Acta Math., 216 (2016), 1–125
DOI: 10.1007/s11511-016-0136-2

The Hodge conjecture and arithmetic quotients of complex balls

by

Nicolas Bergeron
Institut de Mathématiques de Jussieu–Paris Rive Gauche
Paris, France

John Millson
University of Maryland
College Park, MD, U.S.A.

Colette Moeglin
Institut de Mathématiques de Jussieu–Paris Rive Gauche
Paris, France

In memory of Raquel Maritza Gilbert, beloved wife of the second author.

1. Introduction

In this paper we study the Betti cohomology $H^\bullet(S)$ of a smooth projective connected Shimura variety S associated with a standard unitary group. Before stating our main results we recall the construction of these Shimura varieties.

1.1. Shimura varieties associated with standard unitary groups

Let F be a totally real field and E be an imaginary quadratic extension of F. Let V be a vector space defined over E and let $(\cdot,\cdot)$ be a non-degenerate Hermitian form on V. We shall always assume that the Hermitian space $(V,(\cdot,\cdot))$ is *anisotropic*, of signature (p,q), with $p,q>0$, at one Archimedean place and positive definite at all other infinite places. Note that if $p+q>2$ this forces $F\neq\mathbb{Q}$.

Let G be the $\mathbb{Q}$-reductive group obtained from the group of isometries of $(\cdot,\cdot)$ on V, by restricting scalars from F to $\mathbb{Q}$. The real group $G(\mathbb{R})$ is isomorphic to the product $\prod_{j=1}^d U(V_{\tau_j})$ where the V_{τ_j}'s are the completions of V with respect to the different complex embeddings τ_j of E considered up to complex conjugation. By hypothesis,

N. B. is a member of the Institut Universitaire de France.
J. M. was partially supported by NSF grant DMS-1206999.

we therefore have $G(\mathbb{R})\cong U(p,q)\times U(m)^{d-1}$. We denote by K_∞ the maximal compact subgroup of G.

A *congruence subgroup* of $G(\mathbb{Q})$ is a subgroup $\Gamma=G(\mathbb{Q})\cap K$, where K is a compact open subgroup of $G(\mathbb{A}_f)$ of the finite adelic points of G. The (connected) Shimura variety $S=S(\Gamma)=\Gamma\backslash X$ is obtained as the quotient of the Hermitian symmetric space $X=G/K_\infty=U(p,q)/(U(p)\times U(q))$ by the congruence subgroup Γ. We will refer to these Shimura varieties as associated with a standard unitary group $U(p,q)$ (associated with a matrix algebra rather than a general simple algebra). We will be particularly interested in the case $q=1$, when X identifies with the unit ball in $\mathbb{C}^p$.

Since $(\cdot,\cdot)$ is supposed to be anisotropic, the Shimura variety S is a *projective* complex manifold; it has a canonical model, defined over a finite abelian extension of F, that fixes a choice of polarization. See §6 for more details.

1.2. Refined Hodge–Lefschetz decomposition of $H^\bullet(S,\mathbb{C})$

Let $\mathfrak{p}_0$ be the tangent space of X associated with the class of the identity in $U(p,q)$ and let $\mathfrak{p}$ be its complexification.

The group $\mathrm{GL}(p,\mathbb{C})\times\mathrm{GL}(q,\mathbb{C})$, seen as the complexification of the maximal compact subgroup $U(p)\times U(q)$ of $U(p,q)$, acts naturally on $\mathfrak{p}$. As first suggested by Chern [10] the corresponding decomposition of $\bigwedge^\bullet\mathfrak{p}^*$ into irreducible modules induces a decomposition of the exterior algebra

$$\bigwedge\nolimits^\bullet(T^*_{\mathbb{C}}S)=\Gamma\backslash U(p,q)\times_{U(p)\times U(q)}\bigwedge\nolimits^\bullet(\mathfrak{p}^*). \tag{1.1}$$

This decomposition commutes with the Laplacian, giving birth to a decomposition of the cohomology $H^\bullet(S,\mathbb{C})$ refining the Hodge-Lefschetz decomposition, compare [10, bottom of p. 105]. We refer to these spaces of sections as *refined Hodge types*.

The symmetric space X, being of Hermitian type, contains an element c belonging to the center of $U(p)\times U(q)$ such that $\mathrm{Ad}(c)$ induces multiplication by $i=\sqrt{-1}$ on $\mathfrak{p}_0$. Let

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}'\oplus\mathfrak{p}''$$

be the associated decomposition of $\mathfrak{g}=\mathfrak{gl}_{p+q}(\mathbb{C})$—the complexification of $\mathfrak{u}(p,q)$. Thus $\mathfrak{p}'=\{X\in\mathfrak{p}:\mathrm{Ad}(c)X=iX\}$ is the holomorphic tangent space. The exterior algebra $\bigwedge^\bullet\mathfrak{p}$ decomposes as

$$\bigwedge\nolimits^\bullet\mathfrak{p}=\bigwedge\nolimits^\bullet\mathfrak{p}'\otimes\bigwedge\nolimits^\bullet\mathfrak{p}''. \tag{1.2}$$

In the case $q=1$ (then X is the complex hyperbolic space of complex dimension p) it is an exercise to check that the decomposition of (1.2) into refined Hodge type recovers the usual Hodge–Lefschetz decomposition. But in general the decomposition is much finer and it is hard to write down the full decomposition of (1.2) into irreducible modules. Indeed: as a representation of $\mathrm{GL}(p,\mathbb{C})\times\mathrm{GL}(q,\mathbb{C})$ the space $\mathfrak{p}'$ is isomorphic to $V_+\otimes V_-^*$ where $V_+=\mathbb{C}^p$ (resp. $V_-=\mathbb{C}^q$) is the standard representation of $\mathrm{GL}(p,\mathbb{C})$ (resp. $\mathrm{GL}(q,\mathbb{C})$) and the decomposition of $\bigwedge^\bullet\mathfrak{p}'$ is already quite complicated (see [16, equation (19), p. 121]):

$$\bigwedge\nolimits^R(V_+\otimes V_-^*)\cong\bigoplus_{\lambda\vdash R}S_\lambda(V_+)\otimes S_{\lambda^*}(V_-)^*. \tag{1.3}$$

Here we sum over all partitions of R (equivalently Young diagrams of size $|\lambda|=R$) and λ^* is the conjugate partition (or transposed Young diagram).

However, it follows from Vogan–Zuckerman [71] that very few of the irreducible submodules of $\bigwedge^\bullet\mathfrak{p}^*$ can occur as refined Hodge types of non-trivial coholomogy classes.

The ones which can occur (and do occur non-trivially for some Γ) are understood in terms of cohomological representations of $U(p,q)$. We review these cohomological representations in §3. We recall in particular how to associate to each cohomological representation π of $U(p,q)$ a *strongly primitive* refined Hodge type. This refined Hodge type corresponds to an irreducible representation $V(\lambda,\mu)$ of $U(p)\times U(q)$ which is uniquely determined by some special pair of partitions (λ,μ) with λ and μ as in (1.3), see [4]; it is an irreducible submodule of

$$S_\lambda(V_+)\otimes S_\mu(V_+)^*\otimes S_{\mu^*}(V_-)\otimes S_{\lambda^*}(V_-)^*.$$

The first degree where such a refined Hodge type can occur is $R=|\lambda|+|\mu|$. We will use the notation $H^{\lambda,\mu}$ for the space of the cohomology in degree $R=|\lambda|+|\mu|$ corresponding to this special Hodge type; more precisely, it occurs in the subspace $H^{|\lambda|,|\mu|}$.

The group $\mathrm{SL}(q)=\mathrm{SL}(V_-)$ acts on $\bigwedge^\bullet\mathfrak{p}^*$. In this paper we shall concentrate on the subring $\mathrm{SH}^\bullet(S,\mathbb{C})$ of the cohomology $H^\bullet(S,\mathbb{C})$ associated with the subalgebra $(\bigwedge^\bullet\mathfrak{p}^*)^{\mathrm{SL}(q)}$ of $\bigwedge^\bullet\mathfrak{p}^*$—that is elements that are trivial on the V_--side. We will refer to the refined Hodge types occurring in $\mathrm{SH}^\bullet(S,\mathbb{C})$ as *special refined Hodge types*.

In §3.10 we introduce an element

$$c_q\in\left(\bigwedge\nolimits^{2q}\mathfrak{p}^*\right)^{U(p)\times\mathrm{SL}(q)},$$

which defines an invariant form on X and a class in $\mathrm{SH}^{2q}(S,\mathbb{C})$; we shall refer to it as the *Chern class/form*. The class $c_q\in H^{2q}(S,\mathbb{C})$ is the qth power of the class associated with our choice of polarization of S; see e.g. [4].

In particular note that if $q=1$ we have that $\mathrm{SH}^\bullet(S,\mathbb{C})=H^\bullet(S,\mathbb{C})$ and $c_1\in H^2(S,\mathbb{C})$ is the class associated with the polarization.

In general if λ is the partition $q+...+q$ (a times) then $S_{\lambda^*}(V_-)$ is the *trivial* representation of $\mathrm{SL}(V_-)$ and $S_\lambda(V_+)\otimes S_{\lambda^*}(V_-)^*$ occurs in $(\bigwedge^{aq}\mathfrak{p}^+)^{\mathrm{SL}(q)}$; in that case we use the notation $\lambda = a\times q$ and it follows from Proposition 3.12 that[1]

$$\mathrm{SH}^\bullet(S,\mathbb{C}) = \bigoplus_{a,b=0}^{p} \bigoplus_{k=0}^{\min\{p-a,p-b\}} c_q^k H^{a\times q, b\times q}(S,\mathbb{C}). \tag{1.4}$$

(Compare with the usual Hodge–Lefschetz decomposition.) We set

$$\mathrm{SH}^{aq,bq}(S,\mathbb{C}) = H^{aq,bq}(S,\mathbb{C})\cap \mathrm{SH}^\bullet(S,\mathbb{C})$$

so that $\mathrm{SH}^\bullet(S,\mathbb{C}) = \bigoplus_{a,b=0}^p \mathrm{SH}^{aq,bq}(S,\mathbb{C})$. Wedging with c_q corresponds to applying the qth power of the Lefschetz operator associated with our choice of polarization, it therefore follows that the (usual) *primitive* part of $\mathrm{SH}^{aq,bq}(S,\mathbb{C})$ is exactly $H^{a\times q,b\times q}(S,\mathbb{C})$.

1.3. Main results

Vaguely stated our main result (Theorem 1.1) below asserts that the special cohomology $\mathrm{SH}^n(S,\mathbb{C})$ is generated, for n small enough by cup products of three types of classes:

- special classes of type (q,q), that is classes in $\mathrm{SH}^{q,q}(S,\mathbb{C})$;
- holomorphic and anti-holomorphic special cohomology classes, that is classes in $\mathrm{SH}^{\bullet,0}(S,\mathbb{C})$ and $\mathrm{SH}^{0,\bullet}(S,\mathbb{C})$;
- cycle classes of the *special cycles* of Kudla and Millson [46] (these are certain rational linear combinations of Hecke translates of Shimura subvarieties of S).

To state the precise result recall that we may associate to an n-dimensional totally positive Hermitian subspace of V a special cycle of complex codimension nq in S which is a Shimura subvariety associated with a unitary group of type $U(p-n,q)$ at infinity. Since these natural cycles do not behave particularly well under pull-back for congruence coverings, we, following Kudla [43], introduce weighted sums of these natural cycles and show that their cohomology classes form a subring

$$\mathrm{SC}^\bullet(S) = \bigoplus_{n=0}^{p} \mathrm{SC}^{2nq}(S)$$

of $H^\bullet(S,\mathbb{Q})$. We shall show (see Theorem 8.2) that for each n, with $0\leqslant n\leqslant p$, we have

$$\mathrm{SC}^{2nq}(S)\subset \mathrm{SH}^{nq,nq}(S,\mathbb{C})\cap H^\bullet(S,\mathbb{Q}).$$

---

[1] In the body of the paper we will rather write $b\times q, a\times q$ in order to write $\mathrm{U}(a,b)$ instead of $\mathrm{U}(b,a)$.

Note that, in particular, this gives strong restrictions on the possible refined Hodge types that can occur in the cycle classes of special cycles. We furthermore give an intrinsic characterization of the primitive part of the subring $\mathrm{SC}^\bullet(S)$ in terms of automorphic representations. For quotients of the complex 2-ball (i.e. $q=1$ and $p=2$) this was already obtained by Gelbart, Rogawski and Soudry [21]–[23].

We can now state our main theorem.

THEOREM 1.1. *Let a and b be integers such that $3(a+b)+|a-b|<2(p+q)$. First assume that $a\neq b$. Then the image of the natural cup product map*

$$\mathrm{SC}^{2\min\{a,b\}q}(S)\times(\mathrm{SH}^{|a-b|q,0}(S,\mathbb{C})\oplus\mathrm{SH}^{0,|a-b|q}(S,\mathbb{C}))\longrightarrow H^{(a+b)q}(S,\mathbb{C})$$

spans a subspace whose projection into the direct factor $H^{a\times q,b\times q}(S,\mathbb{C})\oplus H^{b\times q,a\times q}(S,\mathbb{C})$ is surjective. If $a=b$ this is no longer true but the image of the natural cup product map

$$\mathrm{SC}^{2(a-1)q}(S)\times\mathrm{SH}^{q,q}(S,\mathbb{C})\longrightarrow H^{2aq}(S,\mathbb{C})$$

spans a subspace whose projection into the direct factor $H^{a\times q,a\times q}(S,\mathbb{C})$ is surjective.

Remark. We shall see that the subrings $\mathrm{SH}^{\bullet,0}(S,\mathbb{C})$, resp. $\mathrm{SH}^{0,\bullet}(S,\mathbb{C})$, are well understood. These are spanned by certain theta series associated with explicit cocycles.

The most striking case of Theorem 1.1 is the case where S is a ball quotient ($q=1$). In this case $\mathrm{SH}^\bullet(S,\mathbb{C})=H^\bullet(S,\mathbb{C})$, and moreover, we prove that if a and b are integers such that $3(a+b)+|a-b|<2(p+1)$, then the space $H^{a+b}(S,\mathbb{Q})$ contains a polarized $\mathbb{Q}$-sub-Hodge structure X such that

$$X\otimes_{\mathbb{Q}}\mathbb{C}=H^{a,b}(S,\mathbb{C})\oplus H^{b,a}(S,\mathbb{C})$$

(see Corollary 6.2). In particular, not only the direct sum $H^{a,b}(S,\mathbb{C})\oplus H^{b,a}(S,\mathbb{C})$ is defined over $\mathbb{Q}$ but

- if $a\neq b$ the direct sum $H^{|a-b|,0}(S,\mathbb{C})\oplus H^{0,|a-b|}(S,\mathbb{C})$ is also defined over $\mathbb{Q}$, and
- if $a,b\geqslant 1$ the subspace $H^{1,1}(S,\mathbb{C})$ is also defined over $\mathbb{Q}$.

Theorem 1.1 therefore implies that every rational class in $H^{a,b}(S,\mathbb{C})\oplus H^{b,a}(S,\mathbb{C})$ is a rational linear combination of cup-products of rational $(1,1)$-classes and push-forwards of holomorphic or anti-holomorphic classes of special cycles.

Remarks. (i) This result cannot hold in degree close to p (the middle degree) as there are not enough cycles. In fact we expect the condition $3(a+b)+|a-b|<2(p+1)$ to be optimal.

(ii) We want to emphasize that the situation here is in two ways much more subtle than in the orthogonal case studied in [7]. First the cohomology groups $H^{a,b}(S,\mathbb{C})$ are in general non-trivial for all possible bi-degrees (a,b). Secondly special cycles *do not* span, even in codimension 1, and one has to consider arbitrary $(1,1)$-classes.

Now recall that the Lefschetz $(1,1)$ theorem implies that any rational $(1,1)$-class is a rational linear combination of classes of codimension one subvarieties of S. As a corollary we obtain the strong form of the generalized Hodge conjecture for S in the corresponding degrees: if c is an integer such that $2n-c<p+1$ then

$$N^c H^n(S,\mathbb{Q}) = H^n(S,\mathbb{Q}) \cap \left(\bigoplus_{\substack{a+b=n \\ a,b\geqslant c}} H^{a,b}(S,\mathbb{C}) \right). \tag{1.5}$$

Here $N^\bullet$ is the coniveau filtration so that by definition $N^c H^\bullet(S,\mathbb{Q})$ is the subspace of $H^\bullet(S,\mathbb{Q})$ which consists of classes that are pushforwards of cohomology classes on a subvariety of S of codimension at least c.

Remarks. (1) The inclusion $\subset$ in (1.5) always holds. In particular $N^c H^n(S,\mathbb{Q})$ is trivial if $n<2c$.

(2) Equation (1.5) confirms Hodge's generalized conjecture in its original formulation (with coefficients in $\mathbb{Q}$). Note however that—as it was first observed by Grothendieck [28]—the right-hand side of (1.5) is not always a Hodge structure. Grothendieck has corrected Hodge's original formulation but in our special case it turns out that the stronger form holds.

Observe that it is a consequence of the hard Lefschetz theorem [27, p. 122], that $N^c H^\bullet(S,\mathbb{Q})$ is stable under duality (the isomorphism given by the hard Lefschetz theorem). Indeed the projection formula states that cohomological push-forward commutes with the actions of $H^\bullet(S,\mathbb{C})$ on the cohomologies of a subvariety V of S and S, and hence with the operators L_V and L_S of the hard Lefschetz theorem. Thus, if $\beta\in H^k(S,\mathbb{C})$ is the push-forward of a class $\alpha\in H^{k-2c}(V,\mathbb{C})$ for a subvariety V of codimension c, then the dual class $L_S^{p-k}(\beta)$ is the push-forward of $L_V^{p-k}(\alpha)$. In conclusion, we have the following result.

Theorem 1.2. *Let S be a connected compact Shimura variety associated with a standard unitary group $U(p,1)$. Let n and c be non-negative integers such that $2n-c<p+1$ or $2n+c>3p-1$, or equivalently $n\in[0,2p]\setminus\left]p-\frac{1}{2}(p-c), p+\frac{1}{2}(p-c)\right[$. Then, we have*

$$N^c H^n(S,\mathbb{Q}) = H^n(S,\mathbb{Q}) \cap \left(\bigoplus_{\substack{a+b=n \\ a,b\geqslant c}} H^{a,b}(S,\mathbb{C}) \right).$$

In particular, we have the following corollary.

Corollary 1.3. *Let S be a connected compact Shimura variety associated with a standard unitary group $U(p,1)$ and let $n\in[0,p]\setminus\left]\frac{1}{3}p, \frac{2}{3}p\right[$. Then every Hodge class in $H^{2n}(S,\mathbb{Q})$ is algebraic.*

Tate [68] investigated the ℓ-adic analogue of the Hodge conjecture. Recall that S is defined over a finite abelian extension M of E. Fix a separable algebraic closure $\overline{M}$ of M. Seeing S as a projective variety over M, we put $\bar{S}=S\times_M \overline{M}$. Given any prime number ℓ we denote the ℓ-adic étale cohomology of $\bar{S}$ by

$$H_\ell^\bullet(\bar{S}) = H^\bullet(\bar{S}, \mathbb{Q}_\ell).$$

Recall that fixing an isomorphism of $\mathbb{C}$ with the completion $\mathbb{C}_\ell$ of an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$ we have an isomorphism

$$H_\ell^\bullet(\bar{S})\otimes\mathbb{C}_\ell \cong H^\bullet(S,\mathbb{C}). \tag{1.6}$$

Given any finite separable extension $L\subset\overline{M}$ of M we let $G_L=\mathrm{Gal}(\overline{M}/L)$ be the corresponding Galois group. Tensoring with $\mathbb{Q}_\ell$ embeds L in $\overline{\mathbb{Q}}_\ell$. The elements of G_L then extend to continuous automorphisms of $\mathbb{C}_\ell$. For $j\in\mathbb{Z}$, let $\mathbb{C}_\ell(j)$ be the vector space $\mathbb{C}_\ell$ with the semi-linear action of G_L defined by $(\sigma,z)\mapsto\chi_\ell(\sigma)^j\sigma(z)$, where χ_ℓ is the ℓ-adic cyclotomic character. We define

$$H_\ell^\bullet(\bar{S})(j) = \varinjlim (H_\ell^\bullet(\bar{S})\otimes\mathbb{C}_\ell(j))^{G_L},$$

where the limit is over finite degree separable extensions L of M. The ℓ-adic cycle map

$$\mathcal{Z}^n(S) \longrightarrow H_\ell^{2n}(\bar{S})$$

maps a subvariety to a class in $H_\ell^{2n}(\bar{S})(n)$; the latter subspace is the space of *Tate classes*. The Tate conjecture states that the ℓ-adic cycle map is surjective, i.e. that every Tate class is algebraic.

Now recall that Faltings [13] has proven the existence of a Hodge–Tate decomposition for the étale cohomology of smooth projective varieties defined over number fields. In particular, the isomorphism (1.6) maps $H_\ell^m(\bar{S})(j)$ isomorphically onto $H^{j,m-j}(S,\mathbb{C})$. From this, Theorem 1.2 and the remark following it, we get the following result.

Corollary 1.4. *Let S be a neat connected compact Shimura variety associated with a standard unitary group $U(p,1)$ and let $n\in[0,p]\setminus]\frac{1}{3}p,\frac{2}{3}p[$. Then every Tate class in $H^{2n}(S,\mathbb{Q}_\ell)$ is algebraic.*

Proof. It follows from the remark following Theorem 1.2 (and Poincaré duality) that the whole subspace $H^{n,n}(S,\mathbb{C})$ is spanned by algebraic cycles as long as $n\in[0,p]\setminus]\frac{1}{3}p,\frac{2}{3}p[$. The corollary is then a consequence of the following commutative diagram.

$$\begin{tikzcd}
H_\ell(\bar{S})(n) \arrow[rr] & & H^{n,n}(\bar{S},\mathbb{C}) \\
& Z^n(S). \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The horizontal arrow is an isomorphism and the two diagonal arrows are the cycle maps. We have proved that the image of the right diagonal arrow spans, and hence the image of the left diagonal arrow spans. □

1.4. General strategy of proof

The proof of Theorem 1.1 relies on the dictionary between cohomology and automorphic forms specific to Shimura varieties. This dictionary allows to translate geometric questions on Shimura varieties into purely automorphic problems.

The first step consists in obtaining an understanding, in terms of automorphic forms, of the special cohomology groups $\mathrm{SH}^n(S,\mathbb{C})$ for n small enough: it is generated by projections of theta series. In other words, we prove the low-degree cohomological surjectivity of the general theta lift (for classes of special refined Hodge type). See Theorem 7.2 which is deduced from Proposition 13.4. The proof goes through the following steps:

- One first argues at the infinite places. By Matsushima's formula the cohomology groups $H^\bullet(S,\mathbb{C})$ can be understood in terms of the appearance in $L^2(\Gamma\backslash U(p,q))$ of certain—called *cohomological*—representations π_∞ of $U(p,q)$. It follows from the Vogan–Zuckerman classification of these cohomological representations that the only cohomological representations π_∞ contributing to $\mathrm{SH}^\bullet(S,\mathbb{C})$ are of very simple type (see Proposition 3.12). We denote by $A(a\times q, b\times q)$, $0\leqslant a,b\leqslant p$, these representations; they define the direct factors $H^{a\times q, b\times q}(S,\mathbb{C})$ of $\mathrm{SH}^{aq,bq}(S,\mathbb{C})$ and induce the refined Hodge–Lefschetz decomposition (1.4).

- Second, one proves that for $n=(a+b)q$ small enough—more precisely for $3(a+b)+|a-b|<2(p+q)$—any cuspidal automorphic representation of $G(\mathbb{A})$ whose local component at infinity is $A(a\times q, b\times q)$ is in the image of the theta correspondence from a smaller unitary group (Proposition 13.4). The proof proceeds as follows: we first prove a precise criterion for a cuspidal automorphic representation π of $G(\mathbb{A})$ whose local component at infinity is sufficiently non-tempered to be in the image of the theta correspondence from a smaller unitary group (Theorem 10.1). This criterion is analogous to a classical result of Kudla and Rallis for the orthogonal-symplectic dual pair (relying on the doubling method of Piatetskii–Shapiro and Rallis, and Rallis' inner product formula). However in the unitary case this criterion does not seem to have been fully worked out. Building on Ichino's regularized Siegel–Weil formula, we prove this criterion in §10. Second, one has to show that the cuspidal automorphic π whose components at infinity are $A(a\times q, b\times q)$, with $3(a+b)+|a-b|<2(p+q)$, do satisfy this criterion. This relies on Arthur's recent endoscopic classification of automorphic representations of classical groups. Arthur's theory relates the classification of G to the classification of the non-connected group

$\mathrm{GL}(N) \rtimes \langle\theta\rangle$ (where θ is some automorphism of $\mathrm{GL}(N)$) through the stabilization of the twisted trace formula recently obtained by Moeglin and Waldspurger [58]. This is the subject of §12 and §13.

The second step shows that not only is $H^{a\times q,b\times q}(S,\mathbb{C})$ generated by theta lifts, but by *special theta lifts*, where the special theta lift restricts the general theta lift to (vector-valued) Schwartz functions that have, at the distinguished infinite place where the unitary group is non-compact, a very explicit expression $\varphi_{aq,bq}$ (see Theorem 9.1, which depends crucially on Theorem 5.24).(2) The Schwartz functions at the other infinite places are (scalar-valued) Gaussians and at the finite places are scalar-valued and otherwise arbitrary. The main point is that the special Schwartz function $\varphi_{aq,bq}$ is a relative Lie algebra cocycle for the unitary group allowing one to interpret the special theta lift cohomologically. In fact we work with the Fock model for the Weil representation and with the cocycle $\psi_{aq,bq}$ with values in the Fock model. This cocycle corresponds to the cocycle $\varphi_{aq,bq}$ with values in the Schrödinger model under the usual intertwiner from the Fock model to the Schrödinger model.

The third step consists, if $b{=}a{+}c$, $c{>}0$, in showing that $\psi_{aq,bq}$ factors as a cup product $\psi_{aq,0}\wedge\psi_{0,bq}$ of the holomorphic and anti-holomorphic cocycles $\psi_{aq,0}$ and $\psi_{0,bq}$. This factorization does not hold for the cocycles $\varphi_{aq,bq}$—see Appendix C. These are local (Archimedean) computations in the Fock model; see Propositions 5.4 and 5.19.

One concludes the proof of Theorem 1.1 by using the result of Kudla–Millson [46] stating that the subspace of the cohomology of S generated by the cycle classes of special cycles is exactly the one obtained from the special theta lift starting with $\varphi_{nq,nq}$ at the distinguished infinite place.

In the paper we do not follow the above order. We rather start with the local computations (describing the cohomological representations and constructing the cocycles $\psi_{aq,bq}$). We then discuss the geometry of the Shimura varieties and reduce the proofs of our main results to purely automorphic statements. We conclude with the proofs of these automorphic results. This will hopefully help a reader willing to accept them to follow more easily the overall structure of the proofs of our main results.

Acknowledgements. We would like to thank Claire Voisin for suggesting that we look at the generalized Hodge conjecture and Don Blasius and Laurent Clozel for useful references. We thank the referees for their work. Their remarks and suggestions helped us improve the exposition.

(2) This step generalizes a special case of a result of Hoffmann and He [32].

Part 1. Local computations

2. Hermitian vector spaces over $\mathbb{C}$

We begin with some elementary linear algebra that will be important to us in what follows. The results we prove are standard, the main point is to establish the notation that will be useful later. For this section the symbol $\otimes$ will mean tensor product over $\mathbb{R}$, in the rest of the paper it will mean tensor product over $\mathbb{C}$.

2.1. Notation

Let V be a complex vector space equipped with a non-degenerate Hermitian form $(\cdot,\cdot)$. Our Hermitian forms will be complex linear in the first argument and complex anti-linear in the second. We will often consider V as a real vector space equipped with the almost complex structure J given by $Jv=iv$. When there are several vector spaces under consideration we write J_V instead of J. We will give V^* the transpose almost complex structure (not the inverse transpose almost complex structure) so

$$(J\alpha)(v)=\alpha(Jv). \tag{2.1}$$

We will sometimes denote this complex structure by J_{V^*}.

Finally recall that we have a real-valued symmetric form B and a real-valued skew-symmetric form $\langle\cdot,\cdot\rangle$ on V considered as a real vector space associated with the Hermitian form $(\cdot,\cdot)$ by the formulas

$$B(v_1,v_2)=\mathrm{Re}(v_1,v_2) \quad \text{and} \quad \langle v_1,v_2\rangle=-\mathrm{Im}(v_1,v_2),$$

so that we have

$$B(v_1,v_2)=\langle v_1,Jv_2\rangle. \tag{2.2}$$

2.2. The complexification of a Hermitian space and the subspaces of type $(1,0)$ and $(0,1)$ vectors

We now form the complexification $V\otimes\mathbb{C}=V\otimes_{\mathbb{R}}\mathbb{C}$ of V, where V is considered as real vector space. The space $V\otimes\mathbb{C}$ has two commuting complex structures namely $J\otimes 1$ and $I_V\otimes i$. We define the orthogonal idempotents p' and p'' in $\mathrm{End}_{\mathbb{R}}(V\otimes\mathbb{C})$ by

$$p'=\tfrac{1}{2}(I_V\otimes 1-J_V\otimes i) \quad \text{and} \quad p''=\tfrac{1}{2}(I_V\otimes 1+J_V\otimes i). \tag{2.3}$$

One readily verifies the equations

$$p'\circ p'=p', \quad p''\circ p''=p'' \quad \text{and} \quad p'\circ p''=p''\circ p'=0. \tag{2.4}$$

In what follows if $v \in V$ is given we will abbreviate $p'(v \otimes 1)$ by v' and $p''(v \otimes 1)$ by v''. We will write zv' for $(1 \otimes z)v'$ and zv'' for $(1 \otimes z)v''$. We note the formulas

$$p'(zv) = zp'(v) \quad \text{and} \quad p''(zv) = \bar{z}p''(v). \tag{2.5}$$

We define $V' = p'(V \otimes \mathbb{C})$ and $V'' = p''(V \otimes \mathbb{C})$. From (2.4) we obtain $V \otimes \mathbb{C} = V' \oplus V''$. An element of V' is said to be of type $(1,0)$ and an element of V'' is said to be of type $(0,1)$. We will identify V with the subspace $V \otimes 1$ in $V \otimes \mathbb{C}$.

2.3. Coordinates on V and the induced coordinates on V' and V''

In this subsection only we will assume that the Hermitian form $(\cdot, \cdot)$ on V is positive definite. Let $\{v_1, ..., v_n\}$ be an orthonormal basis for V over $\mathbb{C}$. Then we obtain induced bases $\{v'_1, ..., v'_n\}$ and $\{v''_1, ..., v''_n\}$ for V' and V'', respectively. For $1 \leqslant j \leqslant n$, we let $z_j(v)$ be the jth coordinate of $v \in V$ relative to the basis $\{v_1, ..., v_n\}$, $z'_j(v')$ be the j coordinate of $v' \in V'$ relative to the basis $\{v'_1, ..., v'_n\}$ and $z''_j(v')$ be the jth coordinate of $v'' \in V''$ relative to the basis $\{v''_1, ..., v''_n\}$. Let $v \in V$. Then, by applying p' and p'' to the equation $v = \sum_{j=1}^n z_j(v) v_j$ and using equation (2.5) we get the following result.

LEMMA 2.1. *We have*

(1) $z'_j(v') = z_j(v) = (v, v_j)$;
(2) $z''_j(v'') = \overline{z_j(v)} = (v_j, v)$.

2.4. The induced Hodge decomposition of V^*

There is a corresponding decomposition $V^* \otimes \mathbb{C} = (V^*)' \oplus (V^*)''$ induced by the almost complex structure J_{V^*}. The complexified canonical pairing $(V^* \otimes \mathbb{C}) \otimes_{\mathbb{C}} (V \otimes \mathbb{C}) \to \mathbb{C}$ given by $(\alpha \otimes z) \otimes_{\mathbb{C}} (v \otimes w) \to (\alpha(v))(zw)$ induces isomorphisms $(V^*)' \to (V')^*$ and $(V^*)'' \to (V'')^*$. We will therefore make the identifications

$$(V^*)' = (V')^* \quad \text{and} \quad (V^*)'' = (V'')^*$$

without further mention. In particular, if $\{v_1, ..., v_n\}$ is a basis for V and $\{f_1, ..., f_n\}$ is the dual basis for V^*, then $\{f'_1, ..., f'_n\}$ is the basis for $(V^*)'$ dual to the basis $\{v'_1, ..., v'_n\}$ for V'.

2.5. The positive almost complex structure J_0 associated with a Cartan involution

We now assume that $(V, (\cdot, \cdot))$ is an indefinite Hermitian space of signature (p, q). We choose once and for all an orthogonal splitting of complex vectors spaces $V = V_+ \oplus V_-$ of

V such that the restriction of $(\cdot,\cdot)$ to V_+ is positive definite and the restriction to V_- is negative definite. Such a splitting is determined by the choice of V_- and consequently corresponds to a point in the symmetric space of V. We can obtain a positive definite Hermitian form $(\cdot,\cdot)_0$ depending on the choice of V_- by changing the sign of $(\cdot,\cdot)$ on V_-. The positive definite form $(\cdot,\cdot)_0$ is called (in classical terminology) a minimal majorant of $(\cdot,\cdot)$. Let θ_{V_-} be the involution which is equal to I_{V_+} on V_+ and to $-I_{V_-}$ on V_-. Since V_+ and V_- are complex subspaces J and θ_{V_-} commute. Then θ_{V_-} is a Cartan involution of V in the sense that it is an order two isometry of $(\cdot,\cdot)$ such that its centralizer in $U(V)$ is a maximal compact subgroup. All Cartan involutions are of the form θ_{V_-} for some splitting of $V=V_+\oplus V_-$ as above. We note that, for $v_1, v_2, v\in V$, we have

$$(v_1,v_2)_0=(v_1,\theta_{V_-}v_2) \quad \text{and} \quad |(v,v)|\leqslant (v,v)_0. \tag{2.6}$$

For this reason $(\cdot,\cdot)_0$ is called a (minimal) majorant of $(\cdot,\cdot)$.

