Non-liftable Calabi–Yau spaces

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Abstract. We construct many new non-liftable three-dimensional Calabi–Yau spaces in positive characteristic. The technique relies on lifting a nodal model to a smooth rigid Calabi–Yau space over some number field as introduced by one of us jointly with D. van Straten.

1. Introduction

In recent years, Calabi–Yau varieties have been studied extensively in the arithmetic context. Following the modularity of elliptic curves, there has been great progress on (singular) K3 surfaces [5] and (rigid) Calabi–Yau threefolds. The latter have two-dimensional H^3 which thus is related to automorphic forms for GL(2) in the number field case. If the variety is defined over $\mathbb{Q}$, this reduces to classical modularity and nowadays follows from Serre's conjecture [11].

It was a recent insight that modularity could be related to non-liftability at some bad primes. After the first results of Hirokado [7] and Schröer [16], there were constructions of non-liftable Calabi–Yau spaces along these lines by Schoen [15] and independently by the first author and van Straten [4]. These constructions are based on specific models as double octics or fiber products of rational elliptic surfaces.

The aim of this paper is to extend the techniques and constructions from [4] to other fiber products. We shall produce a great number of new examples of non-liftable Calabi–Yau spaces. In brief, our results are summarized as follows:

Theorem 1.1. *For each prime $p < 100$, with the exception of* 53 *and* 83, *there is a non-liftable three-dimensional Calabi–Yau space in characteristic p. The same holds for all nodal primes in Tables* 1 *and* 3.

The main idea of the construction is to exhibit a rigid Calabi–Yau space over some number field that reduces to a nodal model of the given Calabi–Yau space X in characteristic p. Then the proof of non-liftability of X follows the lines of [4] which we review in Section 2. We shall work with fiber products of elliptic surfaces. This approach, introduced by Schoen in [14], will be reviewed in Section 3. Sections 4 and 5 introduce the rational elliptic surfaces that we shall be concerned with. The proof of Theorem 1.1 will be completed in Section 6.

As indicated, the rigid Calabi–Yau spaces that we shall construct might also be interesting from the view-point of modularity. Namely, many of them can be defined over $\mathbb{Q}$ and are thus modular. It came a bit as a surprise for us how big the bad primes could actually get (see Tables 1 and 3). Since these primes have to appear in the level of the corresponding modular form, this brings back the question whether up to twisting, there are infinitely many Hecke eigenforms of weight 4 (or any integer weight greater than two) with rational coefficients.

2. Small resolutions in the algebraic setup

In this paper we shall construct non-liftable Calabi–Yau spaces as small resolutions of fiber products of rational elliptic surfaces with section. The method was introduced for semistable elliptic surfaces (all singular fibers of Kodaira type I_n) in [14] with the goal of producing Calabi–Yau threefolds with certain Euler numbers. In [17], twisted fiber products were used to produce new modular varieties over $\mathbb{Q}$. In the present paper we want to apply these techniques, but allow for certain types of unstable fibers. We are especially interested in rational elliptic surfaces with four or five singular fibers, one of them of Kodaira type II and the others semistable.

Let Y and Y' denote elliptic surfaces over $\mathbb{P}^1$ with section. The fiber product

$$X = Y \times_{\mathbb{P}^1} Y'$$

need not be smooth; it is singular exactly at points corresponding to singular points of fibers in both factors. In the case of semistable fibers, the fiber product has type A_1 singular points (nodes) as the only singularities. A three-dimensional node admits a so-called small resolution; here the node is replaced by a smooth variety of codimension two. In the complex analytic setup a node is locally analytically isomorphic to the singularity of the quadratic cone, so it can be resolved by blowing-up an analytic divisor. In [2] Artin gave an algebraic version of the above construction in the category of algebraic spaces [1]. A small resolution of a projective variety in general is not projective. A criterion for projectivity of a small resolution of a fiber product of semistable elliptic surfaces is given in [14].

If the fiber product contains a product of two fibers of type II, then it has a singularity of type D_4, i.e. with a local analytic equation

$$(1) \qquad\qquad x^2 + y^2 + z^3 + t^3 = 0.$$

As in the nodal case, this singularity admits a small resolution. This is achieved by first blowing up the local (analytic) divisor $x + iy = z + t = 0$ and then performing a small resolution of the resulting node. Contrary to the semistable case, this resolution is never projective. In the complex analytic category we obtain a compact (but non-kähler) Moishezon manifold, in the algebraic category we obtain an algebraic space.

In [14], Schoen showed that if Y and Y' are rational and semistable, then X is simply connected with trivial dualizing sheaf. Hence a small resolution gives a Calabi–Yau threefold (or a Calabi–Yau space). This fact was used in [4] to produce non-liftable fiber products due to the following key criterion.

Theorem 2.1. [4, Theorem 4.3 and Corollary 4.4] *Let X be a smooth rigid Calabi–Yau space defined over a finite ring extension R of $\mathbb{Z}$. If $\mathfrak{p}$ is a prime ideal in R such that the reduction $X_{\mathfrak{p}} := X \otimes_R (R/\mathfrak{p})$ of X modulo $\mathfrak{p}$ has a node as the only singularity, then a small resolution of $X_{\mathfrak{p}}$ is a Calabi–Yau space that cannot be lifted to characteristic* 0.

