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Transfinite diameter notions in $\mathbb{C}^N$ and integrals of Vandermonde determinants

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Abstract. We provide a general framework and indicate relations between the notions of transfinite diameter, homogeneous transfinite diameter, and weighted transfinite diameter for sets in $\mathbb{C}^N$. An ingredient is a formula of Rumely (A Robin formula for the Fekete–Leja transfinite diameter, *Math. Ann.* **337** (2007), 729–738) which relates the Robin function and the transfinite diameter of a compact set. We also prove limiting formulas for integrals of generalized Vandermonde determinants with varying weights for a general class of compact sets and measures in $\mathbb{C}^N$. Our results extend to certain weights and measures defined on cones in $\mathbb{R}^N$.

1. Introduction

Given a compact set E in the complex plane $\mathbb{C}$, the *transfinite diameter* of E is the number

$$d(E):=\lim_{n\to\infty}\max_{\zeta_1,...,\zeta_n\in E}|\mathrm{VDM}(\zeta_1,...,\zeta_n)|^{1/\binom{n}{2}}:=\lim_{n\to\infty}\max_{\zeta_1,...,\zeta_n\in E}\prod_{i<j}|\zeta_i-\zeta_j|^{1/\binom{n}{2}}.$$

It is well known that this quantity is equal to the *Chebyshev constant* of E:

$$T(E):=\lim_{n\to\infty}\left[\inf\left\{\|p_n\|_E:p_n(z)=z^n+\sum_{j=0}^{n-1}c_jz^j\right\}\right]^{1/n}$$

(here, $\|p_n\|_E:=\sup_{z\in E}|p(z)|$) and also to $e^{-\rho(E)}$, where

$$\rho(E):=\lim_{|z|\to\infty}[g_E(z)-\log|z|]$$

is the *Robin constant* of E. The function g_E is the *Green function* of logarithmic growth associated with E. Moreover, if w is an admissible weight function on E,

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weighted versions of the above quantities can be defined. We refer the reader to the book of Saff and Totik [20] for the definitions and relationships.

For $E \subset \mathbb{C}^N$ with $N > 1$, multivariate notions of transfinite diameter, Chebyshev constant and Robin-type constants have been introduced and studied by several people. For an introduction to weighted versions of some of these quantities, see Appendix B by Bloom in [20]. In the first part of this paper (Section 2), we discuss a general framework for the various types of transfinite diameters in the spirit of Zaharjuta [22]. In particular, we relate (Theorem 2.9) two weighted transfinite diameters, $d^w(E)$ and $\delta^w(E)$, of a compact set $E \subset \mathbb{C}^N$ using a remarkable result of Rumely [19] which itself relates the (unweighted) transfinite diameter $d(E)$ with a Robin-like integral formula. Theorem 2.9 generalizes a well-known result in one dimension (cf. Theorem 3.1 in Section III.3 of [20]). Recently Robert Berman and Sébastien Boucksom ([2], [3] and [1]) have given a purely analytic proof of the $\mathbb{C}^N$ Rumely formula.

In the second part of the paper (Section 3) we generalize to $\mathbb{C}^N$ some results on strong asymptotics of Christoffel functions proved in [12] in one variable. For a compact subset E of $\mathbb{C}$, an admissible weight function w on E, and a positive Borel measure μ on E such that the triple (E, w, μ) satisfies a weighted Bernstein–Markov inequality (see (3.5)), we take, for each $n = 1, 2, \ldots$, a set of orthonormal polynomials $q_1^{(n)}, \ldots, q_n^{(n)}$ with respect to the varying measures $w(z)^{2n}\, d\mu(z)$, where $\deg q_j^{(n)} = j - 1$, and form the sequence of Christoffel functions $K_n(z) := \sum_{j=1}^n |q_j^{(n)}(z)|^2$. In [12] we showed that

$$\frac{1}{n} K_n(z) w(z)^{2n}\, d\mu(z) \to d\mu_{\text{eq}}^w(z) \tag{1.1}$$

weak-*, where μ_{eq}^w is the potential-theoretic weighted equilibrium measure. The key ingredients to proving (1.1) are, firstly, the verification that

$$\lim_{n\to\infty} Z_n^{1/n^2} = \delta^w(E), \tag{1.2}$$

where

(1.3)

$$Z_n = Z_n(E, w, \mu) := \int_{E^n} |\text{VDM}(\lambda_1, \ldots, \lambda_n)|^2 w(\lambda_1)^{2n} \ldots w(\lambda_n)^{2n}\, d\mu(\lambda_1) \ldots d\mu(\lambda_n);$$

and, secondly, a "large deviation" result in the spirit of Johansson [16]. We generalize these two results to $\mathbb{C}^N$, $N > 1$ (Theorems 3.1 and 3.2). The methods are similar to the corresponding one-variable methods and were announced in [12], Remark 3.1. In particular, $\delta^w(E)$ is interpreted as the transfinite diameter of a circled set in one dimension higher. We also discuss the case where $E=\Gamma$ is an unbounded cone in $\mathbb{R}^N$ for special weights and measures.

Berman, Boucksom and Nystrom ([1]–[6]) have studied wide-ranging generalizations of notions related to strong asymptotics for compact subsets of $\mathbb{C}^N$. We end the paper with a short section which describes some of their results and the relation to the topics of this paper. We are grateful to the authors for making their references available to us. The second author would also like to thank Sione Ma'u and Laura DeMarco for helpful conversations.

2. Transfinite diameter notions in $\mathbb{C}^N$

We begin by considering a function Y from the set of multiindices $\alpha\in\mathbf{N}^N$ to the nonnegative real numbers satisfying

$$Y(\alpha+\beta)\le Y(\alpha)Y(\beta) \quad \text{for all } \alpha,\beta\in\mathbf{N}^N. \tag{2.1}$$

We call a function Y satisfying (2.1) *submultiplicative*; we have three main examples below. Let $e_1(z),...,e_j(z),...$ be a listing of the monomials $\{e_i(z)=z^{\alpha(i)}=z_1^{\alpha_1}...z_N^{\alpha_N}\}$ in $\mathbb{C}^N$ indexed using a lexicographic ordering on the multiindices $\alpha=\alpha(i)=(\alpha_1,...,\alpha_N)\in\mathbf{N}^N$, but with $\deg e_i=|\alpha(i)|$ nondecreasing. We write $|\alpha|:=\sum_{j=1}^N \alpha_j$.

We define the following integers:

(1) $m_d^{(N)}=m_d:=$ the number of monomials $e_i(z)$ of degree at most d in N variables;

(2) $h_d^{(N)}=h_d:=$ the number of monomials $e_i(z)$ of degree exactly d in N variables;

(3) $l_d^{(N)}=l_d:=$ the sum of the degrees of the m_d monomials $e_i(z)$ of degree at most d in N variables.

We have the following relations:

$$m_d^{(N)}=\binom{N+d}{d} \quad\text{and}\quad h_d^{(N)}=m_d^{(N)}-m_{d-1}^{(N)}=\binom{N-1+d}{d}, \tag{2.2}$$

and

$$h_d^{(N+1)}=\binom{N+d}{d}=m_d^{(N)} \quad\text{and}\quad l_d^{(N)}=N\binom{N+d}{N+1}=\frac{N}{N+1}\,dm_d^{(N)}. \tag{2.3}$$

The elementary fact that the dimension of the space of homogeneous polynomials of degree d in $N+1$ variables equals the dimension of the space of polynomials of degree at most d in N variables will be crucial. Finally, we let

$$r_d^{(N)}=r_d:=dh_d^{(N)}=d(m_d^{(N)}-m_{d-1}^{(N)})$$

which is the sum of the degrees of the h_d monomials $e_i(z)$ of degree exactly d in N variables. We observe that

$$l_d^{(N)} = \sum_{k=1}^{d} r_k^{(N)} = \sum_{k=1}^{N} k h_k^{(N)}. \tag{2.4}$$

Let $K \subset \mathbb{C}^N$ be compact. Here are our three natural constructions associated with K.