By taking real and imaginary parts of $(\cdot,\cdot)_0$ we obtain a positive definite symmetric form $B_0(\cdot,\cdot)$ and a symplectic form $\langle\cdot,\cdot\rangle_0$ such that

$$(v_1,v_2)_0=B_0(v_1,v_2)-i\langle v_1,v_2\rangle_0.$$

Define a new complex structure J_0 by

$$J_0=\theta_{V_-}\circ J=J\circ\theta_{V_-}.$$

We note that the new form $(\cdot,\cdot)_0$ is still Hermitian with respect to the old complex structure J,(3) that is

$$(Jv_1,v_2)_0=i(v_1,v_2)_0 \quad \text{and} \quad (v_1,Jv_2)_0=-i(v_1,v_2)_0,$$

and that J_0 is an isometry of $(\cdot,\cdot)_0$, that is

$$(J_0v_1,J_0v_2)_0=(v_1,v_2)_0.$$

We claim that

$$B_0(v_1,v_2)=\langle v_1,J_0v_2\rangle. \tag{2.7}$$

Indeed we have

$$\begin{aligned} B_0(v_1,v_2)&=\mathrm{Re}(v_1,v_2)_0=\mathrm{Re}(v_1,\theta_{V_-}v_2)=\mathrm{Im}\,i(v_1,\theta_{V_-}v_2)=\mathrm{Im}(-(v_1,J\theta_{V_-}v_2))\\ &=-\mathrm{Im}(v_1,J\theta_{V_-}v_2)=\langle v_1,J\theta_{V_-}v_2)\rangle=\langle v_1,J_0v_2\rangle. \end{aligned}$$

The claim follows.

It follows from (2.7) that J_0 is a positive definite almost complex with respect to the symplectic form $\langle\cdot,\cdot\rangle$. For the convenience of the reader we recall this basic definition.

(3) However $(\cdot,\cdot)_0$ is not Hermitian with respect to the new complex structure J_0.

Definition 2.2. Given a symplectic form $\langle\cdot,\cdot\rangle$ and an almost complex structure J_0, we say that $\langle\cdot,\cdot\rangle$ and J_0 are *compatible* if J_0 is an isometry of $\langle\cdot,\cdot\rangle$. We say that J_0 is *positive* (*definite*) with respect to $\langle\cdot,\cdot\rangle$ if J_0 and $\langle\cdot,\cdot\rangle$ are compatible and moreover the symmetric form $B_0(v_1,v_2)=\langle v_1,J_0v_2\rangle$ is positive definite.

It now follows from the above discussion that there is a one-to-one correspondence between minimal Hermitian majorants of $(\cdot,\cdot)$, positive almost complex structures J_0 commuting with J such that the product J_0J is a Cartan involution and points of the symmetric space of $U(V)$ (subspaces Z of dimension q such that the restriction of $(\cdot,\cdot)$ to Z is negative definite). Henceforth, we will call such positive complex structures J_0 *admissible*.

We will therefore have to deal with two different almost complex structures and hence two notions of type $(1,0)$ vectors. To deal with this we use the following notation.

Definition 2.3. (1) If U is a subspace of V which is J-invariant then U', resp. U'', will denote the subspace of type $(1,0)$, resp. type $(0,1)$, vectors for the indefinite almost complex structure J acting on $U\otimes\mathbb{C}$, for example V'_+ is the eigenspace, corresponding to the eigenvalue i, of J acting on $V_+\otimes\mathbb{C}$.

(2) If U is a subspace of V which is J_0-invariant then U'^{0}, resp. U''^{0}, will denote the subspace of type $(1,0)$, resp. type $(0,1)$, vectors for the definite almost complex structure J_0 acting on $U\otimes\mathbb{C}$.

3. Cohomological unitary representations

3.1. Notation

Keep the notation as in §2 and let $m=p+q$. In this section $G=U(V)\cong U(p,q)$ and $K\cong U(p)\times U(q)$ is a maximal compact subgroup of G associated with the Cartan involution $\theta=\theta_{V_-}$ defined in the previous subsection. We let $\mathfrak{g}_0$ be the real Lie algebra of G and $\mathfrak{g}_0=\mathfrak{k}_0\oplus\mathfrak{p}_0$ be the Cartan decomposition associated with the Cartan involution θ. If $\mathfrak{l}_0$ is a real Lie algebra, we denote by $\mathfrak{l}$ its complexification $\mathfrak{l}=\mathfrak{l}_0\otimes\mathbb{C}$.

3.2. Cohomological representations

A unitary representation π of G is *cohomological* if it has non-zero $(\mathfrak{g},K)$-cohomology $H^\bullet(\mathfrak{g},K;V_\pi)$.

Cohomological representations are classified by Vogan and Zuckerman in [71]. Let $\mathfrak{t}_0$ be a Cartan subalgebra of $\mathfrak{k}_0$. A *θ-stable parabolic subalgebra* $\mathfrak{q}=\mathfrak{q}(X)\subset\mathfrak{g}$ is associated with an element $X \in i\mathfrak{t}_0$ and defined as the direct sum

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$$

of the centralizer $\mathfrak{l}$ of X and the sum $\mathfrak{u}$ of the positive eigenspaces of $\mathrm{ad}(X)$. Since $\theta X = X$, the subspaces $\mathfrak{q}$, $\mathfrak{l}$ and $\mathfrak{u}$ are all invariant under θ, so

$$\mathfrak{q} = (\mathfrak{q} \cap \mathfrak{k}) \oplus (\mathfrak{q} \cap \mathfrak{p}),$$

and so on. Let $R = \dim(\mathfrak{u} \cap \mathfrak{p})$.

Let $\mathfrak{h}$ be a theta-stable Cartan subalgebra in $\mathfrak{l}$ (and hence a Cartan subalgebra in $\mathfrak{g}$) containing $\mathfrak{t}$. Choose a system of positive roots $\Delta^+(\mathfrak{l})$ for the roots of $\mathfrak{h}$ in $\mathfrak{l}$. Then the union of the roots in $\Delta^+(\mathfrak{l})$ and the positive roots in $\mathfrak{u}$ is a positive system of roots for the theta-stable Cartan subalgebra $\mathfrak{h}$. We may assume that the resulting system of positive roots for the pair $(\mathfrak{g}, \mathfrak{h})$ includes a positive system for the pair $(\mathfrak{k}, \mathfrak{t})$.

Associated with $\mathfrak{q}$ there is a well-defined and irreducible unitary representation $A_{\mathfrak{q}}$ of G, which is characterized by the following properties. Let $e(\mathfrak{q})$ be a generator of the line $\bigwedge^R(\mathfrak{u} \cap \mathfrak{p})$; we shall refer to such a vector as a *Vogan–Zuckerman vector*. Then $e(\mathfrak{q})$ is the highest weight vector of an irreducible representation $V(\mathfrak{q})$ of K contained in $\bigwedge^R \mathfrak{p}$ (and whose highest weight is thus necessarily $2\varrho(\mathfrak{u} \cap \mathfrak{p})$). The representation $A_{\mathfrak{q}}$ is then uniquely characterized by the following two properties:

(1) $A_{\mathfrak{q}}$ is unitary with trivial central character and with the same infinitesimal character as the trivial representation;

(2) $\mathrm{Hom}_K(V(\mathfrak{q}), A_{\mathfrak{q}}) \neq 0$.

Note that (the equivalence class of) $A_{\mathfrak{q}}$ only depends on the intersection $\mathfrak{u} \cap \mathfrak{p}$ so that two θ-stable parabolic subalgebras $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ and $\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{u}'$ which satisfy $\mathfrak{u} \cap \mathfrak{p} = \mathfrak{u}' \cap \mathfrak{p}$ yield the same (equivalence class of) cohomological representation. Moreover, $V(\mathfrak{q})$ occurs with multiplicity 1 in $A_{\mathfrak{q}}$ and $\bigwedge^R \mathfrak{p}$, and

$$H^\bullet(\mathfrak{g}, K; A_{\mathfrak{q}}) \cong \mathrm{Hom}_{L \cap K}\big(\textstyle\bigwedge^{\bullet - R}(\mathfrak{l} \cap \mathfrak{k}), \mathbb{C}\big). \tag{3.1}$$

Here L is the connected subgroup of G with complexified Lie algebra $\mathfrak{l}$.

In the next paragraphs we give a more explicit parametrization of the cohomological modules of G.

3.3. The Hodge decomposition of the complexified tangent space $\mathfrak{p}$ of the symmetric space of $U(p,q)$ at the basepoint

We first give the standard development of the Hodge decomposition of $\mathfrak{p}$. In what follows we will use a subscript zero to denote a real algebra (subspace of a real algebra) and omit the subscript zero for its complexification. For example we have $\mathfrak{g}_0 = \mathfrak{u}(p,q)$ and $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$.

We start by making the usual identification $\mathfrak{g}\cong\mathrm{End}(V)$ given by

$$A\otimes z\longmapsto zA, \quad A\in\mathfrak{g}_0. \tag{3.2}$$

Rather than using pairs of numbers between 1 and $p+q$ (for example $e_{j,k}$) to denote the usual basis elements of $\mathrm{End}(V)$ we will use the isomorphism $\mathrm{End}(V)\cong V\otimes V^*$; see below. We then have

$$\mathfrak{g}=V\otimes V^*=(V_+\otimes V_+^*)\oplus(V_+\otimes V_-^*)\oplus(V_-\otimes V_+^*)\oplus(V_-\otimes V_-^*).$$

In terms of the above splitting (and identification) we have

$$\mathfrak{k}=(V_+\otimes V_+^*)\oplus(V_-\otimes V_-^*) \quad \text{and} \quad \mathfrak{p}=(V_+\otimes V_-^*)\oplus(V_-\otimes V_+^*). \tag{3.3}$$

3.4. Now consider a basis $\{v_1,...,v_m\}$ for V adapted to the decomposition $V=V_+\oplus V_-$. The following index convention will be useful in what follows. We furthermore suppose that

$$(v_\alpha,v_\beta)=\delta_{\alpha,\beta} \quad \text{and} \quad (v_\mu,v_\nu)=-\delta_{\mu,\nu}.$$

Then the matrix of the Hermitian form $(\cdot,\cdot)$ on V with respect to the basis $\{v_j\}_{j=1}^m$ is the diagonal matrix $\left(\begin{smallmatrix}1_p & \\ & -1_q\end{smallmatrix}\right)$. We therefore end up with the usual matrix realization of the Lie algebra $\mathfrak{g}_0$ of $U(p,q)$, where an $m\times m$ complex matrix $\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)$ belongs to $\mathfrak{g}_0$ if and only if $A^*=-A$, $D^*=-D$ and $B^*=C$. In that realization we have

(1) $\mathfrak{k}_0=\left(\begin{smallmatrix}A & 0\\ 0 & D\end{smallmatrix}\right)$, with A and D skew-Hermitian;
(2) $\mathfrak{k}=\left(\begin{smallmatrix}A & 0\\ 0 & D\end{smallmatrix}\right)$, with A, resp. D, being an arbitrary $p\times p$, resp. $q\times q$, complex matrix;
(3) $\mathfrak{p}_0=\left(\begin{smallmatrix}0 & B\\ B^* & 0\end{smallmatrix}\right)$, with B being an arbitrary $p\times q$ complex matrix;
(4) $\mathfrak{p}=\left(\begin{smallmatrix}0 & B\\ C & 0\end{smallmatrix}\right)$, with B, resp. C, being an arbitrary $p\times q$, resp. $q\times p$, complex matrix.

3.5. For $v\in V$ we define $v^*\in V^*$ by

$$v^*(u)=(u,v), \tag{3.4}$$

and $v_1\otimes v_2^*\in V\otimes V^*=\mathrm{End}\,(V)\cong\mathfrak{g}$ by

$$(v_1\otimes v_2^*)(v)=(v,v_2)v_1. \tag{3.5}$$

If $f\in V^*$ and $v\in V$ one can define $v\otimes f\in\mathrm{End}\,(V)$ in the same way and obtain the canonical identification between $V\otimes V^*$ and $\mathrm{End}(V)$. However, in what follows it will be more useful for us to use the identification of equation (3.5).

We next note that we may identify $V^*\otimes V$ with $(V\otimes V^*)^*=\mathrm{End}(V)^*=\mathrm{End}(V^*)$ by the formula

$$\langle f_1\otimes v_1, v_2\otimes f_2\rangle = f_1(v_2)f_2(v_1).$$

We will denote by ${}^tA$ the element of $\mathrm{End}(V^*)$ corresponding to $A\in\mathrm{End}(V)$, and hence ${}^tA(f)=f\circ A$. Using the above identifications, we have

$${}^t(v\otimes f) = f\otimes v.$$

The adjoint map $A\to A^*$ relative to the Hermitian form $(\cdot,\cdot)$ is the anti-linear map given by

$$(u\otimes v^*)^* = v\otimes u^*.$$

Note that $A\in\mathrm{End}(V)$ is in $\mathfrak{g}_0$ if and only if $A^*=-A$. Hence the conjugation map σ_0 of $\mathrm{End}(V)$ relative to the real form $\mathfrak{g}_0$ is given by $\sigma_0(u\otimes v^*)=-v\otimes u^*$. From either of the two previous sentences we get the following result.

Lemma 3.1. *Let $x, y\in V$. Then $x\otimes y^*-y\otimes x^*$ and $i(x\otimes y^*+y\otimes x^*)$ are in $\mathfrak{g}_0$.*

We now define a basis for $\mathfrak{p}_0$ by defining the basis vectors $e_{\alpha,\mu}$ and $f_{\alpha,\mu}$, $1\leqslant\alpha\leqslant p$ and $p+1\leqslant\mu\leqslant p+q$. We will not need a basis for $\mathfrak{k}_0$. By Lemma 3.1, it follows that the elements below are in fact in $\mathfrak{g}_0$. Here the matrices only show the action on the pair of basis vectors v_j, v_k in the formula immediately to the left of the matrix in the order in which they are given. All other basis vectors are sent to zero

$$e_{\alpha,\mu} = -v_\alpha\otimes v_\mu^* + v_\mu\otimes v_\alpha^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad f_{\alpha,\mu} = i(-v_\alpha\otimes v_\mu^* - v_\mu\otimes v_\alpha^*) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

We now describe the $\mathrm{Ad}(K)$-invariant almost complex structure $J_\mathfrak{p}$ acting on $\mathfrak{p}$ that induces the structure of Hermitian symmetric space on $U(p,q)/(U(p)\times U(q))$. Let $\zeta=e^{i\pi/4}$. Then ζ satisfies $\zeta^2=i$. Let $a(\zeta)$ be the diagonal $m\times m$ block matrix given by

$$a(\zeta) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}.$$

Then $a(\zeta)$ is in the center of $U(p)\times U(q)$ and the adjoint action of $\mathrm{Ad}(a(\zeta))$ on $\mathfrak{p}$ induces the required almost complex structure, that is we have

$$J_\mathfrak{p} = \mathrm{Ad}(a(\zeta)). \tag{3.6}$$

We have

$$a(\zeta)v_\alpha = \zeta v_\alpha, a(\zeta)v_\mu = \zeta^{-1}v_\mu, a(\zeta)v_\alpha^* = \zeta^{-1}v_\alpha^* \quad \text{and} \quad a(\zeta)v_\mu^* = \zeta v_\mu^*. \tag{3.7}$$

In particular, for $1\leqslant\alpha\leqslant p$ and $p+1\leqslant\mu\leqslant p+q$, we have

$$J_\mathfrak{p} e_{\alpha,\mu} = f_{\alpha,\mu} \quad \text{and} \quad J_\mathfrak{p} f_{\alpha,\mu} = -e_{\alpha,\mu}.$$

3.6. We define elements $x_{\alpha,\mu}$ in $\mathfrak{p}'$ and $y_{\alpha,\mu}$ in $\mathfrak{p}''$, and thus also in $\mathfrak{g}$, by

$$x_{\alpha,\mu} = -v_\alpha \otimes v_\mu^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \text{whence } J_{\mathfrak{p}} x_{\alpha,\mu} = i x_{\alpha,\mu},$$

$$y_{\alpha,\mu} = \sigma_0(x_{\alpha,\mu}) = v_\mu \otimes v_\alpha^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{whence } J_{\mathfrak{p}} y_{\alpha,\mu} = -i y_{\alpha,\mu}.$$

The set $\{x_{\alpha,\mu} : 1 \leqslant \alpha \leqslant p \text{ and } p+1 \leqslant \mu \leqslant p+q\}$ is a basis for $\mathfrak{p}'$. In the corresponding matrix realization we have

$$\mathfrak{p}' = V_+ \otimes V_-^* = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} : B \in M_{p\times q}(\mathbb{C}) \right\}.$$

Similarly, the set $\{y_{\alpha,\mu} : 1 \leqslant \alpha \leqslant p,\ p+1 \leqslant \mu \leqslant p+q\}$ is a basis for $\mathfrak{p}''$ and we have

$$\mathfrak{p}'' = V_- \otimes V_+^* = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} : C \in M_{q\times p}(\mathbb{C}) \right\}.$$

Hence we have

$$\sigma_0(\mathfrak{p}') = \mathfrak{p}''.$$

As a consequence of the above computation, we note that we have isomorphisms of $K = U(p) \times U(q)$ modules

$$\mathfrak{p}' \cong M_{p\times q}(\mathbb{C}) \quad \text{and} \quad \mathfrak{p}'' \cong M_{q\times p}(\mathbb{C}),$$

and the above splitting into B and C blocks corresponds to the splitting $\mathfrak{p} = \mathfrak{p}' \oplus \mathfrak{p}''$.

Using the identification $(U \otimes U^*)^* = U^* \otimes U$ we have

$$(\mathfrak{p}')^* = V_+^* \otimes V_- \quad \text{and} \quad (\mathfrak{p}'')^* = V_-^* \otimes V_+. \tag{3.8}$$

Hence the transpose $\phi : V \otimes V^* \to V^* \otimes V$ given by ${}^t(v \otimes f) = f \otimes v$ induces isomorphisms $\mathfrak{p}'' \to (\mathfrak{p}')^*$ and $\mathfrak{p}' \to (\mathfrak{p}'')^*$. On the above basis these maps are given by

$${}^t y_{\alpha,\mu} = {}^t(v_\mu \otimes v_\alpha^*) = v_\alpha^* \otimes v_\mu \quad \text{and} \quad {}^t x_{\alpha,\mu} = {}^t(-v_\alpha \otimes v_\mu^*) = -v_\mu^* \otimes v_\alpha. \tag{3.9}$$

We set $\xi'_{\alpha,\mu} = v_\alpha^* \otimes v_\mu$ and $\xi''_{\alpha,\mu} = -v_\mu^* \otimes v_\alpha$.

We now give two definitions that will be important in what follows. The notation below is chosen to help clarify that the adjoints of the cocycles $\psi_{bq,aq}$, of degree $(a+b)q$, that we construct and study in §5 are *completely decomposable* in the sense that their values at a point of $x \in V^{a+b}$ are wedges of $(a+b)q$ elements of $\mathfrak{p}^*$.

Definition 3.2. Let $x\in V_+$. Then we define $\tilde{x}\in\bigwedge^q\mathfrak{p}'=\bigwedge^q(V_+\otimes(V_-)^*)$ by

$$\tilde{x}=(-1)^q(x\otimes v_{p+1}^*)\wedge\ldots\wedge(x\otimes v_{p+q}^*). \tag{3.10}$$

Remark. By [17, p. 80], there is an equivariant embedding

$$f_q\colon \mathrm{Sym}^q(V_+)\otimes\bigwedge^q(V_-)^*\longrightarrow\bigwedge^q(V_+\otimes(V_-)^*),$$

and hence $\tilde{x}=f(x^{\otimes q}\otimes(v_{p+1}^*\wedge\ldots\wedge v_{p+q}^*))$.

Suppose now that $f\in V_+^*$. We give the following definition.

Definition 3.3. We define $\tilde{f}\in\bigwedge^q\mathfrak{p}''=\bigwedge^q(V_-\otimes(V_+)^*)$ by

$$\tilde{f}=(v_{p+1}\otimes f)\wedge\ldots\wedge(v_{p+q}\otimes f). \tag{3.11}$$

Using the transpose maps of equation (3.9) we obtain ${}^t\tilde{f}\in\bigwedge^q(\mathfrak{p}')^*=\bigwedge^q((V_+)^*\otimes V_-)$ given by

$${}^t\tilde{f}=(f\otimes v_{p+1})\wedge\ldots\wedge(f\otimes v_{p+q}) \tag{3.12}$$

and ${}^t\tilde{x}\in\bigwedge^q((\mathfrak{p}'')^*)=\bigwedge^q((V_-)^*\otimes V_+)$ is given by

$${}^t\tilde{x}=(-1)^q(v_{p+1}^*\otimes x)\wedge\ldots\wedge(v_{p+q}^*\otimes x). \tag{3.13}$$

3.7. Theta-stable parabolic subalgebras

Fix the Borel subalgebra of $\mathfrak{k}$ to be the algebra of matrices in $\mathfrak{k}=\mathfrak{u}(p)\times\mathfrak{u}(q)$ (block diagonal), which are upper-triangular on $V_+=\mathbb{C}^p$ and lower-triangular on $V_-=\mathbb{C}^q$ with respect to these bases. We may take $i\mathfrak{t}_0$ as the algebra of diagonal matrices $(t_1,\ldots,t_{p+q})$.

The roots of $\mathfrak{t}$ occuring in $\mathfrak{p}'$ are the linear forms $t_\alpha-t_\mu$. We now classify all the θ-stable parabolic subalgebras $\mathfrak{q}$ of $\mathfrak{g}$. Let $X=(t_1,\ldots,t_{p+q})$ be such that its eigenvalues on the Borel subalgebra are non-negative. Therefore

$$t_1\geqslant\ldots\geqslant t_p\quad\text{and}\quad t_{p+q}\geqslant\ldots\geqslant t_{p+1}.$$

In [4] we associate two Young diagrams λ_+ and λ_- to X:

- The diagram λ_+ is the subdiagram of $p\times q$ which consists of the boxes of coordinates (α,μ) such that $t_\alpha>t_\mu$.
- The diagram λ_- is the subdiagram of $p\times q$ which consists of the boxes of coordinates (α,μ) such that $t_{p-\alpha+1}<t_{q-\mu+1}$.

A description of the possible pairs (λ_+,λ_-) that can occur is given in [4, Lemma 6].(4)

Recall that we have associated with the parabolic subalgebra $\mathfrak{q}=\mathfrak{q}(X)$ the representations $V(\mathfrak{q})$ and $A_{\mathfrak{q}}$. The equivalence classes of both these representations only depend on the pair (λ_+,λ_-). We will therefore denote by $V(\lambda_+,\lambda_-)$ and $A(\lambda_+,\lambda_-)$ these representations.

3.8. To any Young diagram λ, we associate the irreducible K-representation

$$V(\lambda)=S_\lambda(V_+)\otimes S_{{}^t\lambda}(V_-)^*.$$

Here $S_\lambda(\cdot)$ denotes the Schur functor (see [17]) and ${}^t\lambda\subset q\times p$ is the transposed diagram. The K-representation $V(\lambda)$ occurs with multiplicity one in $\bigwedge^{|\lambda|}(V_+\otimes V_-^*)$, where $|\lambda|$ is the size of λ. The K-representation $V(\lambda_+,\lambda_-)$ is the Cartan product of $V(\lambda_+)$ and $V(\lambda_-)^*$. We recall that the Cartan product of $V(\lambda_+)$ and $V(\lambda_-)^*$ is the irreducible submodule of $V(\lambda_+)\otimes V(\lambda_-)^*$ generated by the tensor product of a highest weight vector for $V(\lambda_+)$ and a highest weight vector for $V(\lambda_-)$. Hence the highest weight of the Cartan product is the sum of the two highest weights of the factors. In our special situation—that of Vogan–Zuckerman K-types—$V(\lambda_+,\lambda_-)$ occurs with multiplicity 1 in $\bigwedge^R\mathfrak{p}$, where $R=|\lambda_+|+|\lambda_-|$.

Note that if $\lambda\subset p\times q$ is a Young diagram, we have

$$S_\lambda(V_+)^*\cong S_{\lambda^\vee}(V_+)\otimes(\textstyle\bigwedge^p V_+)^{-q},$$

where $\lambda^\vee=(q-\lambda_p,...,q-\lambda_1)$ is the complementary diagram of λ in $p\times q$. We conclude that we have

$$V(\lambda_+,\lambda_-)\cong(S_{\lambda_++\lambda_-^\vee}(V_+)\otimes(\textstyle\bigwedge^p V_+)^{-q})\otimes(S_{{}^t\lambda_-+{}^t\lambda_+^\vee}(V_-)\otimes(\textstyle\bigwedge^q V_-)^{-p}). \tag{3.14}$$

3.9. Recall that, as a $\mathrm{GL}(V_+)\times\mathrm{GL}(V_-)$-module, we have

$$\textstyle\bigwedge^R(V_+\otimes V_-^*)\cong\displaystyle\bigoplus_{\lambda\vdash R}S_\lambda(V_+)\otimes S_{{}^t\lambda}(V_-)^*=\bigoplus_{\lambda\vdash R}V(\lambda) \tag{3.15}$$

see [16, equation (19), p. 121]. Here we sum over all partitions of R (equivalently Young diagrams of size $|\lambda|=R$). We will see that, as far as we are concerned with special cycles, we only have to consider the subalgebra $(\bigwedge^\bullet\mathfrak{p})^{\mathrm{special}}$ of $\bigwedge^\bullet\mathfrak{p}$ generated by the submodules $\bigwedge^\bullet(V_+\otimes V_-^*)^{\mathrm{SL}(V_-)}$ of $\bigwedge^\bullet\mathfrak{p}'$, resp. $\bigwedge^\bullet((V_+\otimes V_-^*)^*)^{\mathrm{SL}(V_-)}$ of $\bigwedge^\bullet\mathfrak{p}''$. This amounts

(4) Beware that in this reference μ refers to a subdiagram of $p\times q$ which—in our current notation—corresponds to the complementary diagram of λ_- in $p\times q$.

to considering the submodule of (3.15) which corresponds to the λ of type $b\times q=(q^b)$. We conclude that

$$\begin{aligned}(\textstyle\bigwedge^\bullet\mathfrak{p})^{\mathrm{special}} &= \bigoplus_{a,b=0}^{p} S_{b\times q}(V_+)\otimes S_{a\times q}(V_+)^*\otimes(\textstyle\bigwedge^q V_-)^{a-b}\\ &= \bigoplus_{a,b=0}^{p} S_{b\times q}(V_+)\otimes S_{(p-a)\times q}(V_+)\otimes(\textstyle\bigwedge^p V_+)^{-q}\otimes(\textstyle\bigwedge^q V_-)^{a-b}.\end{aligned} \tag{3.16}$$

This singles out certain parabolic subalgebras that we describe in more detail below. Before that we recall the description of the invariant forms.

3.10. The Chern form

Let $\lambda\subset p\times q$ be a Young diagram. Given a basis $\{z_\ell\}_\ell$ of $V(\lambda)$ we denote by $\{z_\ell^*\}_\ell$ the dual basis of $V(\lambda)^*$ and set

$$C_\lambda=\sum_\ell z_\ell\otimes z_\ell^*\in V(\lambda)\otimes V(\lambda)^*\subset \textstyle\bigwedge^{|\lambda|,|\lambda|}\mathfrak{p}.$$

Here $\bigwedge^{|\lambda|,|\lambda|}\mathfrak{p}$ denotes the subspace of $\bigwedge^*\mathfrak{p}$ of elements of Hodge bidegreee $(|\lambda|,|\lambda|)$.

The element C_λ is independent of the choice of basis $\{z_\ell\}_\ell$ and belongs to $(\bigwedge^\bullet\mathfrak{p})^K$. Now C_λ belongs to $(\bigwedge^\bullet\mathfrak{p})^{\mathrm{special}}$ if and only if $\lambda=n\times q$, for some $n=0,\dots,p$, and $C_{n\times q}=C_q^n$ in $\bigwedge^\bullet\mathfrak{p}$, where $C_q=C_{(q)}$ is the *Chern class*. We conclude the following result.

PROPOSITION 3.4. *The subspace of K-invariants in $(\bigwedge^\bullet\mathfrak{p})^{\mathrm{special}}$ is the subring generated by the Chern class C_q.*

The (q,q)-invariant form on the symmetric space X associated with the Chern class is called the *Chern form* in [46] where it is expressed in terms of the curvature 2-forms $\Omega_{\mu,\nu}=\sum_{\alpha=1}^p\xi''_{\alpha\nu}\wedge\xi'_{\alpha\nu}$ by the formula

$$c_q=\left(\frac{-i}{2\pi}\right)^q\frac{1}{q!}\sum_{\sigma\bar\sigma\in\mathfrak{S}_q}\mathrm{sgn}(\sigma\bar\sigma)\Omega_{p+\sigma(1),p+\bar\sigma(1)}\wedge\ldots\wedge\Omega_{p+\sigma(q),p+\bar\sigma(q)}\in\textstyle\bigwedge^{q,q}\mathfrak{p}^*. \tag{3.17}$$

We now give a detailed description of the modules occuring in (3.16).

3.11. The theta-stable parabolic $Q_{b,0}$ and the Vogan–Zuckerman vector $e(bq,0)$

We first define the theta-stable parabolics $Q_{b,0}$ which will be related to the cohomology of type $(bq,0)$. These parabolics will be maximal parabolics. Suppose that b is a positive integer such that $b<p$. Let $E_b \subset V_+$ be the span of $\{v_1, ..., v_b\}$. We define $Q_{b,0}$ to be the stabilizer of E_b. Equivalently, $Q_{b,0}$ is the theta-stable parabolic corresponding to

$$X = (\underbrace{1, 1, ..., 1}_{b}, 0, ..., 0) \in i\mathfrak{t}_0.$$

We now compute the nilradical of the Lie algebra $\mathfrak{q}_{b,0}$ of $Q_{b,0}$.

Let F_b be the orthogonal complement of E_b in V_+, whence $F_b = \mathrm{Span}\{v_{b+1}, ... v_p\}$. Hence, since $V = V_+ \oplus V_-$, we have

$$V = E_b \oplus F_b \oplus V_-. \tag{3.18}$$

Put $C_b = F_b \oplus V_-$. Thus we have decomposed V into the subspace E_b and its orthogonal complement C_b in V. Let $\mathfrak{u}_{b,0}$ be the nilradical of the Lie algebra $\mathfrak{q}_{b,0}$ of the Lie group $Q_{b,0}$. We now have the following lemma.

Lemma 3.5. *Using the identification* $\mathrm{End}(V) \cong V \otimes V^*$,

$$\mathfrak{u}_{b,0} \cap \mathfrak{p} \cong E_b \otimes V_-^* \subset \mathfrak{p}'.$$

Hence $\{-v_\alpha \otimes v_\mu^* : 1 \leqslant \alpha \leqslant b$ *and* $p+1 \leqslant \mu \leqslant p+q\}$ *is a basis for* $\mathfrak{u}_{b,0} \cap \mathfrak{p}$.

Proof. It is standard that the nilradical of the maximal parabolic subalgebra which is the stabilizer of a complemented subspace E_b, is the space of homomorphisms from the given complement C_b into E_b whence, using the above identification,

$$\mathfrak{u}_{b,0} = E_b \otimes C_b^* = (E_b \otimes F_b^*) \oplus (E_b \otimes V_-^*).$$

Clearly we have

$$\mathfrak{u}_{b,0} \cap \mathfrak{k} = E_b \otimes F_b^* \quad \text{and} \quad \mathfrak{u}_{b,0} \cap \mathfrak{p} = E_b \otimes V_-^*. \qquad \square$$

The next lemma follows immediately from Lemma 3.5.

Lemma 3.6. *The vector* $e(bq, 0) \in \bigwedge^{bq,0} \mathfrak{p} \cong \bigwedge^{bq} \mathfrak{p}'$ *associated with* $\mathfrak{q}_{b,0}$ *by*

$$e(bq, 0) = (-1)^{bq} \tilde{v}_1 \wedge ... \wedge \tilde{v}_b \tag{3.19}$$

is a Vogan–Zuckerman vector for the theta stable parabolic $\mathfrak{q}_{b,0}$.

Note that $S_{b\times q}(V_+)$ is the irreducible representation for $\mathrm{Aut}(V_+)$ which has highest weight $q\varpi_b$, where ϖ_b is the bth fundamental weight (i.e. the highest weight of the bth exterior power of the standard representation). From Lemma 3.6 and the general theory of Vogan–Zuckerman, we get the following lemma.

