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Asymptotic porosity of planar harmonic measure

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Abstract. We study the distribution of harmonic measure on connected Julia sets of unicritical polynomials. Harmonic measure on a full compact set in $\mathbb{C}$ is always concentrated on a set which is porous for a positive density of scales. We prove that there is a topologically generic set $\mathcal{A}$ in the boundary of the Mandelbrot set such that for every $c\in\mathcal{A}$, $\beta>0$, and $\lambda\in(0,1)$, the corresponding Julia set is a full compact set with harmonic measure concentrated on a set which is not β-porous in scale λ^n for n from a set with positive density amongst natural numbers.

1. Introduction

A compact set in the plane is called *full* if its complement is connected. An intuition about the harmonic measure of a full compact set in the plane is that it is supported on "exposed" points of the set. One way to make this notion precise is based on Makarov's theorem, namely that for every parameter $\beta<\frac{1}{2}$ the harmonic measure is supported on the set of points in the neighborhood of which the set is β-porous for a positive density set of scales. It does not seem, however, that such points are very "exposed" and one might conjecture that a stronger statement could be made, for example that a density of scales in which the set is porous is asymptotically 1. We show sets for which the harmonic measure is supported on a set of points around which the set fails to be porous with any given parameter $\beta>0$ in a set of scales of positive density. Intuitively speaking such points can only be accessed by passing through infinitely many increasingly narrow "bottlenecks". This also demonstrates that the positive density statement in Makarov's theorem cannot be improved.

The existence of full compact sets with these properties has not been to our knowledge demonstrated in the literature. Further, we show also that such examples

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are dynamically natural. They occur as Julia sets of quadratic polynomials for parameters from a generic set on the boundary of the connectedness locus. "Generic" here certainly cannot be taken in the sense of the harmonic measure on the boundary of the connectedness locus. It is known from [1] and [9] that this measure is supported on the set of polynomials with porous Julia sets. Genericity here is taken in the topological sense, meaning a residual set. From the dynamical standpoint this is another example of a sharp split between properties which are typical in the sense of the harmonic measure on the boundary of the connectedness locus and topologically generic. Another way in which this contrast manifests itself is in the Hausdorff dimension of the Julia sets. By Shishikura's theorem, the Hausdorff dimension of quadratic Julia sets is 2 on a topologically generic set. But since typical sets in the sense of harmonic measure are porous, their Hausdorff dimensions are always less than 2.

The main idea of this work is to use symbolic dynamics constructed on Yoccoz partitions and Bernoulli probabilistic models, cf. [1] and [9], to control distortion properties of high iterates of unicritical polynomials at almost every point in the sense of harmonic measure. These distortion properties and the renormalization results of [6] allows one to transfer an asymptotic density of the quadratic Feigenbaum Julia set at the critical point to almost all points, in the sense of harmonic measure, of a topologically generic Julia set J_c with c in the boundary of the Mandelbrot set.

Harmonic measure, porosity, and complex dynamics. Let E be a full compact set in $\mathbb{C}$. The *harmonic measure* ω of E with a base point at ∞ can be described in terms of the Riemann map

$$\Psi\colon \widehat{\mathbb{C}}\setminus D(0,1) \longmapsto \widehat{\mathbb{C}}\setminus E,$$

which is tangent to identity at ∞. Namely, Ψ extends radially almost everywhere on the unit circle, with respect to the normalized 1-dimensional Lebesgue measure $d\theta$, and $\omega = \Psi_*(d\theta)$. The probabilistic interpretation of ω is that for any Borel subset U of E, $\omega(U)$ is a hitting probability of a Brownian particle sent from ∞ toward E.

We need also a few basic concepts from complex dynamics. The *Julia set* J_c of a unicritical polynomial $f_c(z) = z^d + c$ is defined as a closure of all repelling periodic points of f_c,

$$J_c := \{z \in \mathbb{C}\colon\ f_c^n(z) = z \text{ and } |(f_c^n)'(z)| > 1 \text{ for some } n \in \mathbb{N}\}.$$

Let $\mathcal{M}_d$ be the set of all $c \in \mathbb{C}$ for which J_c is connected. The set $\mathcal{M} := \mathcal{M}_2$ is usually called the *Mandelbrot set.* The boundary of $\mathcal{M}_d$ is the topological bifurcation locus of J_c, i.e. when c traverses $\partial\mathcal{M}_d$ from the inside of $\mathcal{M}_d$ toward ∞, then the corresponding Julia sets are initially connected and turn totally disconnected.

The distribution of harmonic measure on J_c for certain values of $c\in\partial\mathcal{M}$ shows many 'exotic' features and will be of special interest to us.

Definition 1.1. We will say that a bounded set $E\subset\mathbb{C}$ is *β-porous* at scale ε at z if there is z' such that $|z-z'|=\varepsilon$ and $D(z',\beta\varepsilon)\cap E=\varnothing$.

Clearly, $\beta\in(0,1]$. The concept of porosity has a long history, see [5], but only in the last two decades it was used in a systematic way to study dimensional features of sets, [4]. In the form closely related to that of Theorem 1.2, it appears in [2]. The work [4] shows that any planar set porous at every scale with β tending to 1 is of Hausdorff dimension 1 while in [2] it is proved that a more flexible property of mean porosity at every point of a bounded set E yields $\dim_H(E)<2$.

A set A is *topologically generic* or *residual* in a given topology if it contains an intersection of countably many open dense sets. The boundary of the Mandelbrot set $\mathcal{M}$ has a natural topology inherited from the complex plane.