(1) *Chebyshev constants*: Define the class of polynomials

$$P_i = P(\alpha(i)) := \left\{ e_i(z) + \sum_{j<i} c_j e_j(z) \right\};$$

and the Chebyshev constant

$$Y_1(\alpha) := \inf\{\|p\|_K : p \in P_i\}.$$

We write $t_{\alpha,K} := t_{\alpha(i),K}$ for a Chebyshev polynomial; i.e., $t_{\alpha,K} \in P(\alpha(i))$ and $\|t_{\alpha,K}\|_K = Y_1(\alpha)$.

(2) *Homogeneous Chebyshev constants*: Define the class of homogeneous polynomials

$$P_i^{(H)} = P^{(H)}(\alpha(i)) := \left\{ e_i(z) + \sum_{\substack{j<i \\ \deg(e_j) = \deg(e_i)}} c_j e_j(z) \right\};$$

and the homogeneous Chebyshev constant

$$Y_2(\alpha) := \inf\{\|p\|_K : p \in P_i^{(H)}\}.$$

We write $t_{\alpha,K}^{(H)} := t_{\alpha(i),K}^{(H)}$ for a homogeneous Chebyshev polynomial; i.e.,

$$t_{\alpha,K}^{(H)} \in P^{(H)}(\alpha(i)) \quad \text{and} \quad \|t_{\alpha,K}^{(H)}\|_K = Y_2(\alpha).$$

(3) *Weighted Chebyshev constants*: Let w be an admissible weight function on K (see below) and let

$$Y_3(\alpha) := \inf\{\|w^{|\alpha(i)|} p\|_K : p \in P_i\}$$

be the weighted Chebyshev constant. Note that we use the polynomial classes P_i as in (1). We write $t_{\alpha,K}^w$ for a weighted Chebyshev polynomial; i.e., $t_{\alpha,K}^w$ is of the form $w^{\alpha(i)} p$ with $p \in P(\alpha(i))$ and $\|t_{\alpha,K}^w\|_K = Y_3(\alpha)$.

Let Σ denote the standard $(N-1)$-simplex in $\mathbb{R}^N$; i.e.,

$$\Sigma=\left\{\theta=(\theta_1,...,\theta_N)\in\mathbb{R}^N:\sum_{j=1}^N\theta_j=1 \text{ and } \theta_j\geq 0,\ j=1,...,N\right\},$$

and let

$$\Sigma^0:=\{\theta\in\Sigma:\theta_j>0,\ j=1,...,N\}.$$

Given a submultiplicative function $Y(\alpha)$, define, as with the above examples, a new function

$$\tau(\alpha):=Y(\alpha)^{1/|\alpha|}. \tag{2.5}$$

An examination of Lemmas 1, 2, 3, 5 and 6 in [22] shows that (2.1) is the only property of the numbers $Y(\alpha)$ needed to establish those lemmas. That is, we have the following results for $Y:\mathbf{N}^N\to\mathbb{R}^+$ satisfying (2.1) and the associated function $\tau(\alpha)$ in (2.5).

Lemma 2.1. *For all $\theta\in\Sigma^0$, the limit*

$$T(Y,\theta):=\lim_{\alpha/|\alpha|\to\theta}Y(\alpha)^{1/|\alpha|}=\lim_{\alpha/|\alpha|\to\theta}\tau(\alpha)$$

exists.

Lemma 2.2. *The function $\theta\mapsto T(Y,\theta)$ is log-convex on Σ^0 (and hence continuous).*

Lemma 2.3. *Given $b\in\partial\Sigma$,*

$$\liminf_{\substack{\theta\to b\\ \theta\in\Sigma^0}}T(Y,\theta)=\liminf_{\substack{i\to\infty\\ \alpha(i)/|\alpha(i)|\to b}}\tau(\alpha(i)).$$

Lemma 2.4. *Let $\theta(k):=\alpha(k)/|\alpha(k)|$ for $k=1,2,...$ and let Q be a compact subset of Σ^0. Then*

$$\limsup_{|\alpha|\to\infty}\{\log\tau(\alpha(k))-\log T(Y(\theta(k))):|\alpha(k)|=\alpha \text{ and } \theta(k)\in Q\}=0.$$

Lemma 2.5. *Define*

$$\tau(Y):=\exp\left[\frac{1}{\operatorname{meas}(\Sigma)}\int_{\Sigma^0}\log T(Y,\theta)\,d\theta\right].$$

Then

$$\lim_{d\to\infty}\frac{1}{h_d}\sum_{|\alpha|=d}\log\tau(\alpha)=\log\tau(Y);$$

i.e., using (2.5),

$$\lim_{d\to\infty}\Big[\prod_{|\alpha|=d}Y(\alpha)\Big]^{1/dh_d}=\tau(Y).$$

One can incorporate all of the $Y(\alpha)$'s for $|\alpha|\le d$; this is the content of the next result.

Theorem 2.6. *We have*

$$\lim_{d\to\infty}\Big[\prod_{|\alpha|\le d}Y(\alpha)\Big]^{1/l_d}\qquad \textit{exists and equals } \tau(Y).$$

Proof. Define the geometric means

$$\tau_d^0:=\Big(\prod_{|\alpha|=d}\tau(\alpha)\Big)^{1/h_d},\quad d=1,2,\ldots.$$

The sequence

$$\log\tau_1^0,\log\tau_1^0,\ldots(r_1\text{ times}),\ldots,\log\tau_d^0,\log\tau_d^0,\ldots(r_d\text{ times}),\ldots$$

converges to $\log\tau(Y)$ by the previous lemma; hence the arithmetic mean of the first $l_d=\sum_{k=1}^d r_k$ terms (see (2.4)) converges to $\log\tau(Y)$ as well. Exponentiating this arithmetic mean gives

$$\Big(\prod_{k=1}^d(\tau_k^0)^{r_k}\Big)^{1/l_d}=\Big(\prod_{k=1}^d\prod_{|\alpha|=k}\tau(\alpha)^k\Big)^{1/l_d}=\Big(\prod_{|\alpha|\le d}Y(\alpha)\Big)^{1/l_d}\tag{2.6}$$

and the result follows. □

Returning to our examples (1)–(3), example (1) was the original setting of Zaharjuta [22] which he utilized to prove the existence of the limit in the definition of the *transfinite diameter* of a compact set $K\subset\mathbb{C}^N$. For $\zeta_1,\ldots,\zeta_n\in\mathbb{C}^N$, let

$$\begin{aligned}\mathrm{VDM}(\zeta_1,\ldots,\zeta_n)&=\det[e_i(\zeta_j)]_{i,j=1,\ldots,n}\\&=\det\begin{pmatrix}e_1(\zeta_1)&e_1(\zeta_2)&\ldots&e_1(\zeta_n)\\ \vdots&\vdots&\ddots&\vdots\\ e_n(\zeta_1)&e_n(\zeta_2)&\ldots&e_n(\zeta_n)\end{pmatrix}\end{aligned}\tag{2.7}$$

and for a compact subset $K\subset\mathbb{C}^N$ let

$$V_n = V_n(K) := \max_{\zeta_1,\dots,\zeta_n\in K} |\mathrm{VDM}(\zeta_1,\dots,\zeta_n)|.$$

Then

$$d(K) = \lim_{d\to\infty} V_{m_d}^{1/l_d} \tag{2.8}$$

is the *transfinite diameter* of K; Zaharjuta [22] showed that the limit exists by showing that one has

$$d(K) = \exp\left[\frac{1}{\mathrm{meas}(\Sigma)}\int_{\Sigma^0} \log\tau(K,\theta)\,d\theta\right], \tag{2.9}$$

where $\tau(K,\theta)=T(Y_1,\theta)$ from (1); i.e., the right-hand-side of (2.9) is $\tau(Y_1)$. This follows from Theorem 2.6 for $Y=Y_1$ and the estimate

$$\left(\prod_{k=1}^{d}(\tau_k^0)^{r_k}\right)^{1/l_d} \le V_{m_d}^{1/l_d} \le (m_d!)^{1/l_d}\left(\prod_{k=1}^{d}(\tau_k^0)^{r_k}\right)^{1/l_d}$$

in [22] (compare (2.6)).

