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Local Gromov–Witten invariants of cubic surfaces via nef toric degeneration

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Abstract. We compute local Gromov–Witten invariants of cubic surfaces at all genera. We use a deformation a of cubic surface to a nef toric surface and the deformation invariance of Gromov–Witten invariants.

1. Introduction

A del Pezzo surface S_d of degree d, $1\le d\le 9$,(1) is a smooth surface(2) whose anticanonical divisor $-K_{S_d}$ is ample and such that $(-K_{S_d})^2=d$. For a smooth projective surface X, the local Gromov–Witten (GW) invariant is a rational number defined by the integral of a certain class, which is determined by the canonical divisor K_X, on the moduli stack of stable maps to X, see [4] and [16]. Local GW invariants of del Pezzo surfaces have been intensively studied in physics in relation to the non-critical string by various methods: mirror symmetry, Seiberg–Witten curve technique and so on (see e.g. [22]). In the case of toric del Pezzo surfaces (i.e. for $6\le d\le 9$), a powerful method based on the duality to the Chern–Simons theory enables us to write down an explicit formula for the generating function at all genera, see [1], [6], [7] and [14]. The formula was proved in [31] based on the virtual localization, [11] and [21], together with a formula for Hodge integrals [24]. In a recent interesting work [5], Diaconescu and Florea proposed a closed formula for the generating function of nontoric del Pezzo surfaces S_i, $1\le i\le 5$, for all genera by using the conjectural ruled vertex formalism [8].

(1) In physics literatures, S_d is usually denoted by dP_{9-d} or B_{9-d}. Here we follow the notation used in [26, Section 0]. A brief account of the classification of del Pezzo surfaces can be found there.

(2) In this article, a surface means an algebraic surface over $\mathbb{C}$.

Our modest goal is to obtain a formula for the generating function of local GW invariants of S_3 at all genera. S_3 is isomorphic to $\mathbb{P}^2$ blown-up at 6 points in a general position and it is also realized as a smooth cubic surface in $\mathbb{P}^3$. It is not toric but has a (unique) smooth nef toric degeneration S_3^0 (a smooth toric surface with the nef anticanonical divisor which is deformation equivalent to S_3). Our main idea is to use the deformation invariance of local GW invariants as in [5] and [30] and reduce the computations to those of S_3^0, where we can apply the virtual localization. Here we remark that our results are limited to S_k, $k=3,4,5$, since S_1 and S_2 do not admit nef toric degenerations.

The results of this paper are as follows. We first prove that in the case of a smooth projective surface with the nef anticanonical divisor, local GW invariants are equal to ordinary GW invariants of a projective bundle compactification of the total space of the canonical line bundle (Proposition 2.2). Our proof is based on the virtual localization with respect to the $\mathbb{C}^*$-action in the fiber direction. Then the deformation invariance of the latter, see [23] and [29], implies that of the former (Proposition 2.4). Next we introduce the toric surface S_3^0 and show that it is the nef toric degeneration of S_3 (Proposition 4.1). Then we derive a formula for the generating function of local GW invariants of S_3^0 by the virtual localization (Lemma 5.1). Finally we obtain a formula for the generating function of local GW invariants of S_3 via those of S_3^0 by the deformation invariance (Theorem 5.2).

The organization of the paper is as follows. In Section 2, we give a definition of local GW invariants and show the deformation invariance. In Section 3, we summarize necessary facts about cubic surfaces S_3. In Section 4, we introduce the toric surface S_3^0. For completeness, a proof of the deformation equivalence of S_3 and S_3^0 is included in Appendices A and B. In Section 5, we give formulas for the generating functions of local GW invariants of S_3^0 and S_3. We have computed the formula explicitly for $\beta \in H_2(S_3, \mathbb{Z})$ such that $-K_{S_3} \cdot \beta \leq 6$. The results are listed in Section 6 and Appendix C.

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2. Deformation invariance of local GW invariants

In this article, we call a smooth projective surface X whose anticanonical divisor $-K_X$ is nef (i.e. $-K_X\cdot[C]\geq 0$ for all curves $C\subset X$) a *nef surface.*

Let X be a nef surface and K_X be its canonical divisor. For $\beta\in H_2(X,\mathbb{Z})$ and $g\in\mathbb{Z}_{\geq 0}$, let $\overline{M}_{g,0}(X,\beta)$ (resp. $\overline{M}_{g,1}(X,\beta)$) be the moduli stack of stable maps to X of genus g without marked point (resp. with one marked point) and with the second homology class β. Let $\pi\colon \overline{M}_{g,1}(X,\beta)\to\overline{M}_{g,0}(X,\beta)$ be the forgetful map of the marked point and $\mu\colon \overline{M}_{g,1}(X,\beta)\to X$ be the evaluation at the marked point.