LEMMA 3.7. *The Vogan–Zuckerman vector $e(bq,0)$ is a highest weight vector of the irreducible $K_{\mathbb{C}}\cong\mathrm{GL}(V_+)\times\mathrm{GL}(V_-)$-submodule*

$$V(b\times q):=V(b\times q,0)=S_{b\times q}(V_+)\otimes(\textstyle\bigwedge^q(V_-^*))^b \quad \textit{in } \textstyle\bigwedge^{bq,0}\mathfrak{p}.$$

Remark. As a representation of $K_{\mathbb{C}}=\mathrm{GL}(p)\times\mathrm{GL}(q)$ the representation

$$V(b\times q)\cong S_{b\times q}(\mathbb{C}^p)\otimes(\textstyle\bigwedge^q(\mathbb{C}^q))^{-b}$$

has highest weight

$$(\underbrace{q,...,q}_{b},\underbrace{0,...,0}_{p-b};\underbrace{-b,...,-b}_{q}).$$

3.12. The theta-stable parabolic $Q_{0,a}$ and the Vogan–Zuckerman vector $e(0,aq)$

Suppose that a is a positive integer such that $a<p$. Once again we let E_a be the span of $\{v_1,...,v_a\}$ and F_a be the span of $\{v_{a+1},...,v_p\}$. Let F_a^* be the span of $\{v_{p-a+1}^*,...,v_p^*\}$. We define $Q_{0,a}$ to be the stabilizer of $F_a^*\subset V^*$. We note that the stabilizer of F_a^* is the same as the stabilizer of its annihilator $(F_a^*)^\perp=E_a+V_-\subset V$. Thus $Q_{0,a}$ is the theta-stable parabolic corresponding to

$$X=(\underbrace{1,1,...,1}_{a},\underbrace{0,0,...,0}_{p-a},\underbrace{1,1,...,1}_{q})\in i\mathfrak{t}_0.$$

The proof of the following lemma is similar to that of Lemma 3.5. Let $\mathfrak{u}_{0,a}$ be the nilradical of the Lie algebra $\mathfrak{q}_{0,a}$ of the parabolic $Q_{0,a}$.

LEMMA 3.8. *We have*

$$\mathfrak{u}_{0,a}\cap\mathfrak{p}\cong V_-\otimes F_a^*\subset\mathfrak{p}''.$$

Hence $\{v_\mu\otimes v_\alpha^*: p-a+1\leqslant\alpha\leqslant p$ *and* $p+1\leqslant\mu\leqslant p+q\}$ *is a basis for* $\mathfrak{u}_{0,a}\cap\mathfrak{p}$.

Using (3.12), we obtain the Vogan–Zuckerman vector $e(0,aq)\in\bigwedge^{0,aq}\mathfrak{p}=\bigwedge^{aq}\mathfrak{p}''$ as

$$e(0,aq)=\tilde{v}_{p-a+1}^*\wedge...\wedge\tilde{v}_p^*.$$

We observe that

$$e(0,aq)=\pm\widetilde{w}_0\sigma_0(e(aq,0))$$

where $\widetilde{w}_0$ is the element of $U(p)$ that exchanges the basis vectors v_α and $v_{p+1-\alpha}$, $1\leqslant\alpha\leqslant p$. The reader will also observe that $e(0,aq)$ is a weight vector for the diagonal Cartan subalgebra in $\mathfrak{u}(p)_{\mathbb{C}}$ with weight

$$(\underbrace{0,...,0}_{p-a},\underbrace{-q,...,-q}_{a}),$$

that is a highest weight of the representation $S_{a\times q}((\mathbb{C}^p)^*)$. From Lemma 3.8 we have the following result.

LEMMA 3.9. *The Vogan–Zuckerman vector $e(0,aq)$ is a generator for $\bigwedge^{aq}(\mathfrak{u}_{0,aq}\cap\mathfrak{p})$. As such it is a highest weight vector for $Q_{0,a}$ and (from the above weight formula) it is the highest weight vector for the irreducible $K_{\mathbb{C}}$-submodule $V(0,a\times q)=S_{(a\times q)}(V_+^*)\otimes(\bigwedge^q V_-)^a$ in $\bigwedge^{0,aq}\mathfrak{p}$.*

Remark. As a representation of $K_{\mathbb{C}}=\mathrm{GL}(p)\times\mathrm{GL}(q)$ the representation

$$S_{a\times q}((\mathbb{C}^p)^*)\otimes(\textstyle\bigwedge^q\mathbb{C}^q)^a$$

has highest weight

$$(\underbrace{0,...,0}_{p-a},\underbrace{-q,...,-q}_{a};\underbrace{a,...,a}_{q}).$$

3.13. The theta-stable parabolic $Q_{b,a}$ and the Vogan–Zuckerman vector $e(bq,aq)$

We now define the theta-stable parabolics $Q_{b,a}$, which will shortly be related to the cocycles of Kudla–Millson and their generalization. Here we assume that a and b are positive integers satisfying $a+b\leqslant p$, and hence $b\leqslant p-a$. The associated theta-stable parabolics will be next-to-maximal parabolics, that is stabilizers of 2-step flags.

As before, we let $E_b\subset V_+$ be the span of $\{v_1,...,v_b\}$ and $E_{p-a}=F_{p-a}^*\subset V_+$ be the span of $\{v_1,...,v_{p-a}\}$. Since $b\leqslant p-a$, we find that $E_b\subset E_{p-a}$ and we obtain the 2-step flag

$$\mathcal{F}_{b,a}=E_b\subset E_{p-a}+V_-\subset V.$$

Let $Q_{b,a}$ be the stabilizer of the flag $\mathcal{F}_{b,a}$. Thus $Q_{b,a}$ is the theta-stable parabolic corresponding to

$$X=(\underbrace{1,1,...,1}_{a},\underbrace{0,0,...,0}_{p-a},\underbrace{-1,-1,...,-1}_{q})\in i\mathfrak{t}_0.$$

Since $Q_{b,a}$ is the intersection of stabilizers of the subspaces E_b and $E_{p-a}+V_-$ comprising the flag $\mathcal{F}_{b,a}$, the group $Q_{b,a}$ is the intersection of the two previous ones.

LEMMA 3.10. *We have*

$$Q_{b,a}=Q_{b,0}\cap Q_{0,a}.$$

Let $\mathfrak{u}_{b,a}$ be the nilradical of the Lie algebra $\mathfrak{q}_{b,a}$ of the Lie group $Q_{b,a}$. It is a standard result that if two parabolic subalgebras $\mathfrak{q}_1$ and $\mathfrak{q}_2$ intersect in a parabolic subalgebra $\mathfrak{q}$ then the nilradical of $\mathfrak{q}$ is the sum of the nilradicals of $\mathfrak{q}_1$ and $\mathfrak{q}_2$. Hence, we have

$$\mathfrak{u}_{b,a}\cap\mathfrak{p}\cong(E_b\otimes V_-^*)\oplus(V_-\otimes F_a^*)=(\mathfrak{u}_{b,0}\cap\mathfrak{p})+(\mathfrak{u}_{0,a}\cap\mathfrak{p}). \tag{3.20}$$

We obtain as a corollary that

$$\{v_\alpha \otimes v_\mu^*, v_\mu \otimes v_\beta^* : 1 \leqslant \alpha \leqslant b, p-a+1 \leqslant \beta \leqslant p \text{ and } p+1 \leqslant \mu \leqslant p+q\}$$

is a basis for $\mathfrak{u}_{b,a} \cap \mathfrak{p}$.

Remark. We have

$$\mathfrak{u}_{b,a} \cap \mathfrak{p}' = E_b \otimes V_-^* \quad \text{and} \quad \mathfrak{u}_{b,a} \cap \mathfrak{p}'' = V_- \otimes F_a^*.$$

We then define the Vogan–Zuckerman vector

$$e(bq, aq) \in \bigwedge^{bq,aq} \mathfrak{p}_{\mathbb{C}} \cong \left(\bigwedge^{bq} \mathfrak{p}'\right) \otimes \left(\bigwedge^{aq} \mathfrak{p}'\right)$$

associated with $Q_{b,a}$ by

$$\begin{aligned} e(bq, aq) &= e(bq, 0) \wedge e(0, aq) \\ &= (-1)^{bq} (\tilde{v}_1 \wedge \ldots \wedge \tilde{v}_b) \wedge (\tilde{v}_{p-a+1}^* \wedge \ldots \wedge \tilde{v}_p^*) \in \bigwedge^{bq} (V_+ \otimes (V_-)^*) \otimes \bigwedge^{aq} (V_- \otimes (V_+)^*). \end{aligned} \tag{3.21}$$

For the following lemma, we recall that the Cartan product of two irreducible representations was defined in §3.8. In the case considered below it has highest weight

$$(\underbrace{q, \ldots, q}_{b}, 0, \ldots, 0, \underbrace{-q, \ldots, -q}_{a}; \underbrace{a-b, \ldots, a-b}_{q}).$$

LEMMA 3.11. *The Vogan–Zuckerman vector $e(bq, aq)$ is a generator for*

$$\bigwedge^{(a+b)q} (\mathfrak{u}_{b,a} \cap \mathfrak{p}).$$

As such it is the highest weight vector of the irreducible $K_{\mathbb{C}}$-submodule $V(b \times q, a \times q) \subset \bigwedge^{bq,aq} \mathfrak{p}$, which is isomorphic to the Cartan product of the representations

$$S_{b \times q}(V_+) \otimes \left(\bigwedge^q (V_-^*)\right)^b \quad \textit{and} \quad S_{a \times q}(V_+^*) \otimes \left(\bigwedge^q (V_-)\right)^a.$$

3.14. Wedging with the Chern class defines a linear map in

$$\mathrm{Hom}_K\left(\left(\bigwedge^{\bullet} \mathfrak{p}\right)^{\mathrm{special}}, \left(\bigwedge^{\bullet} \mathfrak{p}\right)^{\mathrm{special}}\right).$$

We denote by C_q also the linear map. It follows from [4, Proposition 10] that $C_q^k(V(b \times q, a \times q))$ is a non-trivial K-type in $\left(\bigwedge^{(b+k)q,(a+k)q} \mathfrak{p}\right)^{\mathrm{special}}$ if and only if $k \leqslant p-(a+b)$. This leads to the following result.

PROPOSITION 3.12. *The irreducible K-types $V(b\times q, a\times q)$, with $a+b\leqslant p$, are the only Vogan–Zuckerman K-types that occur in the subring $(\bigwedge^{\bullet}\mathfrak{p})^{\mathrm{special}}$. Moreover,*

$$\mathrm{Hom}_K(V(b\times q, a\times q), (\textstyle\bigwedge^{nq}\mathfrak{p})^{\mathrm{special}}) = \begin{cases} \mathbb{C}\cdot C_q^k, & \text{if } n-(a+b)=2k, \\ 0, & \text{otherwise.} \end{cases}$$

Here k is any integer in $\{0, ..., p-(a+b)\}$.

Proof. The Vogan–Zuckerman K-types are the representations $V(\lambda_+, \lambda_-)$. Now it follows from (3.14) that if such a K-type occurs in $(\bigwedge^{\bullet}\mathfrak{p})^{\mathrm{special}}$ then $S_{{}^t\lambda_- + {}^t\lambda_+^\vee}(V_-)$ is isomorphic to a power of $\bigwedge^q V_-$ but this can only happen if the diagram ${}^t\lambda_- + {}^t\lambda_+^\vee$ has shape $q\times c$ for some c. This forces both ${}^t\lambda_-$ and ${}^t\lambda_+^\vee$ to be of this shape. We conclude that there exist integers a and b such that $\lambda_+ = b\times q$ and $\lambda_- = a\times q$. This proves the first assertion of the proposition.

We now consider the decomposition (3.16) into irreducibles. It follows from the Littlewood–Richardson rule (see e.g. [16, Corollary 2, p. 121]) that

$$\dim \mathrm{Hom}_{\mathrm{GL}(V_+)}(S_{((2q)^b, q^{p-a-b})}(V_+), S_{(q^B)}(V_+)\otimes S_{(q^{p-A})}(V_+))$$

equals the number of Littlewood–Richardson tableaux of shape $((2q)^b, q^{p-a-b})/(q^B)$ and of weight (q^{p-A}). The latter is 0 if $A\leqslant a$ or $B\leqslant b$. If $A\geqslant a$ the shape $((2q)^b, q^{p-a-b})/(q^B)$ is the disjoint union of two rectangles and it is immediate that there is at most one semistandard filling of $((2q)^b, q^{p-a-b})/(q^B)$ of content (q^{p-A}) that satisfies the reverse lattice word condition; see [16, §5.2, p. 63] (the first row must be filled with ones, the second with twos etc.). We conclude that the multiplicity of $S_{((2q)^b, q^{p-a-b})}(V_+)$ in $S_{(q^B)}(V_+)\otimes S_{(q^{p-A})}(V_+)$ equals 1 if $A=a+k$ and $B=b+k$ for some $k=0, ..., p-(a+b)$, and 0 otherwise. Since, in the former case

$$C_q^k(V(b\times q, a\times q)) \cong S_{((2q)^b, q^{p-a-b})}(V_+)\otimes(\textstyle\bigwedge^p V_+)^{-q}\otimes(\bigwedge^q V_-)^{a-b}$$

in $(\bigwedge^{(a+b+2k)q}\mathfrak{p})^{\mathrm{special}}$, this concludes the proof. $\square$

Remark. Proposition 3.12 implies the decomposition (1.4) of the introduction.

The following proposition shows that the Vogan–Zuckerman types $V(b\times q, a\times q)$ are essentially the only K-types to give small degree cohomology.

PROPOSITION 3.13. *Consider a cohomological module $A_{\mathfrak{q}}$ and let $V(\mathfrak{q})=V(\lambda_+, \lambda_-)$ be the corresponding Vogan–Zuckerman K-type. Suppose that $R=\dim(\mathfrak{u}\cap\mathfrak{p})$ is strictly less than $p+q-2$. Then either $(\lambda_+, \lambda_-)=(b\times q, a\times q)$, for some non-negative integers a and b such that $a+b\leqslant p$, or $(\lambda_+, \lambda_-)=(p\times b, p\times a)$ for some non-negative integers a and b such that $a+b\leqslant q$.*

Proof. See [4, Fait 30]. □

Remark. The cohomological modules corresponding to the second case of Proposition 3.13 correspond to the very same module where we just exchange the roles of p and q.

4. The action of $U(p)\times U(q)\times U(a)\times U(b)$ in the twisted Fock model

In this section we review the construction of the Fock model for the the Weil representation of the dual pair $U(p,q)\times U(a,b)$. We will thereby explain the reversal of a and b in the notation of the preceding sections: relative Lie algebra cohomology for $\mathfrak{u}(p,q)$ of Hodge bidegree (bq,aq) comes from the dual pair $U(p,q)\times U(a,b)$.

It is an important point in what follows that to detect the twist part of the action of $U(p)\times U(q)$ on the Fock model it is enough to determine the action of $U(p)\times U(q)$ on the vacuum vector ψ_0 (the constant polynomial 1 in the Fock model). Thus Lemmas 4.7 and 4.10 and their accompanying corollaries and remarks, which give formulas for the action of the restriction of the Weil representation to $U(p)\times U(q)$ (actually their covers) on ψ_0, will play an important role in what follows.

4.1. The square root of the determinant

In what follows we will need the square root of the determinant and its properties for various unitary groups. Let $U(V)$ be the isometry group of a Hermitian space V and let $\mathfrak{u}(V)$ be its Lie algebra. Hence we have a Lie algebra homomorphism $\mathrm{Tr}\colon \mathfrak{u}(V)\to\mathbb{C}$ (the trace). We define the covering group $\widetilde{U}(V)$ of $U(V)$ to be the pull-back by det of the covering $\pi\colon S^1\to S^1$ given by $\pi(z)=z^2$. Hence

$$\widetilde{U}(V)=\{(g,z): g\in U(V) \text{ and } z\in S^1 \text{with } \det(g)=z^2\}.$$

We then define $\det^{1/2}\colon \widetilde{U}(V)\to S^1$ by

$$\det{}^{1/2}((g,z))=z.$$

For $k\in\mathbb{Z}$ we define the character $\det^{k/2}_{U(V)}$ by $\det^{k/2}_{U(V)}=(\det^{1/2}_{U(V)})^k$. We leave the proofs of the two following lemmas to the reader.

Lemma 4.1. *Suppose $f\colon H\to U(V)$ is a homomorphism. Then the pull-back by f to H of the cover $\widetilde{U}(V)\to U(V)$ is equal to the pull-back of the cover $\pi\colon S^1\to S^1$ above by $\det\circ f\colon H\to S^1$. In particular the pull-back of the cover by f is trivial if and only if H has* *a character* $\chi: H \to S^1$ *such that* $\chi(h)^2 = \det \circ f(h)$, $h \in H$. *There is a lift* $\tilde{f}: \widetilde{H} \to \widetilde{U}(V)$ *of* f *such that*

$$\det^{1/2}_{\widetilde{U}(V)} \circ \tilde{f} = (\det_{U(V)} \circ f)^{1/2}.$$

We will also need the following result.

LEMMA 4.2. *Define* χ *to be the unique character of the universal cover of* $U(V)$ *with derivative* $\frac{1}{2}k\,\mathrm{Tr}$. *Then* χ *descends to the cover* $\widetilde{U}(V)$ *of* $U(V)$ *and the descended character is* $\det^{k/2}_{U(V)}$.

Remark. We see from the lemma that $\det^{1/2}_{U(V)}$ has the same functorial properties as the half-trace.

There are three results concerning the behaviour of $\det^{1/2}_{U(V)}$ under homomorphisms which we will need below. The first two are special cases of Lemma 4.1. The first result is the following.

LEMMA 4.3. *Let* $V_1 \subset V_2$ *be a subspace of a Hermitian space such that the restriction of the form on* V_2 *to* V_1 *is non-degenerate. Then we have an inclusion* $\widetilde{U}(V_1) \to \widetilde{U}(V_2)$ *and*

$$\det^{1/2}_{U(V_2)}|_{\widetilde{U}(V_1)} = \det^{1/2}_{U(V_1)}.$$

The second result concerns the behaviour under the diagonal action of $U(V)$ on the direct sum V^a.

LEMMA 4.4. *Let* V *be a Hermitian space and* a *be a positive integer. Let* $f: U(V) \subset U(V^a)$ *be the diagonal inclusion. Then we have* $\det_{U(V^a)} \circ f = \det^a_{U(V)}$. *Hence the cover* $\widetilde{U}(V^a) \to \widetilde{U}(V^a)$ *pulls back to the non-trivial covering group* $\widetilde{U}(V)$ *if and only if* a *is odd. Furthermore we have*

$$\det^{1/2}_{U(V^a)}|_{\widetilde{U}(V)} = \det^{a/2}_{U(V)}.$$

The third result we need the behaviour of $\det^{1/2}_{U(V)}$ under complex conjugation. Suppose we have chosen a basis $\{v_1, v_2, ..., v_n\}$ for V such that $(v_j, v_k) \in \mathbb{R}$ for all j and k. Let $V_0 \subset V$ be the real form of V given by $V_0 = \mathrm{span}_{\mathbb{R}}\{v_1, ..., v_n\}$. Let τ_{V_0} be complex conjugation of V relative to the real form V_0. The above assumption on the inner products of basis vectors is equivalent to

$$(\tau_{V_0}(x), \tau_{V_0}(y)) = \overline{(x, y)}, \quad x, y \in V. \tag{4.1}$$

Equation (4.1) implies that, if $g \in U(V)$, then $\tau_{V_0} \circ g \circ \tau_{V_0} \in U(V)$ and hence τ_{V_0} induces a conjugation map $\tau_0: U(V) \to U(V)$ given by $\tau_0(g) = \tau_{V_0} \circ g \circ \tau_{V_0}$. We note that the matrix of $\tau_0(g)$ relative to the basis $\{v_1, ..., v_n\}$ is the conjugate of the matrix of g and hence

$$\det(\tau_0(g)) = \overline{\det(g)} = \det^{-1}(g). \tag{4.2}$$

It then follows that τ_0 induces a map of coverings $\tilde{\tau}_0\colon \widetilde{U}(V)\to\widetilde{U}(V)$ given by

$$\tilde{\tau}_0(g,z)=(\tau_0(g),\bar{z}).$$

The next elementary lemma will be important in what follows.

LEMMA 4.5. *Let V and V_0 be as above. Then we have*

$$\det\nolimits_{U(V)}^{1/2}\circ\tilde{\tau}_0=\det\nolimits_{U(V)}^{-1/2}.$$

Proof. We have

$$\det\nolimits^{1/2}(\tilde{\tau}_0(g,z))=\det\nolimits^{1/2}((\tau_0(g),\bar{z}))=\bar{z}=z^{-1}=(\det\nolimits^{1/2}(g,z))^{-1}. \qquad \square$$

4.2. The Fock model of the oscillator representation associated with a point in the symmetric space of $U(p,q)$

We now recall the description of the Fock model of the Weil representation and the associated *polynomial Fock space*. We keep the notation of §2. There is one Fock model for each positive definite complex structure. However, since we will be concerned only with the restriction to the unitary group $U(V)$ we will limit ourselves to the positive definite structures coming from the symmetric space of $U(V)$. We will see below that there is one such model (structure) for each point (splitting $V=V_++V_-$) in the symmetric space of the unitary group. In what follows we will assume that $\dim V_+=p$ and $\dim V_-=q$, and set $m=p+q$.

Recall that there is a canonical positive almost complex structure J_0 associated with the Hermitian majorant $(\cdot,\cdot)_0$ of $(\cdot,\cdot)$ corresponding to the decomposition $V=V_++V_-$, which is given by the formula

$$J_0=J\circ\theta_{V_-}.$$

The eigenspace $V'^{_0}$ corresponding to the eigenvalue i of J_0 acting on $V\otimes\mathbb{C}$ is $V'_++V''_-$.

We define the Gaussian φ_0 on $V'^{_0}$ associated with the majorant $(\cdot,\cdot)_0$ by

$$\varphi_0(v'^{_0})=\exp(-\pi(v'^{_0},v'^{_0})_0).$$

Here we have transferred the majorant $(\cdot,\cdot)_0$ from V to V' using the canonical isomorphism $v\mapsto v'^{_0}=\frac{1}{2}(v\otimes 1-J_0v\otimes i)$. We finally define the Gaussian measure μ on $V'^{_0}$ by

$$\mu=C\varphi_0\mu_0,$$

where μ_0 is Lebesgue measure on $V'^{_0}$ and C is chosen so that the measure of $V'^{_0}$ is 1.

We define the *Fock space* $\mathcal{F}(V)$ to be the space of J_0-holomorphic functions (technically the half-forms) on V'^0 which are square integrable for the Gaussian measure.

We define the *polynomial Fock space* $\mathcal{P}(V)$ to be the subspace of $\mathcal{F}(V)$ consisting of J_0-holomorphic polynomials. Identifying as usual polynomial functions on a space with the symmetric algebra on its dual, we conclude that

$$\mathrm{Pol}(V'^0) \cong \mathrm{Sym}((V'^0)^*) = \mathrm{Sym}((V'_+ + V''_-)^*) \cong \mathrm{Sym}((V'_+)^*) \otimes \mathrm{Sym}((V''_-)^*).$$

It will be important in what follows to note that since V' and V'' are dually paired by (the complex bilinear extension of) the symplectic form A, we have

$$\mathrm{Sym}((V'_+)^*) \cong \mathrm{Sym}(V''_+) \quad \text{and} \quad \mathrm{Sym}((V''_-)^*) \cong \mathrm{Sym}(V'_-).$$

Hence

$$\mathcal{P}(V) = \mathrm{Sym}((V'_+)^*) \otimes \mathrm{Sym}((V''_-)^*) \cong \mathrm{Sym}(V''_+) \otimes \mathrm{Sym}(V'_-). \tag{4.3}$$

Note that the spaces $\mathcal{F}(V)$ and $\mathcal{P}(V)$ depend on the choice of positive almost complex structure J_0.

The point of the previous construction is that there is a unitary representation of the metaplectic group $\omega\colon \mathrm{Mp}(V, \langle\cdot,\cdot\rangle) \to U(\mathcal{F}(V))$. This action provides a model of the Weil representation called the Fock model. In what follows we will abbreviate $\mathrm{Mp}(V, \langle\cdot,\cdot\rangle)$ to Mp and its maximal compact subgroup, given by the 2-fold cover of the unitary group of the positive Hermitian space $(V, (\cdot,\cdot)_0)$, to $\widetilde{U}_0$. Here, we recall that $\langle\cdot,\cdot\rangle$ is the symplectic form on V. In case we have chosen a basis for V then we will write $\mathrm{Mp}(2m, \mathbb{R})$ in place of Mp.

The polynomial Fock space $\mathcal{P}(V)$ is precisely the space of $\widetilde{U}_0$-finite vectors of the Weil representation ω (see e.g. [35] and [9]) and the action of $\widetilde{U}_0$ on the polynomial Fock space is given by the following formula (see [15, Proposition 4.39, p. 184]). If $k \in \widetilde{U}_0$ and $P \in \mathcal{P}(V)$ then

$$(\omega(k)P)(v') = \det_{U_0}(k)^{-1/2} P(k^{-1}v'). \tag{4.4}$$

Remark. The determinant factor comes from the fact we should have multiplied P in the above by the *half-form* (square root of the complex volume form)

$$\sqrt{dz_1 \wedge dz_2 \wedge \ldots dz_m}.$$

We note that the constant polynomial 1 satisfies

$$\omega(k) \cdot 1 = \det_{U_0}(k)^{-1/2} \cdot 1.$$

In general for each model of the Weil representation there is a unique vector ψ_0 which corresponds to 1 traditionally called the *vacuum vector*. The vacuum vector then transforms according to

$$\omega(k)\psi_0 = \det_{U_0}(k)^{-1/2} \psi_0.$$

4.3. The restriction of the Fock model to $\widetilde{U}(p,q)$ and its half-determinant twists

In this section we will study the “restriction of the Weil representation of $\mathrm{Mp}(2m,\mathbb{R})$ to $\widetilde{U}(p,q)$”. We use quotation marks since the map $U(p,q)\to\mathrm{Sp}(2m,\mathbb{R})$ involves a conjugation.

4.3.1. The inclusion $\tilde{\jmath}_{U(p,q)}$ of $\widetilde{U}(p,q)$ in $\mathrm{Mp}(2m,\mathbb{R})$

The conjugation alluded to above comes about because the matrix M' of the symplectic form A relative to the natural basis $\mathcal{B}=\{v_1,...,v_m,iv_1,...,iv_m\}$ for the real vector space $V_{\mathbb{R}}$ underlying V is given by

$$M'=\begin{pmatrix} 0 & I_{p,q}\\ -I_{p,q} & 0\end{pmatrix}\quad\text{instead of } M=\begin{pmatrix} 0 & I_m\\ -I_m & 0\end{pmatrix}. \tag{4.5}$$

Here $I_{p,q}$ is the $m\times m$ matrix given by

$$I_{p,q}=\begin{pmatrix} I_p & 0\\ 0 & -I_q\end{pmatrix}.$$

Hence, the basis $\mathcal{B}'$ is *not* a symplectic basis. Accordingly, we let $\mathcal{B}'$ be the new basis given by

$$\mathcal{B}'=\{v_1,...,v_p,-v_{p+1},...,-v_{p+q},iv_1,...,iv_m\}.$$

Then $\mathcal{B}'$ is a symplectic basis and the change of basis matrix $Z_{p,q}$ is given by

$$Z_{p,q}=\begin{pmatrix} I_{p,q} & 0\\ 0 & I_m\end{pmatrix}. \tag{4.6}$$

In what follows we let $\mathrm{Sp}'(2m,\mathbb{R})$, resp. $\mathfrak{sp}'(2m,\mathbb{R})$, denote the group of isometries of the form M' (so $gM'g^*=M'$), resp. the Lie algebra of this group. We find then that under the canonical map $i_{\mathcal{B}}\colon\mathrm{GL}(n,\mathbb{C})\to\mathrm{GL}(2n,\mathbb{R})$ (associated with the map $\mathrm{GL}(V)\to\mathrm{GL}(V_{\mathbb{R}})$ in the basis $\mathcal{B}$) the image of $U(p,q)$ lies in $\mathrm{Sp}'(2m,\mathbb{R})$. Note that

$$i_{\mathcal{B}}(a+ib)=\begin{pmatrix} a & -b\\ b & a\end{pmatrix}.$$

We let $\jmath_{\mathcal{B}}$ be the restriction of $i_{\mathcal{B}}$ to $U(p,q)$. We define $F_{p,q}\colon\mathrm{Sp}'(2m,\mathbb{R}))\to\mathrm{Sp}(2m,\mathbb{R})$ by $F_{p,q}=\mathrm{Ad}(Z_{p,q})$ and $\jmath_{U(p,q)}$ by

$$\jmath_{U(p,q)}=F_{p,q}\circ\jmath_{\mathcal{B}}.$$

We find that $\jmath_{U(p,q)}$ maps $U(p,q)$ into $\mathrm{Sp}(2m,\mathbb{R})$.

We let $\widetilde{U}(p,q)$ be the pull-back of the metaplectic cover of $\mathrm{Sp}(2m,\mathbb{R})$ by the embedding $\jmath_{U(p,q)}$. It is well known that $\widetilde{U}(p,q)$ is the cover obtained by taking the square root of $\det\colon U(p,q)\to\mathbb{S}^1$; see for example [61, §1.2].

We let $\mathrm{Mp}'(2m,\mathbb{R})$ be the pull-back by $F_{p,q}$ of the metaplectic covering of $\mathrm{Sp}(2m,\mathbb{R})$. Hence we have a lift $\widetilde{F}_{p,q}$ of $F_{p,q}$ such that $\widetilde{F}_{p,q}\colon \mathrm{Mp}'(2m,\mathbb{R})\to\mathrm{Mp}(2m,\mathbb{R})$. Since the covering $\widetilde{U}(p,q)$ of $U(p,q)$ is pulled back from the metaplectic covering of $\mathrm{Sp}(2m,\mathbb{R})$ by the composition $F_{p,q}\circ\jmath_{\mathcal{B}}$, we also have lifts $\tilde{\jmath}_{\mathcal{B}}$ of $j_{\mathcal{B}}$ and $\tilde{\jmath}_{U(p,q)}$ of $\jmath_{U(p,q)}$. Since two coverings of a map that agree at a point agree everywhere (here the point is the identity) we have

$$\tilde{\jmath}_{U(p,q)} = \widetilde{F}_{p,q}\circ\tilde{\jmath}_{\mathcal{B}}. \tag{4.7}$$

4.3.2. The action of $\widetilde{U}(p)\times\widetilde{U}(q)$ on the vacuum vector ψ_0

Let τ_{V_-} be the real linear transformation that is the negative of complex conjugation on V_-. Then in terms of the basis $\mathcal{B}$ the matrix of $I_{V_+}\oplus\tau_{V_-}$ is $Z_{p,q}$. Note that $\tau_q{=}\mathrm{Ad}\tau_{V_-}$ acting on $U(q)$ is complex conjugation and lifts to the operator $\tilde{\tau}_q$ on $\widetilde{U}(q)$ given by

$$\tilde{\tau}_q(g,z) = (\tau_q(g),\bar{z}).$$

In what follows we will need the multiplication map $\tilde{\mu}_{p,q}\colon\widetilde{U}(p)\times\widetilde{U}(q)\to\widetilde{U}(p,q)$ given by

$$\tilde{\mu}_{p,q}(((k_1,1),z_1)((1,k_2),z_2)) = ((k_1,k_2),z_1z_2).$$

Note that $\tilde{\mu}_{p,q}$ factors through the inclusion $\widetilde{U(p)\times U(q)}\to\widetilde{U}(p,q)$ and that it has kernel $\mathbb{Z}/2$. We have an analogous map $\tilde{\mu}_{p+q}\colon\widetilde{U}(p)\times\widetilde{U}(q)\to\widetilde{U}(p+q)$.

We claim that the diagram below commutes. First the induced diagram of maps on the base space commutes; see equation (4.11). Hence the diagram of homomorphisms of covering groups commutes; see the sentence preceding equation (4.7).

$$\begin{array}{ccccc}
\widetilde{U}(p)\times\widetilde{U}(q) & \xrightarrow{\tilde{\mu}_{p,q}} & \widetilde{U}(p,q) & \xrightarrow{\tilde{\jmath}_{U(p,q)}} & \mathrm{Mp}'(2m,\mathbb{R}) \\
{\scriptstyle 1\times\tilde{\tau}_q}\downarrow & & & & \downarrow{\scriptstyle \widetilde{F}_{p,q}} \\
\widetilde{U}(p)\times\widetilde{U}(q) & \xrightarrow{\tilde{\mu}_{p+q}} & \widetilde{U}(p+q) & \xrightarrow{\tilde{\jmath}_{U(p+q)}} & \mathrm{Mp}(2m,\mathbb{R}).
\end{array}$$

Remark. In the diagram we are comparing two different mappings of $\widetilde{U}(p)\times\widetilde{U}(q)$ into $\mathrm{Mp}(2m,\mathbb{R})$. The top mapping factors through $\widetilde{U}(p,q)$ and the bottom mapping factors through $\widetilde{U}(p+q)$.

By Lemma 4.3 we have the following result.

LEMMA 4.6. *We have*

$$\det_{U(p,q)}^{1/2}|_{\widetilde{U}(p)\times\widetilde{U}(q)} = \det_{U(p)}^{1/2}\otimes\det_{U(q)}^{1/2}.$$

We now use Lemmas 4.1 and 4.5, and the above diagram to prove the following result.