2.1. Behavior under reduction

Given a small resolution of a fiber product over some number field, there are two ways for it to attain bad reduction modulo some prime $\mathfrak{p}$. The first possibility is that one of the factors Y and Y' degenerates modulo $\mathfrak{p}$ so that the elliptic structure is lost. This is usually a fairly rare situation. The more common way to attain bad reduction is a fiber degeneration. Here fibers in one or both factors are merged upon reduction. Often this results in an additional node in positive characteristic. However, not every change of structure of the fiber product necessarily implies an additional node. In this paragraph, we comment on this and similar situations.

Consider the situation where one of the surfaces has fibers of types I_1 and I_2 at two points of $\mathbb{P}^1$ that are merged together to a type III fiber under the reduction modulo $\mathfrak{p}$, whereas the second one has fibers of types I_0 and I_1 at these two points. Then on the fiber product X in characteristic zero we have to distinguish two cases.

Assume that X has a fiber of type $I_2 \times I_1$ with two singular points. This situation is sketched as follows:

$$(2) \qquad\qquad \left. \begin{array}{c} I_2 \times I_1 \\ I_1 \times I_0 \end{array} \right\} \ \mathrm{III} \times I_1.$$

Either singularity admits a (non-projective) small resolution by blowing-up the product of one local branch of the I_1 singularity and one of the components of the I_2 resp. III fiber. Here the two local analytic divisors at the two singular points in characteristic zero are merged together to a smooth divisor in the fiber product modulo $\mathfrak{p}$. Blowing-up this smooth divisor yields the required small resolution. Consequently the reduction of the small resolution of the fiber product is a small resolution of the reduction, so the reduction modulo $\mathfrak{p}$ is smooth.

This should be contrasted with the situation where on both surfaces we have singular fibers of the same type as before, but with pairing interchanged:

$$(3) \qquad \left. \begin{array}{c} I_1 \times I_1 \\ I_2 \times I_0 \end{array} \right\} \; \text{III} \times I_1.$$

This time the branch of the I_1 singularity does not reduce to a component of the type III fiber modulo $\mathfrak{p}$. Hence the small resolution of the reduction does not agree with the reduction of the small resolution. Consequently there is bad reduction at $\mathfrak{p}$.

A similar reduction pattern occurs for the merging of I_3 and I_1 to IV. If paired with an unchanged semistable fiber, the reduction is good if and only if the original nodes in characteristic zero come from the I_3 fiber (see for instance [17, p. 222]).

2.2. Application to modularity

The situation from (3) may be observed in [18, Section 3.2]. There the reduction of some fiber product W_2 at the prime 359 has a (non-projective) small resolution, but this resolution does not agree with the reduction of the small resolution in characteristic zero. Consequently there is bad reduction at 359. Curiously this prime does not divide the level of the corresponding weight-four cusp form, but only of the weight-2 form. Note that the precise corresponding cusp form of weight four and level 55, as conjectured in [18, Conjecture 3], can be verified by replacing one of the factors in the fiber product by a 2-isogenous elliptic surface. This construction for correspondences between Calabi–Yau threefolds was introduced in [10]. In the present situation, it exchanges the singular fibers of type I_1 and I_2 above; the new fiber product is rigid of type (3) in the notation of Proposition 3.1. Reduction modulo 359 switches from the second degeneration type to the first, so 359 is a prime of good reduction for the new fiber product. Hence the corresponding Galois representations are unramified at 359. By [18], the computed traces thus suffice to determine the associated cusp form.

3. Fiber products

In this section we employ the notation from [14]. Let $r\colon Y\to\mathbb{P}^1$ and $r'\colon Y'\to\mathbb{P}^1$ be two rational elliptic surfaces with sections. Denote by S and S' the images of the singular fibers of Y and Y' in $\mathbb{P}^1$, and let $S''=S\cap S'$. Assume that the singular fibers of Y and Y' are reduced (Kodaira types I_n, II, III and IV). Then the arguments from [14] for the semistable case generalize directly to this set-up to show that the fiber product

$$X = Y\times_{\mathbb{P}^1} Y'$$

is simply connected and has trivial dualizing sheaf. Hence any crepant resolution of X is a Calabi–Yau threefold. Often one tries to work with small resolutions. For this we fix the following assumptions.

3.1. Assumption

For any $b\in\mathbb{P}^1$, either both fibers Y_b and Y_b' are semistable or they have the same type.

Under this assumption, we shall show that the fiber product X has a smooth model $\widetilde{X}$ which is a Calabi–Yau threefold (as a manifold or as an algebraic space). Recall that the singularities of X correspond to points (x,x'), where x, resp. x', is a singular point of the fiber Y_b, resp. Y_b', at $b=r(x)=r'(x')$. We shall write down a crepant resolution separately for every type of singular fiber. The following arguments hold true in any characteristic different from 2 and 3. In the exceptional characteristics, one might have to pay special attention, for instance because of the presence of wild ramification on the elliptic surfaces (cf. [19]), but we will not need this here.

3.2. Fiber type $I_n\times I_m$

The fiber product has nm nodes on the fiber $Y_b\times Y_b'$. They admit a small crepant resolution. Since the Euler characteristic of a fiber of type I_n is n, we get $e(\widetilde{X}_b)=2nm$. The number of irreducible components of $\widetilde{X}_b$ equals nm.

3.3. Fiber type $II\times II$

The fiber X_b has one singular point. The small resolution from (1) replaces the D_4 singularity with two intersecting lines, and so $e(\widetilde{X}_b)=2\cdot2+1+1=6$. The fiber $\widetilde{X}_b$ is irreducible. (Analytically, this small resolution behaves like resolving $I_1\times IV$.)