Definition 2.1. For $g\in\mathbb{Z}_{\geq 0}$ and $\beta\in H_2(X,\mathbb{Z})$ such that $\int_\beta c_1(K_X)<0$, the local Gromov–Witten invariant $N_{g,\beta}(K_X)$ of X with genus g and the second homology class β is

$$N_{g,\beta}(K_X)=\int_{[\overline{M}_{g,0}(X,\beta)]^{\mathrm{vir}}} c_{\mathrm{top}}(R^1\pi_*\mu^*K_X),$$

where c_{top} denotes the top Chern class which is of degree $(1-g)(\dim X-3)-\int_\beta c_1(K_X)$. (This is equal to the virtual dimension of $\overline{M}_{g,0}(X,\beta)$.)(3)

Let $\mathbb{P}(K_X\oplus\mathcal{O}_X)$ be the projectivization of the total space of the vector bundle $K_X\oplus\mathcal{O}_X$ (here the canonical divisor K_X and the structure sheaf $\mathcal{O}_X$ are regarded as line bundles). This is a $\mathbb{P}^1$-bundle over X. Let $\iota\colon X\hookrightarrow\mathbb{P}(K_X\oplus\mathcal{O}_X)$ be the inclusion as the zero section of $K_X\subset\mathbb{P}(K_X\oplus\mathcal{O}_X)$. We define the (ordinary) GW invariant $N_{g,\iota_*\beta}(\mathbb{P}(K_X\oplus\mathcal{O}_X))$ of $\mathbb{P}(K_X\oplus\mathcal{O}_X)$ of genus g and the second homology class $\iota_*\beta$ by

$$N_{g,\iota_*\beta}(\mathbb{P}(K_X\oplus\mathcal{O}_X))=\int_{[\overline{M}_{g,0}(\mathbb{P}(K_X\oplus\mathcal{O}_X),\iota_*\beta)]^{\mathrm{vir}}} 1.$$

We note that the deformation invariance is established for this ordinary GW invariant in [23] and [29].

Proposition 2.2. *Let X be a nef surface, $\iota\colon X\hookrightarrow\mathbb{P}(K_X\oplus\mathcal{O}_X)$ be the inclusion as the zero section of K_X. For $g\in\mathbb{Z}_{\geq 0}$ and $\beta\in H_2(X,\mathbb{Z})$ such that $\int_\beta c_1(K_X)<0$,*

$$N_{g,\beta}(K_X)=N_{g,\iota_*\beta}(\mathbb{P}(K_X\oplus\mathcal{O}_X)).$$

Consider the natural $\mathbb{C}^*$-action on $\mathbb{P}(K_X\oplus\mathcal{O}_X)$ as the scalar multiplication in the $\mathbb{P}^1$-fiber direction. The action induces an action on $\overline{M}_{g,0}(\mathbb{P}(K_X\oplus\mathcal{O}_X),\iota_*\beta)$ by moving the image curves of stable maps. First we show the following lemma.

(3) The condition $\int_\beta c_1(K_X)<0$ and the nef condition on X imply that $H^0(C,f^*K_X)=0$ for $(f,C)\in\overline{M}_{g,0}(X,\beta)$.

Lemma 2.3. *Let X be a nef surface, $\iota\colon X\hookrightarrow\mathbb{P}(K_X\oplus\mathcal{O}_X)$ be the inclusion as the zero section of K_X. Let $\beta\in H_2(X,\mathbb{Z})$ be a class satisfying $\int_\beta c_1(K_X)<0$. If a stable map $(f,C)\in\overline{M}_{g,0}(\mathbb{P}(K_X\oplus\mathcal{O}_X),\iota_*\beta)$, where C is a connected curve of genus g and $f\colon C\to\mathbb{P}(K_X\oplus\mathcal{O}_X)$ is a morphism such that $[f(C)]=\iota_*\beta$, is fixed by the $\mathbb{C}^*$-action, then the image $f(C)$ is contained in the zero section $\iota(X)$.*

Proof. Denote the $\mathbb{P}^1$-fibration $\mathbb{P}(K_X\oplus\mathcal{O}_X)\to X$ by p, and let $P=[p^{-1}(a)]\in H_2(\mathbb{P}(K_X\oplus\mathcal{O}_X),\mathbb{Z})$ be the class of the fiber $\mathbb{P}^1$, where $a\in X$ is any point. Let $\iota^\infty\colon X\hookrightarrow\mathbb{P}(K_X\oplus\mathcal{O}_X)$ be the inclusion as the zero section of $\mathcal{O}_X$ (the section at the infinity of the $\mathbb{P}^1$-bundle compactification of K_X). Note that for any $\alpha\in H_2(X,\mathbb{Z})$, we have

$$\iota^\infty_*\alpha=\iota_*\alpha-\left(\int_\alpha c_1(K_X)\right)P. \tag{2.1}$$

Let $\gamma\in H_2(\mathbb{P}(K_X\oplus\mathcal{O}_X),\mathbb{Z})$. If a stable map $(f,C)\in\overline{M}_{g,0}(\mathbb{P}(K_X\oplus\mathcal{O}_X),\gamma)$ is fixed by the $\mathbb{C}^*$-action, then the image of an irreducible component C_i of C must be either one of these: (i) $f(C_i)\subset\iota(X)$, (ii) $f(C_i)\subset\iota^\infty(X)$ or (iii) $f(C_i)=p^{-1}(a_i)$, $a_i\in X$, and $C_i\cong\mathbb{P}^1$. So assume that the irreducible components $C_1,\dots,C_k$ of C are of type (i) with $[f(C_i)]=\beta_i\in H_2(X,\mathbb{Z})$, that $C_{k+1},\dots,C_r$ are of type (ii) with $[f(C_i)]=\beta_i\in H_2(X,\mathbb{Z})$, and that $C_{r+1},\dots,C_s$ are of type (iii) with $f\colon C_i\to p^{-1}(a_i)$ being the d_i-fold coverings. Then $[f(C)]=\gamma$ is equivalent to

$$\gamma=\sum_{i=1}^k\iota_*\beta_i+\sum_{i=k+1}^r\iota^\infty_*\beta_i+\sum_{i=r+1}^s d_iP=\sum_{i=1}^r\iota_*\beta_i+\left(\sum_{i=r+1}^s d_i-\sum_{i=k+1}^r\int_{\beta_i}c_1(K_X)\right)P.$$