LEMMA 4.7. *For* $k_1\in\widetilde{U}(p)$ *and* $k_2\in\widetilde{U}(q)$ *we have*

$$\omega(\tilde{\mu}_{p,q}(k_1,k_2))(\psi_0) = (\det_{U(p)}^{-1/2}(k_1)\otimes\det_{U(q)}^{1/2})(k_2)\psi_0.$$

Proof. Let $k=(k_1,k_2)\in K$. In what follows we will abbreviate $\tilde{\tau}_q(k_2)$ to $\bar{k}_2$.

Recall that ψ_0 is the vacuum vector (so 1 in the Fock model). Going around the top of the diagram and noting that the composition of the top right horizontal arrow with the right vertical arrow is $\tilde{\jmath}_{U(p,q)}(k_1,k_2)$, we obtain by definition

$$\omega(\tilde{\jmath}_{U(p,q)}(k_1,k_2))\psi_0 = \omega(k_1,k_2)\psi_0.$$

Going around the diagram the other way we obtain

$$\begin{aligned}
\omega((k_1,k_2))\psi_0 &= (\omega|_{\widetilde{U}(p+q)}((1\otimes\tilde{\tau}_q)(k_1,k_2)))\psi_0\\
&= (\omega|_{\widetilde{U}(p+q)}(k_1,\bar{k}_2))\psi_0\\
&= \det_{U(p+q)}^{-1/2}(k_1,\bar{k}_2)\psi_0\\
&= \det_{U(p)}^{-1/2}(k_1)\det_{U(q)}^{-1/2}(\bar{k}_2)\psi_0\\
&= \det_{U(p)}^{-1/2}(k_1)\det_{U(q)}^{+1/2}(k_2)\psi_0.
\end{aligned}$$ □

Here the second last equality is Lemma 4.3 and the last one is Lemma 4.5. We now have the following consequence.

COROLLARY 4.8. *We have*

$$(\omega\otimes\det_{U(p,q)}^{1/2})|_K(\psi_0) = \det_{U(q)}(k_2)\psi_0. \tag{4.8}$$

Remark. Lemma 4.7 and its corollary give the formula for the twist of the standard action of $U(p)\times U(q)$ in the polynomial Fock model that we will need in §4.6. In particular we need the case $a=1$, $b=0$, where $W=W_+$ is a complex line equipped with a positive unary Hermitian form. Note that in this case $a-b=1$.

We conclude this subsection with the formula for the action of $\widetilde{U}(p)\times\widetilde{U}(q)$ on ψ_0 we will need in §4.6. In particular we need the case $a=0$, $b=1$, where $W=W_-$ is a complex line equipped with a negative unary Hermitian form. In this case tensoring with W has the effect of changing the Hermitian form $(\cdot,\cdot)$ on V to its negative $(\cdot,\cdot)'=-(\cdot,\cdot)$. The matrix M'' of the symplectic form associated with $(\cdot,\cdot)'$ relative to the standard basis $\mathcal{B}=\{v_1,...,v_m,iv_1,...,iv_m\}$ is the negative of the matrix M' above. We identify the isometry groups of the two Hermitian forms $(\cdot,\cdot)$ and $(\cdot,\cdot)'$ with $U(p,q)$ using the standard basis. We obtain a new embedding $j_{U(q,p)}$ of $U(p,q)$ into $\mathrm{Sp}(2m,\mathbb{R})$ which is given by

$$j_{U(q,p)}=\mathrm{Ad}W_{p,q}\circ j_{\mathcal{B}},$$

where

$$W_{p,q}=\begin{pmatrix}-I_{p,q} & 0\\ 0 & I_m\end{pmatrix}. \tag{4.9}$$

We let ω' be the pull-back of the Weil representation to $U(p,q)$ using the embedding $j_{U(q,p)}$.

Recall that $\tau_0\colon U(p,q)\to U(p,q)$ is complex conjugation and $\tilde{\tau}_0$ is its lift to $\widetilde{U}(p,q)$.

Lemma 4.9. *We have*

$$\omega'=\omega\circ\tilde{\tau}_0.$$

Proof. Note first that $Z_{p,q}\circ W_{p,q}=I_{m,m}$, where

$$I_{m,m}=\begin{pmatrix}-I_m & 0\\ 0 & I_m\end{pmatrix}. \tag{4.10}$$

Hence the two embeddings are related by

$$j_{U(q,p)}=\mathrm{Ad}I_{m,m}\circ j_{U(p,q)}.$$

But the embedding $j_{\mathcal{B}}\colon\mathrm{GL}(m,\mathbb{C})\to\mathrm{GL}(2m,\mathbb{R})$ satisfies

$$j_{\mathcal{B}}\circ\tau_0=\mathrm{Ad}I_{m,m}\circ j_{\mathcal{B}}, \tag{4.11}$$

and hence the two embeddings are related by

$$j_{U(q,p)}=j_{U(p,q)}\circ\tau_0.$$

The lemma follows by lifting the previous identity to the 2-fold covers. □

It follows from Lemma 4.5 that the action of $\omega'|_K$ on the vaccum vector is given by the conjugate of the character for the action of $\omega|_K$ and hence from Lemma 4.7 we obtain the following result.

LEMMA 4.10. *We have*

$$\omega'|_K(\psi_0)=(\det_{U(p)}^{1/2}\otimes\det_{U(q)}^{-1/2})\psi_0.$$

COROLLARY 4.11. *The following identity holds*:

$$(\omega'\otimes\det_{U(p,q)}^{-1/2})|_K(\psi_0)=\det_{U(q)}^{-1}(k_2)\psi_0. \tag{4.12}$$

Remark. Lemma 4.10 and its corollary give the formula for the twist of the standard action of $U(p)\times U(q)$ in the polynomial Fock model. In §4.6 we will need the case $a=0$, $b=1$. Note that in this case $a-b=-1$.

4.4. The tensor product of Hermitian vector spaces

Let V be a complex vector space of dimension $m=p+q$ equipped with a Hermitian form $(\cdot,\cdot)_V$ of signature (p,q) and let W be a complex vector space of dimension $a+b$ equipped with a Hermitian form $(\cdot,\cdot)_W$ of signature (a,b). We will regard V as a real vector space equipped with the almost complex structure J_V and W as a real vector space equipped with the almost complex structure J_W. We may regard the tensor product $V\otimes_{\mathbb{C}}W$ as the quotient of the tensor product $V\otimes_{\mathbb{R}}W$ by the relations

$$(J_V(v))\otimes w=v\otimes(J_W(w))$$

for all pairs of vectors $v\in V$ and $w\in W$. Thus we have an almost complex structure $J_{V\otimes W}$ on $V\otimes_{\mathbb{C}}W$ given by

$$J_{V\otimes W}=J_V\otimes I_W=I_V\otimes J_W. \tag{4.13}$$

From now on all tensor products will be over $\mathbb{C}$ unless the contrary is indicated. We will acccordingly abbreviate $V\otimes_{\mathbb{C}}W$ to $V\otimes W$.

We let $\mathcal{S}$ denote the algebra $\mathbb{C}[z_1,z_2]/(z_1^2+1,z_2^2+1)$. An action of $\mathcal{S}$ on a vector space V is given by the choice of two commuting complex structures on V. Hence given a complex vector space V the complexification of V is an $\mathcal{S}$-module and the subspaces V' and V'' are $\mathcal{S}$-submodules.

Let $\iota:(V\otimes W)\otimes_{\mathbb{R}}\mathbb{C}\to(V\otimes_{\mathbb{R}}\mathbb{C})\otimes_{\mathcal{S}}(W\otimes_{\mathbb{R}}\mathbb{C})$ be the map given by

$$\iota_V(v\otimes w\otimes z)=(v\otimes 1)\otimes(w\otimes z).$$

Remark. Note that the map ι is well defined:

$$\iota(J_V(v)\otimes w\otimes 1)=\iota(v\otimes J_W(w)\otimes 1)$$

because we tensored over $\mathcal{S}$.

The following lemma is clear.

Lemma 4.12. *We have*

$$V'\otimes_{\mathbb{S}}W''=0 \quad \text{and} \quad V''\otimes_{\mathbb{S}}W'=0.$$

Accordingly, we have

$$(V\otimes_{\mathbb{R}}\mathbb{C})\otimes_{\mathbb{S}}(W\otimes_{\mathbb{R}}\mathbb{C})=(V'\otimes W')+(V''\otimes W'').$$

We now have the following result.

Proposition 4.13. *The map ι is an isomorphism and consequently induces an isomorphism, which we again denote by $\iota\colon (V\otimes W)\otimes_{\mathbb{R}}\mathbb{C}\cong V'\otimes W'+V''\otimes W''$.*

Proof. Since both the domain and the range of ι have dimension $2\dim V\dim W$ it suffices to prove that ι is onto. But given $(v\otimes z_1)\otimes(w\otimes z_2)\in(V\otimes_{\mathbb{R}}\mathbb{C})\otimes_{\mathbb{S}}(W\otimes_{\mathbb{R}}\mathbb{C})$ we have

$$(v\otimes z_1)\otimes(w\otimes z_2)=(v\otimes 1)\otimes(w\otimes z_1z_2)=\iota(v\otimes w\otimes z_1z_2). \qquad \square$$

The tensor product $(\!(\cdot,\cdot)\!)=(\cdot,\cdot)_V\otimes(\cdot,\cdot)_W$ is a Hermitian form on $V\otimes W$. We let $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ denote the symplectic form on the real vector space underlying $V\otimes W$ (which we will again denote by $V\otimes W$); it is given by the negative of the imaginary part of $(\!(\cdot,\cdot)\!)$. Hence we have

$$\langle\!\langle\cdot,\cdot\rangle\!\rangle=\langle\cdot,\cdot\rangle_V\otimes B_W+B_V\otimes\langle\cdot,\cdot\rangle_W. \tag{4.14}$$

Clearly we have an embedding $U(V)\times U(W)\to\mathrm{Aut}(V\otimes W,\langle\!\langle\cdot,\cdot\rangle\!\rangle)$. It is standard that this product is a dual reductive pair, that is each factor is the full centralizer of the other in the symplectic group $\mathrm{Aut}(\langle\!\langle\cdot,\cdot\rangle\!\rangle)$.

We can now use the direct sum decompositions $V=V_+\oplus V_-$ and $W=W_+\oplus W_-$, and the considerations of §2.5 to change the indefinite almost complex structure $J_{V\otimes W}$ to an admissible positive almost complex structure $(J_{V\otimes W})_0$ on $V\otimes W$. Indeed we can split $V\otimes W$ into a sum of the positive definite space $V_+\otimes W_+ + V_-\otimes W_-$ and the negative definite space $V_+\otimes W_- + V_-\otimes W_+$. The corresponding Cartan involution is $\theta_V\otimes\theta_W$. The Cartan involution $\theta_V\otimes\theta_W$ allows us to define a positive definite Hermitian form $(\!(\cdot,\cdot)\!)_0$ (the corresponding majorant of $(\!(\cdot,\cdot)\!)$) by

$$(\!(v_1\otimes w_1,v_2\otimes w_2)\!)_0=(\!(v_1\otimes w_1,\theta_V(v_2)\otimes\theta_W(w_2))\!)=(v_1,\theta_V(v_2))_V(w_1,\theta_W(w))_W.$$

We define the positive definite complex structure $(J_{V\otimes W})_0$ corresponding to the previous splitting by

$$(J_{V\otimes W})_0=(J_{V\otimes W})\circ(\theta_V\otimes\theta_W)=(J_V\otimes I_W)\circ(\theta_V\otimes\theta_W)=(I_V\otimes J_W)\circ(\theta_V\otimes\theta_W). \tag{4.15}$$

We emphasize that the definite complex structure $(J_{V\otimes W})_0$ depends on the choice of splittings of V and W.

We can now compute the spaces $(V\otimes W)'_0$ of type $(1,0)$ vectors and $(V\otimes W)''_0$ of type $(0,1)$ vectors for $(J_{V\otimes W})_0$ acting on $(V\otimes W)\otimes_{\mathbb{R}}\mathbb{C}$.

Note that

$$\dim_{\mathbb{C}}((V\otimes W)'_0) = \dim_{\mathbb{C}}(V\otimes W) = (a+b)(p+q).$$

LEMMA 4.14. *We have $(U(p)\times U(q))\times(U(a)\times U(b))$-equivariant isomorphisms of complex vector spaces*

$$(V\otimes W)'_0 \cong (V'_+\otimes W'_+)\oplus(V''_+\otimes W''_-)\oplus(V''_-\otimes W''_+)\oplus(V'_-\otimes W'_-),$$
$$(V\otimes W)''_0 \cong (V''_+\otimes W''_+)\oplus(V'_+\otimes W'_-)\oplus(V'_-\otimes W'_+)\oplus(V''_-\otimes W''_-).$$

Under the pairing of $(V\otimes W)'_0$ with $(V\otimes W)''_0$ induced by the symplectic form each of the four subspaces on the right is dually paired with the space immediately below it.

Proof. The space $(V\otimes W)\otimes_{\mathbb{R}}\mathbb{C}$ is the quotient of the space $(V\otimes_{\mathbb{R}}\mathbb{C})\otimes_{\mathbb{C}}(W\otimes_{\mathbb{R}}\mathbb{C})$ by the relation that makes the action of $J_V\otimes 1\otimes 1\otimes 1$ equal to that of $1\otimes 1\otimes J_W\otimes 1$. The operation of passing to the quotient corresponds to setting all tensor products of spaces with superscript prime factors with spaces with superscript double prime factors equal to zero according to Lemma 4.12. Before passing to the quotient we have a direct sum decomposition with $4\times 4=16$ summands, after passing to the quotient we have a direct sum decomposition with eight summands. With the above identification we have

$$\begin{aligned}(V\otimes W)\otimes_{\mathbb{R}}\mathbb{C} = {} & [\underbrace{(V'_+\otimes W'_+)\oplus(V'_+\otimes W'_-)\oplus(V'_-\otimes W'_+)\otimes(V'_-\otimes W'_-)}_{+i}] \\ & \oplus[\underbrace{(V''_+\otimes W''_+)\oplus(V''_+\otimes W''_-)\oplus(V''_-\otimes W''_+)\otimes(V''_-\otimes W''_-)}_{-i}].\end{aligned}$$

Here the subscript $\pm i$ indicates the eigenvalue of $J_{V\otimes W}$ on the summand. Thus the first four summands comprise the subspace of type $(1,0)$ vectors for $J_{V\otimes W}$ and the last four summands comprise the subspace of type $(0,1)$ vectors for $J_{V\otimes W}$. Now the involution $\theta_V\otimes\theta_W$ is also diagonal relative to the above eight summand decomposition with the corresponding eigenvalues $(+1,-1,-1,+1,+1,-1,-1,+1)$. Since $(J_{V\otimes W})_0 = J_{V\otimes W}\circ(\theta_V\otimes\theta_W)$, the second and third summands above move into the space of type $(0,1)$ vectors for $(J_{V\otimes W})_0$ and the sixth and seventh summands move into the space of type $(1,0)$ vectors for $(J_{V\otimes W})_0$. $\square$

4.5. The polynomial Fock model for a unitary dual pair

Our goal in this subsection is to describe the polynomial Fock space $\mathcal{P}(V\otimes W)$. Our main interest will be the subspace $\mathcal{P}(V_+\otimes W)\subset\mathcal{P}(V\otimes W)$ and its description as the algebra of polynomials on the space of $p\times(a+b)$ complex matrices

$$M_{p\times(a+b)}(\mathbb{C})=M_{p\times a}(\mathbb{C})\oplus M_{p\times b}(\mathbb{C}).$$

By Lemma 4.14, we have

$$\mathcal{P}(V\otimes W)=\mathrm{Pol}((V'_+\otimes W'_+)\oplus(V''_+\otimes W''_-))\otimes\mathrm{Pol}((V'_-\otimes W'_-)\oplus(V''_-\otimes W''_+)). \tag{4.16}$$

We will abbreviate the first factor in the tensor product on the right to $\mathcal{P}_+$ and the second factor to $\mathcal{P}_-$. We now choose an orthonormal basis $\{w_1,...,w_a\}$ for W_+ and a basis $\{w_{a+1},...,w_{a+b}\}$ for W_- which is orthonormal with respect to the restriction of $-(\cdot,\cdot)_W$ to W_-.

In this paper we will be primarily concerned with the space $\mathcal{P}_+$. Accordingly we will suppose that $u\in(V_+\otimes W)'_0$. Then there exist unique $x'_1,...,x'_a\in V'_+$ and $y''_1,...,y''_b\in V''_+$ such that

$$u=\sum_{j=1}^{a}x'_j\otimes w'_j+\sum_{k=1}^{b}y''_k\otimes w''_{a+k}. \tag{4.17}$$

We may accordingly represent the element u of $(V_+\otimes W)'$ by

$$(x'_1,...,x'_a;y''_1,...,y''_b)=(\mathbf{x}';\mathbf{y}'')\in(V'_+)^a\oplus(V''_+)^b.$$

Then, by using the basis $\{v'_1,...,v'_p,v''_1,...,v''_p\}$, we may finally represent an element of $(V_+\otimes W)'_0$ as a $p\times(a+b)$ matrix with complex entries. Thus we have

$$(V_+\otimes W)'_0\cong M_{p\times a}(\mathbb{C})\oplus M_{p\times b}(\mathbb{C}).$$

We will think of a point on the right as a $p\times(a+b)$ matrix $Z(\mathbf{x}';\mathbf{y}'')$ divided into a left $p\times a$ block $Z'(\mathbf{x}')=(z'_{\alpha,j}(\mathbf{x}'))$ and a right $p\times b$ block $Z''(\mathbf{y}'')=(z''_{\alpha,k}(\mathbf{y}''))$. We will use these matrix coordinates henceforth (at times we will drop the arguments $\mathbf{x}'$ and $\mathbf{y}''$). By Lemma 2.1, we have

$$z'_{\alpha,j}(\mathbf{x}')=(x_j,v_\alpha),\ 1\leqslant j\leqslant a,\ 1\leqslant\alpha\leqslant p\quad\text{and}\quad z''_{\alpha,k}(\mathbf{y}'')=(v_\alpha,y_j),\ 1\leqslant k\leqslant b,\ 1\leqslant\alpha\leqslant p.$$

Here we have used Greek letter(s) α for the indices belonging to V and Roman letters j,k for the indices belonging to W.

The polynomial Fock model is then the space of polynomials in $z'_{\alpha,j}$ and $z''_{\alpha,k}$ as above. From now on, we will usually work with this matrix description of the polynomial Fock space, and hence we will identify

$$\mathcal{P}_+=\mathrm{Pol}((V'_+\otimes W'_+)\oplus(V''_+\otimes W''_-))\cong\mathrm{Pol}(M_{p\times a}(\mathbb{C})\oplus M_{p\times b}(\mathbb{C})).$$

4.6. The twisted action of $(U(p)\times U(q))\times(U(a)\times U(b))$

Recall that we have a unitary representation of the metaplectic group

$$\omega\colon \mathrm{Mp}(V\otimes W, \langle\!\langle\cdot,\cdot\rangle\!\rangle) \longrightarrow U(\mathcal{F}(V\otimes W)).$$

As above we use $\widetilde{U}_0$ to denote the maximal compact subgroup $\widetilde{U}(V\otimes W, (\!(\cdot,\cdot)\!)_0)$ of $\mathrm{Mp}(V\otimes W, \langle\!\langle\cdot,\cdot\rangle\!\rangle)$. The space of $\widetilde{U}_0$-finite vectors of the Weil representation ω is precisely the polynomial Fock space $\mathcal{P}(V\otimes W)=\mathcal{P}_+\otimes\mathcal{P}_-$, we refer to [35] and [61] for more details. We review how certain subgroups (subalgebras) of $\widetilde{U}(V)\times\widetilde{U}(W)$ act in this model.

We have natural inclusion maps

$$U(V)\times U(W)\longrightarrow U(V\otimes W) \quad \text{and} \quad U(V\otimes W)\longrightarrow \mathrm{Sp}(V\otimes W, \langle\!\langle\cdot,\cdot\rangle\!\rangle).$$

We have previously described 2-fold covers $\widetilde{U}(V)$, $\widetilde{U}(W)$ and $\widetilde{U}(V\otimes W)$ of $U(V)$, $U(W)$ and $U(V\otimes W)$, repspectively, with their respective characters $\det_{U(V)}^{1/2}$, $\det_{U(W)}^{1/2}$ and $\det_{U(V\otimes W)}^{1/2}$. Lemmas 4.3 and 4.4 then imply that

$$\det\nolimits_{U(V\otimes W)}^{1/2}|_{\widetilde{U}(V)} = \det\nolimits_{U(V)}^{(a+b)/2} \quad \text{and} \quad \det\nolimits_{U(V\otimes W)}^{1/2}|_{\widetilde{U}(W)} = \det\nolimits_{U(W)}^{(p+q)/2}. \tag{4.18}$$

If $(k,\ell)\in\mathbb{Z}^2$ the restriction of the Weil representation ω to $\widetilde{U}(V)\times\widetilde{U}(W)$ twisted by the characters $\det_{U(V)}^{k/2}\otimes\det_{U(W)}^{\ell/2}$ will be denoted $\omega_{k,\ell}$. Since the Weil representation of $\widetilde{U}(V\otimes W)$ twisted by $\det^{1/2}$ descends to $U(V\otimes W)$, it follows from equation (4.18) that $\omega_{k,\ell}$ descends to $U(V)\times U(W)$ if and only if $k\equiv a+b \pmod 2$ and $\ell\equiv p+q \pmod 2$.

Note that it follows from (4.4) that the compact subgroup $\widetilde{U}(p)\times\widetilde{U}(q)\times\widetilde{U}(a)\times\widetilde{U}(b)$ acts on $\mathcal{P}$ by the usual action up to a central character. The explicit computation of this central character is given by the following proposition.

Proposition 4.15. *The group $\widetilde{U}(p)\times\widetilde{U}(q)\times\widetilde{U}(a)\times\widetilde{U}(b)$ acts on the line $\mathbb{C}\psi_0$ (so the constant polynomials in the Fock model) under the twisted Weil representation $\omega_{k,\ell}$ by the character $\det_{U(p)}^{(k+b-a)/2}\otimes\det_{U(q)}^{(k+a-b)/2}\otimes\det_{U(a)}^{(\ell+q-p)/2}\otimes\det_{U(b)}^{(\ell+p-q)/2}$.*

The Proposition will follow from Lemma 4.3 and the next lemma (which will be seen to follow from Lemmas 4.7, 4.4 and 4.10).

First by applying Lemma 4.3 to the "block inclusions" $U(W_+)\times U(W_-)\subset U(W)$ and $U(V_+)\times U(V_-)\subset U(V)$ we get

$$\begin{aligned} \det\nolimits_{U(V)}^{k/2}|_{\widetilde{U}(V_+)\times\widetilde{U}(V_-)} &= \det\nolimits_{U(V_+)}^{k/2}\otimes\det\nolimits_{U(V_-)}^{k/2}, \\ \det\nolimits_{U(W)}^{\ell/2}|_{\widetilde{U}(W_+)\times\widetilde{U}(W_-)} &= \det\nolimits_{U(W_+)}^{\ell/2}\otimes\det\nolimits_{U(W_-)}^{\ell/2}. \end{aligned} \tag{4.19}$$

Now the proposition follows from the next lemma.

LEMMA 4.16. *The group $\widetilde{U}(p)\times\widetilde{U}(q)\times\widetilde{U}(a)\times\widetilde{U}(b)$ acts on the line $\mathbb{C}\psi_0$ under the (untwisted) Weil representation ω by the character*

$$\det{}_{U(p)}^{(b-a)/2}\otimes\det{}_{U(q)}^{(a-b)/2}\otimes\det{}_{U(a)}^{(q-p)/2}\otimes\det{}_{U(b)}^{(p-q)/2}.$$

Proof. By the symmetry between V and W it is sufficient to compute the action of $\widetilde{U}(p)\times\widetilde{U}(q)$ under the untwisted Weil representation on $\mathbb{C}\psi_0$. Considering the tensor product of V with a Hermitian space of signature (a,b) amounts to looking at the diagonal action of $U(V)$ on the direct sum of a copies of V and b copies of V with the sign of the Hermitian form changed. Hence, by Lemma 4.4, we are reduced to the special cases $a=1$, $b=0$ and $a=0$, $b=1$. The first case is Lemma 4.7 and the second one is Lemma 4.10. □

4.7. From now on we will always assume that $k=a-b$ and $\ell=p+q$ and will now use the symbol ω to denote the (twisted) representation $\omega_{a-b,p+q}$. The choice of k will turn out to be very important: indeed it follows from Proposition 4.15 that the restriction of ω to the group $U(p)\times U(q)$ then acts on the line $\mathbb{C}\psi_0$ (the constant polynomials in the Fock model) by the character $1\otimes\det_{U(q)}^{a-b}$. As a consequence the group $U(p)\times U(q)$ acts on $\mathcal{P}_+$ by the tensor product of the standard action of $U(p)$ and the character $\det_{U(q)}^{a-b}$. We will later see that the above twist is the correct one to ensure our cocycle $\psi_{bq,aq}$ is $(U(p)\times U(q))$-equivariant; see Lemma 5.18.

To summarize, if we represent the action of the Weil representation ω restricted to $U(p)\times U(q)\times U(a)\times U(b)$ on the subspace $\mathcal{P}_+$ of the Fock model for $U(p,q)\times U(a,b)$, in terms of the $p\times(a+b)$ matrix (sub)representation of the Fock model (see §4.5)

$$\mathcal{P}_+=\mathrm{Pol}(M_{p\times a}(\mathbb{C})\oplus M_{p\times b}(\mathbb{C})),$$

we have the following result.

THEOREM 4.17. (1) *The action of the group $U(a)\times U(b)$ induced by the twisted Weil representation $\omega_{a-b,p+q}$ on polynomials in the matrix variables is the tensor product of the character $\det_{U(a)}^{q}\otimes\det_{U(b)}^{p}$ with the action induced by the natural action on the rows (i.e. from the right) of the matrices. Note that each row has $a+b$ entries. The group $U(a)$ acts on on the first a entries of each row and $U(b)$ acts on the last b entries of each row.*

(2) *The action of the group $U(p)$ is induced by the natural action on the columns (i.e. from the left) of the matrices, acting on the left half of the matrix by the standard action and on the right half by the dual of the standard action so there is no determinant twist.*

(3) *The group $U(q)$ simply scales all polynomials by the central character $\det_{U(q)}^{a-b}$.*

The representation ω yields a correspondence between certain equivalence classes of irreducible admissible representations of $U(a,b)$ and $U(p,q)$. The correspondence between K-types is explicitly described in [61] using the Fock model (and following Howe [35]), we also refer to [39].

5. The special $(\mathfrak{u}(p,q),K)$-cocycles $\psi_{bq,aq}$

In this section we introduce special cocycles

$$\psi_{bq,aq}\in \operatorname{Hom}_K\big(\textstyle\bigwedge^{bq,aq}\mathfrak{p},\mathcal{P}(V\otimes W)\big)$$

with values in the polynomial Fock space.

We will first define the cocycles $\psi_{bq,0}$ of Hodge bidegree $(bq,0)$, and similarly $\psi_{0,aq}$ of Hodge bidegree $(0,aq)$. We will give formulas for their dual maps $\psi^*_{bq,0}$, resp. $\psi^*_{0,aq}$, as the values of these dual maps at $x\in(V\otimes W)'_0$ are decomposable as exterior products of bq, resp. aq, elements of $\mathfrak{p}^*$ depending on x.

5.1. Harmonic and special harmonic polynomials

In this subsection we review the lowering operators coming from the action of the space $\mathfrak{p}''_{U(a,b)}$. We leave to the reader the task of writing out the formulas for $\mathfrak{u}(W)$ analogous to those of §3.3 for $\mathfrak{u}(V)$, in particular of proving $\mathfrak{p}''_{U(W)}\cong W_-\otimes W_+^*$. Hence, in the notation of §3.3 we have a basis $\{w_j\otimes w^*_{a+k}:1\leqslant j\leqslant a \text{ and } 1\leqslant k\leqslant b\}$ for $\mathfrak{p}''_{U(a,b)}$. We define

$$\Delta_{j,k}=\Delta^+_{j,k}=\sum_{\alpha=1}^{p}\frac{\partial^2}{\partial z'_{\alpha,j}\partial z''_{\alpha,k}}\quad \text{for } 1\leqslant j\leqslant a \text{ and } 1\leqslant k\leqslant b.$$

Thus $\Delta_{j,k}$ is a second-order differential operator. Then we have (up to a scalar multiple) the following result.

PROPOSITION 5.1. *We have*

$$\omega(w_j\otimes w^*_{a+k})=\Delta^+_{j,k}.$$

Here ω is the (infinitesimal) oscillator representation for the dual pair $\mathfrak{u}(p)\times\mathfrak{u}(a,b)$. The proposition is a straightforward computation and is implicit in [39, equation 5.1].

Remark. We have analogous Laplace operators $\Delta^-_{j,k}$ on $\operatorname{Pol}((V'_-\otimes W'_-)\oplus(V''_-\otimes W''_+))$.

We define the subspace of *harmonic polynomials*

$$\mathrm{Harm}((V'_+\otimes W'_-)\oplus(V''_+\otimes W''_-))\subset\mathrm{Pol}((V_+\otimes W)'_0)=\mathrm{Pol}((V'_+\otimes W'_+)\oplus(V''_+\otimes W''_-))$$

to be the subspace of polynomials annihilated by the Laplace operators $\Delta_{j,k}, 1\leqslant j,k\leqslant n$.

We will henceforth abbreviate $\mathrm{Harm}((V'_+\otimes W'_-)\oplus(V''_+\otimes W''_-))$ to $\mathcal{H}_+$. The subspace $\mathcal{H}_-=\mathrm{Harm}((V'_-\otimes W'_-)\oplus(V''_-\otimes W''_+))\subset\mathrm{Pol}((V'_-\otimes W'_-)\oplus(V''_-\otimes W''_+))$ is defined analogously to $\mathcal{H}_+$ as the simultaneous kernels of the operators $\Delta^-_{j,k}$. We emphasize that $\mathcal{H}_+$ and $\mathcal{H}_-$ are not closed under multiplication.

Note however that the subalgebra $\mathrm{Pol}(V'_+\otimes W'_+)$ of $\mathcal{P}(V'_+\otimes W')$ is contained in the subspace of harmonic polynomials,

$$\mathrm{Pol}(V'_+\otimes W'_+)\subset\mathrm{Harm}((V'_+\otimes W'_-)\oplus(V''_+\otimes W''_-)).$$

We will call an element of $\mathrm{Pol}(V'_+\otimes W'_+)$ a *special harmonic polynomial*.

Following Kashiwara–Vergne [39] we define elements

$$\Delta_k\in\mathcal{P}(V_+\otimes W)\quad\text{and}\quad\tilde{\Delta}_\ell\in\mathcal{P}(V_+\otimes W)$$

for $1\leqslant k,\ell\leqslant p$ and $a,b\leqslant p$, by

$$\begin{aligned}\Delta_k(\mathbf{x}',\mathbf{y}'')&=\Delta_k(\mathbf{y}'')=\det(z''_{\alpha,j})=\det((v_\alpha,y_j)),\quad 1\leqslant\alpha\leqslant k\text{ and }1\leqslant j\leqslant k,\\ \tilde{\Delta}_\ell(\mathbf{x}',\mathbf{y}'')&=\tilde{\Delta}_\ell(\mathbf{x}')=\det(z'_{\alpha,j})=\det((x_j,v_\alpha)),\quad p-\ell+1\leqslant\alpha\leqslant p\text{ and }1\leqslant j\leqslant\ell.\end{aligned}$$

We note that Δ_k and $\tilde{\Delta}_\ell$ are special harmonic, and hence any power of Δ_k or $\tilde{\Delta}_\ell$ is also special harmonic. One easily verifies the following lemma.

LEMMA 5.2. *Suppose $k+\ell\leqslant p$. Then for any natural numbers ℓ_1 and ℓ_2 the product $\Delta_k^{\ell_1}\cdot(\tilde{\Delta}_\ell)^{\ell_2}$ is harmonic.*

5.2. Some special cocyles

We now give formulas for cocycles which will turn out to be generalizations of the special cocyles constructed by Kudla–Millson.

The domain of the relative Lie algebra cochains is the exterior algebra $\bigwedge^*\mathfrak{p}$, which factors according to

$$\bigwedge{}^*\mathfrak{p}=(\bigwedge{}^*\mathfrak{p}')\otimes(\bigwedge{}^*\mathfrak{p}'')=\bigwedge{}^*(V_+\otimes V_-^*)\otimes\bigwedge{}^*(V_-\otimes V_+^*).\tag{5.1}$$

We will consider only very special cochains whose range is the positive definite Fock model $\mathrm{Pol}((V_+\otimes W)'_0)=\mathcal{P}_+$. Recall that the space $\mathrm{Pol}((V_+\otimes W)'_0)$ factors according to

$$\mathrm{Pol}((V_+\otimes W)'_0)=\mathrm{Pol}(V'_+\otimes W'_+)\otimes\mathrm{Pol}(V''_+\otimes W''_-).\tag{5.2}$$

The key point is that each of the two factorizations has a strong "disjointness property". On the right-hand side of equation (5.1) the only irreducible $U(p)$-representation common to each of the two factors in the tensor product is the trivial representation and the same for equation (5.2).

5.2.1. A restriction on Hodge types

Note that

$$\mathrm{Pol}(V'_+\otimes W'_+)\cong \mathrm{Sym}(V''_+\otimes W''_+) \quad \text{and} \quad \mathrm{Pol}(V''_+\otimes W''_-)\cong \mathrm{Sym}(V'_+\otimes W'_-).$$

Then, since any cochain $\psi_{k,\ell}$ is $U(p)$-equivariant, we obtain the following result.