3.4. Fiber type III × III

The fiber X_b has one singular point. The local (analytic) equation of X near the singularity is

$$x^2 + y^4 - z^2 - t^4 = 0.$$

We blow-up first the surface $x - z = y - t = 0$. The singular point is replaced by a line containing two nodes on it. Small resolutions replace the nodes with two further lines. Hence $e(\widetilde{X}_b) = 3 \cdot 3 + 1 + 1 + 1 = 12$. The fiber $\widetilde{X}_b$ has four irreducible components.

3.5. Fiber type IV × IV

The fiber X_b has one singular point, which is an ordinary triple point. This time we perform a big resolution. Blowing-up a threefold triple point yields a crepant resolution with the singular point replaced by a cubic in $\mathbb{P}^3$. Hence $e(\widetilde{X}_b) = 4 \cdot 4 + 8 = 24$. The fiber $\widetilde{X}_b$ has 10 irreducible components.

3.6. Invariants of fiber products

We have seen that under the above assumption, the fiber product X admits a crepant resolution $\widetilde{X}$. We shall now compute some invariants of the complex Calabi–Yau threefold $\widetilde{X}$. Our arguments follow the line of [14]. We are particularly interested in the dimension of the space of infinitesimal deformations. By Serre duality, this dimension equals $h^{1,2}(\widetilde{X})$. Thus $\widetilde{X}$ is rigid if and only if $h^{1,2}(\widetilde{X}) = 0$.

The explicit crepant resolutions of the singularities of X allow us to compute the Euler characteristic of the complex Calabi–Yau threefold $\widetilde{X}$ explicitly as the sum of Euler characteristics of singular fibers. Denote by n_2 (resp. n_3 and n_4) the number of singular fibers of type II (resp. III and IV). For $b \in \mathbb{P}^1$, let $t(b)$ (resp. $t'(b)$) denote the number of components of the fiber Y_b (resp. Y'_b). Then

$$e(\widetilde{X}) = 2\left(\sum_{b \in \mathbb{P}^1} t(b)t'(b) - n_2 - 3n_3 - 4n_4\right) = 2\left(\sum_{b \in S''} t(b)t'(b) + 2n_2 + 2n_3 + 3n_4\right).$$

We shall compare this with another computation of the Euler characteristic of $\widetilde{X}$ as the alternating sum of Betti numbers. By the exponential sequence, Calabi–Yau threefolds have $b_2 = h^{1,1} = \rho$, where ρ denotes the Picard number. In the present situation an analogue of the Shioda–Tate formula for elliptic surfaces with section [20, Corollary 5.3] shows that

$$\rho(\widetilde{X}) = d + 3 + \mathrm{rk}(\mathrm{MW}(Y)) + \mathrm{rk}(\mathrm{MW}(Y')) + \sum_{B \in S \cup S'} (\#(\text{components of } \widetilde{X}_b) - 1).$$

Here $d=1$ when Y and Y' are isogenous (i.e. their generic fibers are isogenous) and $d=0$ otherwise. Under the fixed assumptions on the singular fibers, the Shioda–Tate formula learns us that the rational elliptic surfaces Y and Y' have

$$\mathrm{rk}(\mathrm{MW}(Y)) = \#S - 4 + n_2 + n_3 + n_4 \quad \text{and} \quad \mathrm{rk}(\mathrm{MW}(Y')) = \#S' - 4 + n_2 + n_3 + n_4.$$

This gives $\rho(\widetilde{X})$. On the other hand, $b_3(\widetilde{X}) = 2(h^{0,3}(\widetilde{X}) + h^{1,2}(\widetilde{X}))$ by complex conjugation and $h^{0,3}(\widetilde{X}) = 1$ by Serre duality. Thus we deduce

$$h^{1,2}(\widetilde{X}) = \rho(\widetilde{X}) - \tfrac{1}{2}e(\widetilde{X}).$$

Since the number of components of the fiber $\widetilde{X}_b$ equals $t(b)t'(b)$ unless both fibers have type IV, where we have to add one for the exceptional divisor, the latest relation can be expanded as

$$(4) \qquad h^{1,2}(\widetilde{X}) = d + \#(S \cup S') - 5 + \sum_{b \in S \setminus S''} (t(b)-1) + \sum_{b \in S' \setminus S''} (t'(b)-1).$$

The above formula for the Hodge number $h^{1,2}(\widetilde{X})$ (under Assumption 3.1) is exactly the same as in the semistable case. A case-by-case analysis as in [14] reveals when $\widetilde{X}$ admits no infinitesimal deformations.

Proposition 3.1. *The complex Calabi–Yau threefold $\widetilde{X}$ is rigid if and only if it contains no fibers of type $I_0 \times I_n$ or $I_n \times I_0$ with $n > 1$ and one of the following cases holds*:
  (1) $S = S'$ and $\#S = 4$ (*then Y and Y' are isogenous*);
  (2) $\#S = \#S' = 4$ and $\#S'' = 3$ (*then Y and Y' are not isogenous*);
  (3) $\#S = 5$ and $\#S' = 4$, $S' \subset S$ (*then Y and Y' are not isogenous*);
  (4) $\#S = 5$ and $S = S' = S''$, *but Y and Y' are not isogenous.*

In the sequel, we shall refer to (rigid) fiber products of types (2) and (3) according to the cases of the proposition. Note that the isogeny conclusion of the first case fails to be valid in positive characteristic. In fact, this failure causes non-liftability as we shall explore in the next sections.