For a compact *circled* set $K\subset\mathbb{C}^N$; i.e., $z\in K$ if and only if $e^{i\phi}z\in K$, $\phi\in[0,2\pi]$, one need only consider homogeneous polynomials in the definition of the directional Chebyshev constants $\tau(K,\theta)$. In other words, in the notation of (1) and (2), $Y_1(\alpha)=Y_2(\alpha)$ for all α so that

$$T(Y_1,\theta) = T(Y_2,\theta) \quad \text{for circled sets } K.$$

This is because for such a set, if we write a polynomial p of degree d as $p=\sum_{j=0}^{d} H_j$, where H_j is a homogeneous polynomial of degree j, then, from the Cauchy integral formula, $\|H_j\|_K\le\|p\|_K$, $j=0,\dots,d$. Moreover, a slight modification of Zaharjuta's arguments prove the existence of the limit of appropriate roots of maximal *homogeneous* Vandermonde determinants; i.e., the homogeneous transfinite diameter $d^{(H)}(K)$ of a compact set (cf. [15]). From the above remarks, it follows that

$$d(K) = d^{(H)}(K) \quad \text{for circled sets } K. \tag{2.10}$$

Since we will be using the homogeneous transfinite diameter, we amplify the discussion. We relabel the standard basis monomials $\{e_i^{(H,d)}(z)=z^{\alpha(i)}=z_1^{\alpha_1}\dots z_N^{\alpha_N}\}$, where $|\alpha(i)|=d$, $i=1,\dots,h_d$, we define the *d-homogeneous Vandermonde determinant*

$$\mathrm{VDMH}_d(\zeta_1,\dots,\zeta_{h_d}) := \det\left[e_i^{(H,d)}(\zeta_j)\right]_{i,j=1,\dots,h_d}. \tag{2.11}$$

Then

$$d^{(H)}(K) = \lim_{d\to\infty}\Big[\max_{\zeta_1,\ldots,\zeta_{h_d}\in K}|\mathrm{VDMH}_d(\zeta_1,\ldots,\zeta_{h_d})|\Big]^{1/dh_d} \tag{2.12}$$

is the *homogeneous transfinite diameter* of K; the limit exists and equals

$$\exp\left[\frac{1}{\mathrm{meas}(\Sigma)}\int_{\Sigma^0}\log T(Y_2,\theta)\,d\theta\right],$$

where $T(Y_2,\theta)$ comes from (2).

Finally, related to example (3), there are similar properties for the weighted version of directional Chebyshev constants and transfinite diameter. To define weighted notions, let $K\subset\mathbb{C}^N$ be closed and let w be an *admissible* weight function on K; i.e., w is a nonnegative, upper semicontinuous function with $\{z\in K : w(z)>0\}$ nonpluripolar. Let $Q:=-\log w$ and define the *weighted pluricomplex Green function* $V^*_{K,Q}(z):=\limsup_{\zeta\to z} V_{K,Q}(\zeta)$, where

$$V_{K,Q}(z):=\sup\{u(z): u\in L(\mathbb{C}^N) \text{ and } u\le Q \text{ on } K\}.$$

Here, $L(\mathbb{C}^N)$ is the set of all plurisubharmonic functions u on $\mathbb{C}^N$ with the property that $u(z)-\log|z|=O(1)$, as $|z|\to\infty$. If K is closed but not necessarily bounded, we require that w satisfies the growth property

$$|z|w(z)\to 0, \quad \text{as } |z|\to\infty,\ z\in K, \tag{2.13}$$

so that $V_{K,Q}$ is well-defined and equals $V_{K\cap B_R,Q}$ for $R>0$ sufficiently large, where $B_R=\{z:|z|\le R\}$ (Definition 2.1 and Lemma 2.2 of Appendix B in [20]). The unweighted case is when $w\equiv 1$ ($Q\equiv 0$); we then write V_K for the pluricomplex Green function. A compact set K is called *regular* if $V_K=V_K^*$; i.e., V_K is continuous; and K is *locally regular* if for each $z\in K$, the sets $K\cap\overline{B(z,r)}$ are regular for $r>0$, where $B(z,r)$ denotes the ball of radius r centered at z. We define the *weighted transfinite diameter*

$$d^w(K):=\exp\left[\frac{1}{\mathrm{meas}(\Sigma)}\int_{\Sigma^0}\log\tau^w(K,\theta)\,d\theta\right]$$

as in [11], where $\tau^w(K,\theta)=T(Y_3,\theta)$ from (3); i.e., the right-hand-side of this equation is the quantity $\tau(Y_3)$.

We remark for future use that if $\{K_j\}_{j=1}^\infty$ is a decreasing sequence of locally regular compact sets with $K_j\downarrow K$, and if w_j is a continuous admissible weight function on K_j with $w_j\downarrow w$ on K, where w is an admissible weight function on K, then the argument in Proposition 7.5 of [11] shows that $\lim_{j\to\infty}\tau^{w_j}(K_j,\theta)=\tau^w(K,\theta)$ for all $\theta \in \Sigma^0$ (we mention that there is a misprint in the statement of this proposition in [11]) and hence

$$\lim_{j\to\infty} d^{w_j}(K_j) = d^w(K). \tag{2.14}$$

In particular, (2.14) holds in the unweighted case ($w\equiv 1$) for any decreasing sequence $\{K_j\}_{j=1}^{\infty}$ of compact sets with $K_j \downarrow K$; i.e.,

$$\lim_{j\to\infty} d(K_j) = d(K) \tag{2.15}$$

(cf. [11], equation (1.13)). Another useful fact is that

$$d(K) = d(\widehat{K}) \quad \text{and} \quad d^{(H)}(K) = d^{(H)}(\widehat{K}) \tag{2.16}$$

for compact K, where

$$\widehat{K} := \{z \in \mathbb{C}^N : |p(z)| \le \|p\|_K \text{ for all polynomials } p\}$$

is the *polynomial hull* of K.

Another natural definition of a weighted transfinite diameter uses weighted Vandermonde determinants. Let $K \subset \mathbb{C}^N$ be compact and let w be an admissible weight function on K. Given $\zeta_1, ..., \zeta_n \in K$, let

$$\begin{aligned} W(\zeta_1, ..., \zeta_n) &:= \mathrm{VDM}(\zeta_1, ..., \zeta_n) w(\zeta_1)^{|\alpha(n)|} ... w(\zeta_n)^{|\alpha(n)|} \\ &= \det \begin{pmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \dots & e_1(\zeta_n) \\ \vdots & \vdots & \ddots & \vdots \\ e_n(\zeta_1) & e_n(\zeta_2) & \dots & e_n(\zeta_n) \end{pmatrix} w(\zeta_1)^{|\alpha(n)|} ... w(\zeta_n)^{|\alpha(n)|} \end{aligned}$$

be a *weighted Vandermonde determinant*. Let

$$W_n := \max_{\zeta_1, ..., \zeta_n \in K} |W(\zeta_1, ..., \zeta_n)| \tag{2.17}$$

and define an *n-th weighted Fekete set* for K and w to be a set of n points $\zeta_1, ..., \zeta_n \in K$ with the property that

$$|W(\zeta_1, ..., \zeta_n)| = \sup_{\xi_1, ..., \xi_n \in K} |W(\xi_1, ..., \xi_n)|.$$

Also, define

$$\delta^w(K) := \limsup_{d\to\infty} W_{m_d}^{1/l_d}. \tag{2.18}$$

We will show in Proposition 2.7 that $\lim_{d\to\infty} W_{m_d}^{1/l_d}$ (the weighted analogue of (2.8)) exists. The question of the existence of this limit if $N>1$ was raised in [11]. Moreover, using a recent result of Rumely, we will show in Theorem 2.9 how $\delta^w(K)$ is related to $d^w(K)$:

$$\delta^w(K) = \left[\exp\left(-\int_K Q\left(dd^c V_{K,Q}^*\right)^N\right)\right]^{1/N} d^w(K), \tag{2.19}$$

where $(dd^c V_{K,Q}^*)^N$ is the complex Monge–Ampère operator applied to $V_{K,Q}^*$. We refer the reader to [17] or Appendix B of [20] for more on the complex Monge–Ampère operator.