LEMMA 5.3. *Suppose that $\psi_{k,0}$ is a cochain of bidegree $(k,0)$ taking values in $\mathcal{P}_+$. Then it must take values in the second factor of the tensor product in (5.2), namely in*

$$\mathrm{Pol}(V''_+\otimes W''_-)\cong \mathrm{Sym}(V'_+\otimes W'_-)\cong \mathrm{Sym}((V'_+)b).$$

Equivalently, suppose that $\psi_{0,\ell}$ is a cochain of bidegree $(0,\ell)$ taking values in $\mathcal{P}_+$. Then it must take values in the first factor of (5.2), namely in

$$\mathrm{Pol}(V'_+\otimes W'_+)\cong \mathrm{Sym}(V''_+\otimes W''_+)\cong \mathrm{Sym}((V''_+)^a).$$

Remark. If we insist on the standard convention $\dim W_+=a$ and $\dim W_-=b$ (as we are going to do) then the Hodge degrees of the special cocycles we construct will be of the form (bq,aq). Thus in our previous notation the cocycle $\psi_{bq,0}$ gives rise to a polynomial function of $y''_1,...,y''_b$ (the right half of the matrix) and the cocycle $\psi_{0,aq}$ gives rise to a polynomial function of $x'_1,...,x'_a$ (the left half of the matrix).

5.3. Our immediate goal now is to give the definitions and establish some properties of the cocycles $\psi_{bq,0}$, of Hodge bidegree $(bq,0)$, and $\psi_{0,aq}$, of Hodge bidegree $(0,aq)$. As pointed out above, these cocycles have the following special properties:

(1) $\psi_{bq,0}\colon \bigwedge^{bq}(V_+\otimes V_-^*)=\bigwedge^{bq}\mathfrak{p}'\to \mathrm{Sym}^{bq}((V''_+\otimes W''_-)^*)$;

(2) $\psi_{0,aq}\colon \bigwedge^{aq}(V_-\otimes V_+^*)=\bigwedge^{bq}\mathfrak{p}''\to \mathrm{Sym}^{aq}((V'_+\otimes W'_+)^*)$.

As stated above, if we first evaluate the cocycles $\psi_{bq,0}$ and $\psi_{0,aq}$ at points in $V''_+\otimes W''_-$ and $V'_+\otimes W'_+$, respectively, the resulting elements of $\bigwedge^\bullet\mathfrak{p}^*$ are completely decomposable. To formalize this decomposability property (which will be very useful for computations) and also to understand how the cocycles transform under $\widetilde{U}(p)\times\widetilde{U}(q)\times\widetilde{U}(a)\times\widetilde{U}(b)$ it is better to give formulas for the the dual maps $\psi^*_{bq,0}$ and $\psi^*_{0,aq}$. These maps will satisfy

(1) $\psi^*_{bq,0}\colon \mathrm{Sym}^{bq}(V''_+\otimes W''_-)\to\bigwedge^{bq}(V_+^*\otimes V_-)=\bigwedge^{bq}(\mathfrak{p}')^*$;

(2) $\psi^*_{0,aq}\colon \mathrm{Sym}^{aq}(V'_+\otimes W'_+)\to\bigwedge^{aq}(V_-^*\otimes V_+)=\bigwedge^{bq}(\mathfrak{p}'')^*$.

Remark. Note that

$$\psi_{bq,0} \in \mathrm{Sym}^{bq}((V''_+ \otimes W''_-)^*) \otimes \bigwedge^{bq}(\mathfrak{p}')^* \quad \text{and} \quad \psi^*_{bq,0} \in \bigwedge^{bq}(\mathfrak{p}')^* \otimes \mathrm{Sym}^{bq}((V''_+ \otimes W''_-)^*)$$

are interchanged by the map that switches the polynomial and exterior factors.

The defining formula for $\psi^*_{bq,0}$ is then the following. Let $\mathbf{y}''=(y''_1, ..., y''_b)$. Then we have

$$\psi^*_{bq,0}(\mathbf{y}'') = {}^t\tilde{y}^*_1 \wedge {}^t\tilde{y}^*_2 \wedge ... \wedge {}^t\tilde{y}^*_b \in \bigwedge^{bq}(V^*_+ \otimes V_-). \tag{5.3}$$

Remark. It is important to observe that the above tensor is a wedge product of bq vectors in $(\mathfrak{p}')^*$ depending on $\mathbf{y}''$, that is

$$\psi^*_{bq,0}(\mathbf{y}'') = [(y^*_1 \otimes v_{p+1}) \wedge ... \wedge (y^*_1 \otimes v_{p+q})] \wedge ... \wedge [(y^*_b \otimes v_{p+1}) \wedge ... \wedge (y^*_b \otimes v_{p+q})].$$

We have a similar formulas for $\psi^*_{0,aq}$. Let $\mathbf{x}'=(x'_1, ..., x'_a)$. Then we have

$$\psi^*_{0,aq}(\mathbf{x}') = {}^t\tilde{x}_1 \wedge ... \wedge {}^t\tilde{x}_a \in \bigwedge^{aq}(V^*_- \otimes V_+) = \bigwedge^{aq}(\mathfrak{p}')^*. \tag{5.4}$$

It is immediate from the above defining formulas that the holomorphic and anti-holomorphic cocycles factor according to the following lemma.

LEMMA 5.4. *Let $a=u+v$ and $b=r+s$. Then we have the following factorizations*:

(1) $\psi_{bq,0}=\psi_{rq,0} \wedge \psi_{sq,0}$;

(2) $\psi_{0,aq}=\psi_{0,uq} \wedge \psi_{0,vq}$.

The exterior product $\wedge$ in Lemma 5.4 is the *outer exterior product* associated with the product in the coefficient ring (in which the cocycles take values)

$$\mathrm{Pol}((V'_+ \otimes W'_+) \otimes \mathrm{Pol}((V''_+ \otimes W''_-) \longrightarrow \mathrm{Pol}((V'_+ \otimes W'_+) \oplus (V''_+ \otimes W''_-)).$$

Here we note that if $V=A \oplus B$ then we have a multiplication map (isomorphism)

$$\mathrm{Pol}(A) \otimes \mathrm{Pol}(B) \longrightarrow \mathrm{Pol}(V).$$

5.3.1. The cocycles of Hodge type $(bq, 0)$

We now give a coordinate formula for $\psi_{q,0}$. Recall that $\xi'_{\alpha,\mu} \in (\mathfrak{p}')^* = V^*_+ \otimes V_-$ is given by $\xi'_{\alpha,\mu}=v^*_\alpha \otimes v_\mu$, $1 \leqslant \alpha \leqslant p$ and $p \leqslant \mu \leqslant p+q$.

LEMMA 5.5. *We have*

$$\psi_{q,0} = \sum_{1 \leqslant \alpha_1, ..., \alpha_q \leqslant p} (z''_{\alpha_1} z''_{\alpha_2} ... z''_{\alpha_q}) \otimes (\xi'_{\alpha_1,p+1} \wedge ... \wedge \xi'_{\alpha_q,p+1}).$$

Proof. Let $y=\sum_{\alpha=1}^{p} z_\alpha v_\alpha$ and hence $y^*=\sum_{\alpha=1}^{p} \bar{z}_\alpha v_\alpha^*$. Therefore

$$\begin{aligned}{}^t\tilde{y}^* &= \sum_{1\leqslant \alpha_1,\dots,\alpha_q\leqslant p} ((v_{\alpha_1}^*\otimes v_{p+1})\wedge\dots\wedge(v_{\alpha_q}^*\otimes v_{p+1}))\otimes(\bar{z}_{\alpha_1}\bar{z}_{\alpha_2}\dots\bar{z}_{\alpha_q})\\ &= \sum_{1\leqslant \alpha_1,\dots,\alpha_q\leqslant p} (\xi'_{\alpha_1,p+1}\wedge\dots\wedge\xi'_{\alpha_q,p+1})\otimes(z''_{\alpha_1}z''_{\alpha_2}\dots z''_{\alpha_q}).\end{aligned}$$

Here the last equation is justified by Lemma 2.1 which states that we have $\overline{z(y)}=z''(y'')$. The lemma then follows because $\psi_{q,0}$ and $\psi_{q,0}^*$ are related by switching the polynomial and exterior tensor factors. □

One can now derive a coordinate formula for $\psi_{bq,0}$ by taking the b-fold outer exterior power of the formula above. We will see later (Proposition 5.19) that $\psi_{bq,0}$ is non-zero.

We will now prove that $\psi_{q,0}$ is closed and hence, by Lemma 5.4, $\psi_{bq,0}$ is closed since the differential d is a graded derivation of the outer exterior product. In this case we have $a=0$ and $b=1$. Hence we let $W=W_-$ be a 1-dimensional complex vector space with basis w_1 equipped with a Hermitian form $(\cdot\,,\cdot)$ such that $(w_1,w_1)=-1$. We apply equation (4.16) with $W_+=0$ to conclude that the Fock model $\mathcal{P}(V\otimes W)$ for the dual pair $U(V)\times U(W)$ is given by

$$\mathcal{P}(V\otimes W)=\mathrm{Pol}(V''_+\otimes W''_-)\otimes\mathrm{Pol}(V'_-\otimes W'_-).$$

We will use z''_α for the coordinates on $V''_+\otimes W''_-$ relative to the basis $\{v''_\alpha\otimes w''_1\}_{\alpha=1}^{p}$, and z'_μ for the coordinates on $V'_-\otimes W'_-$ relative to $\{v'_\mu\otimes w'_1\}_{\alpha=p+1}^{p+q}$. As usual, we let ω denote the (infinitesimal) Weil representation. We then have the following result.

Lemma 5.6. *We have*

$$\omega(x_{\alpha,\mu})=z''_\alpha z'_\mu \quad \textit{and} \quad \omega(y_{\alpha,\mu})=\frac{\partial^2}{\partial z''_\alpha\partial z'_\mu}. \tag{5.5}$$

As usual we define ∂, resp. $\bar\partial$, to be the bidegree $(1,0)$, resp. $(0,1)$, parts of the differential d. It is then an immediate consequence of Lemma 5.6 that

$$\partial=\sum_{\alpha=1}^{p}\sum_{\mu=p+1}^{p+q} z''_\alpha z'_\mu\otimes A(\xi'_{\alpha\mu}) \quad\text{and}\quad \bar\partial=\sum_{\alpha=1}^{p}\sum_{\mu=p+1}^{p+q}\frac{\partial^2}{\partial z''_\alpha\partial z'_\mu}\otimes A(\xi''_{\alpha\mu}). \tag{5.6}$$

It is then clear that $\bar\partial\psi_{q,0}=0$.

Lemma 5.7. *We have* $d\psi_{0,q}=\partial\psi_{q,0}=0$.

Proof. We have

$$\partial\psi_{q,0}=\sum_{\beta=1}^{p}\sum_{\mu=p+1}^{p+q}\sum_{1\leqslant\alpha_1,\ldots,\alpha_q\leqslant p}(z''_\beta z'_\mu)(z''_{\alpha_1}\ldots z''_{\alpha_q})\otimes\xi'_{\beta,\mu}\wedge\xi'_{\alpha_1,p+1}\wedge\ldots\wedge\xi'_{\alpha_q,p+q}.$$

Fix a value $\mu=p+k$ in the second sum. We then have the subsum

$$S_\mu=\sum_{\beta=1}^{p}\sum_{1\leqslant\alpha_1,\ldots,\alpha_q\leqslant p}z''_\beta z''_{\alpha_1}\ldots z''_{\alpha_q}\otimes\xi'_{\beta,\mu}\wedge\xi'_{\alpha_1,p+1}\wedge\ldots\wedge\xi'_{\alpha_q,p+q}.$$

Clearly S_μ may be factored according to

$$S_\mu=\biggl(\sum_{\beta=1}^{p}\sum_{\alpha_k=1}^{p}z''_\beta z''_{\alpha_k}\otimes\xi'_{\beta,\mu}\wedge\xi'_{\alpha_k,\mu}\biggr)\wedge\omega$$

for a certain $(q-2)$-form ω. Clearly the first factor is zero. □

We now study the transformation properties of $\psi^*_{bq,0}$, and hence those of $\psi_{bq,0}$. From formula (5.3) we see that $\psi^*_{bq,0}$ is a homogeneous (of degree q in each y''_j, $1\leqslant j\leqslant b$, and hence of total degree bq) assignment of an element $\psi^*_{bq,0}(\mathbf{y}'')$ in $\bigwedge^{bq}(V^*_+\otimes V_-)$ to a b-tuple $\mathbf{y}''=(y''_1,\ldots,y''_b)\in(V''_+)^b\cong V''_+\otimes W''_-$. From the above formula it is clear that $\psi^*_{bq,0}$ is a $U(V_+)$-equivariant map.

Recall, see [17, p. 80], that there are quotient maps of $U(V_+)$-modules

$$\mathrm{Sym}^{bq}(V_+\otimes W''_-)\longrightarrow S_{b\times q}(V''_+)\otimes S_{b\times q}(W''_-)\cong S_{b\times q}(V''_+)\otimes(\textstyle\bigwedge^b W''_-)^q$$

and

$$\textstyle\bigwedge^{bq}(V^*_+\otimes V_-)\longrightarrow S_{b\times q}(V^*_+)\otimes S_{q\times b}(V_-)\cong S_{b\times q}(V^*_+)\otimes\bigwedge^q(V_-)^b.$$

Now note that $(\bigwedge^b W''_-)^q$ and $(\bigwedge^q V_-)^b$ are 1-dimensional and hence as $U(V_+)$-modules we have

$$S_{b\times q}(V''_+)\otimes(\textstyle\bigwedge^b W''_-)^q\cong S_{b\times q}(V^*_+)\otimes(\bigwedge^q V_-)^b. \tag{5.7}$$

We now prove that $\psi^*_{bq,0}$ induces the above isomorphism (the lower horizontal arrow in the next diagram). In what follows we will use the symbols $U(V_+)$ and $U(p)$ and $U(V_-)$ and $U(q)$ interchangeably.

Lemma 5.8. *The map $\psi^*_{bq,0}$ induces a commutative diagram of $U(V_+)$-modules*

$$\begin{tikzcd}
\mathrm{Sym}^{bq}(V''_+\otimes W''_-) \arrow[r, "\psi^*_{bq,0}"] \arrow[d] & \bigwedge^{bq}(V^*_+\otimes V_-) \arrow[d] \\
S_{b\times q}(V''_+)\otimes\bigl(\bigwedge^b W''_-\bigr)^{\otimes q} \arrow[r, "\psi^*_{bq,0}"] & S_{b\times q}(V^*_+)\otimes\bigl(\bigwedge^q V_-\bigr)^{\otimes b}.
\end{tikzcd}$$

Proof. We first note that the inclusion map (corresponding to the special case $b=1$ of the above quotient map) $\iota: S^q(V_+^*)\otimes\bigwedge^q(V_-)\to\bigwedge^q(V_+^*\otimes V_-)$ is given by

$$\iota(f^{\otimes q}\otimes v_{p+1}\wedge\ldots\wedge v_{p+q})=(f\otimes v_{p+1})\wedge\ldots\wedge(f\otimes v_{p+q})=\tilde{f}.$$

Hence

$$\iota((y^*)^{\otimes q}\otimes v_{p+1}\wedge\ldots\wedge v_{p+q})={}^t\tilde{y}^*.$$

Thus ${}^t\tilde{y}^*$ transforms under $U(V_-)$ according to $\det_{U(V_-)}$, and hence ${}^t\tilde{y}_1^*\wedge\ldots\wedge{}^t\tilde{y}_b^*$ transforms under $U(V_-)$ according to $\det_{U(V_-)}^b$.

Now recall from (1.3) that we have

$$\bigwedge^{bq}(V_+^*\otimes V_-)=\bigoplus S_\lambda(V_+^*)\otimes S_{\lambda'}(V_-).$$

But in order for $\tilde{y}_1^*\wedge\ldots\wedge\tilde{y}_b^*$ to transform under $U(V_-)$ according to $\det_{U(V_-)}^b$ the Young diagram λ' must be a $q\times b$ rectangle, and hence λ must be a $b\times q$ rectangle. But again by [17, p. 80] we have

$$S^{bq}(V_+^*\otimes W_-)=\bigoplus S_\lambda(V_+^*)\otimes S_\lambda(W_-),$$

where the sum is over all Young diagrams λ with bq boxes and at most $\min\{p,b\}$ rows. Since the map $\psi_{bq,0}$ is $U((V_+)^*)$-equivariant, it must factor through the summand where λ is a $b\times q$ rectangle. □

We obtain the following result.

Lemma 5.9. *We have*

$$\psi_{bq,0}\in S_{b\times q}((V_+'')^*)\otimes(\bigwedge^b((W_-'')^*)^{\otimes q})\otimes S_{b\times q}(V_+^*)\otimes(\bigwedge^q(V_-))^{\otimes b}.$$

Proof. By dualizing the result of the previous lemma we obtain

$$\psi_{bq,0}\in\operatorname{Hom}(S_{b\times q}(V_+)\otimes(\bigwedge^q(V_-^*))^{\otimes b}, S_{b\times q}((V_+'')^*)\otimes(\bigwedge^b((W_-'')^*)^{\otimes q}).$$

We then use the isomorphism $\operatorname{Hom}(U_1,U_2)\cong U_2\otimes U_1^*$. □

Corollary 5.10. *The cochain $\psi_{bq,0}$ is invariant under $U(p)$ acting by the standard action and transforms under $U(q)\times U(b)$ according to $\det_{U(q)}^b\otimes\det_{U(b)}^q$.*

5.3.2. The cocycles of Hodge type $(0,aq)$

We first give a coordinate formula for $\psi_{q,0}$. Recall that $\xi''_{\alpha,\mu}\in(\mathfrak{p}'')^*=V_-^*\otimes V_+$ is given by $\xi''_{\alpha,\mu}=v_\mu^*\otimes v_\alpha$, $1\leqslant\alpha\leqslant p$ and $p\leqslant\mu\leqslant p+q$. Let $x\in V_+$ be given by $x=\sum_\alpha z_\alpha v_\alpha$.

The next lemma is proved in the same way as Lemma 5.5.

Lemma 5.11. *We have*

$$\psi_{0,q}(x')=\sum_{1\leqslant\alpha_1,\ldots,\alpha_q\leqslant p}(z'_{\alpha_1}z'_{\alpha_2}\ldots z'_{\alpha_q})\otimes(\xi''_{\alpha_1,p+1}\wedge\ldots\wedge\xi''_{\alpha_q,p+1}).$$

We will now prove that $\psi_{0,q}$ is closed and hence, as before, by Lemma 5.4, $\psi_{0,aq}$ is closed. In this case we have $a=1$ and $b=0$. Hence we let $W=W_+$ be a 1-dimensional complex vector space with basis w_1 equipped with a Hermitian form $(\cdot,\cdot)$ such that $(w_1,w_1)=1$. We apply equation (4.16) with $W_-=0$ to conclude that the Fock model $\mathcal{P}(V\otimes W)$ for the dual pair $U(V)\times U(W)$ is given by

$$\mathcal{P}(V\otimes W)=\mathrm{Pol}(V'_+\otimes W'_+)\otimes\mathrm{Pol}(V''_-\otimes W''_+).$$

We will use z'_α for the coordinates on $V'_+\otimes W'_+$ relative to the basis $\{v'_\alpha\otimes w'_1\}_{\alpha=1}^p$ and z''_μ for the coordinates on $V''_-\otimes W''_+$ relative to $\{v''_\mu\otimes w''_1\}_{\mu=p+1}^{p+q}$. As usual, we let ω be the action of the (infinitesimal) Weil representation.

Lemma 5.12. *We have*

$$\omega(x_{\alpha,\mu})=\frac{\partial^2}{\partial z'_\alpha\partial z''_\mu}\quad\textit{and}\quad\omega(y_{\alpha,\mu})=z'_\alpha z''_\mu. \tag{5.8}$$

It is then an immediate consequence of Lemma 5.12 that we have

$$\partial=\sum_{\alpha=1}^{p}\sum_{\mu=p+1}^{p+q}\frac{\partial^2}{\partial z'_\alpha\partial z''_\mu}\otimes A(\xi'_{\alpha\mu})\quad\text{and}\quad\bar\partial=\sum_{\alpha=1}^{p}\sum_{\mu=p+1}^{p+q}z'_\alpha z''_\mu\otimes A(\xi''_{\alpha\mu}). \tag{5.9}$$

It is then clear that $\partial\psi_{q,0}=0$.

The next lemmas are proved in the same way as Lemmas 5.7–5.9.

Lemma 5.13. *We have* $d\psi_{0,q}=\bar\partial\psi_{0,q}=0$.

Lemma 5.14. *The map* $\psi^*_{0,aq}$ *induces a commutative diagram*

$$\begin{array}{ccc}\mathrm{Sym}^{aq}(V'_+\otimes W'_+) & \xrightarrow{\psi^*_{0,aq}} & \bigwedge^{aq}(V_+\otimes V^*_-)\\ \downarrow & & \downarrow\\ S_{a\times q}(V'_+)\otimes(\bigwedge^a W'_+)^{\otimes q} & \xrightarrow{\psi^*_{0,aq}} & S_{a\times q}(V_+)\otimes(\bigwedge^q V^*_-)^{\otimes a}.\end{array}$$

Lemma 5.15. *We have*

$$\psi_{0,aq}\in S_{a\times q}(V_+)\otimes(\textstyle\bigwedge^q(V^*_-))^{\otimes a}\otimes S_{a\times q}((V'_+)^*)\otimes(\bigwedge^a((W'_+)^*)^{\otimes q}).$$

We then have as before the following consequence.

COROLLARY 5.16. *The cochain $\psi_{0,aq}$ is invariant under $U(p)$ acting by the standard action and transforms under $U(q)\times U(a)$ according to* $\det_{U(q)}^{-a}\otimes\det_{U(a)}^{-q}$.

We now define the general special cocycles $\psi_{bq,aq}$ of type (bq,aq) by

$$\psi_{bq,aq}=\psi_{bq,0}\wedge\psi_{0,aq}.$$

Since these cocycles are wedges of cocycles they are themselves closed. We now summarize the properties of the special cocycles.

PROPOSITION 5.17. *Let $x_1,x_2,...,x_a,y_1,...,y_b\in V_+$ be given. Put*

$$\mathbf{x}'=(x_1',x_2',...,x_a') \quad \textit{and} \quad \mathbf{y}''=(y_1'',y_2'',...,y_b'').$$

Then we have

(1) $\psi_{bq,0}^*(\mathbf{x}',\mathbf{y}'')=\psi_{bq,0}^*(\mathbf{y}'')={}^t\tilde{y}_1^*\wedge{}^t\tilde{y}_2^*\wedge...\wedge{}^t\tilde{y}_b^*\in\bigwedge^{bq}(V_+^*\otimes V_-)\cong\bigwedge^{bq}(\mathfrak{p}')^*$.

(2) $\psi_{0,aq}(\mathbf{x}',\mathbf{y}'')=\psi_{aq,0}^*(\mathbf{x}')={}^t\tilde{x}_1\wedge{}^t\tilde{x}_2\wedge...\wedge{}^t\tilde{x}_a\in\bigwedge^{aq}(V_-^*\otimes V_+)\cong\bigwedge^{aq}(\mathfrak{p}'')^*$.

(3) $\psi_{bq,aq}^*(\mathbf{x}',\mathbf{y}'')=({}^t\tilde{y}_1^*\wedge{}^t\tilde{y}_2^*\wedge...\wedge{}^t\tilde{y}_b^*)\wedge({}^t\tilde{x}_1\wedge{}^t\tilde{x}_2\wedge...\wedge{}^t\tilde{x}_a)$, *which belongs to*

$$\bigwedge^{aq}(V_+^*\otimes V_-)\otimes\bigwedge^{bq}(V_-^*\otimes V_+)\cong\bigwedge^{aq}(\mathfrak{p}')^*\otimes\bigwedge^{bq}(\mathfrak{p}'')^*.$$

(4) *The cochain $\psi_{bq,aq}$ is a cocycle.*

(5) *The cocycle $\psi_{aq,aq}$ is the representation in the Fock model of the cocycle $\varphi_{aq,aq}$ in the Schrödinger model of Kudla and Millson.*

(6) *The cocycle $\psi_{bq,aq}$ transforms under $U(q)$ according to* $\det_{U(q)}^{b-a}$.

(7) *The cocycle $\psi_{bq,aq}$ is invariant under* $\mathrm{SL}(q)$.

(8) *The cocycle $\psi_{bq,aq}$ transforms under $U(a)\times U(b)$ according to* $\det_{U(a)}^{-q}\otimes\det_{U(b)}^{q}$.

Proof. The only item that is not yet proved is (5). Note that (6) and (7) follow from Corollaries 5.10 and 5.16. We will prove (5) in Appendix C. □

Remark. By Lemma 5.4 the general cocycle $\psi_{bq,aq}$ factors as a product of the basic holomorphic and anti-holomorphic coycles $\psi_{q,0}$ and $\psi_{0,q}$.

5.4. In §4.6 we pointed out that if a and b had opposite parity then to descend the Weil representation restricted to $\widetilde{U}(p,q)$ we needed to twist by an odd power of $\det_{U(p,q)}^{1/2}$. The following lemma shows that this odd power is uniquely determined by the condition that the special cocycles $\psi_{bq,aq}$ is a relative Lie algebra cochain with values in the Weil representation. (This holds even in the case where a and b have the same parity.)

LEMMA 5.18. *The cocycle $\psi_{bq,aq}$, considered as a linear map from $\bigwedge^* \mathfrak{p}$ to $\mathcal{P}_+$, is $(U(p)\times U(q))$-equivariant if and only if we twist the restriction of the Weil representation to $\widetilde{U}(p,q)$ by the character $\det_{U(p,q)}^{(a-b)/2}$. In this case the action of $\widetilde{U}(p)$ on polynomials will factor through the action induced by the standard action of $U(p)$ on V_+, and the action of $\widetilde{U}(V_-)$ will simply scale all polynomials by $\det_{U(q)}^{b-a}$.*

Proof. It is clear that there exists at most one twist such that $\psi_{bq,aq}$ is equivariant. Hence, it suffices to prove the if part of the lemma. The if part follows from Theorem 4.17 and (6) of Proposition 5.17. □

5.5. The values of the special cocycles on $e(bq,0)$, $e(0,aq)$ and $e(bq,aq)$

We now evaluate our special cocycles in the Vogan–Zuckerman vectors.

PROPOSITION 5.19. *We have*

(1) $\psi_{bq,0}(e(bq,0))(\mathbf{x}',\mathbf{y}'')=\Delta_b(\mathbf{y}'')^q=\Delta_b(z''_{\alpha,k})^q$;

(2) $\psi_{0,aq}(e(0,aq))(\mathbf{x}',\mathbf{y}'')=\tilde{\Delta}_a(\mathbf{x}')^q=\tilde{\Delta}_a(z'_{\alpha,j})^q$;

(3) $\psi_{bq,aq}(e(bq,aq))(\mathbf{x}',\mathbf{y}'')=\psi_{bq,0}(e(bq,0))(\mathbf{y}'')\ \psi_{0,aq}(e(0,aq))(\mathbf{x}')=\tilde{\Delta}_a(\mathbf{x}')^q\ \Delta_b(\mathbf{y}'')^q$ $=\tilde{\Delta}_a(z'_{\alpha,k})^q\Delta_b(z''_{\alpha,k})^q$.

Proof. We first prove (1). By equation (3.21), we have

$$e(bq,0)=(-1)^{bq}\tilde{v}_1\wedge\tilde{v}_2\wedge\ldots\wedge\tilde{v}_b.$$

Combining this formula with the first formula in Proposition 5.17, we have

$$\begin{aligned}\psi_{bq,0}(e(bq,0))(\mathbf{x}',\mathbf{y}'')&=(\psi^*_{bq,0}(\mathbf{x}',\mathbf{y}''))(e(bq,0))\\&=(-1)^{bq}({}^t\tilde{y}_1^*\wedge{}^t\tilde{y}_2^*\wedge\ldots\wedge{}^t\tilde{y}_b^*)(\tilde{v}_1\wedge\tilde{v}_2\wedge\ldots\wedge\tilde{v}_b).\end{aligned}\tag{5.10}$$

Recall the definitions

$${}^t\tilde{y}_1^*\wedge{}^t\tilde{y}_2^*\wedge\ldots\wedge{}^t\tilde{y}_b^*=[(y_1^*\otimes v_{p+1})\wedge\ldots\wedge(y_1^*\otimes v_{p+q})]\wedge\ldots\wedge[(y_b^*\otimes v_{p+1})\wedge\ldots\wedge(y_b^*\otimes v_{p+q})]$$

and

$$\tilde{v}_1\wedge\tilde{v}_2\wedge\ldots\wedge\tilde{v}_b=[(v_1\wedge v_{p+1}^*)\wedge\ldots\wedge(v_1\wedge v_{p+q}^*)]\wedge\ldots\wedge[(v_a\wedge v_{p+1}^*)\wedge\ldots\wedge(v_b\wedge v_{p+q}^*)].$$

From equation (5.10) and the two equations immediately above, we see that

$$(-1)^{bq}\psi_{bq,0}(e(bq,0))(\mathbf{y}'')$$

is the determinant of the $bq\times bq$ matrix $A(\mathbf{y}'')$ with entries

$$(y_i^*\wedge v_{p+j})(v_k\wedge v_{p+\ell}^*)=(v_k,y_i)\delta_{j,\ell}$$

arranged in some order (with more work we could show that $A(\mathbf{y}'')=-I_q\otimes Z''(\mathbf{y}'')$ but we prefer to avoid this computation and procede more invariantly). By definition, $(v_k,y_i)=z''_{i,k}(\mathbf{y}'')$. Hence the above matrix entry is either $z''_{i,k}$ or zero, and hence $\psi_{bq,0}(e(bq,0))(\mathbf{y}'')$ is a polynomial of degree at most bq in the entries $z''_{i,k}$ of the $b\times b$ matrix $Z''(\mathbf{y}'')$. Hence $\psi_{bq,0}(e(bq,0))(\mathbf{y}'')$ is a polynomial $p(Z'')$ on the space of $b\times b$ matrices Z''. But by Corollary 5.10 and Proposition 5.17 we have, for $g\in U(b)$,

$$\psi_{bq,0}(e(bq,0))(\mathbf{y}''g)=\det{}^q_{U(b)}(g)\psi_{bq,0}(e(bq,0))(\mathbf{y}''),$$

and hence

$$p(Z''g)=\det{}^q_{U(b)}(g)p(Z'').$$

Thus, in case $\det(Z''(\mathbf{y}''))\neq 0$, we have

$$p(Z''(\mathbf{y}''))=p(I_b)\det(Z''(\mathbf{y}''))^q.$$

By Zariski density of the invertible matrices (and the fact that every $n\times n$ matrix Z'' may be written as $Z''(\mathbf{y}'')$ for a suitable $\mathbf{y}''$) the above equation holds for all $b\times b$ matrices Z''. It remains to evaluate the value of p on the identity matrix I_b. This follows by setting $y_j=v_j$, $1\leqslant j\leqslant b$, and observing that each successive term in the bq-fold product ${}^t\tilde{y}_1^*\wedge{}^t\tilde{y}_2^*\wedge\ldots\wedge{}^t\tilde{y}_b^*$ is the negative of the dual basis vector for the corresponding term in $e(bq,0)=\tilde{v}_1\wedge\tilde{v}_2\wedge\ldots\wedge\tilde{v}_b$. Hence we obtain the determinant of $-I_{bq}$.

The proof of (2) is similar.

We now observe that formula (3) follows from (1) and (2). By equation (3.21) we have

$$e(bq,aq)=e(bq,0)\wedge e(0,aq),$$

and by Proposition 5.17 we have

$$\psi_{bq,aq}(\mathbf{x}',\mathbf{y}'')=\psi_{bq,0}(\mathbf{y}'')\wedge\psi_{0,aq}(\mathbf{x}').$$

Hence we have

$$\begin{aligned}\psi_{bq,aq}(e(bq,aq))(\mathbf{x}',\mathbf{y}'')&=(\psi_{bq,0}(\mathbf{y}'')\wedge\psi_{0,aq}(\mathbf{x}'))(e(bq,0)\wedge e(0,aq))\\&=\psi_{bq,0}(e(bq,0))(\mathbf{y}'')\psi_{0,aq}(e(0,aq))(\mathbf{x}').\end{aligned}$$

The proposition follows. $\square$

We conclude that the polynomials $\psi_{bq,0}(e(bq,0))$ and $\psi_{0,aq}(e(0,aq))$ are (special) harmonic for all a and b, and if $a+b\leqslant p$ then $\psi_{bq,aq}(e(bq,aq))$ is harmonic. The polynomial $\psi_{bq,0}(e(bq,0))$ transforms under $U(a)\times U(b)$ according to the 1-dimensional representation $1\otimes\det^{-q}$. The polynomial $\psi_{0,bq}(e(0,aq))$ transforms under $U(a)\times U(b)$ according to the 1-dimensional representation $\det^q\otimes 1$. The polynomial $\psi_{bq,aq}(e(bq,aq))$ transforms under $U(a)\times U(b)$ according to the 1-dimensional representation $\det^q\otimes\det^{-q}$.