Now take $\gamma=\iota_*\beta$ with $\beta\in H_2(X,\mathbb{Z})$ satisfying $\int_\beta c_1(K_X)<0$ and solve the above equation. The assumption that X is nef implies that the coefficient of P in the last line is always nonnegative. Therefore it is zero if and only if there is no irreducible components of type (iii) and $\int_{\beta_i}c_1(K_X)=0$ for those of type (ii). Then connectedness of the domain curve C implies that either $f(C)\subset\iota(X)$ or $f(C)\subset\iota^\infty(X)$. For the latter case, $\int_{[f(C)]}c_1(K_X)=0$ and this contradicts the assumption $\int_\beta c_1(K_X)<0$. Thus $f(C)\subset\iota(X)$. □

Proof of Proposition 2.2. By Lemma 2.3, the $\mathbb{C}^*$-fixed point set is isomorphic to $\overline{M}_{g,0}(X,\beta)$. Then, by the virtual localization [11],

$$N_{g,\iota_*\beta}(\mathbb{P}(K_X\oplus\mathcal{O}_X))=\int_{[\overline{M}_{g,0}(X,\beta)]^{\mathrm{vir}}}e_{\mathbb{C}^*}(R^1\pi_*\mu^*K_X).$$

Here $e_{\mathbb{C}^*}$ is the equivariant Euler class. (In the equation below [11, (24)], the nontrivial contribution comes only from the factor $e(B_5^m)$; $e(B_2^m)$ does not contribute because $\int_\beta c_1(K_X)<0$.) Since the left-hand side is independent of the weight, so is the right-hand side and we can replace it with the nonequivariant integral. □

Proposition 2.4. *Let X be a nef surface and X' be a nef surface which is deformation equivalent to X. Let $\beta\in H_2(X,\mathbb{Z})$ be a class satisfying $\int_\beta c_1(K_X)<0$ and $\beta'\in H_2(X',\mathbb{Z})$ be the class corresponding to β under a deformation. Then $N_{g,\beta}(K_X)=N_{g,\beta'}(K_{X'})$ for $g\in\mathbb{Z}_{\geq 0}$.*

Proof. Since the surfaces X and X' are deformation equivalent, $\mathbb{P}(K_X\oplus\mathcal{O}_X)$ and $\mathbb{P}(K_{X'}\oplus\mathcal{O}_{X'})$ are also deformation equivalent. Let $\iota\colon X\hookrightarrow\mathbb{P}(K_X\oplus\mathcal{O}_X)$ and $\iota'\colon X'\hookrightarrow\mathbb{P}(K_{X'}\oplus\mathcal{O}_{X'})$ be the inclusions as the zero sections of K_X and $K_{X'}$, respectively.

We have

$$N_{g,\beta}(K_X)=N_{g,\iota_*\beta}(\mathbb{P}(K_X\oplus\mathcal{O}_X))=N_{g,\iota'_*\beta'}(\mathbb{P}(K_{X'}\oplus\mathcal{O}_{X'}))=N_{g,\beta'}(K_{X'}).$$

The middle equality follows from the deformation invariance of ordinary GW invariants [23], [29]. The first and the third equalities follow from Proposition 2.2. □

3. Cubic surfaces S_3

Here we summarize some facts on cubic surfaces, see e.g. [12, Section V 4] for details.

Let S_3 be a cubic surface. S_3 is realized as a blowing up $\pi\colon S_3\to\mathbb{P}^2$ at six points in a general position. Let $e_1,...,e_6$ be the classes of the exceptional curves of π and let l be the class of a line in $\mathbb{P}^2$ pulled back by π. Then $l,e_1,...,e_6$ form a basis of $\mathrm{Pic}(S_3)$. Their intersections are

$$l^2=1,\quad e_i^2=-1,\quad l\cdot e_i=0,\quad \text{and}\quad e_i\cdot e_j=0 \text{ if } i\neq j.$$

Let h be the class of the hyperplane section of $\mathbb{P}^3$. Then we have

$$h=-K_{S_3}=3l-\sum_{i=1}^{6}e_i.$$

It is a classical fact that S_3 contains exactly twenty-seven lines which are given as follows:

$$e_i,\ i=1,...,6,\quad l-e_i-e_j,\ 1\leq i<j\leq 6,\quad \text{and}\quad 2l-\sum_{i\neq j}e_i,\ j=1,...,6.$$

Each one of these is an exceptional curve of the first kind. These twenty-seven lines are the minimal generators of the Mori cone (the cone generated by effective divisors on X modulo numerical equivalence) (cf. [26, (0.6)]).

It is well-known that the Weyl group W_{E_6} of type E_6 acts on $\mathrm{Pic}(S_3)$ as symmetries of configurations of twenty seven lines. Its generators are given as follows.

$$s_i : e_i \longleftrightarrow e_{i+1},\ 1 \leq i \leq 5,$$
$$s_6 : e_1 \longmapsto l-e_2-e_3, \quad e_2 \longmapsto l-e_1-e_3, \quad e_3 \longmapsto l-e_1-e_2, \quad l \longmapsto 2l-e_1-e_2-e_3.$$

It is known [9, Section 4] that W_{E_6} coincides with the group of automorphisms of $\mathrm{Pic}(S_3)$ which preserve the intersection form, the canonical class, and the semigroup of effective classes.

Hereafter we identify $\mathrm{Pic}(S_3)$ with $H^2(S_3,\mathbb{Z}) \cong H_2(S_3,\mathbb{Z})$.