The purpose of this article is to prove the following theorem.

Theorem 1.2. *For every full compact $E\subset\mathbb{C}$ and every choice of positive $\beta<\frac{1}{2}$ and $\lambda>1$, there exist $\sigma>0$ and a set of full harmonic measure such that for every z from this set*

$$\liminf_{N\to\infty}\frac{1}{N}\#\{n\in(0,N):E \textit{ is } \beta\textit{-porous at some scale } \varepsilon\in(\lambda^{-n-1},\lambda^{-n}) \textit{ at } z\}\geq\sigma.$$

Moreover, there exists a topologically generic set $\mathcal{A}$ in $\partial\mathcal{M}$ such that for every $c\in\mathcal{A}$ and every choice of positive numbers $\lambda>1$ and $\beta>0$ there exists $\sigma>0$ and a set of full harmonic measure in J_c such that for every z in this set,

$$\liminf_{N\to\infty}\frac{1}{N}\#\{n\in(0,N):J_c \textit{ is not } \beta\textit{-porous at any scale } \varepsilon\in(\lambda^{-n-1},\lambda^{-n}) \textit{ at } z\}\geq\sigma$$

and

$$\lim_{N\to\infty}\frac{1}{N}\max\{n\in(0,N):J_c \textit{ is not } \beta\textit{-porous at any scale } \varepsilon\in(\lambda^{-n-1},\lambda^{-n}) \textit{ at } z\}=1.$$

The proof of the asymptotic porosity of harmonic measure at a positive density of scales is standard, although the limiting value of $\beta=\frac{1}{2}$ falls short of the upper bound 1. We do not know what happens for β between $\frac{1}{2}$ and 1. The calculation is based on Makarov's law of iterated logarithm [3]. The main difficulty lies in the second part of Theorem 1.2 which shows that the estimate cannot be improved in that the density σ is only positive, but not 1.

Corollary 1.3. *Let $\mathcal{A}$ be the set defined in Theorem 1.2. There exists a topologically generic set $\mathcal{A}'\subset\mathcal{A}$ such that for every $c\in\mathcal{A}$ the corresponding Julia set J_c is locally connected of Hausdorff dimension* 2.

Proof. By [8], we know that there exists a residual set in $\mathcal{M}$ such that for every c from this set, $\dim_H(J_c)=2$. Also, by the work of Yoccoz, there is another topologically generic set in $\partial\mathcal{M}$ such that for every c from this set J_c is locally connected. Intersecting these three topologically generic sets, we obtain the claim of Corollary 1.3. □

2. Asymptotic Lipschitz accessibility

Let Ω be a domain in $\widehat{\mathbb{C}}$ with $\partial\Omega\subset\mathbb{C}$. Let $\delta(z)$ denote the Euclidean distance $\operatorname{dist}(z,\partial\Omega)$. Let $d\rho(z)$ be a hyperbolic metric in Ω of constant curvature -1. The hyperbolic distance between $x,y\in\Omega$ is defined by $\rho(x,y)=\inf\int_\gamma|d\rho|$, where the infimum is taken over all curves $\gamma\subset\Omega$ joining x and y.

We say that Ω is *Hölder* accessible with an exponent α at $y\in\partial\Omega$ if there exist a positive constant $C>0$ and a hyperbolic geodesic Γ joining a base point $x_0\in\Omega$ and y such that for every $w\in\Gamma$,

$$\rho(w,x_0)\le\frac{1}{\alpha}\log\frac{1}{|w-y|}+C.$$

A domain Ω is *asymptotically Lipschitz accessible* at $y\in\partial\Omega$ if it is Hölder accessible at y with an exponent $\alpha(w)$ tending to 1 when w tends to y along Γ.

The distribution of planar harmonic measure is governed by Makarov's law of iterated logarithm [3].

Theorem 2.1. (Makarov) *If g maps $D(0,1)$ conformally into $\mathbb{C}$ then*

$$\limsup_{r\to1}\frac{|\log g'(r\zeta)|}{\sqrt{\log\dfrac{1}{1-r}\log\log\log\dfrac{1}{1-r}}}\le 6$$

for almost all ζ from the unit circle in the sense of 1*-dimensional Lebesgue measure.*

From Theorem 2.1, we know that almost every point in J_c, with respect to the harmonic measure, is asymptotically Lipschitz accessible from the outside of J_c. Indeed, we can assume that $\zeta=1$ is a typical point in the sense of Theorem 2.1.

Hence, for every $\alpha\in(0,1)$ there exists r_0 such that for every $r\in(r_0,1)$, $|g'(r)|\le(1-r)^{\alpha-1}$. Let $w=g(r)$, $y=g(1)$, and $z_0=g(0)$. If w is close enough to y then

$$|w-y|\le\int_r^1|g'(r)|\,dr\le\frac{1}{\alpha}(1-r)^{\alpha}. \tag{1}$$

The conformal invariance of the hyperbolic distance and (1) imply that

$$\rho(w,x_0)=\rho(r,0)=\log\frac{1+r}{1-r}\le\frac{1}{\alpha}\log\frac{1}{|w-y|}+C,$$

where the constant C does not depend on α provided w is near y.

The first estimate of Theorem 1.2 follows immediately from Proposition 2.2 below.