5.6. The cocycle $\psi_{bq,0}$ generates the $S_{b\times q}(V_+)\otimes(\bigwedge^q V_-^*)^b$ isotypic component for the action of $U(p)\times U(q)$ on the polynomial Fock space

In this section we will abbreviate the space of harmonic polynomials $\mathrm{Harm}(V_+\otimes W)$ to $\mathcal{H}_+$, and the space $\mathrm{Harm}(V_-\otimes W)$ to $\mathcal{H}_-$. The goal of this subsection is to prove the following theorem.

Theorem 5.20. *We have*

$$\mathrm{Hom}_K(S_{b\times q}(V_+)\otimes(\textstyle\bigwedge^q V_-^*)^b,\mathcal{P}(V\otimes W))=\mathcal{U}(\mathfrak{u}(a,b)_{\mathbb{C}})\psi_{bq,0}.$$

Theorem 5.20 will be a consequence of the next three lemmas. We will henceforth abbreviate the representation $S_{b\times q}(\mathbb{C}^p)\otimes(\bigwedge^q V_-^*)^b$ to $V(bq)$.

Recall that $e(bq,0)=\tilde v_1\wedge\ldots\wedge\tilde v_b\in\bigwedge^{bq}(V_+\otimes V_-^*)$ is the Vogan–Zuckerman vector. We have seen in Proposition 5.19 that

$$\psi_{bq,0}(e(bq))=\Delta_b(\mathbf{y}'')^q,$$

and consequently the value of $\psi_{bq,0}$ on the Vogan Zuckerman vector $e(bq,0)$ is a (special) harmonic polynomial, that is

$$\psi_{bq,0}(e(bq,0))\in\mathrm{Pol}(V_+''\otimes W_-'')\subset\mathcal{H}_+.$$

We now have the following lemma.

Lemma 5.21. *We have*

$$\mathrm{Hom}_K(V(bq),\mathcal{H}_+\otimes\mathcal{H}_-)=\mathbb{C}\psi_{bq,0}.$$

Clearly Lemma 5.21 follows from the following one.

Lemma 5.22. *The representation $V(bq)$ of $U(p)$ occurs once in $\mathcal{H}_+$. Moreover, the 1-dimensional representation $\det_{U(q)}^{-b}$ of $U(q)$ occurs once in $\mathcal{H}_-$. Hence*

$$\mathrm{Hom}_{U(p)\times U(q)}(V(bq),\mathcal{H}_+\otimes\mathcal{H}_-)=\mathbb{C}\psi_{bq,0}.$$

Proof. We first prove that $S_{b\times q}(V_+)$ of $U(p)$ occurs once in $\mathcal{H}_+$. Indeed, the actions of the groups $U(p)$ and $U(a)\times U(b)$ on $\mathcal{H}_+$ form a dual pair. Furthermore the correspondence of unitary representations $\tau: U(p)^\vee \to U(a)^\vee \times U(b)^\vee$ is given in [39, Theorem 6.3]. From their formula we see that $\lambda = b\times q$ corresponds to the 1-dimensional representation $1\otimes\det^q$ of $U(a)\times U(b)$. Since the multiplicity of the representation with highest weight λ of $U(p)$ corresponds to the dimension of the corresponding representation $\tau(\lambda)$, which is 1 in this case, we have proved that $S_{b\times q}(V_+)$ occurs once as claimed. We note that we may realize this occurrence explicity as follows. First note that

$$\mathrm{Sym}^{bq}(V'_+\otimes W'_-) \cong \mathcal{P}^{bq}(V''_+\otimes W''_-) \subset \mathcal{H}_+.$$

But by [17, p. 80], we have

$$S_{b\times q}(V'_+)\otimes S_{b\times q}(W'_-) \subset \mathrm{Sym}^{bq}(V'_+\otimes W'_-).$$

We note that, since $\dim(W_-)=b$, the bth exterior power of W'_- is the top exterior power and we have

$$S_{b\times q}(W'_-) \cong (\textstyle\bigwedge^b W'_-)^q.$$

Consequently $U(W_-)$ acts on the second factor by $\det^q_{U(b)}$.

It remains to prove that the representation $\det^{-b}_{U(q)}$ of $U(q)$ occurs once in the oscillator representation action on the harmonic polynomials $\mathcal{H}_-$ in

$$\mathcal{P}_- = \mathrm{Pol}((V'_-\otimes W'_-))\otimes \mathrm{Pol}((V''_-\otimes W''_+)).$$

Since the oscillator representation action of $U(q)$ is the standard action twisted by $\det^{-b}_{U(q)}$, this is equivalent to showing that the trivial representation of $U(q)$ occurs once in the standard action of $U(q)$ on $\mathcal{H}_-$. It occurs at least once because the constant polynomials are harmonic. But as above, by [39, Theorem 6.3], the trivial representation of $U(q)$ corresponds to the trivial representation of $U(a)\times U(b)$ and consequently it has multiplicity 1 and the lemma follows. □

Theorem 5.20 is now a consequence of the following result of Howe, see [35, Proposition 3.1].

LEMMA 5.23. *We have*

$$\mathrm{Hom}_K(V(bq), \mathcal{P}(V\otimes W)) = \mathcal{U}(\mathfrak{u}(a,b)_{\mathbb{C}})\,\mathrm{Hom}_K(V(bq), \mathcal{H}_+\otimes\mathcal{H}_-).$$

Hence, combining Lemmas 5.23 and 5.21, we obtain

$$\mathrm{Hom}_K(V(bq), \mathcal{P}(V\otimes W)) = \mathcal{U}(\mathfrak{u}(a,b)_{\mathbb{C}})\psi_{bq,0}.$$

Theorem 5.20 is now proved.

Remark. Applying reasoning similar to that immediately above, one deduces that the cocycle $\psi_{0,aq}$ generates (over $\mathcal{U}(\mathfrak{u}(a,b)_{\mathbb{C}})$) the $S_{a\times q}(V_+^*)\otimes(\bigwedge^q V_-)^a$ isotypic component of the polynomial Fock space.

5.7. The cocycle $\psi_{bq,aq}$ generates the $V(bq,aq)$ isotypic component of the polynomial Fock space

We recall that $V(bq,aq)$ is the representation of $U(p)\times U(q)$ with highest weight being the sum of the two previous highest weights:

$$(\underbrace{q,q,...,q}_{b},0,0,...,0,\underbrace{-q,-q,...,-q}_{a};\underbrace{a-b,a-b,...,a-b}_{q}).$$

We also note that this representation is the Cartan product of $S_{b\times q}(\mathbb{C}^p)\otimes\det_{U(q)}^{-b}$ and $S_{a\times q}((\mathbb{C}^p)^*)\otimes\det_{U(q)}^{a}$. The Cartan product was defined in §3.8.

THEOREM 5.24. *We have*

$$\operatorname{Hom}_K(V(bq,aq),\mathcal{P}(V\otimes W))=\mathcal{U}(\mathfrak{u}(a,b)_{\mathbb{C}})\psi_{bq,aq}.$$

Theorem 5.24 is proved the same way as Theorem 5.20. Once again we have a multiplicity one result in $\mathcal{H}_+\otimes\mathcal{H}_-$.

LEMMA 5.25. *We have*

$$\operatorname{Hom}_K(V(bq,aq),\mathcal{H}_+\otimes\mathcal{H}_-)=\mathbb{C}\psi_{bq,aq}.$$

Proof. The product group $U(p)\times(U(a)\times U(b))$ acts as a dual pair on

$$\mathcal{H}(V_+\otimes(W'_+\oplus W''_-)).$$

Hence the dual representation of $U(a)\times U(b)$ has highest weight

$$(\underbrace{q,q,...,q}_{a},\underbrace{-q,-q,...,-q}_{b}),$$

and hence it is $\det^q\otimes\det^{-q}$. Thus, it is 1-dimensional so the Cartan product of $S_{b\times q}(\mathbb{C}^p)$ and $S_{a\times q}(\mathbb{C}^p)^*$ has multiplicity 1 in $\mathcal{H}_+$.

We leave the proof that the 1-dimensional representation $\det_{U(q)}^{a-b}$ of $U(q)$ has multiplicity 1 in $\mathcal{H}_-$ to the reader (once again the constant polynomials transform under $U(q)$ by this twist). The lemma follows. $\square$

Now Theorem 5.24 follows from the result of Howe; see Lemma 5.23.

5.8. The polynomial Fock space

We refer the reader to Appendix C for the notation used in the following paragraph and further details.

In the study of the global theta correspondence beginning in §10 we will need to consider the cocycles $\varphi_{bq,aq}$ with values in the Schrödinger model for the oscillator representation of $U(V)\times U(W)$ corresponding to the cocycles $\psi_{bq,aq}$ defined above with values in the Fock model. In order to give a precise statement of the relation between them we recall there is an intertwining operator, the Bargmann transform $B_{V\otimes W}$, from the Schrödinger model of the oscillator representation of $U(V)\times U(W)$ to the Fock model; see [15, p. 40 and p. 180]. We define the polynomial Fock space $\mathbf{S}(V\otimes E)\subset\mathcal{S}(V\otimes E)$ to be the image of the holomorphic polynomials in the Fock space under the inverse Bargmann transform $B^{-1}_{V\otimes W}$. Here $\mathcal{S}(V\otimes E)$ is the Schwartz space. We then have

$$\varphi_{bq,aq}=(B^{-1}_{V\otimes W}\otimes 1)\psi_{bq,aq}. \tag{5.11}$$

Part 2. The geometry of Shimura varieties

6. Shimura varieties and their cohomology

6.1. Notation

Let E be a CM-field with totally real maximal subfield F satisfying $[F:\mathbb{Q}]=d$. We assume that $d>1$. We denote by $\mathbb{A}_{\mathbb{Q}}$ the ring of adèles of $\mathbb{Q}$, and by $\mathbb{A}$ the ring of adèles of F. We fix d non-conjugate complex embeddings $\tau_1,...,\tau_d: E\to\mathbb{C}$, and denote by $x\mapsto\bar{x}$ the non-trivial automorphism of E induced by the complex conjugation of $\mathbb{C}$ with respect to any of these embeddings. We identify F, resp. E, with a subfield of $\mathbb{R}$, resp. $\mathbb{C}$, via τ_1.

Let $(V,(\cdot\,,\cdot))$ be a non-degenerate anisotropic Hermitian vector space over E with $\dim_E V=m$. We let $V_{\tau_j}=V\otimes_{E,\tau_j}\mathbb{C}$ be the complex Hermitian vector space obtained as the completion of V with respect to the complex embedding τ_j. Choosing a suitable isomorphism $V_{\tau_j}\cong\mathbb{C}^m$, we may write

$$(u,v)={}^t u H_{p_j,q_j}\bar{v} \quad \text{for all } u,v\in\mathbb{C}^m,$$

where

$$H_{p_j,q_j}=\begin{pmatrix} 1_{p_j} & \\ & -1_{q_j}\end{pmatrix},$$

and (p_j,q_j) is the signature of V_{τ_j}. We will consider in this paper only those $(V,(\cdot\,,\cdot))$ such that $q_2=...=q_d=0$ and let $(p,q)=(p_1,q_1)$. By replacing $(\cdot\,,\cdot)$ with $-(\cdot\,,\cdot)$ we can, and will, assume that $p\geqslant q$.

6.2. Unitary group (of similitudes) of V

We view the unitary group in m variables $U(V)$ as a reductive algebraic group over F. We let $G_1 = \mathrm{Res}_{F/\mathbb{Q}} U(V)$, so that for any $\mathbb{Q}$-algebra A

$$\begin{aligned} G_1(A) &= \{g \in \mathrm{End}_E(V) \otimes_{\mathbb{Q}} A : (gu, gv) = (u, v) \text{ for all } u, v \in V \otimes_{\mathbb{Q}} A\} \\ &= \{g \in \mathrm{End}_E(V) \otimes_{\mathbb{Q}} A : gg^* = 1\}. \end{aligned}$$

The embeddings $\tau_j : E \to \mathbb{C}$ in particular induce an isomorphism

$$G_1(\mathbb{R}) = U(p, q) \times U(m)^{d-1}.$$

The group of unitary similitudes $\mathrm{GU}(V)$ is the algebraic group over F whose points in any F-algebra A are given by

$$\begin{aligned} \mathrm{GU}(V)(A) &= \{g \in \mathrm{End}_E(V) \otimes_F A : (g\,\cdot, g\,\cdot) = \lambda(g)(\,\cdot\,,\,\cdot\,) \text{ for some } \lambda(g) \in A^\times\} \\ &= \{g \in \mathrm{End}_E(V) \otimes_F A : gg^* = \lambda(g) \in A^\times\}. \end{aligned}$$

Here λ is the *similitude norm*. We let $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GU}(V)$.(5) Consider the rational torus $\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m}F}$ whose group of rational points is $F^\times$. By abuse of notation, we let $\lambda : G \to \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m}F}$ be the homomorphism of algebraic groups over $\mathbb{Q}$ induced by the similitude norm. Set

$$\mathrm{GU}(a, b) = \{A \in \mathrm{GL}_{a+b}(\mathbb{C}) : {}^t A H_{a,b} \bar{A} = c(A) H_{a,b} \text{ and } c(A) \in \mathbb{R}^\times\}.$$

The embeddings $\tau_j : E \to \mathbb{C}$ induce an isomorphism

$$G(\mathbb{R}) \cong \mathrm{GU}(p, q) \times \mathrm{GU}(m)^{d-1}.$$

It is useful to point out that the group of E-points in $\mathrm{GU}(V)$ is isomorphic to $\mathrm{GL}(V) \times E^\times$, inside which the F-group $\mathrm{GU}(V)$ is defined as

$$\{(g, t) \in \mathrm{GL}(V) \times E^\times : (t(g^*)^{-1}, \bar{t}\,) = (g, t)\}.$$

In this formulation the similitude norm λ is the projection on the second factor. The determinant on $U(V)$ induces—by restriction of scalars—a character

$$\det : G_1 \longrightarrow \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_{\mathrm{m}E}.\text{(6)}$$

By the above discussion λ and det generate the character group of G, and on the rational group these characters are related by $\lambda(g)^m = \mathrm{N}_{E/F}(\det(g))$.

(5) We warn the reader that in this part of the paper G refers to the unitary similitude group and that we now refer to the usual unitary group as G_1.

(6) Here, by definition, the group of rational points of $\mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_{\mathrm{m}E})$ is $E^\times$.

6.3. Shimura data

The symmetric space X associated with $G_{\mathrm{der}}(\mathbb{R})$—or equivalently the symmetric space associated with $G(\mathbb{R})$ (modulo its center)—is also the space

$$X = U(p,q)/(U(p)\times U(q))$$

of negative q-planes in V_{τ_1}. It is isomorphic to a bounded symmetric domain in $\mathbb{C}^{pq}$. Following the general theory of Deligne [12], [53]—see also Kottwitz [40] for our particular case—the pair (G,X) defines a Shimura variety $\mathrm{Sh}(G,X)$ which has a canonical model over the reflex field $E(G,X)$. More precisely, let $\mathbb{S}$ be the real algebraic group $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_{\mathrm{m}\mathbb{C}}$, so that $\mathbb{S}(\mathbb{R})=\mathbb{C}^\times$, and define a homomorphism of real algebraic groups $h_0\colon \mathbb{S}\to G$ as follows. Since

$$G(\mathbb{R}) \cong \mathrm{GU}(p,q)\times \mathrm{GU}(m)^{d-1},$$

it suffices to define the components h_j, $j=1,...,d$, of h_0.

For $j>1$, we take h_j to be the trivial homomorphism. For $j=1$, fix a base point $x_0\in X$; then x_0 corresponds to a negative q-plane V_- in V_{τ_1}. Let us simply write V for V_{τ_1} in the remaing part of this paragraph. Now let $V_+\subset V$ denote the orthogonal complement of V_- with respect to $(\cdot,\cdot)$. As in §2, we associate with the decomposition $V=V_++V_-$ a positive definite Hermitian form $(\cdot,\cdot)_{x_0}$—the associated minimal majorant—by changing the sign of $(\cdot,\cdot)$ on V_-. By taking the real part of $(\cdot,\cdot)_{x_0}$ we obtain a positive definite symmetric form $B(\cdot,\cdot)_{x_0}$. Let θ_{x_0} be the Cartan involution which acts as the identity on V_+ and as $-\mathrm{id}$ on V_-, and let $J_{x_0}=\theta_{x_0}\circ J$ be the corresponding positive almost complex structure on V. We then have

$$B(u,v)_{x_0} = \langle J_{x_0}u, v\rangle = -\langle u, J_{x_0}v\rangle.$$

For $a+ib\in\mathbb{C}$, let

$$h(a+ib) = a+bJ_{x_0} \in \mathrm{End}(V).$$

The map h defines an $\mathbb{R}$-algebra homomorphism such that

- $h(z)^*=h(\bar{z})$, where again $*$ is the involution on $\mathrm{End}(V)$ determined by $(\cdot,\cdot)$,
- the form $\langle h(i)u,v\rangle$ is symmetric and positive definite on V.

Note that $h(z)h(z)^*=|z|^2$. We conclude that the restriction of h to $\mathbb{C}^\times$ defines a homomorphism of real algebraic groups $h_1\colon\mathbb{C}^\times\to\mathrm{GU}(V)$.

Let $h_0=(h_1,...,h_d)$. Then h_0 defines a homomorphism of real algebraic groups $h_0\colon\mathbb{S}\to G$. The space X may then be viewed as the space of conjugates of h_1 by $\mathrm{GU}(V)$ or, equivalently, of h_0 by $G(\mathbb{R})$.

Now we have $\mathbb{S}(\mathbb{C})=\mathbb{C}^\times \times \mathbb{C}^\times$, where we order the factors such that the first factor corresponds to the identity embedding $\mathbb{C}\to\mathbb{C}$. Recall that we have decomposed $V=V_{\tau_1}$ as $V=V_+\oplus V_-$, so that $h_1(z)$ acts by z, resp. $\bar{z}$, on V_+, resp. V_-. With respect to this decomposition the Hermitian matrix of $(\cdot,\cdot)$ is diagonal and equal to $H_{p,q}$. Identifying the complexification of $\mathrm{GU}(V)$ with $\mathrm{GL}_m(\mathbb{C})\times\mathbb{C}^d$, the complexified homomorphism $h_1\colon\mathbb{S}(\mathbb{C})\to\mathrm{GL}_m(\mathbb{C})\times\mathbb{C}^d$ is given by

$$h_{1\mathbb{C}}(z,w)=\begin{pmatrix} z1_p & \\ & w1_q \end{pmatrix}\times zw.$$

Let $\mu\colon\mathbb{C}^\times\to G(\mathbb{C})\cong(\mathrm{GL}_m(\mathbb{C})\times\mathbb{C}^\times)^d$ be the restriction of the complexification $h_{0\mathbb{C}}$ of h_0 to the first factor. Up to conjugation, we may assume that the image of μ is contained in a maximal torus of G defined over $\mathbb{Q}$. Therefore it defines a cocharacter of G. By definition the *reflex field* $E(G,X)=E(G,h_0)$ is the subfield of $\overline{\mathbb{Q}}$ corresponding to the subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of elements fixing the conjugacy class of μ. It is a subfield of any extension of $\mathbb{Q}$ over which G splits. In particular $E(G,X)$ is a subfield of E. We can decompose $V=V_{\tau_1}$ as $V=V_+\oplus V_-$, so that $h(z)$ acts by z, resp. $\bar{z}$, on V_+, resp. V_-, and $E(G,X)$ is precisely the field of definition of the representation V_+ of E. Since the Hermitian space V is F-anisotropic, we conclude that $E(G,X)=E$.

6.4. The complex Shimura variety

The pair (G,X), or (G,h_0), gives rise to a Shimura variety $\mathrm{Sh}(G,X)$ which is defined over the reflex field E. In particular if $K=\prod_p K_p\subset G(\mathbb{A}_\mathbb{Q}^f)$, with $K_p\subset G(\mathbb{Q}_p)$, is an open compact subgroup of the finite adelic points of G, we can consider $\mathrm{Sh}_K(G,X)$. This is a projective variety over E whose set of complex points is identified with

$$S(K)=\mathrm{Sh}_K(G,X)(\mathbb{C})=G(\mathbb{Q})\backslash(X\times G(\mathbb{A}_\mathbb{Q}^f))/K. \tag{6.1}$$

We will always choose K to be *neat* in the following sense: For every $k\in K$, there exists some prime p such that the semisimple part of the p-component of k has no eigenvalues which are roots of unity other than 1. Every compact open subgroup of $G(\mathbb{A}_\mathbb{Q}^f)$ contains a neat subgroup of finite index.

In general $S(K)$ is not connected, but instead a disjoint union of spaces of the type $S(\Gamma)$ discussed in the introduction, for various arithmetic subgroups $\Gamma\subset G_1(\mathbb{Q})=U(V)(F)$. Since we have assumed K to be neat, these arithmetic subgroups are all torsion free.

The connected components of $S(K)$ can be described as follows. Write

$$G(\mathbb{A}_\mathbb{Q}^f)=\bigsqcup_j G(\mathbb{Q})g_jK, \tag{6.2}$$

with $g_j \in G(\mathbb{A}_f)$. Then

$$S(K) \cong \bigsqcup_j S(\Gamma_j), \tag{6.3}$$

where $S(\Gamma_j) = \Gamma_j \backslash X$ and Γ_j is the image in the adjoint group $G_{\mathrm{ad}}(\mathbb{R})$ of the subgroup

$$\Gamma'_j = g_j K g_j^{-1} \cap G(\mathbb{Q}) \tag{6.4}$$

of $G(\mathbb{Q})$ (see [53, Lemma 5.13]).

6.5. The structure of $S(K)$

In this paragraph we provide some more details on the structure of the connected components of $S(K)$. Let $G_{\mathrm{der}} = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{SU}(V)$ be the derived subgroup of G. This subgroup is connected and simply connected as an algebraic group. It therefore follows, from e.g. [53, p. 311], that the set of connected components of the complex Shimura variety $S(K)$ can be identified with the double coset

$$T(\mathbb{Q}) \backslash (Y \times T(\mathbb{A}_{\mathbb{Q}}^f))/\nu(K).$$

Here $T = G/G_{\mathrm{der}}$ is the maximal torus quotient of G, ν is the projection $G \to T$ and we define

$$Y = T(\mathbb{R}) / \operatorname{Im}(Z(\mathbb{R}) \to T(\mathbb{R})),$$

where Z is the center of G and the homomorphism $Z \to T$ is obtained by composing the inclusion $Z \hookrightarrow G$ with ν.

Let us finally describe the group T. First consider the rational torus

$$\mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_{\mathrm{m}E} \times \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m}F},$$

whose group of rational points is $E^\times \times F^\times$. The group T can be described as the rational subtorus defined by the equation $\mathrm{N}_{E/F}(x) = t^m$, with $x \in \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_{\mathrm{m}E}$ and $t \in \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m}F}$. See Kottwitz [40, §7] for a slightly different situation.

Let T_1 be the rational group defined as the kernel of the norm homomorphism $\mathrm{N}_{E/F} = \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_{\mathrm{m}E} \to \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m}F}$. Then $T_1 = G_1/G_{\mathrm{der}}$ is the maximal abelian quotient of G_1 and the quotient map $G_1 \to T_1$ is induced by det.

If m is even, say $m = 2k$, then the torus T is isomorphic to $T_1 \times \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m}F}$, the isomorphism being given by $(x,t) \mapsto (xt^{-k}, t)$ for $(x,t) \in T$.

If m is odd, say $m = 2k+1$, then T is isomorphic to $\mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_{\mathrm{m}E}$, the isomorphism being given by $(x,t) \mapsto xt^{-k}$ for $(x,t) \in T$.

In any case the quotient T/T_1 is isomorphic to the rational torus $\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m}F}$.

It follows that $T(\mathbb{R}) = (\mathbb{C}^\times)^d$ if m is odd, and $T(\mathbb{R}) = (\mathbb{R}^\times \times U_1)^d$ if m is even, with U_1 being the complex unit circle. The image of $Z(\mathbb{R})$ is $(\mathbb{C}^\times)^d$, resp. $(\mathbb{R}_{>0} \times U_1)^d$. Therefore, $Y = \{1\}$ if m is odd, and $Y = \{\pm 1\}^d$ if m is even.

6.6. Cohomology of Shimura varieties

We are interested in the cohomology groups $H^\bullet(S(K),R)$ where R is a $\mathbb{Q}$-algebra. If $K'\subset K$ is another compact subgroup of $G(\mathbb{A}_\mathbb{Q}^f)$ we let $\mathrm{pr}\colon S(K')\to S(K)$ be the natural projection. It induces a map

$$\mathrm{pr}^*\colon H^\bullet(S(K),R)\longrightarrow H^\bullet(S(K'),R).$$

Passing to the direct limit over K via the maps pr^*, we obtain

$$H^\bullet(\mathrm{Sh}(G,X),R)=\varinjlim_K H^\bullet(S(K),R).$$

The cohomology groups $H^\bullet(\mathrm{Sh}(G,X),\mathbb{C})$ are $G(\mathbb{A}_\mathbb{Q}^f)$-modules. For any character ω of $Z(\mathbb{A}_\mathbb{Q}^f)$, we denote by $H^\bullet(\mathrm{Sh}(G,X),\mathbb{C})(\omega)$ the ω-eigenspace. Denote also by $\widetilde{\omega}$ the character of $Z(\mathbb{A}_\mathbb{Q})$ trivial on $Z(\mathbb{R})Z(\mathbb{Q})$ and with finite part ω. Then one knows (see e.g. [9]) that

$$H^\bullet(\mathrm{Sh}(G,X),\mathbb{C})(\omega)\cong H^\bullet(\mathfrak{g},K_\infty;L^2(G,\widetilde{\omega})), \tag{6.5}$$

where $\mathfrak{g}$ is the Lie algebra of $G(\mathbb{R})$, K_∞ is the stabilizer of a point in the symmetric space X and $L^2(G,\widetilde{\omega})$ is the Hilbert space of measurable functions f on $G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})$ such that, for all $g\in G(\mathbb{A}_\mathbb{Q})$ and $z\in Z(\mathbb{A}_\mathbb{Q})$, $f(gz)=f(g)\widetilde{\omega}(z)$ and $|f|$ is square-integrable on $G(\mathbb{Q})Z(\mathbb{A}_\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})$.

Since G is anisotropic each $L^2(G,\widetilde{\omega})$ decomposes as a direct sum of irreducible unitary representations of $G(\mathbb{A}_\mathbb{Q})$ with finite multiplicities. A representation π which occurs in this way is called an *automorphic representation* of G and is factorizable as a restricted tensor product of admissible representations [14]. We shall write $\pi=\pi_\infty\otimes\pi_f$, where π_∞ is a unitary representation of $G(\mathbb{R})$ and π_f is a representation of $G(\mathbb{A}_\mathbb{Q}^f)$. We denote by $\chi(\pi)$, resp. $\chi(\pi_f)$, its *central character* $\widetilde{\omega}$, resp. ω, and by $\mathrm{m}(\pi)$ its multiplicity in $L^2(G,\widetilde{\omega})$.

6.7. Representations with cohomology

Let Coh_∞ be the set of unitary representations π_∞ of $G(\mathbb{R})$ (up to equivalence) such that

$$H^\bullet(\mathfrak{g},K_\infty;\pi_\infty)\neq 0, \tag{6.6}$$

where $\mathfrak{g}$ is the Lie algebra of $G(\mathbb{R})$ and K_∞ is the stabilizer of a point in the symmetric space X. Note that K_∞ is the centralizer in G of the maximal compact subgroup K_1 of $G_1(\mathbb{R})$. Given a representation π of G, we denote by π_1 its restriction to G_1 and say that

π is *essentially unitary* if π_1 is unitary. Recall from §3 that cohomological representations of $G_1(\mathbb{R})$ are classified by Vogan and Zuckerman in [71].

The representation theory of G is substantially identical to that of G_1. Let Z and Z_1 denote the centers or G and G_1, respectively. Then $G = ZG_1$, and every representation (local or global) of G_1 extends to G; it suffices to extend its central character.

Since we only consider cohomological representations of $G(\mathbb{R})$ having trivial central character the classification of Coh_∞ amounts to the Vogan–Zuckerman classification. In particular, the set Coh_∞ is finite. For any π_f, set

$$\mathrm{Inf}(\pi_f) = \{\pi_\infty \in \mathrm{Coh}_\infty : \mathrm{m}(\pi_\infty \otimes \pi_f) \neq 0\}.$$

Let Coh_f be the set of π_f such that $\mathrm{Inf}(\pi_f)$ is non-empty.

We will be particularly interested in the cohomological representations $A(b\times q, a\times q)$. We denote by $\mathrm{Coh}_f^{b,a}$ the set of π_f such that $\mathrm{Inf}(\pi_f)$ contains $A(b\times q, a\times q)$.

6.8. Let $\mathcal{H}_K$ be the Hecke algebra of $\mathbb{Q}$-linear combinations of K-double cosets in $G(\mathbb{A}_{\mathbb{Q}}^f)$. If π_f is a representation of $G(\mathbb{A}_{\mathbb{Q}}^f)$, we let π_f^K denote the representation of $\mathcal{H}_K$ on the space of K-fixed vectors of π_f.

Over $\mathbb{C}$ it follows from Matsushima's formula (see [52], [9]) that there is an $\mathcal{H}_K$-isomorphism

$$H^\bullet(S(K), \mathbb{C}) \longrightarrow \bigoplus_{\pi_f \in \mathrm{Coh}_f} H^\bullet(\pi_f, \mathbb{C}) \otimes \pi_f^K, \tag{6.7}$$

where

$$H^\bullet(\pi_f, \mathbb{C}) = \bigoplus_{\pi_\infty \in \mathrm{Inf}(\pi_f)} \mathrm{m}(\pi_\infty \otimes \pi_f) H^\bullet(\mathfrak{g}, K_\infty; \pi_\infty).$$

Given two integers a and b, we denote by $H^{b\times q, a\times q}(S(K), \mathbb{C})$ the part of $H^\bullet(S(K), \mathbb{C})$ which corresponds to the cohomological representation $\pi_\infty = A(b\times q, a\times q)$, so that the $\mathcal{H}_K$-isomorphism induces the isomorphism

$$H^{b\times q, a\times q}(S(K), \mathbb{C}) \to \bigoplus_{\pi_f \in \mathrm{Coh}_f^{b,a}} \mathrm{m}(A(b\times q, a\times q) \otimes \pi_f) H^{(a+b)q}(\mathfrak{g}, K_\infty; A(b\times q, a\times q)) \otimes \pi_f^K. \tag{6.8}$$

6.9. Rational subspaces of the cohomology groups

Since the action of $\mathcal{H}_K$ is defined on $H^\bullet(S(K), \mathbb{Q})$, we obtain a $\overline{\mathbb{Q}}$-form of (6.7):

$$H^\bullet(S(K), \overline{\mathbb{Q}}) \longrightarrow \bigoplus_{\pi_f \in \mathrm{Coh}_f} H^\bullet(\pi_f, \overline{\mathbb{Q}}) \otimes \pi_f^K(\overline{\mathbb{Q}}), \tag{6.9}$$

where $H^\bullet(\pi_f, \overline{\mathbb{Q}})$ and $\pi_f^K(\overline{\mathbb{Q}})$ are $\overline{\mathbb{Q}}$-forms of $H^\bullet(\pi_f, \mathbb{C})$ and π_f^K, respectively. By considering arbitrary small K, we obtain a $\overline{\mathbb{Q}}$-form $\pi_f(\overline{\mathbb{Q}})$ of any $\pi_f \in \mathrm{Coh}_f$. Moreover, since $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $H^\bullet(S(K), \overline{\mathbb{Q}})$ via its action on the coefficients $\overline{\mathbb{Q}}$, it permutes the summands in (6.9) and therefore induces an action $(\sigma, \pi_f) \mapsto \pi_f^\sigma$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on Coh_f. We let $A(\pi_f) = \{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : \pi_f^\sigma \cong \pi_f\}$ be the stabilizer of π_f and denote by $\mathbb{Q}(\pi_f)$ the corresponding number field. Given $[\pi_f] \in \mathrm{Coh}_f / \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we define

$$\begin{aligned} W([\pi_f]) &= \bigoplus_{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/A(\pi_f)} H^\bullet(\pi_f^\sigma, \overline{\mathbb{Q}}) \otimes \pi_f^\sigma(\overline{\mathbb{Q}}) \\ &= \bigoplus_{\sigma \in \mathrm{Hom}(\mathbb{Q}(\pi_f), \overline{\mathbb{Q}})} \mathrm{Hom}_{G(\mathbb{A}_\mathbb{Q}^f)}(\pi_f^\sigma, H^\bullet(S(K), \overline{\mathbb{Q}})) \otimes \pi_f^\sigma(\overline{\mathbb{Q}}), \end{aligned}$$

so that

$$H^\bullet(S(K), \overline{\mathbb{Q}}) \cong \bigoplus_{[\pi_f] \in \mathrm{Coh}_f / \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} W([\pi_f])^K.$$

THEOREM 6.1. *Suppose that $\pi_f \in \mathrm{Coh}_f$ contributes to $H^\bullet(S(K), \mathbb{C})$. Then the following conditions hold*:

(1) *The subspace $W([\pi_f])^K \subset H^\bullet(S(K), \overline{\mathbb{Q}})$ is a polarized $\mathbb{Q}$-sub-Hodge structure of $H^\bullet(S(K), \mathbb{Q})$.*

(2) *If moreover $\pi_f \in \mathrm{Coh}_f^{b,a}$ with $3(a+b) + |a-b| < 2m$, then we have*(7)

$$(W([\pi_f])^K \otimes_\mathbb{Q} \mathbb{C}) \cap \mathrm{SH}^{(a+b)q}(S(K), \mathbb{C}) \subset H^{b \times q, a \times q}(S(K), \mathbb{C}) \oplus H^{a \times q, b \times q}(S(K), \mathbb{C}).$$

Proof. The first part is classical. It follows from the fact that the Hecke algebra $\mathcal{H}_K$ acts as algebraic correspondences on $S(K)$ which yield morphisms of the rational Hodge structure $H^\bullet(S(K), \mathbb{Q})$. The polarization then comes from the cup product and Poincaré duality, both of which are functorial for algebraic correspondences.