Lemma 3.1. *$N_{g,\beta}(K_{S_3}) = N_{g,w(\beta)}(K_{S_3})$ for $w \in W_{E_6}$.*

Proof. See e.g. [13, Section 2.4]. □

4. Nef toric surfaces deformation equivalent to S_3, S_4 and S_5

Let S_3^0, S_4^0 and S_5^0 be the nef toric surfaces whose fans are given in Figure 4.1. Here the nine one-dimensional cones of S_3^0 are generated by

$$v_1 = (1,0), \quad v_2 = (0,1), \quad v_3 = (-1,2),$$
$$v_4 = (-1,1), \quad v_5 = (-1,0), \quad v_6 = (-1,-1),$$
$$v_7 = (0,-1), \quad v_8 = (1,-1), \quad v_9 = (2,-1).$$

Let the fan of the toric del Pezzo surface S_6 be given in Figure 4.1 and let p_1, p_2 and p_3 be the torus fixed points of S_6 corresponding to the two-dimensional cones generated by (v_5,v_7), (v_8,v_1) and (v_2,v_4). S_3^0 (resp. S_4^0 and S_5^0) is obtained by blowing up S_6 at p_1, p_2 and p_3 (resp. p_1, p_2, and p_1). S_k^0 contains (-2)-curves and its anticanonical divisor is nef but not ample.

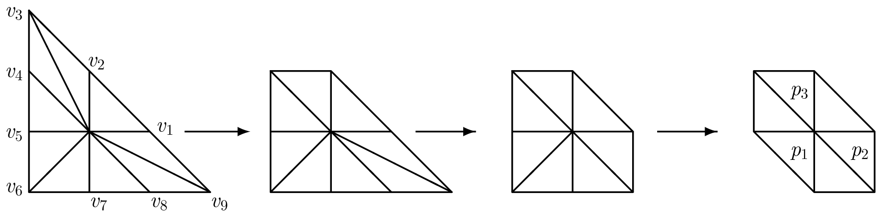

Figure 4.1. $S_3^0 \to S_4^0 \to S_5^0 \to S_6$.

Proposition 4.1. *S_k^0, $k=3,4,5$, is deformation equivalent to S_k.*

A proof will be given in Appendix A (see Proposition A.2).

Now let us explain the geometry of the nef toric surface S_3^0. The torus-invariant divisors C_i, $1\le i\le 9$, corresponding to v_i have the intersections:

$$C_i\cdot C_{i+1}=1, \quad C_i\cdot C_j=0,\ j\neq i, i\pm 1, \quad C_i^2=\begin{cases} -1 & i=3,6,9, \\ -2 & i=1,2,4,5,7,8, \end{cases} \tag{4.1}$$

and the canonical divisor $K_{S_3^0}$ is rationally equivalent to $-C_1-...-C_9$. The Mori cone is generated by $C_1, ..., C_9$ [27, Proposition 2.26].

Note that $\mathrm{Pic}(S_3^0)\cong\mathrm{Pic}(S_3)$ and an isomorphism is given by

$$\begin{aligned} &C_1\longmapsto e_2-e_5, \quad C_2\longmapsto l-e_2-e_3-e_6, \quad C_3\longmapsto e_6, \\ &C_4\longmapsto e_3-e_6, \quad C_5\longmapsto l-e_1-e_3-e_4, \quad C_6\longmapsto e_4, \\ &C_7\longmapsto e_1-e_4, \quad C_8\longmapsto l-e_1-e_2-e_5, \quad C_9\longmapsto e_5. \end{aligned} \tag{4.2}$$

This is explained as follows. First, in S_6, we regard the torus-invariant divisors C_1', C_4' and C_7' corresponding to v_1, v_4 and v_7 as the exceptional curves of blowing up of $\mathbb{P}^2$ and identify them with e_2, e_3 and e_1. The torus-invariant divisors C_2', C_5' and C_8' corresponding to v_2, v_5 and v_8 are identified with the proper transforms $l-e_2-e_3$, $l-e_1-e_3$ and $l-e_1-e_2$ of lines in $\mathbb{P}^2$. Then in S_3^0, C_3, C_6 and C_9 are exceptional curves of the blowup at p_3, p_1 and p_2 and we identify them with e_6, e_4 and e_5. For $i=1,2,4,5,7,8$, C_i is the proper transform of C_i'. (This identification can be seen from the construction of a deformation in the proof of Proposition A.2.)

From here on, we identify $\mathrm{Pic}(\mathrm{S}_3^0)$ with $H^2(S_3^0,\mathbb{Z})\cong H_2(S_3^0,\mathbb{Z})$.

Theorem 4.2. *For $g\in\mathbb{Z}_{\ge 0}$ and $\beta\in H_2(S_3,\mathbb{Z})$ such that $K_{S_3}\cdot\beta<0$,*

$$N_{g,\beta}(K_{S_3})=N_{g,\beta'}(K_{S_3^0}),$$

where $\beta'\in H_2(S_3^0,\mathbb{Z})$ is the class corresponding to β by (4.2).

Proof. This follows from Propositions 2.4 and 4.1. □

Remark 4.3. The statements similar to Theorem 4.2 hold for S_4 and S_5: local GW invariants of S_4 and S_5 are the same as those of S_4^0 and S_5^0. Their generating functions also have expressions analogous to the formula for S_3 (which will be stated in Theorem 5.2). Local GW invariants of S_4 and S_5 appear among those of S_3 with a natural identification of second homology classes $H_2(S_3,\mathbb{Z})=H_2(S_4,\mathbb{Z})\oplus\mathbb{Z}e_6=H_2(S_5,\mathbb{Z})\oplus\mathbb{Z}e_5\oplus\mathbb{Z}e_6$. See [20, §6].