Proposition 2.2. *Suppose that $\Omega\subset\widehat{\mathbb{C}}$, with $\partial\Omega\subset\mathbb{C}$, is Hölder accessible at y with an exponent $\alpha>0$. Then for every $\beta\in\left(0,\frac{1}{2}\right)$ and $\lambda>1$ we have that*

$$\liminf_{N\to\infty}\frac{1}{N}\#\{n\in(0,N):\partial\Omega\text{ is }\beta\text{-porous at scale }\varepsilon\in(\lambda^{-n-1},\lambda^{-n})\}\ge\frac{1-2\alpha/\beta}{1-\beta}.$$

Proof. Suppose a hyperbolic geodesic Γ joins a base point $x_0\in\Omega$ and $y\in\partial\Omega$. For every non-negative integer n, set $A_n:=\{z\in\mathbb{C}:\lambda^{-n-1}\le|z-y|\le\lambda^{-n}\}$. Let $w_n\in\Gamma$ be a point of last intersection of $\Gamma(w)$ with the circle $|z-y|=\lambda^n$ when w runs from x_0 to y along Γ. We use the Hölder accessibility of Ω at y and the fact that the density $d\rho(z)$ of the hyperbolic metric in every simply connected domain is bigger than or equal to $1/2\delta(z)$ to obtain that

$$\sum_{i=0}^{n-1}\int_{\Gamma\cap A_n}\frac{1}{\delta(z)}\,|dz|\le2\sum_{i=0}^{n-1}\int_{\Gamma\cap A_n}|d\rho(z)|\le2\rho(w_n,x_0)\le\frac{2}{\alpha}n\log\lambda+C. \tag{2}$$

Put $\chi_i=0$ if K at y is not β-porous at any scale $\varepsilon\in(\lambda^{-i-1},\lambda^{-i})$ and $\chi_i=1$ otherwise. We want to estimate the dependence of $\int_{\Gamma\cap A_i}\delta(z)^{-1}\,|dz|$ on the value of χ_i.

If $\chi_i=0$ then

$$\int_{\Gamma\cap A_i}\frac{1}{\delta(z)}\,|dz|\ge\int_{\Gamma\cap A_i}\frac{1}{\beta|y-z|}\,|dz|\ge\int_{\lambda^{-i-1}}^{\lambda^{-i}}\frac{1}{\beta r}\,dr=\frac{\log\lambda}{\beta}.$$

If $\chi_i=1$ then

$$\int_{\Gamma\cap A_i}\frac{1}{\delta(z)}\,|dz|\ge\int_{\lambda^{-i-1}}^{\lambda^{-i}}\frac{1}{r}\,dr=\log\lambda.$$

Let $s_n = \sum_{i=0}^{n} \chi_i$. Combining these estimates with (2), we obtain that

$$\frac{\log\lambda}{\beta}(n-s_n)+s_n\log\lambda \le \frac{2n}{\alpha}\log\lambda+C.$$

Dividing both sides by $n\log\lambda$ and letting n tend to ∞, we have that

$$\liminf_{n\to\infty}\frac{s_n}{n} \ge \frac{1-2\beta/\alpha}{1-\beta}. \quad \square$$

3. Dynamics on the Julia set and the harmonic measure

As a preparation for the proof of the second part of Theorem 1.2, we conduct certain probabilistic considerations, closely following [1]. The goal is to describe the behavior of a typical point with respect to the harmonic measure.

3.1. Bernoulli model

Consider a Bernoulli shift on p symbols denoted $0, 1, ..., p-1$, with its natural Tikhonov topology and product probability measure. Let Ω denote the space of the shift and S the shift map. A point $\omega\in\Omega$ is identified with a sequence $x_0(\omega)=1, x_1(\omega), ..., x_n(\omega), ...$ with $x_i(\omega)\in\{0, 1, ..., p-1\}$ for $i>0$.

Let us introduce a metric on Ω which induces the Tikhonov topology: namely if $\omega_1, \omega_2\in\Omega$ we find the least $i\ge 0$ for which $x_i(\omega_1)\ne x_i(\omega_2)$ and set $d(\omega_1, \omega_2)=2^{-i}$. Note the improved triangle inequality

$$d(\omega_1, \omega_2) \le \max(d(\omega_1, \omega_3), d(\omega_3, \omega_2)).$$

Furthermore, S is 2-Lipschitz with respect to d.

Let $\mathbb{N}$ denote the set of positive integers, and $\mathbb{Z}_+=\mathbb{N}\cup\{0\}$.

Definition 3.1. Given $\omega, \nu\in\Omega$, let us define a function

$$\rho_{\omega,\nu} : \mathbb{N} \longrightarrow \mathbb{N}$$

as

$$\rho_{\omega,\nu}(k) = \inf\{j>0 : d(S^j\omega, \nu) \le 2^{j-k}\}.$$

Informally speaking, $\rho_{\omega,\nu}(k)=j$ means that starting from j at least through $k-1$ the code $x_i(\omega)$ repeats the initial sequence of ν starting with $x_0(\nu)$ and j is the smallest positive number with this property. For $j=k$ this requirement becomes vacuous, so $\rho_{\omega,\nu}(k)\le k$.

We are now ready to state the result.

Theorem 3.2. *Let* (S,Ω,p) *be a Bernoulli shift on finitely many symbols. There exists* $L>0$ *and for every* $\nu\in\Omega$ *not periodic under* S *with period less than or equal to* L, *there are a constant* $\varkappa$ *and a set* Ω_ν *of full measure in* Ω *such that for every* $\omega\in\Omega_\nu$ *one can find an increasing infinite sequence of integers* k_i *so that*

(1) $k_i-\rho_{\omega,\nu}(k_i)<\varkappa$ *for every* i;

(2) $\limsup(k_i/i)\le 3$;

(3) $k_{i+1}<k_i+\sqrt{k_i}$ *for all but finite number of* i.