We begin by proving the existence of the limit in the definition of $\delta^w(E)$ in (2.18) for a set $E \subset \mathbb{C}^N$ and an admissible weight w on E (see also [2]).

Proposition 2.7. *Let $E \subset \mathbb{C}^N$ be a compact set with an admissible weight function w. The limit*

$$\delta^w(E) := \lim_{d\to\infty} \left[\max_{\lambda^{(i)}\in E} |\mathrm{VDM}(\lambda^{(1)}, ..., \lambda^{(m_d^{(N)})})| w(\lambda^{(1)})^d ... w(\lambda^{(m_d^{(N)})})^d\right]^{1/l_d^{(N)}}$$

exists.

Proof. Following [8], we define the circled set

$$F = F(E, w) := \{(t, z) = (t, t\lambda) \in \mathbb{C}^{N+1} : \lambda \in E \text{ and } |t| = w(\lambda)\}.$$

We first relate weighted Vandermonde determinants for E with homogeneous Vandermonde determinants for the compact set

$$F(D) := \{(t, z) = (t, t\lambda) \in \mathbb{C}^{N+1} : \lambda \in E \text{ and } |t| \le w(\lambda)\}. \tag{2.20}$$

Note that $F \subset \overline{F} \subset F(D) \subset \widehat{\overline{F}}$ (cf. [8], (2.4)) where $\widehat{\overline{F}}$ is the polynomial hull of $\overline{F}$ (recall (2.16)); thus

$$d^{(H)}(\overline{F}) = d^{(H)}(F(D)). \tag{2.21}$$

To this end, for each positive integer d, choose

$$m_d^{(N)} = \binom{N+d}{d}$$

(recall (2.2)) points $\{(t_i, z^{(i)})\}_{i=1,...,m_d^{(N)}} = \{(t_i, t_i\lambda^{(i)})\}_{i=1,...,m_d^{(N)}}$ in $F(D)$ and form the d-homogeneous Vandermonde determinant

$$\mathrm{VDMH}_d((t_1, z^{(1)}), ..., (t_{m_d^{(N)}}, z^{(m_d^{(N)})})).$$

We extend the lexicographical order of the monomials in $\mathbb{C}^N$ to $\mathbb{C}^{N+1}$ by letting t precede any of $z_1,...,z_N$. Writing the standard basis monomials of degree d in $\mathbb{C}^{N+1}$ as

$$\{t^{d-j}e_k^{(H,d)}(z): j=0,...,d \text{ and } k=1,...,h_j\};$$

i.e., for each power $d-j$ of t we multiply by the standard basis monomials of degree j in $\mathbb{C}^N$, and dropping the superscript (N) in $m_d^{(N)}$, we have the d-homogeneous Vandermonde matrix

$$\begin{pmatrix} t_1^d & t_2^d & \cdots & t_{m_d}^d \\ t_1^{d-1}e_2(z^{(1)}) & t_2^{d-1}e_2(z^{(2)}) & \cdots & t_{m_d}^{d-1}e_2(z^{(m_d)}) \\ \vdots & \vdots & \ddots & \vdots \\ e_{m_d}(z^{(1)}) & e_{m_d}(z^{(2)}) & \cdots & e_{m_d}(z^{(m_d)}) \end{pmatrix}$$

$$= \begin{pmatrix} t_1^d & t_2^d & \cdots & t_{m_d}^d \\ t_1^{d-1}z_1^{(1)} & t_2^{d-1}z_1^{(2)} & \cdots & t_{m_d}^{d-1}z_1^{(m_d)} \\ \vdots & \vdots & \ddots & \vdots \\ (z_N^{(1)})^d & (z_N^{(2)})^d & \cdots & (z_N^{(m_d)})^d \end{pmatrix}.$$

Factoring t_i^d out of the ith column, we obtain

$$\mathrm{VDMH}_d((t_1,z^{(1)}),...,(t_{m_d},z^{(m_d)})) = t_1^d...t_{m_d}^d \mathrm{VDM}(\lambda^{(1)},...,\lambda^{(m_d)});$$

thus, writing $|A|:=|\det A|$ for a square matrix A,

$$(2.22)\quad \begin{vmatrix} t_1^d & t_2^d & \cdots & t_{m_d}^d \\ t_1^{d-1}z_1^{(1)} & t_2^{d-1}z_1^{(2)} & \cdots & t_{m_d}^{d-1}z_1^{(m_d)} \\ \vdots & \vdots & \ddots & \vdots \\ (z_N^{(1)})^d & (z_N^{(2)})^d & \cdots & (z_N^{(m_d)})^d \end{vmatrix}$$

$$= |t_1|^d...|t_{m_d}|^d \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1^{(1)} & \lambda_1^{(2)} & \cdots & \lambda_1^{(m_d)} \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_N^{(1)})^d & (\lambda_N^{(2)})^d & \cdots & (\lambda_N^{(m_d)})^d \end{vmatrix},$$

where $\lambda_k^{(j)} = z_k^{(j)}/t_j$ provided $t_j \neq 0$. By definition of $F(D)$, as $(t_i,z^{(i)})=(t_i,t_i\lambda^{(i)})\in F(D)$, we have $|t_i|\leq w(\lambda^{(i)})$. Clearly the maximum of

$$|\mathrm{VDMH}_d((t_1,z^{(1)}),...,(t_{m_d},z^{(m_d)}))|$$

over points in $F(D)$ will occur when all $|t_j|=w(\lambda^{(j)})>0$ (recall that w is an admissible weight) so that from (2.22),

$$\max_{(t_i,z^{(i)})\in F(D)} |\mathrm{VDMH}_d((t_1,z^{(1)}),\dots,(t_{m_d},z^{(m_d)}))|$$
$$= \max_{\lambda^{(i)}\in E} |\mathrm{VDM}(\lambda^{(1)},\dots,\lambda^{(m_d)})| w(\lambda^{(1)})^d \dots w(\lambda^{(m_d)})^d.$$

As mentioned in the discussion of (2.12) the limit

$$\lim_{d\to\infty}\Big[\max_{(t_i,z^{(i)})\in F(D)} |\mathrm{VDMH}_d((t_1,z^{(1)}),\dots,(t_{m_d},z^{(m_d)}))|\Big]^{1/dh_d^{(N+1)}} =: d^{(H)}(F(D))$$

exists [15]; thus the limit

$$\lim_{d\to\infty}\Big[\max_{\lambda^{(i)}\in E} |\mathrm{VDM}(\lambda^{(1)},\dots,\lambda^{(m_d)})| w(\lambda^{(1)})^d \dots w(\lambda^{(m_d)})^d\Big]^{1/l_d^{(N)}} := \delta^w(E)$$

exists. □

Corollary 2.8. *For a nonpluripolar compact set $E\subset\mathbb{C}^N$ with an admissible weight function w and*

$$\begin{aligned} F=F(E,w)&:=\{(t,z)=(t,t\lambda)\in\mathbb{C}^{N+1}:\lambda\in E \text{ and } |t|=w(\lambda)\}, \\ \delta^w(E)&=d^{(H)}(\overline{F})^{(N+1)/N}=d(\overline{F})^{(N+1)/N}. \end{aligned} \tag{2.23}$$