5. A formula for the generating function of local GW invariants of S_3

5.1. First we consider the generating function of local GW invariants of S_3^0 with $\beta \in H_2(S_3^0,\mathbb{Z})$ such that $K_{S_3^0}\cdot\beta<0$. Take a basis $c_1,\dots,c_7$ of $H_2(S_3^0,\mathbb{Z})$ and let $X_1,\dots,X_7$ be associated formal variables. For $\beta=a_1c_1+\dots+a_7c_7\in H_2(S_3^0,\mathbb{Z})$, denote $X_1^{a_1}\dots X_7^{a_7}$ by X^β. We write the generating function as

$$F_{S_3^0}=\sum_{\substack{\beta\in H_2(S_3^0,\mathbb{Z})\\ K_{S_3^0}\cdot\beta<0}}\sum_{g\geq 0} N_{g,\beta}(K_{S_3^0})\lambda^{2g-2}X^\beta.$$

Let $t_i=X^{[C_i]}$, $1\leq i\leq 9$, and $s_i=C_i^2$ (see (4.1)). Define $Z_{S_3^0}$ by

$$Z_{S_3^0}=\prod_{i=1}^{9}\sum_{\nu^i}((-1)^{s_i}t_i)^{|\nu^i|}e^{\sqrt{-1}\lambda s_i\varkappa(\nu^i)/2}W_{\nu^i,\nu^{i+1}}(e^{\sqrt{-1}\lambda}).$$

Here each ν^i, $1\leq i\leq 9$, runs over the set of partitions and $\nu^{10}=\nu^1$ is assumed. For partitions $\mu=(\mu_1,\mu_2,\dots)$ and $\nu=(\nu_1,\nu_2,\dots)$,

$$W_{\mu,\nu}(q)=s_\mu(q^\rho)s_\nu(q^{\mu+\rho})\in\mathbb{Q}(q^{1/2}),\quad |\mu|=\sum_{i\geq 1}\mu_i,\quad \varkappa(\mu)=\sum_{i\geq 1}\mu_i(\mu_i-2i+1),$$

where $q^{\mu+\rho}=(q^{\mu_i-i+1/2})_{i\geq 1}$, $q^\rho=(q^{-i+1/2})_{i\geq 1}$ and s_μ denotes the Schur function. Define $Z_{(-2)}(t)$ by

$$Z_{(-2)}(t)=\exp\left[-\sum_{j\geq 1}\frac{1}{j}\left(2\sin\frac{j\lambda}{2}\right)^{-2}t^j\right].$$

Lemma 5.1.

$$\exp(F_{S_3^0})=\frac{Z_{S_3^0}}{\prod_{i=1,4,7}Z_{(-2)}(t_i)Z_{(-2)}(t_{i+1})Z_{(-2)}(t_it_{i+1})}.$$

Proof. Recall that S_3^0 has a canonical $T=(\mathbb{C}^*)^2$-action determined by its fan. Let $K_{S_3^0}^T=-C_1-\dots-C_9$ be a T-invariant divisor. For any $\beta\in H_2(S_3^0,\mathbb{Z})$ and $g\in\mathbb{Z}_{\geq 0}$, define $N_{g,\beta}^T(S_3^0)$ by the following equivariant integral:

$$N_{g,\beta}^T(S_3^0)=\int_{[\overline{M}_{g,0}(S_3^0,\beta)^T]^{\mathrm{vir}}}\frac{e_T(R^1\pi_*\mu^*K_{S_3^0}^T)}{e_T(R^0\pi_*\mu^*K_{S_3^0}^T)}\frac{1}{e_T(\mathrm{Norm})}.$$

Here $\overline{M}_{g,0}(S_3^0,\beta)^T$ is the fixed point set of the induced T-action, e_T denotes the equivariant Euler class and Norm is the virtual normal bundle determined by the obstruction theory [11, (23) and (24)]. Note that $N^T_{g,\beta}(S^0_3)=0$ if there is no effective divisors of the form $\sum_{1\le i\le 9} a_i[C_i]$, $a_i\in\mathbb{Z}_{\ge 0}$, which are rationally equivalent to β, because $\overline{M}_{g,0}(S^0_3,\beta)^T$ is empty.

Consider the exponential of the generating function for *all* classes

$$\exp\Bigg[\sum_{\beta\in H_2(S^0_3,\mathbb{Z})}\sum_{g\ge 0} N^T_{g,\beta}(S^0_3)\lambda^{2g-2}X^\beta\Bigg]. \tag{5.1}$$

Carrying out the localization calculation in the same way as [31]([4]) and using the formula for Hodge integrals [24, Theorem 1], we see that (5.1) is equal to $Z_{S^0_3}$.