4. Rational elliptic surfaces with one fiber of type II and three semistable fibers

In the following sections we shall construct non-liftable Calabi–Yau spaces in positive characteristic. We achieve this by considering fiber products of rational elliptic surfaces which have combinations of singular fibers which do not occur in characteristic zero. Our main tool to construct these Calabi–Yau spaces are rational elliptic surfaces with section and four singular fibers. Over $\mathbb{C}$ these surfaces

have been classified by Beauville [3] (the semistable ones, equivalently with finite Mordell–Weil group) and Herfurtner [6]. In positive characteristic, rational elliptic surfaces with finite Mordell–Weil group have been classified by Lang [12], [13].

In [4], non-liftable fiber products arising from semistable rational elliptic surfaces were studied. Here we shall consider surfaces with one fiber of type II and three semistable singular fibers. There are four such surfaces. Möbius transformations (or twists) send the fiber of type II to ∞ and two other singular fibers to 0 and 1. Then the remaining cusp runs through the set

$$\left\{\lambda, 1-\lambda, \frac{1}{\lambda}, 1-\frac{1}{\lambda}, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}\right\}$$

and the precise value determines the twist for all present surfaces. Equations can be obtained from the families in Section 5 by specialization. The following table lists the surfaces by the short-hand $[n_1, n_2, n_3, \text{II}]$ for the configuration of singular fibers and by places $0, 1, \lambda$ of the semistable singular fibers in the given order

$[1, 1, 8, \text{II}]$	α	1	0
$[1, 2, 7, \text{II}]$	$\frac{32}{81}$	1	0
$[1, 4, 5, \text{II}]$	$\frac{1}{81}$	1	0
$[2, 3, 5, \text{II}]$	$\frac{27}{32}$	1	0

where $\alpha = \dfrac{17 + 56\sqrt{-2}}{81}$.

Let Y and Y' be two different twists of elliptic surfaces from the above list, determined by their loci of singular fibers

$$S = \{0, 1, \infty, \lambda\} \quad \text{and} \quad S' = \{0, 1, \infty, \lambda'\}.$$

Since $\lambda \neq \lambda'$, the elliptic surfaces Y and Y' are not isogenous. By Section 3, the fiber product $X := Y \times_{\mathbb{P}^1} Y'$ has a small resolution $\widetilde{X}$ which is a (non–projective) Calabi–Yau manifold. From (4) we get

$$h^{1,2}(\widetilde{X}) = t(\lambda) + t'(\lambda') - 2,$$

so $\widetilde{X}$ is rigid if and only if both fibers S_λ and $S'_{\lambda'}$ have Kodaira type I_1. When $\widetilde{X}$ is defined over $\mathbb{Q}$, it is modular even if it is non-rigid, since the compatible system of Galois of $H^3(\widetilde{X})$ (for ℓ-adic cohomology in the category of algebraic spaces, or for a partial big projective resolution) can be split into two-dimensional subrepresentations by [8].

The primes of bad reduction of $\widetilde{X}$ (or a big resolution of X) are primes at which either one of the factors degenerates or the fiber product changes the configuration of singular fibers. The latter corresponds to the merging of some elements of the set $\{0, 1, \infty, \lambda, \lambda'\}$ upon reduction. These primes can be identified as the divisors of the numerator and denominator of $(\lambda - \lambda')\lambda\lambda'(\lambda - 1)(\lambda' - 1)$.

λ	λ'	Bad primes		$h^{1,2}$	λ	λ'	Bad primes		$h^{1,2}$
		Nodal	Degenerate				Nodal	Degenerate	
$\frac{5}{32}$	$\frac{32}{5}$	37	$5,3,2$	5	$\frac{1}{81}$	$\frac{32}{81}$	31	$7,5,3,2$	0
$\frac{5}{32}$	$-\frac{27}{5}$	$127,7$	$5,3,2$	5	$\frac{1}{81}$	$-\frac{32}{49}$	$139,19$	$7,5,3,2$	6
$\frac{5}{32}$	$-\frac{5}{27}$	59	$5,3,2$	3	$\frac{1}{81}$	$-\frac{49}{32}$	4001	$7,5,3,2$	1
$\frac{5}{32}$	$\frac{80}{81}$	431	$5,3,2$	1	$\frac{32}{81}$	$\frac{81}{32}$	113	$7,3,2$	1
$\frac{5}{32}$	81	$199,13$	$5,3,2$	4	$\frac{32}{81}$	$\frac{81}{49}$	4993	$7,3,2$	6
$\frac{5}{32}$	$\frac{81}{80}$	137	$5,3,2$	5	$\frac{32}{81}$	$\frac{49}{81}$	17	$7,3,2$	0
$\frac{5}{32}$	$\frac{1}{81}$	373	$5,3,2$	1	$\frac{32}{81}$	$-\frac{32}{49}$	$13,5$	$7,3,2$	6
$\frac{5}{32}$	$\frac{49}{81}$	1163	$7,5,3,2$	1	α	$\frac{32}{5}$	$1187,67$	$5,3,2$	4
$\frac{5}{32}$	$\frac{32}{81}$	619	$7,5,3,2$	1	α	$\frac{32}{27}$	$97,43$	$5,3,2$	2
$\frac{5}{32}$	$-\frac{32}{49}$	47	$7,5,3,2$	7	α	$\frac{1}{81}$	17	$5,3,2$	0
$\frac{5}{32}$	$\frac{81}{49}$	2347	$7,5,3,2$	7	α	-80	521201	$5,3,2$	3
$\frac{1}{81}$	81	41	$5,3,2$	3	α	$\frac{80}{81}$	$19,11,7$	$5,3,2$	0
$\frac{1}{81}$	$-\frac{1}{80}$	$23,7$	$5,3,2$	4	α	$\frac{32}{81}$	$89,73$	$7,5,3,2$	0
$\frac{1}{81}$	-80	6481	$5,3,2$	3	α	$-\frac{49}{32}$	$281,107,11$	$7,5,3,2$	1
$\frac{1}{81}$	$\frac{80}{81}$	79	$5,3,2$	0	α	$1-\alpha$	337	$3,2$	0
$\frac{1}{81}$	$\frac{81}{32}$	6529	$7,5,3,2$	1					