We postpone the proof of the second part until §13. One important ingredient is the global theta correspondence that we review in the next section. □

In the special $q=1$ case we have $\mathrm{SH}^\bullet(S(K), \mathbb{C}) = H^\bullet(S(K), \mathbb{C})$, and we get the following result.

COROLLARY 6.2. *Let $q=1$ and let a and b be integers such that $3(a+b) + |a-b| < 2m$. Then, the space $H^{a+b}(S(K), \mathbb{Q})$ contains a polarized $\mathbb{Q}$-sub-Hodge structure X such that*

$$X \otimes_\mathbb{Q} \mathbb{C} = H^{a,b}(S(K), \mathbb{C}) \oplus H^{b,a}(S(K), \mathbb{C}).$$

(7) Recall that $\mathrm{SH}^\bullet$ is defined in the introduction.

Remark. In particular, the subspace $H^{1,1}(S(K),\mathbb{C})\subset H^2(S(K),\mathbb{C})$ is defined over $\mathbb{Q}$ as long as $p>2$. Note that this is not the case if $p=2$; see [8]. We will see that, when $p>2$, the subspace $H^{1,1}(S(K),\mathbb{C})\subset H^2(S(K),\mathbb{C})$ is generated by theta lifts from unitary groups of signature $(1,1)$ at infinity. This is no more true when $p=2$, but the subspace which is generated by classes obtained by theta lifts—or equivalently the subspace associated with endoscopic representations—is indeed defined over $\mathbb{Q}$; see [8]. Granted this, another proof that $H^{1,1}(S(K),\mathbb{C})\subset H^2(S(K),\mathbb{C})$ is defined over $\mathbb{Q}$ when $p>2$ was proposed to us by M. Harris. Indeed classes obtained by theta lift restrict to classes obtained by theta lifts to any sub-Shimura variety associated with the smaller unitary group $U(2,1)$. The general result now reduces to the theorem of Blasius and Rogawski via Oda's trick using the restriction theorem of Harris and Li [31].

7. The global theta correspondence

7.1. The theta correspondence

We keep F, E, V and $(\cdot,\cdot)$ as in §6.1 and let W be an n-dimensional vector space over E equipped with a skew-Hermitian $\langle\cdot,\cdot\rangle$, which is conjugate linear in the first argument. We take V to be a left E vector space and W to be a right E vector space. These conventions come into play when considering the tensor product $\mathbb{W}=W\otimes_E V$; as an F-vector space, it is endowed with the symplectic form

$$[\cdot,\cdot]=\mathrm{tr}_{E/F}(\langle\cdot,\cdot\rangle\otimes\overline{(\cdot,\cdot)}),$$

where $\mathrm{tr}_{E/F}$ denotes the usual trace of E over F. We let $\mathrm{Sp}(\mathbb{W})$ be the corresponding symplectic F-group. Then $(U(V),U(W))$ forms a reductive dual pair in $\mathrm{Sp}(\mathbb{W})$, in the sense of Howe [34].

Remark. We can define a Hermitian space W' by $W'=W$ (viewed as a left E vector space via $aw=wa$) and

$$(w_1,w_2)=\alpha^{-1}\langle w_2,w_1\rangle.$$

We will sometimes abusively refer to W as a Hermitian space; note however that this involves the choice of α made in §6.1.

Let $\mathrm{Mp}(\mathbb{W})$ be the metaplectic 2-fold cover of $\mathrm{Sp}(\mathbb{W})$ (see Weil [76]). Fix a choice of a non-trivial character ψ of $\mathbb{A}/F$ and denote by $\omega=\omega_\psi$ the corresponding (automorphic) Weil representation of $\mathrm{Mp}(\mathbb{W})$, as in [34].

A *complete polarization* $\mathbb{W}=\mathbb{X}+\mathbb{Y}$, where $\mathbb{X}$ and $\mathbb{Y}$ are maximal totally isotropic subspaces of $\mathbb{W}$, leads to the realization of ω on $L^2(\mathbb{X})$. This is known as the Schrödinger model for ω; see Gelbart [19]. In that way ω is realized as an automorphic representation of $\mathrm{Mp}(\mathbb{W})$. The maximal compact subgroup of $\mathrm{Sp}(\mathbb{W})$ is $U=U_{nm}$, the unitary group in nm variables. We denote by $\widetilde{U}$ its preimage in $\mathrm{Mp}(\mathbb{W})$. The associated space of smooth vectors of ω is the Bruhat–Schwartz space $\mathcal{S}(\mathbb{X}(\mathbb{A}))$. The $(\mathfrak{sp},\widetilde{U})$-module associated with ω is made explicit by the realization of ω in the Fock model. Using it, one sees that the subspace of $\widetilde{U}$-finite vectors in ω is the subspace $\mathbf{S}(\mathbb{X}(\mathbb{A}))\subset\mathcal{S}(\mathbb{X}(\mathbb{A}))$ obtained by replacing, at each infinite place, the Schwartz space by the *polynomial Fock space* $\mathbf{S}(\mathbb{X})\subset\mathcal{S}(\mathbb{X})$, i.e. the image of holomorphic polynomials on $\mathbb{C}^{nm}$ under the intertwining map from the Fock model of the oscillator representation to the Schrödinger model.

7.2. We denote by $U_m(\mathbb{A})$, $U_n(\mathbb{A})$, $\mathrm{Sp}_{2nm}(\mathbb{A})$ and $\mathrm{Mp}_{2nm}(\mathbb{A})$ the adelic points of $U(V)$, $U(W)$, $\mathrm{Sp}(\mathbb{W})$ and $\mathrm{Mp}(\mathbb{W})$, respectively. According to Rao, Perrin and Kudla [42], for any choice of a pair of characters $\chi=(\chi_1,\chi_2)$ of $\mathbb{A}_E^\times/E^\times$ whose restrictions to $\mathbb{A}^\times$ satisfy $\chi_1|_{\mathbb{A}^\times}=\varepsilon_{E/F}^m$ and $\chi_2|_{\mathbb{A}^\times}=\varepsilon_{E/F}^n$, where we denote by $\varepsilon_{E/F}$ the character of $\mathbb{A}^\times/F^\times$ associated with the quadratic extension E/F by classfield theory, there exists a homomorphism

$$\tilde{\imath}_\chi\colon U_m(\mathbb{A})\times U_n(\mathbb{A})\longrightarrow \mathrm{Mp}_{2nm}(\mathbb{A}) \tag{7.1}$$

lifting the natural map

$$\imath\colon U_m(\mathbb{A})\times U_n(\mathbb{A})\longrightarrow \mathrm{Sp}_{2nm}(\mathbb{A}),$$

and so, we obtain a representation ω_χ of $U_m(\mathbb{A})\times U_n(\mathbb{A})$ on $\mathbf{S}(\mathbb{X}(\mathbb{A}))$.(8)

The global metaplectic group $\mathrm{Mp}_{2nm}(\mathbb{A})$ acts on $\mathcal{S}(\mathbb{X}(\mathbb{A}))$ via ω and preserves the dense subspace $\mathbf{S}(\mathbb{X}(\mathbb{A}))$. For each $\phi\in\mathbf{S}(\mathbb{X}(\mathbb{A}))$ we form the theta function

$$\theta_{\psi,\phi}(x)=\sum_{\xi\in\mathbb{X}(F)}\omega_\psi(x)(\phi)(\xi) \tag{7.2}$$

on $\mathrm{Mp}_{2nm}(\mathbb{A})$. Pulling the oscillator representation ω_ψ back to $U_m(\mathbb{A})\times U_n(\mathbb{A})$ using the map (7.1) we get a a smooth, slowly increasing function $(g,g')\mapsto\theta_{\psi,\chi,\phi}(g',g)=\theta_{\psi,\phi}(\tilde{\imath}_\chi(g',g))$ on $U(V)\backslash U_m(\mathbb{A})\times U(W)\backslash U_n(\mathbb{A})$; see [76], [34].

Remark. Let $\chi'=(\chi_1',\chi_2')$ be another pair of characters of $\mathbb{A}_E^\times/E^\times$ whose restrictions to $\mathbb{A}^\times$ satisfy $\chi_1'|_{\mathbb{A}^\times}=\varepsilon_{E/F}^m$ and $\chi_2'|_{\mathbb{A}^\times}=\varepsilon_{E/F}^n$, and put $\mu=\chi_1'\chi_1^{-1}$ and $\nu=\chi_2'\chi_2^{-1}$. Since $\mu|_{\mathbb{A}^\times}=\nu|_{\mathbb{A}^\times}=1$, we can define characters μ' and ν' of $\mathbb{A}_E^1$—the adelic points of the kernel of the norm $\mathrm{N}_{E/F}$—by setting $\mu'(x/\bar{x})=\mu(x)$ and $\nu'(x/\bar{x})=\nu(x)$. Let $\mu_n=\mu'\circ\det$ and

(8) At infinity the choices of χ_1 and χ_2 correspond to the choice of a pair of integers (k,ℓ) with $k\equiv m \pmod 2$, and $\ell\equiv n \pmod 2$ and ω_χ yields $\omega_{k,\ell}$ as in §4.6.

$\nu_m = \nu' \circ \det$ be the associated characters of $U_n(\mathbb{A})$ and $U_m(\mathbb{A})$, respectively. Then it follows from the explicit formulas contained in [42] that

$$\omega_\psi(\tilde{\imath}_{\chi'}(g,g')) = \omega_\psi(\tilde{\imath}_\chi(g,g'))\nu_m(g)\mu_n(g').$$

7.3. The global theta lifting

We denote by $\mathcal{A}^c(U(W))$ the set of irreducible cuspidal automorphic representations of $U_n(\mathbb{A})$, which occur as irreducible subspaces in the space of cuspidal automorphic functions in $L^2(U(W)\backslash U_n(\mathbb{A}))$. As in [47] we will denote by $[U_n]$ the quotient $U(W)\backslash U_n(\mathbb{A})$. For a $\pi' \in \mathcal{A}^c(U(W))$, the integral

$$\theta^f_{\psi,\chi,\phi}(g) = \int_{[U_n]} \theta_{\psi,\chi,\phi}(g,g') f(g')\, dg', \tag{7.3}$$

with $f \in H_{\pi'}$ (the space of π'), defines an automorphic function on $U_m(\mathbb{A})$: the integral (7.3) is well defined, and determines a slowly increasing function on $U(V)\backslash U_m(\mathbb{A})$. We denote by $\Theta^V_{\psi,\chi,W}(\pi')$ the space of the automorphic representation generated by all $\theta^f_{\psi,\chi,\phi}(g)$ as ϕ and f vary, and call $\Theta^V_{\psi,\chi,W}(\pi')$ the (ψ,χ)-theta lifting of π' to $U_m(\mathbb{A})$. Note that, since $\mathbf{S}(\mathbb{X}(\mathbb{A}))$ is dense in $\mathcal{S}(\mathbb{X}(\mathbb{A}))$ we may as well let ϕ vary in the subspace $\mathbf{S}(\mathbb{X}(\mathbb{A}))$.

We can similarly define $\mathcal{A}^c(U(V))$ and $\Theta^W_{\psi,\chi,V}$ the (ψ,χ)-theta correspondence from $U(V)$ to $U(W)$.

Definition 7.1. We say that a representation $\pi \in \mathcal{A}^c(U(V))$ *is in the image of the cuspidal ψ-theta correspondence from a smaller group* if there exists a skew-Hermitian space W with $\dim W \leqslant m$, a representation $\pi' \in \mathcal{A}^c(U(W))$ and a pair of characters χ such that

$$\pi = \Theta^V_{\psi,\chi,W}(\pi').$$

7.4. Local signs

Given a representation $\pi \in \mathcal{A}^c(U(V))$ in the image of the cuspidal ψ-theta correspondence from a smaller group $U(W)$, we associate with π local signs in the following way: Let v be a finite place of F. We first assume that $E \otimes_F F_v$ is a field. By a theorem of Landherr [49], for each n there are exactly two different classes of isomorphism of n-dimensional Hermitian spaces over E_v:

(1) For $n=2r+1$ odd, let $W_{r,r}$ denote the Hermitian space of dimension $2r$ over E_v with maximal isotropic subspaces of dimension r, then the two classes are represented

by $W^{\pm}=W_{r,r}\oplus W_1^{\pm}$ where $W_1^{\pm}\cong E_v$ is the 1-dimensional Hermitian space over E_v with Hermitian form $(x,y)=\alpha\bar{x}y$, where $\alpha\in F_v^{\times}$ with $\varepsilon_{E_v/F_v}(\alpha)=\pm 1$.

(2) For $n=2r$ even, then the two classes are represented by $W^+=W_{r,r}$ and $W^-=W_{r-1,r-1}\oplus W_2^-$, where W_2^- is an anisotropic space of dimension 2.

Now we associate to π the local sign $\varepsilon(\pi_v)=\pm 1$ depending on whether $W\otimes_F F_v\cong W^+$ or $W\otimes_F F_v\cong W^-$.

If $E\otimes_F F_v$ is not a field, we define $\varepsilon(\pi_v)=1$.

The conservation relation conjecture of Harris, Kudla and Sweet [30, Speculations 7.5 and 7.6]—whose relevant part to us has been proved by Gong and Grenié [26]—implies that this local sign is well defined and only depends on π.

Note that the local sign $\varepsilon(\pi_v)$ is equal to 1 at all but finitely many places and that we have

$$\prod_{v<\infty}\varepsilon(\pi_v)=1.$$

7.5. Extension of the theta correspondence to similitude groups

The extension of the theta correspondence to unitary similitude groups has been worked out in details by Michael Harris in [29, §3.8]. It is based on the observation that the map

$$i\colon \mathrm{GU}(W)\times \mathrm{GU}(V)\longrightarrow \mathrm{GL}(W\otimes_E V),\quad i(g',g)(w\otimes v)=wg'\otimes g^{-1}v$$

takes the algebraic subgroup

$$\mathrm{G}(U(V)\times U(W)):=\{(g,g')\in \mathrm{GU}(V)\times \mathrm{GU}(W):\lambda(g)=\lambda(g')\}$$

into $\mathrm{Sp}(W\otimes_E V)$.

Note that an automorphic representation π of G is in the image of the extension to unitary similitude groups of the theta correspondence from a smaller group $\mathrm{GU}(W)$ if π_1 is in the image of the ψ-theta correspondence from $U(W)$. In that case we will loosely say that π is in the image of the ψ-theta correspondence from $U(W)$.

The main automorphic ingredient of our paper is the following theorem. It is a corollary of Proposition 13.4 below whose proof is the goal of Part 3.

THEOREM 7.2. *Let a and b be integers such that $3(a+b)+|a-b|<2m$ and let $\pi_f\in \mathrm{Coh}_f^{b,a}$. Set $\pi=A(b\times q,a\times q)\otimes\pi_f$. Then π is in the image of the ψ-theta correspondence from a smaller group $U(W)$ of signature (a,b) at infinity.*

Reduction to Proposition 13.4. Proposition 13.4 implies that π_1 is in the image of the ψ-theta correspondence from $U(W)$, where W is some $(a+b)$-dimensional skew-Hermitian space over E. It remains to prove that over the infinite place v of F such that $U(V)(F_v) \cong U(p,q)$, the signature of W is (a,b). But this follows from the explicit description of the Archimedean theta correspondence obtained by Annegret Paul [61]: the cohomological representation $A(b\times q, a\times q)$ is the image of the local theta correspondence from a group $U(W, \mathbb{C}/\mathbb{R})$ with $\dim W = a+b$ if and only if the signature of W is (a,b). □

Since $Z(\mathbb{A}_{\mathbb{Q}}^f)$ maps into $T(\mathbb{A}_{\mathbb{Q}}^f)$ via the map ν, it acts on the disconnected Shimura variety $S(K)$ by permuting the connected components as described in §6.5. We conclude with the following corollary.

COROLLARY 7.3. *Let S be any connected component of $S(K)$ and let a and b be integers such that $3(a+b)+|a-b|<2m$. Then $H^{b\times q, a\times q}(S,\mathbb{C})$ is generated by classes of theta lifts from unitary groups of signature (a,b) at infinity.*

8. Special cycles

8.1. Notation

We keep the notation as in §6.1 and follow the adelization [43] of the work of Kudla–Millson. Let n be an integer with $0 \leqslant n \leqslant p$. Given an n-tuple $\mathbf{x}=(x_1, ..., x_n) \in V^n$ we let $U=U(\mathbf{x})$ be the F-subspace of V spanned by the components of $\mathbf{x}$. We write $(\mathbf{x},\mathbf{x})$ for the $n\times n$ Hermitian matrix with (j,k) entry equal to (x_j, x_k). Assume that $(\mathbf{x},\mathbf{x})$ is totally positive semidefinite of rank t, i.e. over each infinite place the Hermitian matrix $(\mathbf{x},\mathbf{x})$ is semidefinite and non-negative. Equivalently, as a sub-Hermitian space $U\subset V$ is totally positive definite of dimension t. In particular, $0\leqslant t\leqslant p$ (and $t\leqslant n$). The constructions of the preceding section can therefore be made with the space $U^\perp$ in place of V. Set $H=\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{G}(U(U)\times U(U^\perp))$. There is a natural morphism $H\to G$ and (the image of) H is isomorphic to G_U, the stabilizer of U in G. By abuse of notation, we will use both H and G_U to denote the stabilizer of U in G. Recall that we can realize the symmetric space X as the set of negative q-planes in V_{v_0}. We then let X_H be the subset of X consisting of those q-planes which lie in $U_{v_0}^\perp$.

8.2. Shimura subvarieties

There is a natural morphism $i_U\colon \mathrm{Sh}(H,X_H)\to\mathrm{Sh}(G,X)$ which is defined over the reflex field E. If K is an open compact subgroup of $G(\mathbb{A}_{\mathbb{Q}}^f)$ we set $K_H=H(\mathbb{A}_{\mathbb{Q}}^f)\cap K$. The variety $\mathrm{Sh}_{K_H}(H,X_H)$ is projective, defined over E and the set of its complex points identifies with

$$S_H(K_H)=H(\mathbb{Q})\backslash(X_H\times H(\mathbb{A}_{\mathbb{Q}}^f))/K_H.$$

Now given an element $g\in G(\mathbb{A}_{\mathbb{Q}}^f)$, we may shift the natural morphism i_U by g to get

$$\begin{aligned} i_{U,g,K}\colon H(\mathbb{Q})\backslash(X_H\times H(\mathbb{A}_{\mathbb{Q}}^f))/K_{H,g} &\longrightarrow G(\mathbb{Q})\backslash(X\times G(\mathbb{A}_{\mathbb{Q}}^f))/K, \\ H(\mathbb{Q})(z,h)K_{H,g} &\longmapsto G(\mathbb{Q})(z,hg)K, \end{aligned} \tag{8.1}$$

where $K_{H,g}=H(\mathbb{A}_{\mathbb{Q}}^f)\cap gKg^{-1}$.

8.3. Connected cycles

Suppose that K is neat and let $g\in G(\mathbb{A}_{\mathbb{Q}}^f)$. Set

$$\Gamma_g'=gKg^{-1}\cap G(\mathbb{Q}) \quad \text{and} \quad \Gamma_{g,U}'=gKg^{-1}\cap H(\mathbb{Q})=\Gamma_g'\cap H(\mathbb{Q}).$$

Now let Γ_g, resp. $\Gamma_{g,U}$, denote the image of Γ_g', resp. $\Gamma_{g,U}'$, in the adjoint group $G_{\mathrm{ad}}(\mathbb{R})$, resp. $H_{\mathrm{ad}}(\mathbb{R})$. Then the natural map $\Gamma_{g,U}z\mapsto\Gamma_g z$ yields a (totally geodesic) immersion of $\Gamma_{g,U}\backslash X_H$ into $\Gamma_g\backslash X$. We will denote the corresponding (connected) cycle by $c(U,g,K)$.

We now introduce composite cycles that may be seen as composed of the connected cycles $c(U,g,K)$.

8.4. Special cycles

Given $\beta\in\mathrm{Her}_n(E)$ an $n\times n$ totally positive Hermitian matrix—we use the notation $\beta\gg 0$ for such a Hermitian matrix—we define

$$\Omega_\beta=\left\{\mathbf{x}\in V^n : \tfrac{1}{2}(\mathbf{x},\mathbf{x})=\beta\right\}. \tag{8.2}$$

The natural action of $G(\mathbb{A}_{\mathbb{Q}}^f)$ on $V(\mathbb{A}_{\mathbb{Q}}^f)^n$ restricts to an action on $\Omega_\beta(\mathbb{A}_{\mathbb{Q}}^f)$. Then, any K-invariant compact subset of $\Omega_\beta(\mathbb{A}_{\mathbb{Q}}^f)$ decomposes as a union of at most finitely many disjoint K-orbits.

Now let $\varphi\in\mathcal{S}(V(\mathbb{A}_{\mathbb{Q}}^f)^n)$ be a K-invariant Schwartz function on $V(\mathbb{A}_{\mathbb{Q}}^f)^n$ with values in $\mathbb{C}$. For β as above, with $\Omega_\beta(F)\neq\varnothing$, let

$$Z(\beta,\varphi,K)=\sum_j \sum_{\substack{\mathbf{x}\in\Omega_\beta(F)\\ \mathrm{mod}\ \Gamma_{g_j}'}} \varphi(g_j^{-1}\mathbf{x})c(U(\mathbf{x}),g_j,K). \tag{8.3}$$

Here the g_j's in $G(\mathbb{A}_{\mathbb{Q}}^f)$ are those in (6.2).

Write $g\mapsto\omega(g)$ for the natural action of $g\in G(\mathbb{A}_{\mathbb{Q}}^f)$ on $\mathcal{S}(V(\mathbb{A}_{\mathbb{Q}}^f)^n)$ given by

$$(\omega(g)\varphi)(x)=\varphi(g^{-1}x).$$

As in [43, Propositions 5.9 and 5.10], we have the following result.

PROPOSITION 8.1. (1) *For any* $g\in G(\mathbb{A}_{\mathbb{Q}}^f)$, *we have*

$$Z(\beta,\omega(g)\varphi,gKg^{-1})=Z(\beta,\varphi,K)g^{-1}.$$

(2) *Suppose that* $K'\subset K$ *is another open compact subgroup of* $G(\mathbb{A}_{\mathbb{Q}}^f)$, *and let*

$$\mathrm{pr}\colon S(K')\longrightarrow S(K)$$

be the natural projection. Then we have $\mathrm{pr}^*(Z(\beta,\varphi,K))=Z(\beta,\varphi,K')$.

It follows from (2) that $Z(\beta,\varphi,K)$ gives a well-defined element $Z(\beta,\varphi)$ in $\mathrm{Sh}(G,X)$.

8.5. The ring of special cycles

Let $\mathcal{S}(V(\mathbb{A}_{\mathbb{Q}}^f)^n)_{\mathbb{Z}}$ be the space of locally constant functions on $V(\mathbb{A}_{\mathbb{Q}}^f)^n$ with compact support and values in $\mathbb{Z}$. For any commutative ring R, let

$$\mathcal{S}(V(\mathbb{A}_{\mathbb{Q}}^f)^n)_R=(\mathcal{S}(V(\mathbb{A}_{\mathbb{Q}}^f)^n)_{\mathbb{Z}})\otimes_{\mathbb{Z}}R.$$

Note that the natural action ω turns it into a $G(\mathbb{A}_{\mathbb{Q}}^f)$-module. It follows from Proposition 8.1 that for any Hermitian $n\times n$ matrix $\beta\gg 0$ we get a $G(\mathbb{A}_{\mathbb{Q}}^f)$-equivariant map

$$\begin{aligned}\mathcal{S}(V(\mathbb{A}_{\mathbb{Q}}^f)^n)_{\mathbb{Q}}&\longrightarrow H^{2qn}(\mathrm{Sh}(G,X),\mathbb{Q}),\\ \varphi&\longmapsto[\beta,\varphi]:=[Z(\beta,\varphi)].\end{aligned}\tag{8.4}$$

Following Kudla [43], in order to study the $G(\mathbb{A}_{\mathbb{Q}}^f)$-submodule which is the image of this map, we first extend this construction to the case where β is only *totally positive semidefinite*. The expression (8.3) is still well defined. Denoting by t the rank of β, one obtains a class

$$[\beta,\varphi]^0=[Z(\beta,\varphi)]\in H^{2qt}(\mathrm{Sh}(G,X),\mathbb{Q}).$$

Now recall that the symmetric domain X has a natural Kähler form Ω and that, for any compact open subgroup $K\subset G(\mathbb{A}_{\mathbb{Q}}^f)$, $(1/2\pi i)\Omega$ induces a $(1,1)$-form on $S(K)$ which is the Chern form of the canonical bundle of $S(K)$. The cup product with Ω^q—or equivalently with the Chern form c_q introduced above—induces the qth power of the Lefschetz operator:

$$L^q\colon H^\bullet(\mathrm{Sh}(G,X),\mathbb{Q})\longrightarrow H^{\bullet+2q}(\mathrm{Sh}(G,X),\mathbb{Q})$$

on cohomology which commutes with the action of $G(\mathbb{A}_{\mathbb{Q}}^f)$. We then set

$$[\beta,\varphi]=L^{q(n-t)}([\beta,\varphi]^0)\in H^{2qn}(\mathrm{Sh}(G,X),\mathbb{Q}).\tag{8.5}$$

For each n, with $0\leqslant n\leqslant p$, let

$$\mathrm{SC}^{2nq}(\mathrm{Sh}(G,X))\subset H^{2qn}(\mathrm{Sh}(G,X),\mathbb{Q})$$

be the subspace spanned by the classes $[\beta,\varphi]$, where β is any Hermitian (totally) positive semidefinite $n\times n$ matrix ($\beta\geqslant 0$). The subspace $\mathrm{SC}^{2nq}(\mathrm{Sh}(G,X))$ is defined over $\mathbb{Q}$ and is Hecke stable. We therefore have a direct sum decomposition into π_f-isotypical components:

$$\mathrm{SC}^{2nq}(\mathrm{Sh}(G,X))=\bigoplus_{\pi_f\in\mathrm{Coh}_f}\mathrm{SC}^{2nq}(\mathrm{Sh}(G,X),\pi_f). \tag{8.6}$$

We can now state our main result on special cycles.

THEOREM 8.2. *The following statements hold*: (1) *The space*

$$\mathrm{SC}^{\bullet}(\mathrm{Sh}(G,X))=\bigoplus_{n=0}^{p}\mathrm{SC}^{2nq}(\mathrm{Sh}(G,X))$$

is a subring of $H^{\bullet}(\mathrm{Sh}(G,X),\mathbb{Q})$.

(2) *For each* n, *with* $0\leqslant n\leqslant p$, *we have*

$$\mathrm{SC}^{2nq}(\mathrm{Sh}(G,X))\subset\mathrm{SH}^{nq,nq}(\mathrm{Sh}(G,X),\mathbb{C})\cap H^{2qn}(\mathrm{Sh}(G,X),\mathbb{Q}).$$

(3) *If we furthermore assume that* $3n<p+q$, *then the subspace* $\mathrm{SC}^{2nq}_{\mathrm{prim}}(\mathrm{Sh}(G,X))$ *spanned by the projection of* $\mathrm{SC}^{2nq}(\mathrm{Sh}(G,X))$ *into the primitive part*

$$\mathrm{SH}^{n\times q,n\times q}(\mathrm{Sh}(G,X),\mathbb{C})$$

of $\mathrm{SH}^{nq,nq}(\mathrm{Sh}(G,X),\mathbb{C})$ *is defined over* $\mathbb{Q}$ *and we have a direct sum decomposition*

$$\mathrm{SC}^{2nq}_{\mathrm{prim}}(\mathrm{Sh}(G,X))=\bigoplus_{\pi_f}H^{2nq}(\pi_f,\mathbb{C})\otimes\pi_f, \tag{8.7}$$

where the sum runs over all $\pi_f\in\mathrm{Coh}_f^{n,n}$ *such that* $\varepsilon(\pi_v)=1$ *for all finite places* v.

Remark. The proof of Theorem 8.2 is based on Kudla–Millson's theory [44]–[46] that give an explicit construction of Poincaré dual forms to the special cycles. The first part of Theorem 8.2 immediately follows from their theory as was already pointed out by Kudla in [43]. The last part is the real new part; it will follow from Theorem 7.2 and the results of §5.

Before proving Theorem 8.2, which we do in the next section, we review the relevant results of the Kudla–Millson theory.

8.6. The forms of Kudla–Millson

Recall that in equation (5.11) of §5 we have defined an element

$$\varphi_{nq,nq}\in \mathrm{Hom}_K\big(\textstyle\bigwedge^{nq,nq}\mathfrak{p},\mathbf{S}(V^n)\big)$$

for each n such that $0\leqslant n\leqslant p$. Here $V=V_{\tau_1}$ is the completion of V with respect to the τ_1-embedding. With $G=\mathrm{GU}(p,q)$ and K denoting the stabilizer of a fixed base point $x_0\in X$, we have

$$X\cong G/K\cong U(p,q)/(U(p)\times U(q)),$$

and the space of differential forms on X of type (a,b) is

$$\Omega^{a,b}(X)\cong \mathrm{Hom}_K\big(\textstyle\bigwedge^{a,b}\mathfrak{p},C^\infty(G)\big).$$

Evaluation at x_0 therefore yields an isomorphism

$$[\mathbf{S}(V^n)\otimes\Omega^{nq,nq}(X)]^G\cong \mathrm{Hom}_K\big(\textstyle\bigwedge^{nq,nq}\mathfrak{p},\mathbf{S}(V^n)\big).$$

We will abusively denote by $\varphi_{nq,nq}$ the corresponding element in $[\mathbf{S}(V^n)\otimes\Omega^{nq,nq}(X)]^G$.

8.7. Let V be a positive definite Hermitian space of dimension $m=p+q$ over $\mathbb{C}$. Let

$$\varphi_0(\mathbf{x})=\exp(-\pi\,\mathrm{trace}(\mathbf{x},\mathbf{x}))$$

be the standard Gaussian. Then, under the Weil representation ω of $U(n,n)$ associated with V, we have

$$\omega(k',k'')\varphi_0=\det(k')^m\det(k'')^{-m}\varphi_0,\quad (k',k'')\in U(n)\times U(n).$$

If $\mathbf{x}\in V^n$ with $\frac{1}{2}(\mathbf{x},\mathbf{x})=\beta$, then for $g'\in U(n,n)$ we set

$$W_\beta(g')=\omega(g')\varphi_0(\mathbf{x}). \tag{8.8}$$

8.8. Now we return to the global situation. Let n be an integer with $1\leqslant n\leqslant p$. For $\varphi\in\mathcal{S}(V(\mathbb{A}_{\mathbb{Q}}^f)^n)$, we define

$$\phi=\varphi_{nq,nq}\otimes\Bigg(\bigotimes_{j=2}^{d}\varphi_0\Bigg)\otimes\varphi\in[\mathbf{S}(V(\mathbb{A}_{\mathbb{Q}})^n)\otimes\Omega^{nq,nq}(X)]^{G(\mathbb{R})}, \tag{8.9}$$

where $\varphi_{nq,nq}$ is the Schwartz form for V_{τ_1} and φ_0 is the Gaussian for V_{τ_j}, $j>1$.

Let W be a $2n$-dimensional vector space over E equipped with a *split* ι-skew-Hermitian form and let $G'=\mathrm{Res}_{F/\mathbb{Q}}U(W)$. The splitting of W gives rise to a polarization $\mathbb{X}+\mathbb{Y}$ of $W\otimes_E V$ with $\mathbb{X}\cong V^n$. The global group $G'(\mathbb{A}_\mathbb{Q})$ acts in $\mathbf{S}(V(\mathbb{A}_\mathbb{Q})^n)$ via the global Weil representation associated with this polarization, our fixed additive character ψ of $\mathbb{A}/F$ and some choice character χ of $\mathbb{A}_E^\times/E^\times$ whose restrictions to $\mathbb{A}^\times$ satisfy $\chi|_{\mathbb{A}^\times}=\varepsilon_{E/F}^m$. If φ is K-invariant, then for $g'\in G'(\mathbb{A}_\mathbb{Q})$ and $g\in G(\mathbb{A}_\mathbb{Q})$ we may then form the theta function $\theta_{\psi,\chi,\phi}(g,g')$ as in §7.3. As a function of g, it defines a closed (nq,nq)-form on $S(K)$ which we abusively denote by $\theta_n(g',\varphi)$. Let $[\theta_n(g',\varphi)]$ be the corresponding class in

$$H^{nq,nq}(S(K),\mathbb{R})\subset H^{nq,nq}(\mathrm{Sh}(G,X),\mathbb{R}).$$

For $g'=(g_1',...,g_d')\in G'(\mathbb{R})=U(n,n)^d\subset G'(\mathbb{A}_\mathbb{Q})$ and for $\beta\geqslant 0$ Hermitian in $\mathrm{Her}_n(E)$, we set

$$W_\beta(g')=\prod_{j=1}^d W_{\beta^{\tau_j}}(g_j').$$

The following result is the main theorem of [46], rephrased here in the adelic language following Kudla [43]. It relates the cohomology class $[\theta_n(g',\varphi)]$ to those of the algebraic cycles $Z(\beta,\varphi)$ via Fourier decomposition as in the classical work or Hirzebruch–Zagier [33].