Theorem 3.2 and its proof are analogous to Theorem 2.1 in [1].

4. Combinatorial considerations

This section is devoted to the proof of Theorem 3.2 which proceeds entirely inside the Bernoulli model. We will prove certain properties of the function $\rho_{\omega,\nu}$. We give formal proofs based on the metric d. Alternative proofs can be constructed by using the interpretation of the function $\rho_{\omega,\nu}$ in terms of repeating codes.

Lemma 4.1. *If* $\nu\in\Omega$ *is not periodic with period less than* L *then*

$$\lim_{k\to\infty}\rho_{\nu,\nu}(k)\ge L.$$

Proof. The function $\rho_{\nu,\nu}$ is non-decreasing, so let us suppose that it stabilizes at some j. Then Definition 3.1 implies that

$$d(S^j\nu,\nu)\le 2^{j-q}$$

for q arbitrarily large, so $S^j\nu=\nu$ and ν is periodic with period $j\ge L$. $\square$

Lemma 4.2. *Let* $\omega,\nu\in\Omega$. *If* $\rho_{\omega,\nu}(k+1)>\rho_{\omega,\nu}(k)$, *then*

$$\rho_{\omega,\nu}(k+1)\ge\rho_{\omega,\nu}(k)+\rho_{\nu,\nu}(k-\rho_{\omega,\nu}(k)).$$

Proof. Let $j=\rho_{\omega,\nu}(k)$ and $J=\rho_{\omega,\nu}(k+1)$. Then $d(S^J\omega,\nu)\le 2^{J-k-1}$. Next,

$$d(S^{J-j}\nu,\nu)\le\max(d(S^J\omega,\nu),d(S^J\omega,S^{J-j}\nu)) \tag{3}$$

and

$$d(S^J\omega,S^{J-j}\nu)=d(S^{J-j}(S^j\omega),S^{J-j}\nu)\le 2^{J-j}d(S^j\omega,\nu)\le 2^{J-k}.$$

Now, the estimate (3) leads to

$$d(S^{J-j}\omega,\nu)\le\max(2^{J-k-1},2^{J-k})=2^{J-k}=2^{(J-j)-(k-j)}.$$

In view of Definition 3.1 this means that

$$\rho_{\omega,\nu}(k-j)\leq J-j,$$

which is precisely what the lemma claims. □

4.1. The key argument

From now we regard ν as fixed and we will write ρ_ω instead of $\rho_{\omega,\nu}$. In the first step of the proof of Theorem 3.2, we can now specify $\varkappa$. It will still satisfy $\rho_{\nu,\nu}(\varkappa)\geq L+1$. In view of Lemma 4.1, we can choose $\varkappa$ with this property.

Proposition 4.3. *For a certain choice of a positive integer L the following holds true. Let G_ω be defined as the set of all $k\in\mathbb{N}$ for which $k-\rho_\omega(k)<\varkappa$.*

Then almost surely for $\omega\in\Omega$:

(1) $\varkappa<\infty$;

(2)

$$\liminf_{n\to\infty}\frac{|G_\omega\cap\{1,\ldots,n\}|}{n}\geq\frac{1}{3};$$

(3) *except for finitely many n, if $n\in G_\omega$, then*

$$G_\omega\cap\big(n,n+\sqrt{n}\big)\neq\varnothing.$$

The first property holds whenever ω is not periodic under S, which is true almost everywhere regardless of L. Since $\varkappa$ is locally constant, we can restrict the attention to a cylinder Ω_0 on which $\varkappa$ is finite and constant. In particular, we will talk of probabilities conditioned onto Ω_0.

We will now prove Proposition 4.3 which implies Theorem 3.2.

Lemma 4.4. *Consider a sequence of m independent Bernoulli trials, each with the probability of success at most $P<1$. For some $M>0$ let X_i be 1 if the i-th trial is a success, and $-M$ if it is a failure.*

There is a constant M_0 only depending on P so that if $M\geq M_0$, then $\sum_{i=1}^m X_i\leq -m$ with probability at least $1-\exp(-m(1-P)/4)$.

Proof. This is Lemma 3.3 from [1]. □

We now indicate the idea of the proof of the remaining part of Proposition 4.3. We watch a non-negative function $r_\omega(k):=k-\rho_\omega(k)$. As k grows by 1, then $r_\omega(k)$ may increase at most by 1, and that only happens if $x_{k+1}(\omega)=x_{k-\rho_\omega(k)+1}(\omega)$. Call the event $r_\omega(k+1)(\omega)>r_\omega(k)$ a success at $k+1$. Clearly, this defines independent trials with probabilities of success all bounded by $P<1$, where P is the maximum of probabilities of any symbol. If $k\notin G_\omega$ meaning that $r_\omega(k)\geq\varkappa$, then failure at $k+1$ means that we get $p_\omega(r_\omega(k))\geq L+1$ and $r_\omega(k+1)-r_\omega(k)\leq -L$.

If we count only the trials which follow $k\notin G_\omega\cap[n,n+m)$, then by Lemma 4.4, with overwhelming probability r_ω will jointly drop by the number of such k. Regardless of the outcome of trials following $k\in G_\omega$, $r_\omega(k)$ may grow at most by 1. If the number of $k\notin G_\omega$ is more than $2/3m$, this implies a drop by $m/3$. But on the other hand, $r_\omega(k)$ is non-negative which yields a lower bound on the density of G_ω in $[n,n+m)$.