Next we have to subtract the contributions coming from the classes β which do not satisfy the inequality $K_{S^0_3}\cdot\beta<0$. Note that such effective classes are of the forms $a[C_1]+b[C_2]$, $a[C_4]+b[C_5]$ or $a[C_7]+b[C_8]$, $a,b\in\mathbb{Z}_{\ge 0}$. Therefore

$$\begin{aligned}(5.2)\ \exp\Bigg[&\sum_{\substack{\beta\in H_2(S^0_3,\mathbb{Z})\\ K_{S^0_3}\cdot\beta\ge 0}}\sum_{g\ge 0} N^T_{g,\beta}(S^0_3)\lambda^{2g-2}X^\beta\Bigg]\\ &=\prod_{i=1,4,7}\exp\Bigg[\sum_{a,b\in\mathbb{Z}_{\ge 0}}\sum_{g\ge 0} N^T_{g,a[C_i]+b[C_{i+1}]}(S^0_3)\lambda^{2g-2}t_i^a t_{i+1}^b\Bigg].\end{aligned}$$

The $i=1$ factor is easily obtained by setting $t_3=t_4=...=t_9=0$ in (5.1). It is equal to

$$Z_{S^0_3}|_{t_3=t_4=\ldots=t_9=0}=Z_{(-2)}(t_1)Z_{(-2)}(t_2)Z_{(-2)}(t_1t_2).$$

The $i=4,7$ factors are similar. Dividing (5.1) by (5.2), we obtain that

$$\begin{aligned}\exp\Bigg[&\sum_{\substack{\beta\in H_2(S^0_3,\mathbb{Z})\\ K_{S^0_3}\cdot\beta<0}}\sum_{g\ge 0} N^T_{g,\beta}(S^0_3)\lambda^{2g-2}X^\beta\Bigg]\\ &=\frac{Z_{S^0_3}}{\prod_{i=1,4,7} Z_{(-2)}(t_i)Z_{(-2)}(t_{i+1})Z_{(-2)}(t_it_{i+1})}.\end{aligned}$$

By the virtual localization [11], $N^T_{g,\beta}(S^0_3)=N_{g,\beta}(S^0_3)$ for β such that $K_{S^0_3}\cdot\beta<0$. Thus we complete our proof. □

([4]) The contribution to $N^T_{g,\beta}(S^0_3)$ from a fixed locus turns out to be completely the same as [31, (13) and (16)]. Thus the summation over genera, second homology classes and fixed loci proceeds in the same manner.

5.2. Next we study the generating function of local GW invariants of S_3. Let $Q=(Q_1,\dots,Q_6,Q_7)$ be a set of formal variables and denote $Q_1^{a_1}Q_2^{a_2}\dots Q_7^{a_7}$ by Q^β for $\beta=a_1e_1+\dots+a_6e_6+a_7l\in H_2(S_3,\mathbb{Z})$. Define

$$F_d=\sum_{\substack{\beta\in H_2(S_3,\mathbb{Z})\\ -K_{S_3}\cdot\beta=d}}\sum_{g\in\mathbb{Z}_{\geq 0}} N_{g,\beta}(K_{S_3})\lambda^{2g-2}Q^\beta,\quad d\in\mathbb{Z}_{\geq 1},$$

and $F_{S_3}:=\sum_{d\geq 1}F_d$.

Theorem 5.2. *With the following identification of the parameters*

$$\begin{aligned}
&t_1=Q^{e_2-e_5}, && t_2=Q^{l-e_2-e_3-e_6}, && t_3=Q^{e_6},\\
&t_4=Q^{e_3-e_6}, && t_5=Q^{l-e_1-e_3-e_4}, && t_6=Q^{e_4},\\
&t_7=Q^{e_1-e_4}, && t_8=Q^{l-e_1-e_2-e_5}, && t_9=Q^{e_5},
\end{aligned}\tag{5.3}$$

we have

$$\exp(F_{S_3})=\exp(F_{S_3^0}).$$

Proof. This follows from Theorem 4.2 and Lemma 5.1. The identification (5.3) is determined by (4.2). □

Remark 5.3. In [5], Diaconescu and Florea obtained a formula for F_{S_3} which is different from ours ((3.14) for $k=5$ in [5]). It would be an interesting problem to show that these two formulas are equivalent.

Define $m(\beta)$ for $\beta\in H_2(S_3,\mathbb{Z})$ by

$$m(\beta)=\frac{1}{\#\{w\in W_{E_6}\mid w(\beta)=\beta\}}\sum_{w\in W_{E_6}}Q^{w(\beta)}.$$

By Lemma 3.1, F_d should be written in terms of these numbers. F_d up to $d=6$ are shown in Appendix C.

6. Gopakumar–Vafa invariants

Let $n^g_\beta(K_{S_3})$, $g\in\mathbb{Z}_{\geq 0}$, $\beta\in H_2(S_3,\mathbb{Z})$, be numbers defined by

$$F_{S_3}=\sum_{\beta\in H_2(S_3,\mathbb{Z})}\sum_{g\in\mathbb{Z}_{\geq 0}}\sum_{k\geq 1}\frac{n^g_\beta(K_{S_3})}{k}\left(2\sin\frac{k\lambda}{2}\right)^{2g-2}Q^{k\beta}.$$

$n^g_\beta(K_{S_3})$ are called *Gopakumar–Vafa invariants* [10]. They are listed in Table 6.1.

Table 6.1. Gopakumar–Vafa invariants $n^g_\beta(K_{S_3})$.

d	β	$\#\mathcal{O}(\beta)$	genus	g 0	1	2	3	4	5
1	e_6	27	0	1					
2	$-e_1+l$	27	0	−2					
3	l	72	0	3					
	$-e_1-e_2-e_3-e_4-e_5-e_6+3l$	1	1	27	−4				
4	$-e_1-e_2+2l$	216	0	−4					
	$-e_1-e_2-e_3-e_4-e_5+3l$	27	1	−32	5				
5	$-e_1+2l$	432	0	5					
	$-e_1-e_2-e_3-e_4+3l$	216	1	35	−6				
	$-2e_1-e_2-e_3-e_4-e_5-e_6+4l$	27	2	205	−68	7			
6	$-2e_1-e_2+3l$	432	0	−6					
	$2l$	72	0	−6					
	$-e_1-e_2-e_3+3l$	720	1	−36	7				
	$-2e_1-e_2-e_3-e_4-e_5+4l$	270	2	−198	72	−8			
	$-e_1-e_2-e_3-e_4-e_5-e_6+4l$	72	3	−936	498	−108	9		
	$-2e_1-2e_2-2e_3-2e_4-2e_5-2e_6+6l$	1	4	−3780	2636	−846	141	−10	

Remark 6.1. (a) Gopakumar–Vafa invariants $n^g_\beta(K_{S_3})$ of S_3 are integers. Moreover, for each β, $n^g_\beta(K_{S_3})$ is equal to zero for all but a finite number of g. This follows from the same statement for the toric surface S^0_3 (see [28] and [19]).

