non-smooth coefficients: part II, *Duke Math. J.* **81** (1995), 227–250.
11. STEIN, E. M., *Harmonic Analysis*: *Real-Variable Methods*, *Orthogonality*, *and Oscillatory Integrals*, Princeton Mathematical Series **43**. Princeton University Press, Princeton, NJ, 1993.

Tuomas Hytönen
Department of Mathematics and Statistics
P.O. Box 68 (Gustaf Hällströms gata 2b)
FI-00014 Helsingfors universitet
Finland
tuomas.hytonen@helsinki.fi

Andreas Rosén
Department of Mathematics
Linköpings universitet
SE-581 83 Linköping
Sweden
andreas.rosen@liu.se