Table 1. Fiber products specified by λ and λ' with bad primes and $h^{1,2}$.

We are interested in the situation when the reduction X_p of $\widetilde{X}$ modulo p is a nodal Calabi–Yau space. In our setting, this happens exactly when $\lambda=\lambda'\notin\{0,1,\infty\}$ modulo the given prime. Equivalently the prime number p is a divisor of the numerator of $\lambda-\lambda'$, but it is not a divisor of the numerator or denominator of $\lambda\lambda'(\lambda-1)(\lambda'-1)$. In the opposite situation, when the prime p divides the numerator or the denominator of $\lambda\lambda'(\lambda-1)(\lambda'-1)$, we get certain kinds of degenerate reduction.

Table 1 collects the bad primes for various fiber products of twists of elliptic surfaces from the previous list. In the table "nodal bad prime" refers to a prime divisor of the numerator of $\lambda-\lambda'$, whereas "degenerate bad prime" means a prime dividing the numerator or denominator of $\lambda\lambda'(\lambda-1)(\lambda'-1)$. The last column gives the Hodge number $h^{1,2}(\widetilde{X})$ from (4). Note that the same primes can be obtained for several different pairs λ,λ'. In order to keep the length of the table reasonable, we list only the "simplest" example for any given prime.

Theorem 4.1. *Let X be a fiber product as specified in Table 1. Let p denote a nodal prime. Then any small resolution of X over $\overline{\mathbb{F}}_p$ is a non-liftable Calabi–Yau space.*

We conclude this section with a partial proof of Theorem 4.1. Let $\widetilde{X}$ be a rigid fiber product from Table 1 ($h^{1,2}(\widetilde{X})=0$). Let p be a nodal prime. By Theorem 2.1, the small resolution of the reduction $\widetilde{X}_p$ is a Calabi–Yau space in characteristic p that does not lift to characteristic 0. Consequently we get non-liftable Calabi–Yau spaces in characteristics 7, 11, 17, 19, 31, 73, 79, 89 and 337.

For the non-rigid fiber products from Table 1, we shall develop an alternative approach in the next section that will allow us to apply Theorem 2.1 and deduce non-liftability at the nodal primes.

5. Rigid fiber products

For some of the fiber products from Table 1, we have been able to apply Theorem 2.1 to deduce non-liftability at the nodal primes directly, namely in the cases where the corresponding complex Calabi–Yau threefold is already rigid ($h^{1,2}=0$). In this section, we prove non-liftability at the nodal primes for all other fiber products from Table 1. The general idea is to lift a partial small resolution to a rigid smooth Calabi–Yau space over some number field. Equivalently, we exhibit a rigid Calabi–Yau threefold over some number field that reduces modulo a suitable prime to the given fiber product with an additional node. Here we will work with fiber products of type (3) in terms of Proposition 3.1. To achieve this, we shall keep one of the elliptic surfaces of the original fiber product from Table 1 unchanged, whereas we replace the second one with an elliptic surface with five singular fibers, four semistable and one of type II. We shall first introduce the rational elliptic surfaces that we will use for this construction. Then we will explain how to complete the proof of Theorem 4.1.

5.1. Rational elliptic surfaces with one fiber of type II and four semistable singular fibers

We shall study rational elliptic surfaces with five singular fibers under the assumption that one fiber has type II and all others are semistable. These surfaces come in seven families, each depending on one parameter. Here we shall only introduce four families which suffice for our purposes. Each family will be given in terms of an extended Weierstrass form over $\mathbb{Q}$ with parameter m. We also list the parameter choices where the family degenerates to one of the surfaces from the previous section or other interesting surfaces (see Section 6). By way of reduction, the equations stay valid outside characteristics 2 and 3. Those characteristics play a special role since fibers of type II do necessarily come with wild ramification by [19]. Hence modulo 2 and 3 there has to occur some degeneration of singular fibers.

As exploited in [19], expanded Weierstrass forms are very useful when one wants to find elliptic surfaces with specific singular fibers. In the present situation, the equations can be found without much difficulty, so we decided to omit the details.

The given families have Mordell–Weil rank two. This follows from the Shioda–Tate formula [20, Corollary 5.3] since the Picard number is ten while the contribution from singular fibers and the zero section amounts to 8. From the specific Weierstrass model, it is easy to read off a section of the elliptic fibration by setting $x=0$. This section has infinite order since fibers of type II do not accommodate torsion sections of order relatively prime to the characteristic. Unless otherwise noted, all listed degenerate members of the family share the property that the Mordell–Weil rank drops to one due to the fiber degeneration. The given section specialises to a generator of the Mordell–Weil group up to finite index.