Proof. The first equality follows from the proof of Proposition 2.7 using (2.21) and the relation

$$l_d^{(N)}=\frac{N}{N+1}dh_d^{(N+1)}$$

(see (2.3)). The second equality is (2.10). □

We next relate $\delta^w(E)$ and $d^w(E)$ but we first recall the remarkable formula of Rumely [19] (see also [2]). For a plurisubharmonic function u in $L(\mathbb{C}^N)$ we can define the *Robin function* associated with u:

$$\rho_u(z):=\limsup_{|\lambda|\to\infty}[u(\lambda z)-\log|\lambda|].$$

This function is plurisubharmonic (cf. [7], Proposition 2.1) and logarithmically homogeneous:

$$\rho_u(tz)=\rho_u(z)+\log|t| \quad \text{for } t\in\mathbb{C}.$$

For $u=V_{E,Q}^*$ (V_E^*) we write $\rho_u=\rho_{E,Q}$ (ρ_E). Rumely's formula relates ρ_E and $d(E)$:

$$(2.24)\quad -\log d(E)=\frac{1}{N}\biggl[\int_{\mathbb{C}^{N-1}}\rho_E(1,t_2,...,t_N)(dd^c\rho_E(1,t_2,...,t_N))^{N-1}$$
$$+\int_{\mathbb{C}^{N-2}}\rho_E(0,1,t_3,...,t_N)(dd^c\rho_E(0,1,t_3,...,t_N))^{N-2}+...$$
$$+\int_{\mathbb{C}}\rho_E(0,...,0,1,t_N)(dd^c\rho_E(0,...,0,1,t_N))+\rho_E(0,...,0,1)\biggr].$$

Here we make the convention that $dd^c=(i/\pi)\partial\bar\partial$ so that in any dimension $d=1,2,...$,

$$\int_{\mathbb{C}^d}(dd^cu)^d=1$$

for any $u\in L^+(\mathbb{C}^d)$; i.e., for any plurisubharmonic function u in $\mathbb{C}^d$ which satisfies

$$C_1+\log(1+|z|)\le u(z)\le C_2+\log(1+|z|)$$

for some C_1 and C_2.

We begin by rewriting (2.24) for *regular circled* sets E using an observation of Sione Ma'u. Note that for such sets, $V_E^*=\rho_E^+:=\max(\rho_E,0)$ (cf. [13], Lemma 5.1). If we intersect E with a hyperplane $\mathcal{H}$ through the origin, e.g., by rotating coordinates, we take $\mathcal{H}=\{z=(z_1,...,z_N)\in\mathbb{C}^N:z_1=0\}$, then $E\cap\mathcal{H}$ is a regular, compact, circled set in $\mathbb{C}^{N-1}$ (which we identify with $\mathcal{H}$). Moreover, we have

$$\rho_{\mathcal{H}\cap E}(z_2,...,z_N)=\rho_E(0,z_2,...,z_N)$$

since each side is logarithmically homogeneous and vanishes for $(z_2,...,z_N)\in\partial(\mathcal{H}\cap E)$. Thus the terms

$$\int_{\mathbb{C}^{N-2}}\rho_E(0,1,t_3,...,t_N)(dd^c\rho_E(0,1,t_3,...,t_N))^{N-2}$$
$$+...+\int_{\mathbb{C}}\rho_E(0,...,0,1,t_N)(dd^c\rho_E(0,...,0,1,t_N))+\rho_E(0,...,0,1)$$

in (2.24) are seen to equal

$$(N-1)d^{\mathbb{C}^{N-1}}(\mathcal{H}\cap E)$$

(where we temporarily write $d^{\mathbb{C}^{N-1}}$ to denote the transfinite diameter in $\mathbb{C}^{N-1}$ for emphasis) by applying (2.24) in $\mathbb{C}^{N-1}$ to the set $\mathcal{H}\cap E$. Hence we have

$$(2.25)\qquad -\log d(E)=\frac{1}{N}\int_{\mathbb{C}^{N-1}}\rho_E(1,t_2,...,t_N)(dd^c\rho_E(1,t_2,...,t_N))^{N-1}$$
$$+\frac{N-1}{N}[-\log d^{\mathbb{C}^{N-1}}(\mathcal{H}\cap E)].$$

Theorem 2.9. *For a nonpluripolar compact set* $E\subset\mathbb{C}^N$ *with an admissible weight function* w,

$$\delta^w(E)=\left[\exp\left(-\int_E Q(dd^cV_{E,Q}^*)^N\right)\right]^{1/N} d^w(E). \tag{2.26}$$

Proof. We first assume that E is locally regular and Q is continuous. It is known in this case that $V_{E,Q}=V_{E,Q}^*$ (cf. [21], Proposition 2.16). As before, we define the circled set

$$F=F(E,w):=\{(t,z)=(t,t\lambda)\in\mathbb{C}^{N+1}:\lambda\in E \text{ and } |t|=w(\lambda)\}.$$

We claim that this is a regular compact set; i.e., that V_F is continuous. First of all, $V_F^*(t,z)=\max(\rho_F(t,z),0)$ (cf. Proposition 2.2 of [8]) so that it suffices to verify that $\rho_F(t,z)$ is continuous. From Theorem 2.1 and Corollary 2.1 of [8],

$$V_{E,Q}(\lambda)=\rho_F(1,\lambda) \quad \text{on } \mathbb{C}^N \tag{2.27}$$

which implies, by the logarithmic homogeneity of ρ_F, that $\rho_F(t,z)$ is continuous on $\mathbb{C}^{N+1}\setminus\{(t,z):t=0\}$. Corollary 2.1 and (2.6) in [8] give that

$$\rho_F(0,\lambda)=\rho_{E,Q}(\lambda) \quad \text{for } \lambda\in\mathbb{C}^N \tag{2.28}$$

and $\rho_{E,Q}$ is continuous by Theorem 2.5 of [11]. Moreover, the limit exists in the definition of $\rho_{E,Q}$:

$$\rho_{E,Q}(\lambda):=\limsup_{|t|\to\infty}[V_{E,Q}(t\lambda)-\log|t|]=\lim_{|t|\to\infty}[V_{E,Q}(t\lambda)-\log|t|];$$

and the limit is uniform in λ (cf. Corollary 4.4 of [13]) which implies, from (2.27) and (2.28), that $\lim_{t\to 0}\rho_F(t,\lambda)=\rho_F(0,\lambda)$ so that $\rho_F(t,z)$ is continuous. In particular, from Theorem 2.5 in Appendix B of [20],

$$V_{E,Q}(\lambda)=Q(\lambda)=\rho_F(1,\lambda) \quad \text{on the support of } (dd^cV_{E,Q})^N$$

so that

$$\int_E Q(\lambda)(dd^cV_{E,Q}(\lambda))^N=\int_{\mathbb{C}^N}\rho_F(1,\lambda)(dd^c\rho_F(1,\lambda))^N. \tag{2.29}$$

On the other hand, $E_\rho^w:=\{\lambda\in\mathbb{C}^N:\rho_{E,Q}(\lambda)\le 0\}$ is a circled set, and, according to (3.14) in [11], $d^w(E)=d(E_\rho^w)$. But

$$\begin{aligned}\rho_{E,Q}(\lambda)&=\limsup_{|t|\to\infty}[V_{E,Q}(t\lambda)-\log|t|]\\&=\limsup_{|t|\to\infty}[\rho_F(1,t\lambda)-\log|t|]=\limsup_{|t|\to\infty}\rho_F(1/t,\lambda)=\rho_F(0,\lambda).\end{aligned}$$

Thus

$$E_\rho^w = \{\lambda \in \mathbb{C}^N : \rho_F(0,\lambda) \le 0\} = F \cap \mathcal{H},$$

where $\mathcal{H}=\{(t,z)\in\mathbb{C}^{N+1}:t=0\}$ and hence

$$d^w(E) = d(E_\rho^w) = d(F \cap \mathcal{H}). \tag{2.30}$$

From (2.25) applied to $F\subset\mathbb{C}^{N+1}$,

$$-\log d(F) = \frac{1}{N+1}\int_{\mathbb{C}^N} \rho_F(1,\lambda)(dd^c\rho_F(1,\lambda))^N + \frac{N}{N+1}[-\log d(F\cap\mathcal{H})]. \tag{2.31}$$

Finally, from (2.23) (note that $F=\overline{F}$ since w is continuous),

$$\delta^w(E) = d(F)^{(N+1)/N}; \tag{2.32}$$

putting together (2.29), (2.30), (2.31) and (2.32) gives the result if E is locally regular and Q is continuous.