Let us state this reasoning formally.

Lemma 4.5. *Fix L and consider ω in a cylinder, where $\varkappa$ takes a constant finite value, choose integers $n>\varkappa$ and m in such a way that $10r_\omega(n)\leq m$. Let P denote the maximum of probabilities of any single symbol. There is a constant L_0 which depends only on P so that if $L\geq L_0$, then with probability at least $1-\exp(-m(1-P)/6)$*

$$|G_\omega\cap[n,n+m)|>\frac{m}{3}.$$

Proof. Let βm be the number of integers in the set $B=[n,n+m)\setminus G_\omega$. For any $k\in B$ let us call it a success when $p_\omega(k+1)=p_\omega(k)$. This defines a sequence of Bernoulli trials with the probability of success at most P. On the other hand, if $k\notin G_\omega$, then $r_\omega(k)\geq\varkappa$ and by Lemma 4.2 and the definition of $\varkappa$, $r_\omega(k+1)-r_\omega(k)\leq -L$. We apply Lemma 4.4 to this sequence of βm trials. We get that there is $L_0:=M_0$ depending only on P, so that if $L\geq L_0$, then

$$\sum_{k\in B} r_\omega(k+1)-r_\omega(k)\leq -m\beta \tag{4}$$

with probability at least $1-\exp(-m\beta(1-P)/4)$. For any $k\in[n,n+m)$,

$$r_\omega(k+1)-r_\omega(k)\leq 1.$$

Let $B^c:=[n,n+m)\cap G_\omega$. Hence, assuming estimate (4),

$$\begin{aligned} r_\omega(n+m)-r_\omega(n) &= \sum_{k\in B} r_\omega(k+1)-r_\omega(k)+\sum_{k\in B^c} r_\omega(k+1)-r_\omega(k)\\ &\leq -m\beta+m(1-\beta)\leq m(1-2\beta). \end{aligned}$$

But $r_\omega(n+m)\geq 0$ while $r_\omega(n)\leq 0.1m$ by the hypothesis of the lemma, so

$$m(1-2\beta)\geq r_\omega(n+m)-r_\omega(n)\geq -0.1m.$$

Thus, estimate (4) implies $\beta \leq 0.55$. Hence, if $\beta \geq \frac{2}{3}$, then estimate (4) fails, which happens only with probability not exceeding

$$\exp\left(-m\frac{1-P}{6}\right). \quad \square$$

Now Proposition 4.3 follows easily. The claim regarding the asymptotic density of G_ω follows if we apply Lemma 4.5 with $n=\varkappa$ and m tending to ∞ and invoke Borel–Cantelli's lemma. If we now apply the lemma with $m=\left[\sqrt{n}\right]$, we get that almost surely for almost all n either $r_\omega(n) > 0.1\left[\sqrt{n}\right]$ or

$$\left|G_\omega \cap [n, n+\sqrt{n})\right| \geq \frac{[\sqrt{n}]}{3}.$$

(Here $[x]$ stands for the integer part of x.) For all sufficiently large n the first condition implies that $n \notin G_\omega$ and the second one that $G_\omega \cap \left(n, n+\sqrt{n}\right) \neq \varnothing$ as needed.

5. Yoccoz pieces

5.1. Construction of symbolic dynamics

Ray-sectors and facets. Let us define a self-map of the circle $T(x)=dx \pmod 1$. For this section, we assume that $c \in \mathcal{M}_d$, all of its periodic orbits are repelling and the critical orbit is infinite. In particular, f_c has a fixed point $q(c)$ which attracts a ray with external angle λ. This ray is periodic under T with period $p>1$. Rays with external angles $\lambda, ..., T^{p-1}(\lambda)$ divide the plane into p ray-sectors. Corresponding to each ray sector there is an arc on the circle consisting of all external arguments of rays belonging to this ray-sector.

The mapping f_c is univalent on ray-sectors which do not contain 0. We can label the ray sectors in such a way that Q_0 contains 0, Q_1 contains c, and $f_c(Q_i) = Q_{i+1 \pmod p}$ for $i=1, ..., p-1$. The set $f_c(Q_0)$ is the union of all ray-sectors. We will denote by s_i the arc of the circle which corresponds to Q_i. We will call these open arcs *facets* of order 0.

The Yoccoz partition. Let us continue to develop the picture of ray-sectors and the corresponding gaps. Consider the topological disk $\Delta := \{z : G_c(z) < 1\}$, where

$$G_c(z) = \lim_{n\to\infty} \frac{\log|f_c^n(z)|}{d^n}. \tag{5}$$

Observe that Δ intersects each Q_i along a "curvilinear triangle" Δ_i. The collection of these Δ_i's is sometimes referred to as the *Yoccoz partition* for f_c.

Induced maps. The action of f_c on each Q_i, $i>0$, is boring: each is mapped univalently onto Q_{i+1} modulo p. As the result, the codes k and ℓ are quite redundant, since every symbol is predicted by the previous one, provided that one is not 0.

Definition 5.1. An induced map Φ_c is defined on the union of all ray-sectors Q_i. On Q_i, $i>0$, we set $\Phi_c:=f_c^{p-i+1}$. On Q_0, $\Phi_c=f_c$.