5.2. $[1, 1, 1, 7, \mathrm{II}]$

Extended Weierstrass form

$$y^2 = x^3 + (t^2 + mt + 3)x^2 + (2t + 3)x + 1.$$

I_1	I_1	I_1	I_7	II
$4mt^3 - 12t^3 + 8m^2t^2 - 24mt^2 + 3t^2 + 4m^3t - 12m^2t$ $+ 30mt - 76t + 27m^2 - 108m + 108 = 0$			∞	0

m	3	-6	2
Degeneration	$[1, 1, 8, \mathrm{II}]$	$[1, 2, 7, \mathrm{II}]$	$[1, 1, 7, \mathrm{III}]$

5.3. $[1, 1, 2, 6, \mathrm{II}]$

Extended Weierstrass form

$$y^2 = x^3 + (t^2 + 3t + 3m)x^2 + ((3m - 1)t + 3m^2)x + m^3.$$

I_1	I_1	I_2	I_6	II
$4mt^2 - t^2 + 18mt - 4t + 27m^2 = 0$		$\dfrac{1}{m-1}$	∞	0

m	1	$\frac{1}{4}$	0
Degeneration	$[1, 1, 8, \mathrm{II}]$	$[1, 2, 7, \mathrm{II}]$	$[1, 2, 6, \mathrm{III}]$

5.4. $[1, 1, 3, 5, \mathrm{II}]$

Extended Weierstrass form

$$y^2 + (mt - 27m + 18)xy + 16ty = x^3 + tx^2.$$

I_1	I_1	I_3	I_5	II
	$m^4t^2 - 2m^2(27m^2 - 28m - 4)t + (27m - 2)(3m - 2)^3 = 0$	0	∞	27

m	$\frac{2}{27}$	1	$\frac{1}{9}$
Degeneration	$[1, 4, 5, \mathrm{II}]$	$[2, 3, 5, \mathrm{II}]$	$[1, 3, 5, \mathrm{III}]$

5.5. $[1, 2, 3, 4, \mathrm{II}]$

Extended Weierstrass form

$$y^2 + 4(9t + m - 1)xy + 27t(t-1)(8t+m)y = x^3 - 3(72t^2 + 9(2m-1)t + (m-1)^2)x^2.$$

	I_1	I_2	I_3	I_4	II
	$-\dfrac{4(m-1)^3}{81(m+2)^2}$	$-\dfrac{m}{8}$	0	1	∞

m	0	$-\frac{5}{4}$	-2	1	$-\frac{2}{7}$
Degeneration	$[1, 4, 5, \mathrm{II}]$	$[2, 3, 5, \mathrm{II}]$	$[2, 3, 4, \mathrm{III}]$	$[2, 4, \mathrm{II}, \mathrm{IV}]$	$[3, 4, \mathrm{II}, \mathrm{III}]$

Note that the last two specializations do not cause a drop of the Mordell–Weil rank.

5.6. Illustration of the method

We shall construct fiber products of type (3) in the notation of Proposition 3.1 from

• Y: the family of rational elliptic surfaces of type $[1, 2, 3, 4, \mathrm{II}]$,

• Y': the elliptic surfaces with four singular fibers from the previous section.

In our usual notation, we have

$$S = \left\{ 0, 1, \infty, -\frac{m}{8}, -\frac{4(m-1)^3}{81(m+2)^2} \right\} \quad \text{and} \quad S' := \{0, 1, \infty, \lambda\}.$$

After a further Möbius transformation for the second elliptic surface, we can assume that the fiber of type II sits at ∞ for both elliptic surfaces Y and Y' (as in the previous section).

We now choose $m=-8\lambda$. Then the fiber product $X:=Y\times_{\mathbb{P}^1}Y'$ has a small resolution $\widetilde{X}$ which is a non-projective rigid Calabi–Yau manifold by Proposition 3.1. In order to find the primes of bad reductions we have to find the primes at which the construction degenerates. In the ring of integers of the field of definition of the algebraic space $\widetilde{X}$, the candidates for bad reduction are exactly the prime divisors of the numerators and denominators of the numbers

$$-\frac{m}{8}, \quad -\frac{m}{8}-1, \quad -\frac{4(m-1)^3}{81(m+2)^2}, \quad -\frac{4(m-1)^3}{81(m+2)^2}-1 \quad \text{and} \quad -\frac{4(m-1)^3}{81(m+2)^2}+\frac{m}{8}.$$

For the reduction we distinguish four cases:

(1) If $-m/8$ equals $0,1$ or ∞ modulo a given prime $\mathfrak{p}$, then Y' reduces to an elliptic surface with only three singular fibers. In this case either there is a non-reduced singular fiber not allowing a small resolution (cf. Assumption 3.1) or the characteristic is at most 7 (with configurations such as $[7,\mathrm{II},\mathrm{III}]$ in characteristic 7, or $[5,5,\mathrm{II}]$ and $[5,\mathrm{III},\mathrm{IV}]$ in characteristic 5). In particular, we always get a bad prime, but we cannot get a non-liftable Calabi–Yau space in characteristic greater then 7.

(2) If $-4(m-1)^3/81(m+2)^2=1$, i.e. $m=\frac{5}{4}$, the surface Y degenerates to $[2,3,5,\mathrm{II}]$. This imposes an additional node on the fiber product. By Theorem 2.1, reduction gives a Calabi–Yau space in positive characteristic p that does not lift to characteristic zero (one of the examples predicted in Table 1).