The general case follows from approximation. Take a sequence of locally regular compact sets $\{E_j\}_{j=1}^\infty$ decreasing to E and a sequence of weight functions $\{w_j\}_{j=1}^\infty$ with w_j continuous and admissible on E_j and $w_j \downarrow w$ on E (cf. Lemma 2.3 of [8]). From (2.14) we have

$$\lim_{j\to\infty} d^{w_j}(E_j) = d^w(E). \tag{2.33}$$

Also, by Corollary 2.8 we have

$$\delta^{w_j}(E_j) = d(F_j)^{(N+1)/N}, \tag{2.34}$$

where

$$F_j = F_j(E_j,w_j) = \{t(1,\lambda) : \lambda\in E_j \text{ and } |t| = w_j(\lambda)\}.$$

Since $E_{j+1}\subset E_j$ and $w_{j+1}\le w_j$, the sets

$$F_j(D) = \{t(1,\lambda) : \lambda\in E_j \text{ and } |t| \le w_j(\lambda)\}$$

satisfy $F_{j+1}(D)\subset F_j(D)$ and hence

$$d(F_{j+1}(D)) = d(F_{j+1}) \le d(F_j(D)) = d(F_j).$$

Since $F_j(D)\downarrow F(D)$ (see (2.20)), we conclude from (2.15), (2.21), (2.10) and (2.34) that

$$\lim_{j\to\infty} \delta^{w_j}(E_j) = \delta^w(E). \tag{2.35}$$

Applying (2.26) to E_j, w_j, Q_j and using (2.33) and (2.35), we conclude that

$$\int_{E_j} Q_j \big(dd^c V_{E_j,Q_j}\big)^N \to \int_E Q\big(dd^c V^*_{E,Q}\big)^N,$$

from Lemma 3.6.2 of [17] completing the proof of (2.26). □

3. Integrals of Vandermonde determinants

In this section, we first state and prove the analogue of an "unweighted" generalization to $\mathbb{C}^N$ of Theorem 2.1 of [12] as it has a self-contained proof utilizing Chebyshev polynomials discussed in Section 2. We first recall some terminology. Given a compact set $E\subset\mathbb{C}^N$ and a measure ν on E, we say that (E,ν) satisfies a Bernstein–Markov inequality for holomorphic polynomials in $\mathbb{C}^N$ if, given $\varepsilon>0$, there exists a constant $M=M(\varepsilon)$ such that for all such polynomials Q_n of degree at most n,

$$\|Q_n\|_E \le M(1+\varepsilon)^n \|Q_n\|_{L^2(\nu)}. \tag{3.1}$$

Theorem 3.1. *Let (E,μ) satisfy a Bernstein–Markov inequality. Then*

$$\lim_{d\to\infty} Z_d(E,\mu)^{1/2l_d^{(N)}} = d(E),$$

where

$$Z_d(E,\mu) := \int_{E^{m_d^{(N)}}} |\mathrm{VDM}(\lambda^{(1)},...,\lambda^{m_d^{(N)}})|^2\, d\mu(\lambda^{(1)})...d\mu(\lambda^{m_d^{(N)}}). \tag{3.2}$$

Proof. Since $\mathrm{VDM}(\zeta_1,...,\zeta_n)=\det[e_i(\zeta_j)]_{i,j=1,...,n}$ for any positive integer n, if we apply the Gram–Schmidt procedure to the monomials $e_1,...,e_{m_d^{(N)}}$ to obtain orthogonal polynomials $q_1,...,q_{m_d^{(N)}}$ with respect to μ, where $q_j\in P_j$ has minimal $L^2(\mu)$-norm among all such polynomials, we get, upon using elementary row operations on $\mathrm{VDM}(\zeta_1,...,\zeta_{m_d^{(N)}})$ and expanding the determinant (cf. [14], Chapter 5 or Section 2 of [12])

$$\int_{E^{m_d^{(N)}}} |\mathrm{VDM}(\zeta_1,...,\zeta_{m_d^{(N)}})|^2\, d\mu(\zeta_1)...d\mu(\zeta_{m_d^{(N)}}) = m_d^{(N)}! \prod_{j=1}^{m_d^{(N)}} \|q_j\|^2_{L^2(\mu)}. \tag{3.3}$$

Let $t_{\alpha,E}\in P(\alpha)$ be a Chebyshev polynomial; i.e., $\|t_{\alpha,E}\|_E=Y_1(\alpha)$. Then Theorem 2.6 shows that

$$\lim_{d\to\infty}\Big(\prod_{|\alpha|\le d}\|t_{\alpha,E}\|_E\Big)^{1/l_d} = \tau(Y_1)$$

since

$$\lim_{d\to\infty}(m_d^{(N)}!)^{1/l_d^{(N)}}=1.$$

Zaharjuta's theorem (2.9) shows that $\tau(Y_1)=d(E)$ so we need to show that

$$\lim_{d\to\infty}\bigg(\prod_{|\alpha|\le d}\|t_{\alpha,E}\|_E\bigg)^{1/l_d}=\lim_{d\to\infty}\bigg(\prod_{|\alpha|\le d}\|q_\alpha\|_{L^2(\mu)}\bigg)^{1/l_d}. \tag{3.4}$$

This follows from the Bernstein–Markov property. First note that

$$\|q_\alpha\|_{L^2(\mu)}\le\|t_{\alpha,E}\|_{L^2(\mu)}\le\mu(E)\|t_{\alpha,E}\|_E$$

from the $L^2(\mu)$-norm minimality of q_α; then, given $\varepsilon>0$, the Bernstein–Markov property and the sup-norm minimality of $t_{\alpha,E}$ give

$$\|t_{\alpha,E}\|_E\le\|q_\alpha\|_E\le M(1+\varepsilon)^{|\alpha|}\|q_\alpha\|_{L^2(\mu)}$$

for some $M=M(\varepsilon)>0$. Taking products of these inequalities over $|\alpha|\le d$; taking l_d-th roots; and letting $\varepsilon\to 0$ gives the result. This reasoning is adapted from the proof of Theorem 3.3 in [9]. □

A weighted polynomial on E is a function of the form $w(z)^n p_n(z)$, where p_n is a holomorphic polynomial of degree at most n. Let μ be a measure with support in E such that (E,w,μ) satisfies a Bernstein–Markov inequality for weighted polynomials (referred to as a *weighted B–M inequality* in [8]): given $\varepsilon>0$, there exists a constant $M=M(\varepsilon)$ such that for all weighted polynomials $w^n p_n$,

$$\|w^n p_n\|_E\le M(1+\varepsilon)^n\|w^n p_n\|_{L^2(\mu)}. \tag{3.5}$$

Generalizing Theorem 3.1, we have the following result.

Theorem 3.2. *Let (E,w,μ) satisfy a Bernstein–Markov inequality (3.5) for weighted polynomials. Then*

$$\lim_{d\to\infty}Z_d^{1/2l_d^{(N)}}=\delta^w(E),$$

where

$$Z_d=Z_d(E,w,\mu):=\int_{E^{m_d^{(N)}}}|\mathrm{VDM}(\lambda^{(1)},\dots,\lambda^{(m_d^{(N)})})|^2 \times w(\lambda^{(1)})^{2d}\dots w(\lambda^{(m_d^{(N)})})^{2d}\,d\mu(\lambda^{(1)})\dots d\mu(\lambda^{(m_d^{(N)})}). \tag{3.6}$$

Our original proof of Theorem 3.2 followed along the lines of Section 3 of [12]; we provide a brief sketch. Let $E\subset\mathbb{C}^N$ be a nonpluripolar compact set with an admissible weight function w and let μ be a measure with support in E such that (E,w,μ) satisfies a Bernstein–Markov inequality for weighted polynomials. The integrand

$$|\mathrm{VDM}(\lambda^{(1)},...,\lambda^{(m_d^{(N)})})|^2 w(\lambda^{(1)})^{2d}...w(\lambda^{(m_d^{(N)})})^{2d}$$

in the definition of Z_d in (3.6) has a maximal value on $E^{m_d^{(N)}}$ whose $1/2l_d^{(N)}$ root tends to $\delta^w(E)$. To show that the integrals themselves have the same property, we begin by constructing the circled set $F\subset\mathbb{C}^{N+1}$ defined as in Section 4:

$$F=F(E,w):=\{(t,z)=(t,t\lambda)\in\mathbb{C}^{N+1}:\lambda\in E \text{ and } |t|=w(\lambda)\}.$$