(b) One could observe that $n^g_\beta(K_{S_3})$ in Table 6.1 are zero if g is larger than the genus $\beta\cdot(\beta+K_{S_3})/2+1$ of a nonsingular curve which belongs to β.

(c) The results are in agreement with the previous results in [4, Table 7, $X_3(1,1,1,1)$], [22, Table 1, $n{=}6$] and [25, Table 3] obtained by the B-model calculation of mirror symmetry. Also compare with [15, Table 7].

A. Nef toric surfaces and their deformations

The following classification is given in the preprint version of [3] (see also [4, Table 1]).

Lemma A.1. *There are exactly* 16 *nef toric surfaces, whose fans are shown in Figure* A.1.

We will refer to the nef toric surfaces using the numbers shown in the frames in Figure A.1.

Proof. The minimal nef toric surfaces are $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$ and the Hirzebruch surface $\mathbb{F}_2$, which are nos. 1, 2, and 4, respectively. Nef toric surfaces are obtained from them by blowing up at torus-fixed points successively. By the nef condition, we must blow-up at torus-fixed points which is not on a torus-fixed (-2)-curve. All possible patterns of blowing-ups are listed in Figure A.1. Note that nos. 13, 15

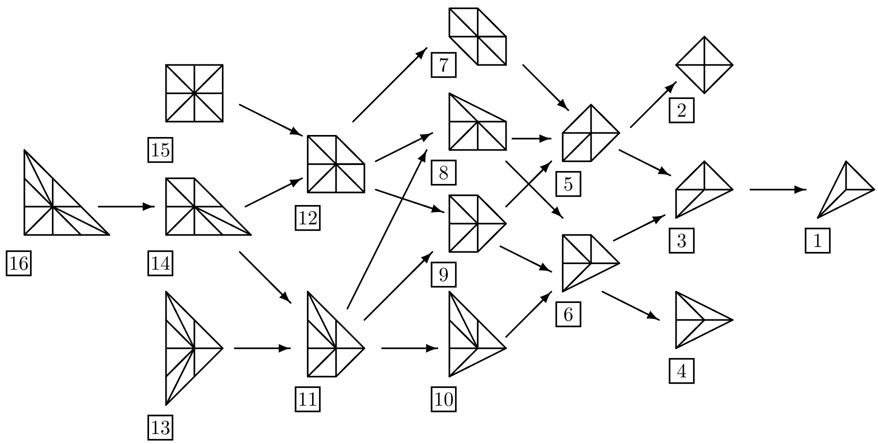

Figure A.1. Classification of nef toric surfaces. The arrows indicate blow-downs. The numbers in the frames are reference numbers. Note that S_3^0, S_4^0 and S_5^0 introduced in Section 4 are nos. 16, 14, and 12, respectively.

and 16 can no longer be blown-up to nef toric surfaces, since all of their torus-fixed points are on a torus-fixed (-2)-curve. This completes the classification. □

Proposition A.2. *A nef toric surface has a smooth versal deformation family of dimension $h^1(\Theta)$, where Θ is the sheaf of germs of holomorphic vector fields, whose general member is a del Pezzo surface of degree c_1^2.*

$h^1(\Theta)$ and c_1^2 are given in Table A.1.

Table A.1. Eight deformation types and $h^1(\Theta)(=-(7c_1^2-5c_2)/6+h^0(\Theta)+h^2(\Theta))$.

Deformation type	I	II	III		IV		V				VI		VII			VIII
no.	1	3	2	4	5	6	7	8	9	10	11	12	13	14	15	16
c_1^2	9	8	8		7		6				5		4			3
c_2	3	4	4		5		6				7		8			9
$-(7c_1^2-5c_2)/6$	−8	−6	−6		−4		−2				0		2			4
$h^0(\Theta)$	8	6	6	7	4	5	2	4	3	5	3	2	3	2	2	2
$h^2(\Theta)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$h^1(\Theta)$	0	0	0	1	0	1	0	2	1	3	3	2	5	4	4	6

Proof. Note that $h^2(\Theta)=0$ for any smooth compact toric surface (Corollary B.2). This implies smoothness of a versal deformation family [18].