(3) If $-4(m-1)^3/81(m+2)^2=\infty$, i.e. $m=-2$, then the elliptic surface Y degenerates to $[2,3,4,\mathrm{III}]$. Since a fiber of type $\mathrm{II}\times\mathrm{III}$ has no small resolution by [9], $\mathfrak{p}$ is a bad prime, but we cannot apply Theorem 2.1.

(4) If $-4(m-1)^3/81(m+2)^2=-m/8$ (resp. $-4(m-1)^3/81(m+2)^2=0$), the surface Y degenerates to $[\mathrm{III},3,4,\mathrm{II}]$ (resp. $[2,\mathrm{IV},4,\mathrm{II}]$). By Section 2.1, the fiber of type $I_n\times\mathrm{III}$ (resp. $I_n\times\mathrm{IV}$) admits a small resolution in a way that is compatible with the small resolution of the original fiber of type $I_n\times I_2$ (resp. $I_n\times I_3$). In this case, $\mathfrak{p}$ is a prime of good reduction.

Table 2 lists choices of λ and m such that the fiber product of Y and Y' is rigid (type (3)). We then give nodal and degenerate primes. By our previous considerations, we obtain non-projective rigid Calabi–Yau spaces with all bad primes from both columns and non-liftable Calabi–Yau spaces in all nodal characteristics.

Remark 5.1. Although the singular Calabi–Yau threefolds from Table 2 are defined over $\mathbb{Q}$, as well as the big resolution (non-crepant), it is a delicate question if the small resolution can be defined over $\mathbb{Q}$. For modularity questions, however, it suffices to consider the (partial) big resolution, a smooth projective variety with $b_3=2$. Modularity then follows from Serre's conjecture [11]. We found the big bad primes of the threefolds remarkable. Necessarily they appear in the level of the

λ	m	Bad primes	
		Nodal	Degenerate
$-\frac{27}{5}$	$\frac{216}{5}$	$127,7$	$113,5,3,2$
$\frac{32}{5}$	$-\frac{256}{5}$	37	$41,5,3,2$
$-\frac{5}{27}$	$\frac{40}{27}$	59	$47,5,3,2$
$\frac{32}{27}$	$-\frac{256}{27}$	$127,7$	$101,5,3,2$
$\frac{1}{81}$	$-\frac{8}{81}$	373	$11,7,5,3,2$
81	-648	$199,13$	$19,17,5,3,2$
$\frac{81}{80}$	$-\frac{81}{10}$	137	$61,5,3,2$

λ	m	Bad primes	
		Nodal	Degenerate
$\frac{80}{81}$	$-\frac{640}{81}$	431	$239,5,3,2$
$\frac{32}{81}$	$-\frac{256}{81}$	619	$47,7,3,2$
$\frac{49}{81}$	$-\frac{392}{81}$	1163	$23,7,5,3,2$
$\frac{81}{32}$	$-\frac{81}{4}$	19	$73,7,3,2$
$\frac{81}{49}$	$-\frac{648}{49}$	2347	$11,7,5,3,2$
$-\frac{32}{49}$	$\frac{256}{49}$	47	$59,7,3,2$

Table 2. Rigid fiber products involving $[1,2,3,4,\mathrm{II}]$.

associated modular form. In comparison, it seems that for all examples so far the biggest bad prime (up to twists) was 73 (cf. [17]).

5.7. Proof of Theorem 4.1

We shall now show how to set up other rigid fiber products in order to prove the non-liftability of all nodal examples from the first table and thus complete the proof of Theorem 4.1.

Let Y and Y' denote the rational elliptic surfaces giving a fiber product X from Table 1 specified by λ and λ'. Assume that $h^{1,2}(\widetilde{X})>0$ and that p denotes a nodal prime. Then reduction modulo p imposes additional nodes on $\widetilde{X}_p$, but Theorem 2.1 does not apply since the original Calabi–Yau threefold $\widetilde{X}$ is not rigid. To prove non-liftability, we shall extend the construction from Section 5.6.

Proposition 5.2. *Let $\widetilde{X}$ be as above. Then there is a rigid fiber product of type* (3) *over some number field which reduces to $\widetilde{X}_p$ modulo some prime $\mathfrak{p}$ above p.*

As a corollary of the proposition, we deduce from Theorem 2.1 that the reduction of $\widetilde{X}$ modulo $\mathfrak{p}$ has a non-liftable small resolution. This completes the proof of Theorem 4.1.

The main idea in the proof of Proposition 5.2 is to replace Y or Y' in the fiber product X by a suitable member of one of the families from Sections 5.2–5.5, as we did in Section 5.6. First we need the following observation.

Lemma 5.3. *Each rational elliptic surface from Section* 4 *arises from one of the families in Sections* 5.2–5.5 *by degeneration where an I_1 fiber is merged with another fiber.*

Proof of Proposition 5.2. Since we only consider fiber products with $h^{1,2}(\widetilde{X})>0$, an inspection of Table 1 reveals that we can assume that Y does not have configuration $[1,1,8,\mathrm{II}]$ after exchanging Y and Y' if necessary. This will be convenient because then all singular fibers of Y sit above rational points of $\mathbb{P}^1$, i.e. $\lambda \in \mathbb{Q}$.