We construct a measure ν on $\overline{F}$ associated with μ such that $(\overline{F},\nu)$ satisfies the Bernstein–Markov property for holomorphic polynomials in $\mathbb{C}^{N+1}$; i.e., (3.1) holds. Define

$$\nu:=m_\lambda\otimes\mu,\quad \lambda\in E,$$

where m_λ is normalized Lebesgue measure on the circle $|t|=w(\lambda)$ in the complex t-plane given by

$$C_\lambda:=\{(t,t\lambda)\in\mathbb{C}^{N+1}:t\in\mathbb{C}\}.$$

That is, if ϕ is continuous on $\overline{F}$, then

$$\int_{\overline{F}}\phi(t,z)\,d\nu(t,z)=\int_E\left[\int_{C_\lambda}\phi(t,t\lambda)\,dm_\lambda(t)\right]d\mu(\lambda).$$

Equivalently, if $\pi:\mathbb{C}^{N+1}\to\mathbb{C}^N$ via $\pi(t,z)=z/t:=\lambda$, then $\pi_*(\nu)=\mu$. One shows that

$$\begin{aligned}\widetilde{Z}_d:=&\int_{\overline{F}^{m_d^{(N)}}}|\mathrm{VDMH}_d((t_1,z^{(1)}),...,(t_{m_d^{(N)}},z^{(m_d^{(N)})}))|^2\\ &\times d\nu(t_1,z^{(1)})...d\nu(t_{m_d^{(N)}},z^{(m_d^{(N)})})\end{aligned}$$

satisfies $\widetilde{Z}_d=Z_d$, where $m_d^{(N)}=\binom{N+d}{d}$ and Z_d is defined in (3.6). Finally one relates the homogeneous and weighted Vandermonde determinants to verify that

$$\lim_{d\to\infty}\widetilde{Z}_d^{1/2dm_d^{(N)}}=d^{(H)}(\overline{F}).$$

Given the existence of the limit in the definition of $\delta^w(E)$ (Proposition 2.7), a short proof of Theorem 3.2 due to Berman and Boucksom ([2] or [5]) runs as follows: first, as in the proof of Theorem 3.1, using elementary row operations in the integrand of Z_d, we can replace the monomials $e_1,...,e_{m_d^{(N)}}$ by the orthogonal polynomials $q_1,...,q_{m_d^{(N)}}$ with respect to $L^2(w^{2d}\mu)$ to obtain

$$Z_d = m_d^{(N)}! \prod_{j=1}^{m_d^{(N)}} \|q_j\|^2_{L^2(w^{2d}\mu)}.$$

On the other hand, taking points $x_1,...,x_{m_d^{(N)}}$ achieving the maximum in W_{m_d} (recall (2.17)), we have, upon applying the weighted Bernstein–Markov property to the weighted polynomial

$$\zeta_1 \longmapsto \mathrm{VDM}(\zeta_1, x_2, ..., x_{m_d^{(N)}}) w(\zeta_1)^d ... w(x_{m_d^{(N)}})^d,$$

$$\begin{aligned} W^2_{m_d} &= |\mathrm{VDM}(x_1,...,x_{m_d^{(N)}})|^2 w(x_1)^{2d} ... w(x_{m_d^{(N)}})^{2d} \\ &\leq [M(1+\varepsilon)]^2 \int_E ... \int_E |\mathrm{VDM}(\zeta_1, x_2, ..., x_{m_d^{(N)}})|^2 w(\zeta_1)^{2d} ... w(x_{m_d^{(N)}})^{2d} d\mu(\zeta_1). \end{aligned}$$

Repeating this argument in each variable we obtain

$$W^2_{m_d} \leq [M(1+\varepsilon)]^{2m_d^{(N)}} Z_d$$

and the result follows.

As a corollary, we get a "large deviation" result, which follows easily from Theorem 3.2. Define a probability measure $\mathcal{P}_d$ on $E^{m_d^{(N)}}$ via, for a Borel set $A \subset E^{m_d^{(N)}}$,

$$\mathcal{P}_d(A) := \frac{1}{Z_d} \int_A |\mathrm{VDM}(z_1,...,z_{m_d^{(N)}})|^2 w(z_1)^{2d} ... w(z_{m_d^{(N)}})^{2d}\, d\mu(z_1)...d\mu(z_{m_d^{(N)}}).$$

Proposition 3.3. *Given* $\eta > 0$, *define*

$$\begin{aligned} A_{d,\eta} := \{ &(z_1,...,z_{m_d^{(N)}}) \in E^{m_d^{(N)}} : \\ &|\mathrm{VDM}(z_1,...,z_{m_d^{(N)}})|^2 w(z_1)^{2d} ... w(z_n)^{2d} \geq (\delta^w(E) - \eta)^{2l_d} \}. \end{aligned}$$

Then there exists $d^* = d^*(\eta)$ *such that for all* $d > d^*$,

$$\mathcal{P}_d(E^{m_d^{(N)}} \setminus A_{d,\eta}) \leq \left(1 - \frac{\eta}{2\delta^w(E)}\right)^{2l_d}.$$

Proof. From Theorem 3.2, given $\varepsilon>0$,

$$Z_d \geq [\delta^w(E)-\varepsilon]^{2l_d}$$

for $d\geq d(\varepsilon)$. Thus

$$\begin{aligned}\mathcal{P}_d(E^{m_d^{(N)}}\setminus A_{d,\eta}) &= \frac{1}{Z_d}\int_{E^{m_d^{(N)}}\setminus A_{d,\eta}} |\mathrm{VDM}(z_1,...,z_{m_d^{(N)}})|^2 \\ &\quad\times w(z_1)^{2d}...w(z_{m_d^{(N)}})^{2d}\,d\mu(z_1)...d\mu(z_{m_d^{(N)}}) \\ &\leq \frac{[\delta^w(E)-\eta]^{2l_d}}{[\delta^w(E)-\varepsilon]^{2l_d}}\end{aligned}$$

if $d\geq d(\varepsilon)$. Choosing $\varepsilon<\eta/2$ and $d^*=d(\varepsilon)$ gives the result. $\square$

Finally, we state a version of (1.2) for an unbounded cone Γ in $\mathbb{R}^N$ with $\Gamma=\overline{\operatorname{int}\Gamma}$. Precisely, our set-up is the following: Let $R(x)=R(x_1,...,x_N)$ be a polynomial in N (real) variables $x=(x_1,...,x_N)$ and let

$$d\mu(x) := |R(x)|\,dx = |R(x_1,...,x_N)|\,dx_1...dx_N. \tag{3.7}$$

Next, let $w(x)=\exp(-Q(x))$, where $Q(x)$ satisfies the inequality

$$Q(x)\geq c|x|^\gamma \tag{3.8}$$

for all $x\in\Gamma$ for some $c,\gamma>0$.