Versal deformation families of nef toric surfaces are constructed inductively as follows. Let $\pi\colon \widetilde{S}\to S$ be one of the blowing-ups in Figure A.1. Let $P\in S$ be the center of the blowing-up π which is the intersection of two torus-fixed curves C_1 and C_2 (see Figure A.2). By comparing Table A.1 with Figure A.1, we have

$$h^1(\widetilde{S},\Theta)=\begin{cases} h^1(S,\Theta), & \text{if } C_1^2>-1, \text{ and } C_2^2>-1,\\ h^1(S,\Theta)+1, & \text{if } C_1^2=-1, \text{ and } C_2^2>-1,\\ h^1(S,\Theta)+2, & \text{if } C_1^2=C_2^2=-1. \end{cases} \tag{A.1}$$

Since smooth rational curves on complex surfaces with self-intersection ≥ -1 are stable under small deformations [17, example on p. 86] (see also [2, Chapter IV, Proposition (3.1)]), a complete deformation family of $\widetilde{S}$ can be found as a simultaneous blowing-up of a complete deformation family of S. Furthermore, by (A.1), we can find a versal deformation family of $\widetilde{S}$ as follows. First, we consider a versal deformation family $\mathcal{S}$ of S on which C_1 and C_2 deform holomorphically. If both C_1 and C_2 have self-intersection >-1, simultaneous blowing up of $\mathcal{S}$ at P gives a versal deformation family of $\widetilde{S}$ which is of dimension $h^1(S,\Theta)$. If $C_1^2=-1$ and $C_2^2>-1$, we move the center P in the C_2 direction (see Figure A.2) and blow $\mathcal{S}$ up simultaneously to get a versal deformation family of $\widetilde{S}$ which is of dimension $h^1(S,\Theta)+1$. If $C_1^2=C_2^2=-1$, we move the center P in the whole direction and blow $\mathcal{S}$ up simultaneously to get a versal deformation family of $\widetilde{S}$ which is of dimension $h^1(S,\Theta)+2$.

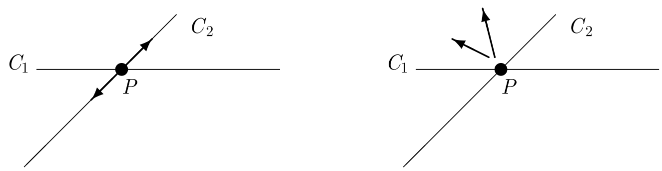

Figure A.2. The center P of a blowing-up (C_1 and C_2 are torus-fixed curves) and its moving. The left is the case with $C_1^2=-1$ and $C_2^2\geq 0$ and the right is the case with $C_1^2=C_2^2=-1$.

Thus we can find versal deformation families of nef toric surfaces inductively. It is easy to see that their general members are del Pezzo surfaces. □

B. Unobstructedness

Let X be a smooth compact toric surface, $D:=D_1+...+D_r$ be the sum of all torus invariant divisors $D_1,...,D_r$, and $\Theta(-\log D)$ be the sheaf of germs of holomorphic vector fields with logarithmic zeros along D.

Lemma B.1. $H^2(X,\Theta(-\log D))=0$.

Proof. Since $\Theta(-\log D)=\mathcal{O}\otimes_{\mathbb{Z}}N$ (cf. [27, Proposition 3.1]), where N is the 2-dimensional lattice such that the fan of X sits in $N\otimes\mathbb{R}$. $H^2(X,\Theta(-\log D))=H^2(X,\mathcal{O}\otimes_{\mathbb{Z}}N)=H^2(X,\mathcal{O}\oplus\mathcal{O})=0$, as $H^2(X,\mathcal{O})=0$ (cf. [27, Corollary 2.8]). □

Corollary B.2. $H^2(X,\Theta)=0$.

Proof. From the exact sequence (cf. [27, Theorem 3.12])

$$0\longrightarrow\Theta(-\log D)\longrightarrow\Theta\longrightarrow\bigoplus_{i=1}^{r}\mathcal{O}(D_i)|_{D_i}\longrightarrow 0,$$

and Lemma B.1, we have $H^2(X,\Theta)=0$. □

C. F_d, $1\leq d\leq 6$

Let $b[k]:=(2\sin(k\lambda/2))^2$.

$$F_1=\frac{1}{b[1]}m(e_6),$$
$$F_2=\frac{1}{2b[2]}m(2e_6)+\frac{-2}{b[1]}m(-e_1+l),$$
$$F_3=\frac{1}{3b[3]}m(3e_6)+\frac{3}{b[1]}m(l)+\left(-4+\frac{27}{b[1]}\right)m(-e_1-e_2-e_3-e_4-e_5-e_6+3l),$$
$$F_4=\frac{1}{4b[4]}m(4e_6)+\frac{-2}{2b[2]}m(-2e_1+2l)+\frac{-4}{b[1]}m(-e_1-e_2+2l)$$
$$+\left(5+\frac{-32}{b[1]}\right)m(-e_1-e_2-e_3-e_4-e_5+3l)$$
$$F_5=\frac{1}{5b[5]}m(5e_6)+\frac{5}{b[1]}m(-e_1+2l)+\left(-6+\frac{35}{b[1]}\right)m(-e_1-e_2-e_3-e_4+3l)$$
$$+\left(7b[1]-68+\frac{205}{b[1]}\right)m(-2e_1-e_2-e_3-e_4-e_5-e_6+4l),$$
$$F_6=\frac{1}{6b[6]}m(6e_6)-\frac{2}{3b[3]}m(-3e_1+3l)+\left(\frac{3}{2b[2]}-\frac{6}{b[1]}\right)m(2l)$$
$$+\left(7-\frac{36}{b[1]}\right)m(-e_1-e_2-e_3+3l)$$
$$+\left(-8b[1]+72-\frac{198}{b[1]}\right)m(-2e_1-e_2-e_3-e_4-e_5+4l)$$
$$+\left(9b[1]^2-108b[1]+498-\frac{936}{b[1]}\right)m(-e_1-e_2-e_3-e_4-e_5-e_6+4l)$$
$$+\left(\frac{1}{2}\left(-4+\frac{27}{b[2]}\right)-10b[1]^3+141b[2]^2-846b[1]+2636-\frac{3780}{b[1]}\right)$$
$$\times m(-2e_1-2e_2-2e_3-2e_4-2e_5-2e_6+6l).$$

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