Analogously we can construct a mapping ϕ induced by T on the circle which corresponds to Φ_c. Thus, $\phi=T$ on s_0 and $\phi=T^{p-i+1}$ on any other s_i.

Lemma 5.2. *The following statements hold*:
– Φ_c *maps any* Δ_i *over the union of all* Δ_j;
– *If* $K\subset\Delta_j$ *is relatively compact in* Δ_j, *then* $\Phi_c^{-1}(K)$ *is relatively compact in* $\bigcup_{i=0}^{p-1}\Delta_i$;
– *If* $j\neq1$, *then* $\overline{\Phi_c^{-1}(\Delta_j)}$ *is relatively compact in* $\bigcup_{i=0}^{p-1}\Delta_i$;
– *The only critical value of* Φ_c *is at* c *and this is a branching point of degree* d.

Proof. To get $\Phi_c(z)$ we first map z to Q_0 and then one more time. The properties of Φ_c depend on this last iteration, which can be easily understood in terms of the map T acting on the external angles of rays. The proof is then easy and mostly standard. □

Itineraries. We define for every $z\in\mathbb{C}$ its itinerary $\omega(z)=\omega_0,...,\omega_n,...$ by the condition $\Phi_c^i(z)\in\Delta_{\omega_i}$. The transformation $z\mapsto\omega(z)$ semi-conjugates f_c to the full shift on p symbols. Similarly, we define, for γ in the circle, an itinerary $\ell(\gamma)=\ell_0,...,\ell_n,...$ by the condition $\phi^i(\gamma)=s_{\ell_i}$. Both itineraries could be finite if the point cannot be iterated or leaves the Yoccoz partition. However, if $z\in J_c$ is the closure of a ray with external argument γ then $\omega(z)=\ell(\gamma)$.

5.2. Properties of Yoccoz pieces

Consider an itinerary $x_0,...,x_k$. A *Yoccoz piece* of order k following this itinerary is any maximal connected set of points z for which the first $k+1$ symbols of their itineraries $\omega(z)$ are the same as $x_0,...,x_k$. Similarly, a *facet* of order k following this itinerary is a maximal connected set (arc) of points γ for which the first $k+1$ symbols of their itineraries $\ell(\gamma)$ are the same as $x_0,...,x_k$.

Lemma 5.3. *If a Yoccoz piece* D *of order* k *follows the itinerary* $x_0,...,x_k$, *then* Φ_c^k *restricted to the piece is a proper holomorphic map onto* Δ_{x_k}.

Proof. Obviously, $F := \Phi_c^k$ is a holomorphic map from the Yoccoz piece into Δ_{x_k}. We have to show that F is proper. The proof proceeds by induction with respect to k. For $k=0$, F is the identity map. In general, $\Phi_c^k = \Phi_c \circ \Phi_c^{k-1}$, where both Φ_c and Φ_c^{k-1} are proper by the induction hypothesis. □

As a consequence of Lemma 5.3, Φ_c^k is onto Δ_{x_k} and is a finitely branched cover. In particular, D is a topological disk. Here is another lemma.

Lemma 5.4. *Consider the two Yoccoz pieces D of order k which follows an itinerary $x_0, ..., x_k$ and $D' \subset D$ which follows the itinerary $x_0, ..., x_k, x_{k+1}$ with $x_{k+1} \neq 1$. Then $\overline{D}'$ is contained in D.*

Proof. Observe that

$$D' \subset \Phi_c^{-k}(\Phi_c^{-1}(\Delta_{x_{k+1}})).$$

By the properties of Φ_c, listed following its definition, $\Phi^{-1}(\Delta_{x_{k+1}})$ is precompact in Δ_{x_k} and so the claim follows by Lemma 5.3. □

The next lemma establishes a connection between the dynamics of Φ_c on Yoccoz pieces and the combinatorial function $\rho_{\omega(z),\omega(c)}$, see Definition 3.1.

Lemma 5.5. *Suppose that D is a Yoccoz piece of order k which contains a point z. Then Φ_c^j are univalent on D for all $j < \rho_{\omega(z),\omega(c)}(k)$.*

Proof. Choose the smallest j for which Φ_c^j is not univalent. Then $\Phi_c^j(D) \ni c$. But then the $k-j$ consecutive symbols of $\omega(c)$ starting from the beginning and from $\omega_j(z)$ are the same, or $d(S^j\omega(z), \omega(c)) \leq 2^{j-k}$, which implies that $j \geq \rho_{\omega(z),\omega(c)}(k)$ by Definition 3.1. □

Lemma 5.6. *For every Yoccoz piece of order k, the set of external arguments whose rays enter D is a union of finitely many facets of order k.*

Proof. This immediately follows by induction with respect to k. □

5.3. Visits to large scale

Theorem 5.7. *Let $f_c(z) = z^d + c$ and suppose that $c \in \mathcal{M}_d$, all periodic orbits of f_c are repelling and the orbit of c is infinite. On the set of such c there are two integer-valued continuous functions $L(c)$ and $\varkappa(c)$ such that if the itinerary $\omega(c)$ is not periodic with period less than or equal to $L(c)$, then for every Yoccoz piece D of* *order at least $\varkappa(c)$ there exist a constant K and a set of full harmonic measure in J_c such that for every z in this set the following holds*:

There exists an infinite increasing sequence of integers m_i and sequence of Yoccoz pieces $U_i \ni z$ such that f^{m_i} maps U_i univalently onto a Yoccoz piece of order $\varkappa(c)$ while $f^{m_i}(z) \in D$ for every i. The sequence m_i is dense in the following sense:

(1) $\limsup_{i\to\infty} m_i/i \le K$;

(2) $\lim_{i\to\infty} m_{i+1}/m_i = 1$.