Theorem 3.4. *With μ and Q as in (3.7) and (3.8), let $S_w:=\operatorname{supp}(dd^cV_{\Gamma,Q})^N$. Then*

$$\lim_{d\to\infty} Z_d(\Gamma,w,\mu)^{1/2l_d^{(N)}} = \delta^w(S_w),$$

where

$$\begin{aligned}Z_d(\Gamma,w,\mu) &:= \int_{\Gamma^{m_d^{(N)}}} \mathrm{VDM}(\lambda^{(1)},...,\lambda^{(m_d^{(N)})})^2 \\ &\quad\times w(\lambda^{(1)})^{2d}...w(\lambda^{(m_d^{(N)})})^{2d}\,d\mu(\lambda^{(1)})...d\mu(\lambda^{(m_d^{(N)})}).\end{aligned} \tag{3.9}$$

Remark. The integrals considered in Theorem 3.4 may be considered as multivariate versions (i.e., with a multivariable Vandermonde determinant in the integrand rather than a one-variable Vandermonde determinant) of integrals of the form

$$\int_{\mathbb{R}^d} \mathrm{VDM}(\lambda_1,...,\lambda_d)^2 e^{-dQ(\lambda_1)}...e^{-dQ(\lambda_d)}\,d\lambda_1...d\lambda_d$$

considered in [14], Chapter 6, arising in the joint probability distribution of eigenvalues of certain random matrix ensembles. They are also multivariate versions of Selberg integrals of Laguerre type (cf. [18], (17.6.5)) which, after rescaling by a factor of d, are of the form, for $\Gamma=[0,\infty)\subset\mathbb{R}$ and $\alpha>0$,

$$\int_{\Gamma^d} \mathrm{VDM}(\lambda_1,...,\lambda_d)^2 e^{-d\lambda_1}...e^{-d\lambda_d}\left(\prod_{j=1}^{d}\lambda_j^{\alpha}\right)d\lambda_1...d\lambda_d.$$

Proof. We begin by observing that

$$\mathrm{VDM}(\lambda^{(1)},...,\lambda^{(m_d^{(N)})})^2 w(\lambda^{(1)})^{2d}...w(\lambda^{(m_d^{(N)})})^{2d} \tag{3.10}$$

(the integrand in (3.9)) becomes, if all but one of the $m_d^{(N)}-1$ variables are fixed, a weighted polynomial in the remaining variable. Since $w(x)$ is continuous, by Theorem 2.6 in Appendix B of [20], a weighted polynomial attains its maximum on $S_w\subset\Gamma$. Hence the maximum value of (3.10) on $\Gamma^{m_d^{(N)}}$ is attained on $(S_w)^{m_d^{(N)}}$. Since S_w has compact support (cf. Lemma 2.2 of Appendix B of [20]), we can take $T>0$ sufficiently large with $S_w\subset\Gamma\cap B_T$, where $B_T:=\{x\in\mathbb{R}^N:|x|\leq T\}$ and

$$\delta^w(S_w)=\delta^w(\Gamma\cap B_T).$$

We need the following result.

Lemma 3.5. *For all sufficiently large $T>0$, there exists $M=M(T)>0$ with*

$$\|w^dp\|_{L^2(\Gamma,\mu)}\leq M\|w^dp\|_{L^2(\Gamma\cap B_T,\mu)}$$

if $p=p(x)$ is a polynomial of degree d.

Proof. By Theorem 2.6(ii) in Appendix B of [20], we have

$$|w(x)^dp(x)|\leq\|w^dp\|_{S_w}e^{d(V_{\Gamma,Q}(x)-Q(x))}$$

for all $x\in\Gamma$. Since $V_{\Gamma,Q}\in L(\mathbb{C}^N)$ and $Q(x)\geq c|x|^{\gamma}$ for $x\in\Gamma$, there is a $c_0>0$ with

$$|w(x)^dp(x)|\leq\|w^dp\|_{S_w}e^{-c_0d|x|^{\gamma}}$$

for all x in Γ with $|x|\geq T$ for T sufficiently large. Hence

$$\|w^dp\|_{L^2(\Gamma,\mu)}\leq\|w^dp\|_{L^2(\Gamma\cap B_T,\mu)}+\|w^dp\|_{S_w}\left[\int_{\{x\in\Gamma:|x|\geq T\}}e^{-2c_0d|x|^{\gamma}}|R(x)|\,dx\right]^{1/2}.$$

Now $(\Gamma\cap B_T,\mu)$ satisfies the Bernstein–Markov property ([10], Theorem 2.1); thus by [8], Theorem 3.2, the triple $(\Gamma\cap B_T,w,\mu)$ satisfies the weighted Bernstein–Markov property. Hence, given $\varepsilon>0$, there is $M_1=M_1(\varepsilon)>0$ with

$$\|w^dp\|_{S_w}=\|w^dp\|_{\Gamma\cap B_T}\le M_1(1+\varepsilon)^d\|w^dp\|_{L^2(\Gamma\cap B_T,\mu)}.$$

A simple estimate shows that

$$\left[\int_{\{x\in\Gamma:|x|\ge T\}} e^{-2c_0d|x|^\gamma}|R(x)|\,dx\right]^{1/2}\le e^{-c'd}$$

for some $c'>0$. The result now follows by choosing ε sufficiently small. $\square$

We now expand the integrands in the formulas for $Z_d(\Gamma):=Z_d(\Gamma,w,\mu)$ and in $Z_d(\Gamma\cap B_T):=Z_d(\Gamma\cap B_T,w|_{\Gamma\cap B_T},\mu|_{\Gamma\cap B_T})$ as a product of L^2-norms of orthogonal polynomials as in (3.3), and then proceed as in the proof of Corollary 2.1 in Section 5 of [12] to conclude that

$$\lim_{d\to\infty}Z_d(\Gamma)^{1/2l_d^{(N)}}=\lim_{d\to\infty}Z_d(\Gamma\cap B_T)^{1/2l_d^{(N)}}=\delta^w(\Gamma\cap B_T)=\delta^w(S_w).\quad\square$$

4. Final remarks

In this section we discuss some results from the Berman and Boucksom papers [1], [2], [3], [4] and [6]. Recall from Section 2 that a dth weighted Fekete set for a compact set $E\subset\mathbb{C}^N$ and an admissible weight w on E is a set of m_d points $\zeta_1^{(d)},...,\zeta_{m_d}^{(d)}\in E$ with the property that

$$|W(\zeta_1,...,\zeta_{m_d})|=\sup_{\xi_1,...,\xi_{m_d}\in E}|W(\xi_1,...,\xi_{m_d})|,$$

where W is defined in (2.17). In [11] we asked if the sequence of probability measures

$$\mu_d:=\frac{1}{m_d}\sum_{j=1}^{m_d}\langle\zeta_j^{(d)}\rangle,\quad d=1,2,...,$$

where $\langle z\rangle$ denotes the point mass at z and $\{\zeta_1^{(d)},...,\zeta_{m_d}^{(d)}\}$ is a dth weighted Fekete set for E and w, has a unique weak-* limit, and, if so, whether this limit is the Monge–Ampère measure, $\mu_{\mathrm{eq}}^w:=(dd^cV_{E,Q}^*)^N$. In [4] and [6] a positive answer is provided.

Suppose now that μ is a measure on E such that (E,w,μ) satisfies a Bernstein–Markov inequality for weighted polynomials. Define the probability measures

$$\mu_d(z):=\frac{1}{Z_d}R_1^{(d)}(z)w(z)^{2d}\,d\mu(z),$$

where Z_d is defined in (3.6) and

$$R_1^{(d)}(z) := \int_{E^{m_d-1}} |\mathrm{VDM}(\lambda^{(1)}, ..., \lambda^{(m_d-1)}, z)|^2 \times w(\lambda^{(1)})^{2d} ... w(\lambda^{(m_d-1)})^{2d}\, d\mu(\lambda^{(1)}) ... d\mu(\lambda^{(m_d-1)}). \tag{4.1}$$

We observe that with the notation in (4.1) and (3.6),

$$\frac{R_1^{(d)}(z)}{Z_d} = \frac{1}{m_d} \sum_{j=1}^{m_d} |q_j^{(d)}(z)|^2, \tag{4.2}$$

where $q_1^{(d)}, ..., q_{m_d}^{(d)}$ are orthonormal polynomials with respect to the measure $w(z)^{2d}\, d\mu(z)$ forming a basis for the polynomials of degree at most d. To verify (4.2), we refer the reader to the argument in Remark 2.1 of [12]. Forming the sequence of Christoffel functions $K_d(z) := \sum_{j=1}^{m_d} |q_j^{(d)}(z)|^2$, in [12], Theorem 2.2, it was shown that if $N=1$ then $\mu_d(z) \to \mu_{\mathrm{eq}}^w(z)$ weak-*; i.e.,

$$\frac{1}{m_d} K_d(z) w(z)^{2d}\, d\mu(z) \to \mu_{eq}^w(z) \quad \text{weak-*}. \tag{4.3}$$

In [5] it is shown that (4.3) holds in $\mathbb{C}^N$ for $N>1$ as well.

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