Let $\mathcal{Y}(m)$ be a family degenerating to Y' as in Lemma 5.3, say $Y'=\mathcal{Y}(m_0)$. By choosing an appropriate m, we want to match the singular fibers of Y and $\mathcal{Y}(m)$ in such a way that their fiber product has type (3) and the non-degenerate singular fibers for the specialization at m_0 agree with those of $\widetilde{X}$. For the $[1,2,3,4,\mathrm{II}]$ family, this was easily achieved in Section 5.6 since all cusps are rational. For the other families, we have to be a little more careful.

Let $\mathcal{Y}(m)$ be one of the families from Sections 5.3 and 5.4. A Möbius transformation sends the reducible fibers to 0 and 1 and the fiber of type II to ∞ (accordingly for Y and Y'). This locates the two fibers of type I_1 at the zeroes of the quadratic polynomial $\Delta(t,m)$ with coefficients in $\mathbb{Q}(m)$. For $Y \times_{\mathbb{P}^1} \mathcal{Y}(m)$ to have type (3), we require one of these zeroes to be λ. In other words, m has to solve the equation

$$(5) \qquad \Delta(\lambda, m)=0.$$

By reduction, we can also consider (5) as an equation over $\mathbb{F}_p$. Since $\lambda \equiv \lambda'$ modulo p, the reduced equation has the solution m_0 modulo p. This solution lifts to a solution m_1 of (5) over some number field. Then, by construction, the fiber product $Y \times_{\mathbb{P}^1} \mathcal{Y}(m_1)$ has type (3) and reduces to X modulo some prime above p.

Finally we consider the case where $\mathcal{Y}(m)$ has type $[1,1,1,7,\mathrm{II}]$. As before we can take the singular fibers of types I_7 and II to 0 and ∞ by a Möbius transformation. Then the three I_1 fibers of $\mathcal{Y}(m)$ are located at the zeroes of the cubic polynomial $\Delta(t,m)$ with coefficients in $\mathbb{Q}(m)$. In order to set up a fiber product with Y of type (3), we need that two of these zeroes are multiples of λ. This condition is encoded in the product of irreducible factors of the resultant $\mathrm{Res}(\Delta(t,m),\Delta(\lambda t,m))$ with respect to t which do not divide $\Delta(0,m)$. Equivalently, for any sufficiently large N,

$$F(\lambda, m) = \frac{\mathrm{Res}(\Delta(t,m),\Delta(\lambda t,m))}{\gcd(\mathrm{Res}(\Delta(t,m),\Delta(\lambda t,m)),\Delta(0,m)^N)}$$

We continue as before by considering this polynomial equation over $\mathbb{F}_p$ and lifting the solution m_0 modulo p to some number field. The resulting fiber product has the required properties. $\square$

Remark 5.4. The above degeneration technique was not required in [4] for the following reason: For a semistable rational elliptic surface with finite Mordell–Weil group, one can construct an isogenous surface with an I_1 fiber at any given position.

λ	λ'	Bad primes	
		Nodal	Degenerate
$\frac{1}{4}$	$\frac{3}{128}$	29	$5, 3, 2$
$\frac{3}{4}$	$\frac{3}{128}$	31	$5, 3, 2$
$\frac{3}{128}$	$\frac{125}{128}$	61	$5, 3, 2$

λ	λ'	Bad primes	
		Nodal	Degenerate
ω	$\frac{1}{4}$	631	$7, 5, 3, 2$
ω	$\frac{3}{4}$	1061	$7, 5, 3, 2$
ω	$\frac{3}{128}$	757, 11	$7, 3, 2$
ω	$\frac{125}{128}$	151, 71	$7, 3, 2$

Table 3. Rigid fiber products with III×III specified by λ and λ'.

Hence one can replace fiber products of such rational elliptic surfaces of type (2) in the notation of Proposition 3.1 by an isogenous rigid fiber product.

6. Proof of Theorem 1.1

Theorem 4.1 implies the second statement of Theorem 1.1 about Table 1. For the first statement, only the primes 2, 3, 29, 61 and 71 are missing from the list of nodal primes in Table 1. As explained before, it is known that there are non-liftable projective Calabi–Yau threefolds in characteristics 2 and 3 (cf. [7] and [16]). It remains to prove the corresponding fact for Calabi–Yau spaces in characteristics 29, 61 and 71.

To achieve this, we pursue the same approach as in Section 4, but this time for rational elliptic surfaces with one singular fiber of type III. We shall use the specializations of the families in Sections 5.2–5.4 to construct fiber products of type (2). With cusps normalized as before (III at ∞) we find the following surfaces,

$[1, 1, 7, \text{III}]$	ω	1	0
$[1, 2, 6, \text{III}]$	$\frac{1}{4}$	1	0
$[1, 3, 5, \text{III}]$	$\frac{3}{128}$	1	0

, where $\omega = \left(\dfrac{1 - \sqrt{-7}}{1 + \sqrt{-7}} \right)^{7}$.

From the twists we obtain several interesting fiber products of type (2). As before they are specified by λ and λ' and yield the nodal primes given by Table 3.

By Theorem 2.1 the nodal primes give non-liftable fiber products. This completes the proof of Theorem 1.1.

We conclude this paper by noting that our techniques can be applied to other rational elliptic surfaces with four singular fibers. For instance, one could consider the remaining twisted fiber products of the above surfaces. This would give further nodal primes 131, 137, 503, 509, 1019, 2287 and 2671. However, our approach does not result in any other small primes (<100) or big primes (>7000). Hence we decided not to include these computations in this paper.

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