Proof. We start with defining a probabilistic setting. We will regard the circle with its normalized Lebesgue measure as a probabilistic space with the measure preserving transformation ϕ. Let $\mathcal{P}_k$ denote a partition of the circle into facets of order k. The key observation is that ϕ is then a random variable independent of $\mathcal{P}_0$. Since ϕ^k maps a facet of order k linearly onto a facet of order 0, it follows that more generally ϕ^{k+1} is independent of $\mathcal{P}_k$ for every $k \ge 0$. In particular, the map $\gamma \mapsto \ell(\gamma)$, which conjugates ϕ to the one-sided shift S, is a measure isomorphism. Also, if $q_i > k$ then the partitions $\mathcal{P}_k$, $\phi^{-q_1}(\mathcal{P}_k), ..., \phi^{-q_i}(\mathcal{P}_k), ..., \phi^{-\sum_{i=1}^n q_i}(\mathcal{P}_k), ...$ are all independent.

In addition we have a map χ from the circle into J_c which maps γ to the landing point of the corresponding external ray and transports the Lebesgue measure to the harmonic measure of J_c.

As a starting point we can take a sequence k_i constructed in Theorem 3.2. Theorem 3.2 will also fix the values of $L(c)$ and $\varkappa(c)$. We know from Lemma 5.5 that for z from a set of full harmonic measure and every i, z has a neighborhood which is mapped univalently by $\Phi_c^{k_i-\varkappa(c)}$ onto a Yoccoz piece of order $\varkappa(c)$. We would like to choose a further subsequence k_{i_j} so that $\Phi_c^{k_{i_j}-\varkappa(c)}(z) \in D$. First, by setting $i_j = j(k+1)$ we guarantee that $k_{i_{j+1}} - k_{i_j} \ge k$ and hence the events consisting of $\phi^{k_{i_j}-\varkappa(c)}$ belonging to any facet of order k are independent. The facet here should be chosen corresponding to D as in Lemma 5.6 so that whenever $\phi^{k_{i_j}-\varkappa(c)}(\gamma)$ belongs to this facet then $\Phi_c^{k_{i_j}-\varkappa(c)}(\chi(\gamma)) \in D$. Events consisting of $\phi^{k_{i_j}-\varkappa(c)(\gamma)}$ belonging to this facet can be viewed as independent Bernoulli trials with a positive probability of success t. The final subsequence k_{i_j} is obtained by eliminating all j for which such a trial results in a failure. The resulting sequence preserves the density properties of k_{i_j}, namely $\limsup_{j\to\infty} k_{i_j}/j \le K'$, where K' depends on t and k, and $\limsup_{j\to\infty} k_{i_{j+1}}/k_{i_j} = 1$. The proof of this is standard and follows closely the reasoning in Section 3.3 in [1].

Since Φ_c is an iterate of f_c of order no more than p, if we define m_j by the condition $f_c^{m_j} = \Phi_c^{k_{i_j}-\varkappa(c)}$, the sequence m_j satisfies the same density conditions with $K=pK'$ and by its construction satisfies the other claims of Theorem 5.7 as well. □

6. Examples of non-porosity

6.1. Construction of the residual set

The property opposite to porosity is hairiness. Let us recall that a set E is *hairy* at z_0 if and only if for every $\beta>0$ there exists $\varepsilon>0$ such that whenever $|z-z_0|<\varepsilon$, then $D(z,\beta\varepsilon)\cap E\neq\varnothing$. By [6], the Julia set of the Feigenbaum polynomial is hairy at its critical point, and hence at the critical value. Additionally, it can be asserted that the following holds.

Proposition 6.1. *There exists a dense set $\mathcal{Y}$ in $\partial\mathcal{M}$ such that for every $c\in\mathcal{Y}$ the Julia set of z^2+c is hairy at c.*

Proposition 6.1 follows from Theorem 1 in [7] which asserts that every open subset of $\mathcal{M}$ in the induced topology contains a complete copy of the Mandelbrot set. It means that for every $c\in\mathcal{M}$ there exists c' in that neighborhood such that a certain iterate of z^2+c' is quasiconformally conjugate to z^2+c on a neighborhood U of 0 which is mapped over $\overline{U}$ as a proper holomorphic map by that iterate. The neighborhood U contains the image K of the Julia set J_c, by the quasiconformal conjugacy, and necessarily $K\subset J_{c'}$. If J_c is hairy at c, then so is K at c' and thus $J_{c'}$ itself is hairy at c'. Taking c to be the Feigenbaum polynomial, we obtain Proposition 6.1.

Proposition 6.2. *There exist a residual set $\mathcal{R}$ in $\mathcal{M}$ such that for every $c\in\mathcal{R}$,*
(1) *all periodic orbits of f_c are repelling;*
(2) *the itinerary $\omega(c)$ is infinite and aperiodic;*
(3) *for every $\theta>0$ there exists $R>0$ such that if $|z-c|\leq R$ then*

$$D(z,\theta R)\cap J_c\neq\varnothing.$$

Proof. Fix an integer m. Consider the set $S_{1,m}$ of all $c\in\partial\mathcal{M}$ such that any non-repelling periodic orbit of f_c has period greater than m. This set is clearly open. Its density is a direct consequence of the density of copies of the Mandelbrot set used in the proof of Proposition 6.1. Next, we consider the set $S_{2,m}$ of $c\in\partial\mathcal{M}$ for which the critical itinerary is well defined up to length m and not periodic with any period less than m. This set contains an open and dense subset. Indeed, there is a dense set of c with infinite and aperiodic critical itineraries, and some open neighborhood of each of them belongs to $S_{2,m}$. Finally, $S_{3,m}$ is the set of all c such that the claim of Proposition 6.2 is satisfied for all $\theta > 1/m$. This again contains an open and dense set since every element of the set $\mathcal{Y}$ from Proposition 6.1 contains an open neighborhood with this property.

By intersecting $\bigcap_{m=1}^{\infty}(S_{1,m}\cap S_{2,m}\cap S_{3,m})$ we get the set $\mathcal{R}$. $\square$

6.2. Proof of the last claim of Theorem 1.2

Let $c\in\partial\mathcal{M}$ be a parameter supplied by Proposition 6.2. We fix positive $\beta<\frac{1}{2}$ and $\lambda>1$. For any $\theta>0$, to be specified later, find $R>0$ so that if $|z-c|\leq R$ then $D(z,\theta R)$ intersects J_c. By the decay of geometry, there exists a Yoccoz piece $W\ni 0$ such that $W\subset D(0,R/2)$. This means that for every $y\in W$, J_c is not 2θ-porous at scale $R/2$ at y.

Theorem 5.7 guarantees the existence of a set of full harmonic measure in J_c and positive constants $\varkappa$ and K such that for every z from this set there exists an increasing sequence $m_i\in\mathbb{N}$ and a collection of nested neighborhoods U_i of z such that for every i, f^{m_i} maps U_i on some Yoccoz piece U of order $\varkappa$ while $f^{m_i}(z)\in W$. Also, $\limsup_{i\to\infty} m_i/i\leq K$ and $\lim_{i\to\infty} m_{i+1}/m_i=1$.

Choose $R>0$ so small that $\operatorname{mod}(U\setminus D(0,R))\geq 10$. Using uniformly bounded distortion of f^{-m_i} on $D(0,R)$ we can choose $\theta>0$ so that J_c is not β/λ-porous at z, and scales $\varepsilon_i:=|(f^{m_i}(z))'|^{-1}R$. By the Markov property of Yoccoz pieces, $f^{m_i}(U_{i+1})\subset W$ and hence the conformal invariance of modulus yields

$$\operatorname{mod}(U_i\setminus U_{i+1})\geq \operatorname{mod}(U\setminus D(0,R))\geq 10. \tag{6}$$

From the super-additivity of modulus, $\operatorname{mod}(U\setminus U_i)\geq 10i$. The Teichmüller module theorem asserts that if the modulus of a topological annulus is $v>5\log 2$ then the topological annulus contains a geometric ring of modulus $v-5\log 2$. Since $J_c\subset D(0,2)$, we may assume that $\operatorname{diam} U<2^5$ and consequently, for every $i\in\mathbb{N}$ large enough,

$$\operatorname{diam} U_i\leq e^{-10i+5\log 2}\operatorname{diam} U.$$

For the same $i\in\mathbb{N}$, the Schwarz lemma implies that

$$\frac{|(f^{m_i}(z))'|}{R}\geq 2^{-5}e^{10i-5\log 2}\geq 2^{-10}e^{5m_i/K},$$

which in terms of ε_i becomes

$$\log\frac{1}{\varepsilon_i}\geq -10\log 2+\frac{5m_i}{K}\geq\frac{m_i}{K}. \tag{7}$$

We want to show that the density of the sequence $[\log(1/\varepsilon_i)]$ is positive, where $[\,x\,]$ stands for the integer part of x. By compactness, there exists a positive constant M so that $|f_c'(z)|<M$ for every $z\in J_c$ and every $c\in\mathcal{M}$. Therefore,

$$\log\frac{1}{\varepsilon_i}\le m_i\log M. \tag{8}$$

The estimate (6) implies that there exists $L>0$ such that at most L consecutive numbers $\log(1/\varepsilon_i)$ could yield the same integer $[\log(1/\varepsilon_i)]$. Using this observation, the inequality (8), and $\limsup_{i\to\infty} m_i/i\le K$, we have that

$$\begin{aligned}\frac{1}{N}\#\left\{i\in[1,N]:\left[\log\frac{1}{\varepsilon_i}\right]\le N\right\} &\ge \frac{1}{NL}\#\{i\in[1,N]:m_i\log M\le N\}\\ &\ge \frac{1}{NL}\#\{i\in[1,N]:2Ki\log M\le N\}\\ &\ge \frac{1}{2KL\log M}\end{aligned}$$

provided N is large enough.

The last estimate of Theorem 1.2 follows from (7) and a direct calculation. Let i be the largest integer such that $\log(1/\varepsilon_i)\le N$. Since for every $y\in J_c$, $|(f^{m_{i+1}-m_i})'(y)|\le M^{m_{i+1}-m_i}$,

$$\frac{\log\varepsilon_i-\log\varepsilon_{i+1}}{|\log\varepsilon_{i+1}|}\le K\frac{\log M^{m_{i+1}-m_i}}{m_{i+1}}\le K\log M\frac{m_{i+1}-m_i}{m_{i+1}}$$

and therefore,

$$\lim_{N\to\infty}\frac{1}{N}\log\frac{1}{\varepsilon_i}=1.$$

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