Proposition 8.3. *For $g'\in G'(\mathbb{R})\subset G'(\mathbb{A}_\mathbb{Q})$ and $\varphi\in\mathcal{S}(V(\mathbb{A}_\mathbb{Q}^f)^n)$, the Fourier expansion of $g'\mapsto[\theta_n(g',\varphi)]$ is given by*

$$[\theta_n(g',\varphi)]=\sum_{\beta\geqslant 0}[\beta,\varphi]W_\beta(g').$$

8.9. Proof of Theorem 8.2 I

We first prove Theorem 8.2 (1). Let $0\leqslant n_1,n_2\leqslant p$ and choose W_1 and W_2 split skew-Hermitian vector spaces over E of dimensions $2n_1$ and $2n_2$. Write $G'_{n_j}=\mathrm{Res}_{F/\mathbb{Q}}U(W_j)$, $j=1,2$. Given two Schwartz functions $\varphi_j\in\mathcal{S}(V(\mathbb{A}_\mathbb{Q}^f)^{n_j})$ and two Hermitian matrices $\beta_j\geqslant 0$ in $\mathrm{Her}_{n_j}(E)$, $j=1,2$, we want to prove that the cup product of $[\beta_1,\varphi_1]$ and $[\beta_2,\varphi_2]$ belongs to $\mathrm{SC}^{2nq}(\mathrm{Sh}(G,X))$ where $n=n_1+n_2$. It is now natural to introduce the skew-Hermitian vector space $W=W_1\oplus W_2$, the group $G_n'=\mathrm{Res}_{F/\mathbb{Q}}U(W)$ and the Schwartz function $\varphi=\varphi_1\otimes\varphi_2\in\mathcal{S}(V(\mathbb{A}_\mathbb{Q}^f)^n)$. We then have a natural homomorphism

$$G'_{n_1}(\mathbb{A}_\mathbb{Q})\times G'_{n_2}(\mathbb{A}_\mathbb{Q})\xrightarrow{\iota}G'_n(\mathbb{A}_\mathbb{Q}),$$

and the local product formula (Propositions 5.4 and 5.19) implies that for $g_j'\in G'_{n_j}(\mathbb{R})$, $j=1,2$, we have

$$\theta_n(\iota(g_1',g_2'),\varphi)=\theta_{n_1}(g_1',\varphi_1)\wedge\theta_{n_2}(g_2',\varphi_2). \tag{8.10}$$

Taking cohomology classes, applying Proposition 8.3 and comparing Fourier coefficients yields that $[\beta_1,\varphi_1]\cup[\beta_2,\varphi_2]$ decomposes as the sum $\sum_{\beta\geqslant 0}[\beta,\varphi]$ over the β's such that

$$W_\beta(\iota(g_1',g_2'))=W_{\beta_1}(g_1')W_{\beta_2}(g_2').$$

This proves the first part of Theorem 8.2.

To prove the second part it is enough to prove that the cohomology classes $[\theta_n(g',\varphi)]$ belong to $\mathrm{SH}^{nq,nq}(\mathrm{Sh}(G,X),\mathbb{R})$. This in turn follows from the fact that $\varphi_{nq,nq}$, seen as an (nq,nq)-form on X with values in $\mathbf{S}(V^n)$ is $\mathrm{SL}(q)$-invariant. But this can be read out from the explicit formula for $\varphi_{nq,nq}$; see Proposition 5.17 and Appendix C.

The last part of Theorem 8.2 will be deduced from our main theorem, which we state and prove in the next section.

9. Main theorem

9.1. Notation

We keep the notation as in the previous section except that we will now always assume that a and b are integers such that $3(a+b)+|a-b|<2m$ (we recall that $m=\dim V=p+q$).

9.2. The special lift

It follows from Theorem 7.2 that if $\pi_f\in\mathrm{Coh}_f^{b,a}$ then $\pi=A(b\times q,a\times q)\otimes\pi_f$ is in the image of the ψ-theta correspondence from a smaller group $U(W)$ of signature (a,b) at infinity. In particular the whole cohomology group $H^{b\times q,a\times q}(\mathrm{Sh}(G),\mathbb{C})$ is generated by the automorphic functions $\theta^f_{\psi,\chi,\phi}$ as in (7.3) where ϕ and f vary. Here f is an automorphic function of $\mathrm{GU}(W)$ and ϕ is a Schwartz function in the space $\mathbf{S}(\mathbb{X}(\mathbb{A}))$ associated with a choice of a complete (global) polarization of the symplectic space $\mathbb{W}$. We may furthermore restrict to functions ϕ that are decomposable as $\phi_\infty\otimes\phi_f$.

Now at infinity the Schwartz space $\mathbf{S}(\mathbb{X}(F_\infty))$ is a model for the Weil representation. We will abuse notation and denote by $\varphi_{bq,aq}$ the Schwartz function in $\mathbf{S}(\mathbb{X}(F_\infty))$ which is the tensor product of $\varphi_{bq,aq}$ of §5, equation (5.11), at the infinite place where the real group is non-compact and Gaussians at the other infinite places. We finally denote by $H^{b\times q,a\times q}(\mathrm{Sh}(G),\mathbb{C})_{\mathrm{special}}$ the subspace of *special lifts* that are generated by the projections in $H^{b\times q,a\times q}(\mathrm{Sh}(G),\mathbb{C})$ of the automorphic functions $\theta^f_{\psi,\chi,\varphi_{bq,aq}\otimes\phi_f}$ as ϕ_f, χ and f vary.

We now prove that special lifts span the whole refined Hodge type $a\times q,b\times q$ in the cohomology of $\mathrm{Sh}(G)$.

THEOREM 9.1. *We have*

$$H^{b\times q,a\times q}(\mathrm{Sh}(G),\mathbb{C})_{\mathrm{special}} = H^{b\times q,a\times q}(\mathrm{Sh}(G),\mathbb{C}).$$

Proof. The proof follows the same lines as that of [7, Theorem 10.5]. First recall the following simple general observation. Let K be a group and let A, B, U and V be K-modules. Suppose that we have K-module homomorphisms $\Phi: U\to V$ and $\Psi: B\to A$. Then we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_K(A,U) \arrow[r, "\Phi_*"] \arrow[d, "\Psi^*"'] & \mathrm{Hom}_K(A,V) \arrow[d, "\Psi^*"] \\ \mathrm{Hom}_K(B,U) \arrow[r, "\Phi_*"] & \mathrm{Hom}_K(B,V). \end{tikzcd} \tag{9.1}$$

Here Φ_* is postcomposition with Φ and Ψ^* is precomposition with Ψ.

In what follows K will be the group K_∞. We now define K_∞-module homomorphisms Φ and Ψ that will concern us here. We begin with the $(\mathfrak{g},K_\infty)$-module homomorphism Φ. Let $\pi_f\in\mathrm{Coh}_f^{b,a}$. Denote by H_{π_f} the $A(b\times q,a\times q)\otimes\pi_f$-isotypical subspace in $L^2(G,\chi(\pi_f))$ and let

$$H=\bigoplus_{\pi_f\in\mathrm{Coh}_f^{b,a}} H_{\pi_f}.$$

Recall that we have

$$H^{b\times q,a\times q}(\mathrm{Sh}(G),\mathbb{C})\cong H^{(a+b)q}(\mathfrak{g},K_\infty;H)\cong \mathrm{Hom}_{K_\infty}(V(b,a),H). \tag{9.2}$$

Theorem 7.2 implies that each automorphic representation $\pi=A(b\times q,a\times q)\otimes\pi_f$, with $\pi_f\in\mathrm{Coh}_f^{b,a}$ is in the image of the ψ-theta correspondence from a smaller group $U(W)$ of signature (a,b). We realize the oscillator representation as a $(\mathfrak{g},K_\infty)\times G(\mathbb{A}_f)$-module in the subspace

$$\mathbf{S}(\mathbb{X}(F_{v_0}))\times\mathcal{S}(\mathbb{X}(\mathbb{A}_f))\subset\mathbf{S}(\mathbb{X}(\mathbb{A})).$$

Here the inclusion maps an element (φ_∞,φ) of the right-hand side to

$$\phi=\varphi_\infty\otimes\Bigg(\bigotimes_{\substack{v\mid\infty\\ v\neq v_0}}\varphi_0\Bigg)\otimes\varphi,$$

where the factors φ_0, at the infinite places v not equal to v_0, denote the unique elements (up to scalar multiples) that are fixed by the compact group $U(m)$. We abusively write elements of $\mathbf{S}(\mathbb{X}(F_{v_0}))\times\mathcal{S}(\mathbb{X}(\mathbb{A}_f))$ as $\varphi_\infty\otimes\varphi$.

Fix some pair of characters χ as in §7.3 and let H' be the direct sum of the spaces of cuspidal automorphic representations π' of $\mathrm{GU}(W)$ such that

$$\Theta^V_{\psi,\chi,W}(\pi') = A(b\times q, a\times q)\otimes\pi_f$$

for some $\pi_f\in\mathrm{Coh}_f^{b,a}$. From now on, we abbreviate

$$\mathbf{S} = \mathbf{S}(\mathbb{X}(F_{v_0}))\times\mathcal{S}(\mathbb{X}(\mathbb{A}_f)).$$

It follows from the definition of the global theta lift (see §7.3) that for any $f\in H'$ and $\phi\in\mathbf{S}$ the map $f\otimes\phi\mapsto\theta^f_{\psi,\chi,\phi}$ is a $(\mathfrak{g}, K_\infty)$-module homomorphism from $H'\otimes\mathbf{S}$ to the space H. And Theorem 7.2 implies that the images of these, as χ and ψ vary, span the whole space H. We will drop the dependence of ψ and χ henceforth and abbreviate this map to θ, whence $f\otimes\phi\mapsto\theta(f\otimes\phi)$. Then in the diagram (9.1) we set $U=H'\otimes\mathbf{S}$ and $V=H$ and $\Phi=\theta$.

We now define the map Ψ. We will take A as above to be the vector space $\bigwedge^{nq}\mathfrak{p}$, the vector space B to be the submodule $V(b,a)$ and Ψ to be the inclusion $i_{b,a}\colon V(b,a)\to\bigwedge^{nq}\mathfrak{p}$ (note that there is a unique embedding up to scalars and the scalars are not important here).

From the general diagram (9.1) we obtain the desired commutative diagram

$$\begin{tikzcd} H'\otimes\mathrm{Hom}_{K_\infty}\big(\bigwedge^{nq}\mathfrak{p},\mathbf{S}\big) \arrow[r, "\theta_*"] \arrow[d, "i^*_{b,a}"'] & \mathrm{Hom}_{K_\infty}\big(\bigwedge^{nq}\mathfrak{p}, H\big) \arrow[d, "i^*_{b,a}"] \\ H'\otimes\mathrm{Hom}_{K_\infty}(V(b,a),\mathbf{S}) \arrow[r, "\theta_*"] & \mathrm{Hom}_{K_\infty}(V(b,a), H) = H^{b\times q, a\times q}(\mathrm{Sh}(G),\mathbb{C}). \end{tikzcd} \tag{9.3}$$

We now examine the diagram. Since $V(b,a)$ is a summand, the map on the left is onto. Also, by (3.1), the map on the right is an isomorphism.

We now define $U_{\varphi_{bq,aq}}$ to be the 1-dimensional subspace of $\mathrm{Hom}_{K_\infty}(\bigwedge^{nq}\mathfrak{p},\mathbf{S})$ generated by $\varphi_\infty=\varphi_{bq,aq}$.

The theorem will then follow from the equation

$$i^*_{b,a}\circ\theta_*(H'\otimes U_{\varphi_{bq,aq}}) = \mathrm{Image}(i^*_{b,a}\circ\theta_*). \tag{9.4}$$

Since the above diagram is commutative, equation (9.4) holds if and only if we have

$$\theta_*\circ i^*_{b,a}(H'\otimes U_{\varphi_{bq,aq}}) = \mathrm{Image}(i^*_{b,a}\circ\theta_*). \tag{9.5}$$

Put $\overline{U}_{\varphi_{bq,aq}}=i^*_{b,a}(U_{\varphi_{bq,aq}})$. As the left vertical arrow $i^*_{b,a}$ is onto, equation (9.5) holds if and only if

$$\theta_*(H'\otimes\overline{U}_{\varphi_{bq,aq}}) = \theta_*(H'\otimes\mathrm{Hom}_{K_\infty}(V(b,a),\mathbf{S})). \tag{9.6}$$

We now prove equation (9.6). To this end let $\xi\in\theta_*(H'\otimes\mathrm{Hom}_{K_\infty}(V(b,a),\mathbf{S}))$. Hence, by definition, there exists $\phi=\varphi_\infty\otimes\varphi\in\mathrm{Hom}_{K_\infty}(V(b,a),\mathbf{S})$ and $f\in H'$ such that

$$\theta_*(f\otimes\phi)=\xi. \tag{9.7}$$

We claim that in equation (9.7) (up to replacing the component f_{v_0}) we may replace the factor φ_∞ of ϕ by $\varphi_{bq,aq}$ without changing the right-hand side ξ of equation (9.7). Indeed by Theorem 5.24 there exists $Z\in\mathcal{U}(\mathfrak{u}(b,a)_{\mathbb{C}})$ such that

$$\varphi_\infty=Z\varphi_{bq,aq}. \tag{9.8}$$

Now by [32, Lemma 6.9] (with slightly changed notation) we have

$$\theta_*(f\otimes Z\phi)=\theta_*(Z^*f\otimes\phi). \tag{9.9}$$

Here $Z\mapsto Z^*$ is the involution of $\mathcal{U}(\mathfrak{u}(b,a)_{\mathbb{C}})$ induced by the map $g\mapsto g^{-1}$ of $U(a,b)$.

Hence setting $f'=Z^*f$ we obtain, for all $f\in H'$,

$$\begin{aligned}\xi&=\theta_*(f\otimes(\varphi_\infty\otimes\varphi))=\theta_*(f\otimes(Z\varphi_{bq,aq}\otimes\varphi))\\&=\theta_*(Z^*f\otimes(\varphi_{bq,aq}\otimes\varphi))=\theta_*(f'\otimes(\varphi_{bq,aq}\otimes\varphi)).\end{aligned} \tag{9.10}$$

We conclude that the image of the space $H'\otimes U_{\varphi_{bq,aq}}$ under θ_* coincides with the image of $H'\otimes\mathrm{Hom}_{K_\infty}(V_{bq,aq},\mathbf{S})$ as required. □

We now prove the last part of Theorem 8.2.

PROPOSITION 9.2. *Let n be an integer such that $3n<m$. We then have a direct sum decomposition*

$$\mathrm{SC}^{2nq}_{\mathrm{prim}}(\mathrm{Sh}(G,X))=\bigoplus_{\pi_f}H^{2nq}(\pi_f,\mathbb{C})\otimes\pi_f,$$

where the sum runs over all $\pi_f\in\mathrm{Coh}_f^{n,n}$ such that $\varepsilon(\pi_v)=1$ for all finite places v.

Proof. As $3n<m$, it follows from Theorem 7.2 that, if $\pi_f\in\mathrm{Coh}_f^{n,n}$, the automorphic representation $\pi=A(q^n,q^n)\otimes\pi_f$ is in the image of the ψ-theta correspondence from a smaller group $U(W)$ of signature (n,n) at infinity. By local theta dichotomy the global Hermitian space is completely determined by π; it is split if and only if $\varepsilon(\pi_v)=1$ for every finite place v. By Theorem 9.1, we may, in the statement of Proposition 9.2, therefore replace the right-hand side by the subspace of $H^{n\times q,n\times q}(\mathrm{Sh}(G),\mathbb{C})$ generated by the projections of the cohomology classes $[\theta_n(g',\varphi)]$, where $\varphi\in\mathcal{S}(V(\mathbb{A}_{\mathbb{Q}}^f)^n)$ and $g'\in G'(\mathbb{A}_{\mathbb{Q}})$. Let us denote this subspace by $\mathcal{H}$.

Denote by $\langle\cdot,\cdot\rangle$ the Petersson scalar product restricted to the primitive part of $H^{2nq}(\mathrm{Sh}(G),\mathbb{C})$. Letting $\mathrm{SC}^{2nq}_{\mathrm{prim}}(\mathrm{Sh}(G),\mathbb{C})^{\perp}$ and $\mathcal{H}^{\perp}$ denote the respective annihilators in $H^{2nq}_{\mathrm{prim}}(\mathrm{Sh}(G),\mathbb{C})$ it suffices to prove

$$\mathrm{SC}^{2nq}_{\mathrm{prim}}(\mathrm{Sh}(G),\mathbb{C})^{\perp}=\mathcal{H}^{\perp}. \tag{9.11}$$

Now consider $\eta\in H^{2nq}_{\mathrm{prim}}(\mathrm{Sh}(G),\mathbb{C})$. Assume η is K-invariant for some level K. It follows from proposition 8.3 that for $g'\in G'(\mathbb{R})\subset G'(\mathbb{A}_{\mathbb{Q}})$ the Fourier expansion of the modular form

$$\theta_{\varphi}(\eta):=\langle[\theta_n(g',\varphi)],\eta\rangle\left(=\int_{X_K}\theta_n(g',\varphi)\wedge{*\eta}\right)$$

is given by

$$\theta_{\varphi}(\eta)=\sum_{\beta\geqslant 0}\langle[\beta,\varphi],\eta\rangle W_{\beta}(g')=\sum_{\beta\gg 0}\langle[\beta,\varphi],\eta\rangle W_{\beta}(g'),$$

since η is primitive. In particular, the form η is orthogonal to $\mathrm{SC}^{nq}_{\mathrm{prim}}(\mathrm{Sh}(G),\mathbb{C})$ if and only if all the Fourier coefficients of all the modular forms $\theta_{\varphi}(\eta)$ vanish and therefore $\theta_{\varphi}(\eta)=0$.

On the other hand, since G is anisotropic, the theta lift

$$\Theta^{W}_{\psi,\chi,V}(A(n\times q,n\times q)\otimes\pi_f)$$

is well defined for each $\pi_f\in\mathrm{Coh}^{n,n}_f$, and we get an automorphic representation of the group $U(W)$. The latter is isotropic (recall that W is split) but it follows from a general principle—see e.g. Lemma 11.3 below—that $\Theta^{W}_{\psi,\chi,V}(A(n\times q,n\times q)\otimes\pi_f)$ is either $\{0\}$ or a cuspidal automorphic representation of $U(W)$. Indeed by Rallis' theta tower property if $\Theta^{W}_{\psi,\chi,V}(A(n\times q,n\times q)\otimes\pi_f)$ is non-zero and non-cuspidal, there exists a proper subspace $W'\subset W$ such that $\Theta^{W'}_{\psi,\chi,V}(A(n\times q,n\times q)\otimes\pi_f)$ is non-zero and cuspidal. Localizing at the place at infinity we get a contradiction: the cohomological representation $A(b\times q,a\times q)$ is not in the image of the local theta correspondence from a group $U(W,\mathbb{C}/\mathbb{R})$ with $\dim W<a+b$; see [61].

In particular, each modular form $\theta_{\varphi}(\eta)$ is either zero or a cuspidal automorphic form. Now η belongs to $\mathcal{H}^{\perp}$ if and only if, for any $f\in H_{\sigma'}$ $(\sigma'\in\mathcal{A}^c(U(W)))$, we have

$$\int_{X_K}\theta(f,\varphi)[\varphi_{nq,nq}]\wedge{*\eta}=0.$$

Since

$$\int_{X_K}\theta(f,\varphi)[\varphi_{nq,nq}]\wedge{*\eta}=\int_{[U(W)]}\theta_{\varphi}(\eta)f(g')\,dg',$$

we get that η belongs to $\mathcal{H}^{\perp}$ if and only if $\theta_{\varphi}(\eta)=0$. This concludes the proof. $\square$

We can now prove our main theorem.

THEOREM 9.3. *Assume that* $b=a+c$ *with* $c>0$. *Then the natural cup product map*

$$\mathrm{SC}^{2aq}(\mathrm{Sh}(G),\mathbb{C})\times\mathrm{SH}^{cq,0}(\mathrm{Sh}(G),\mathbb{C})\longrightarrow H^{b\times q,a\times q}(\mathrm{Sh}(G),\mathbb{C})$$

is surjective. If $c=0$, *this is no longer true but the natural cup product map*

$$\mathrm{SC}^{2(b-1)q}(\mathrm{Sh}(G),\mathbb{C})\times\mathrm{SH}^{q,q}(\mathrm{Sh}(G),\mathbb{C})\longrightarrow H^{b\times q,b\times q}(\mathrm{Sh}(G),\mathbb{C})$$

is surjective.

Remark. There is a similar statement when $a=b+c$ with $c>0$ except that the natural cup product map becomes

$$\mathrm{SC}^{2bq}(\mathrm{Sh}(G),\mathbb{C})\times H^{0,cq}(\mathrm{Sh}(G),\mathbb{C})\longrightarrow H^{b\times q,a\times q}(\mathrm{Sh}(G),\mathbb{C}).$$

Theorem 1.1 follows.

Proof. Write $b=a+c$ with $c\geqslant 0$. It follows from Theorem 9.1 that $H^{b\times q,a\times q}(\mathrm{Sh}(G),\mathbb{C})$ is generated by the automorphic functions $\theta^{f}_{\psi,\chi,\varphi_{bq,aq}\otimes\phi_f}$ as ϕ_f, χ and f vary. These data depend on a global Hermitian space W of signature (a,b) at infinity. Recall from §7.4 that if v is a finite place of F the local Hermitian space $W=W_v$ decomposes as a sum

$$W=\begin{cases} W_{a,a}\oplus W_c, & \text{if } c>0,\\ W_{a-1,a-1}\oplus W_2, & \text{otherwise,}\end{cases} \tag{9.12}$$

where $W_{r,r}$ denotes the Hermitian space of dimension $2r$ with maximal isotropic subspaces of dimension r, and W_k denotes the unique Hermitian space of dimension k with the same local sign as W.[9] Since the W_v are localizations of a global space, there exists a corresponding global W_c, or W_2 in case $c=0$, and the corresponding decomposition (9.12) holds globally. If $c>0$, resp. $c=0$, we write $G_1'=\mathrm{Res}_{F/\mathbb{Q}}\,U(W_{b,b})$, resp. $G_1'=\mathrm{Res}_{F/\mathbb{Q}}\,U(W_{b-1,b-1})$, and $G_2'=\mathrm{Res}_{F/\mathbb{Q}}\,U(W_c)$, resp. $G_2'=\mathrm{Res}_{F/\mathbb{Q}}\,U(W_2)$. We have a natural homomorphism

$$G_1'(\mathbb{A}_\mathbb{Q})\times G_2'(\mathbb{A}_\mathbb{Q})\xrightarrow{\ \iota\ } G'(\mathbb{A}_\mathbb{Q}).$$

We may now decompose the space $\mathbb{W}=W\otimes_E V$ accordingly:

$$\mathbb{W}=\begin{cases} (W_{a,a}\otimes_E V)\oplus\mathbb{W}_c, & \text{if } c>0,\\ (W_{a-1,a-1}\otimes_E V)\oplus\mathbb{W}_2, & \text{otherwise.}\end{cases} \tag{9.13}$$

---

(9) At infinite places one should require that W_c is positive definite if $c>0$, and that W_2 is of signature $(1,1)$ if $c=0$.

Here $\mathbb{W}_k = W_k \otimes_E V$. We finally choose a complete polarization

$$\mathbb{W}_k = \mathbb{X}_k + \mathbb{Y}_k,$$

and consider the associated polarization $\mathbb{X}+\mathbb{Y}$ of $\mathbb{W}$, where

$$\mathbb{X} \cong \begin{cases} V^a \oplus \mathbb{X}_c, & \text{if } c>0, \\ V^{a-1} \oplus \mathbb{X}_2, & \text{otherwise.} \end{cases} \tag{9.14}$$

The polynomial Fock space $\mathbf{S}(\mathbb{X}(\mathbb{A}))$ then contains

$$\mathbf{S}(V(\mathbb{A})^a) \otimes \mathbf{S}(\mathbb{X}_c(\mathbb{A})), \quad \text{resp. } \mathbf{S}(V(\mathbb{A})^{a-1}) \otimes \mathbf{S}(\mathbb{X}_2(\mathbb{A})),$$

as a dense subspace. The local product formula at infinity (Propositions 5.4 and 5.19) decomposes $\varphi_{bq,aq}$ as a cup product $\varphi_{aq,aq} \wedge \varphi_{cq,0}$, resp. $\varphi_{(a-1)q,(a-1)q} \wedge \varphi_{q,q}$. As in the proof of Theorem 8.2 (1) (see in particular (8.10)), we conclude that if $\phi_f = \phi_{1f} \otimes \phi_{2f}$ belongs to $\mathbf{S}(V(\mathbb{A})^a) \otimes \mathbf{S}(\mathbb{X}_c(\mathbb{A}))$, resp. $\mathbf{S}(V(\mathbb{A})^{a-1}) \otimes \mathbf{S}(\mathbb{X}_2(\mathbb{A}))$, we have the following cup product of differential forms:

$$\theta_{\psi,\chi,\varphi_{bq,aq}\otimes\phi_f}(\iota(g_1',g_2'),\cdot) = \begin{cases} \theta_a(g_1',\phi_{1f}) \wedge \theta_{\psi,\chi,\varphi_{cq,0}\otimes\phi_{2f}}(g_2',\cdot), & \text{if } c>0, \\ \theta_{a-1}(g_1',\phi_{1f}) \wedge \theta_{\psi,\chi,\varphi_{q,q}\otimes\phi_{2f}}(g_2',\cdot), & \text{otherwise.} \end{cases} \tag{9.15}$$

Recall that $H^{b\times q, a\times q}(\mathrm{Sh}(G),\mathbb{C})$ is generated by the projections of the cohomology classes of the differential forms $\theta_{\psi,\chi,\phi}(g',\cdot)$. Since $\theta_{\psi,\chi,\omega(g_1')\phi}(g',\cdot) = \theta_{\psi,\chi,\phi}(g_1'g',\cdot)$ and the space $\mathbf{S}(V(\mathbb{A})^a) \otimes \mathbf{S}(\mathbb{X}_c(\mathbb{A}))$, resp. $\mathbf{S}(V(\mathbb{A})^{a-1}) \otimes \mathbf{S}(\mathbb{X}_2(\mathbb{A}))$, is a dense subspace of $\mathbf{S}(\mathbb{X}(\mathbb{A}))$, we get that $H^{b\times q, a\times q}(\mathrm{Sh}(G),\mathbb{C})$ is generated by the projections of the cohomology classes of the differential forms (9.15). Theorem 9.3 now follows from Proposition 9.2. $\square$

Part 3. Automorphic forms

10. On the global theta correspondence

10.1. In this part of the paper we prove Proposition 13.4 that was used in the proof of Theorem 7.2. We keep the notation as in §3. The main technical point is to prove that if $\pi \in \mathcal{A}^c(U(V))$ is such that its local component at infinity is sufficiently non-tempered (this has to be made precise), then the global representation π is in the image of the cuspidal ψ-theta correspondence from a smaller group.

As usual we encode local components of π into an L-function. In fact we only consider its *partial L-function* $L^S(s,\pi) = \prod_{v\notin S} L(s,\pi_v)$ where S is a sufficiently big finite set of places such that π_v is unramified for each $v \notin S$. For such a v we define the local factor $L(s, \pi_v)$ by considering the Langlands parameter of π_v.

We may generalize these definitions to form the partial (Rankin–Selberg) L-functions $L^S(s, \pi \times \eta)$ for any automorphic character η.

The goal of this and the next section is to prove the following theorem, which is a first important step toward the proof that a sufficiently non-tempered automorphic representation is in the image of the cuspidal ψ-theta correspondence from a smaller group. A large part of it is not new. The proof follows the pioneering work of Kudla and Rallis for the usual orthogonal-symplectic dual pair (see also [55] and [24, Theorem 1.1 (1)] and generalized in [18]). Most of the steps that appear to be different in the unitary case can now be found in the literature; in that respect one key ingredient is due to Ichino [36].

THEOREM 10.1. *Let* $\pi \in \mathcal{A}^c(U(V))$ *and let* η *be a character of* $\mathbb{A}_E^\times / E^\times$. *We assume that there exists some integer* $a > 1$ *such that the partial* L*-function* $L^S(s, \pi \times \eta)$ *is holomorphic in the half-plane* $\mathrm{Re}(s) > \frac{1}{2}(a-1)$ *and has a pole at* $s = \frac{1}{2}(a-1)$.

Then, we have $\eta|_{\mathbb{A}^\times} = \varepsilon_{E/F}^{m-a}$ *and there exists some* n*-dimensional skew-Hermitian space over* E, *with* $n = m - a$, *such that* π *is in the image of the cuspidal* ψ*-theta correspondence from the group* $U(W)$.

Remark. It will follow from the proof that letting $\chi_2 = \eta$ and fixing some arbitrary choice of character χ_1 such that $\chi_1|_{\mathbb{A}^\times} = \varepsilon_{E/F}^m$, there exists a representation $\pi' \in \mathcal{A}^c(U(W))$ such that

$$\pi = \Theta_{\psi, \chi, W}^V(\pi').$$

Note that the remark on p. 63 implies that changing our choice of χ_1 amounts to twisting π' by a character of $\mathbb{A}_E^1$. Similarly, replacing ψ by any other additive character $\psi_t(x) = \psi(tx)$ for some $t \in F$, amounts to rescaling the Hermitian space W.

10.2. Strategy of proof of Theorem 10.1

The proof of Theorem 10.1 is based on Rallis' inner product formula. Let π be as in Theorem 10.1. We want to construct an n-dimensional skew-Hermitian space W over E and some pair of characters $\chi = (\chi_1, \chi_2)$ of $\mathbb{A}_E^\times / E^\times$ (as in §7.1) such that

$$\Theta_{\psi, \chi, V}^W(\pi) \neq 0. \tag{10.1}$$

Theorem 10.1 will then follow by duality. Such a W will be of the form

$$W = W_r = W_0 \oplus \mathbb{H}^r,$$

where $\mathbb{H}$ denotes the hyperbolic plane, i.e. the split skew-Hermitian space of dimension 2 and W_0 is anisotropic. According to this decomposition we write $n=n_0+2r$. One says that the family of spaces $\{W_r : r\geqslant 0\}$ forms a *Witt tower* of non-degenerate skew-Hermitian spaces. Replacing $W=W_r$ by W_{r-1}, we may assume that $\Theta^{W_{r-1}}_{\psi,\chi,V}(\pi)=0$, so that $\theta^f_{\psi,\chi,\phi}$ is a cusp form (possibly zero). To prove (10.1), we shall compute the square of its Petersson norm

$$\|\theta^f_{\psi,\chi,\phi}\|^2=\int_{[U_n]}\theta^f_{\psi,\chi,\phi}(g')\overline{\theta^f_{\psi,\chi,\phi}(g')}\,dg' \tag{10.2}$$

using Rallis' inner product formula. In our range $n\leqslant m$ the latter takes the rough form

$$\|\theta^f_{\psi,\chi,\phi}\|^2=c\,\mathrm{Res}_{s=(m-n)/2}\,L^S\big(s+\tfrac{1}{2},\pi\times\eta\big),$$

where c is a non-zero constant involving local coefficients of the oscillator representation. We will now provide the details of the proof.

We first recall the doubling method introduced by Piatetskii–Shapiro and Rallis [20] in order to relate L-functions, as the function $L^S(s,\pi\times\eta)$ in Theorem 10.1, to the Weil representation.

10.3. Doubling the group

Equip $V\oplus V$ with the split form $(\cdot\,,\cdot)\oplus-(\cdot\,,\cdot)$. Let $U(V\oplus V)$ be the corresponding isometry group. We denote the subspace $V\oplus\{0\}$ by V and the subspace $\{0\}\oplus V$ by $-V$. There is a canonical embedding

$$\mathrm{J}\colon U(V)\times U(-V)\longrightarrow U(V\oplus V).$$

Let P be the Siegel parabolic of $U(V\oplus V)$ preserving the maximal isotropic subspace

$$V^d=\{(v,v):v\in V\}\subset V\oplus V.$$

Let $P=MU$ be the Levi decomposition of P. Here U is the unipotent radical of P, and M is the subgroup which preserves both V^d and

$$V_d=\{(v,-v):v\in V\}\subset V\oplus V.$$

Thus $M(F)$ is isomorphic to $\mathrm{GL}_m(E)$ via restriction to $V_d\cong E^m$, and we write $\mu(a)\in M$ for the element corresponding to $a\in\mathrm{GL}_m(E)$. Note that

$$P\cap(U(V)\times U(-V))=U(V)^d,$$

where $U(V)^d$ is the image of $U(V)$ under the diagonal embedding $U(V)\mapsto U(V)\times U(-V)$.

10.4. Siegel Eisenstein series

Define the homomorphism $\det\colon M(F)\to E^\times$ by $\mu(a)\mapsto\det a$ and denote by $|\cdot|_E$ the norm map $E^\times\to F^\times$. Then det and $|\det|_E$ uniquely extend to homomorphisms $\det\colon M(\mathbb{A})\to\mathbb{A}_E^\times$ and $|\det|_E\colon M(\mathbb{A})\to\mathbb{A}^\times$. Given a character η of $E^\times\backslash\mathbb{A}_E^\times$, we define the quasi-character $\eta|\cdot|^s$, $s\in\mathbb{C}$, of $M(\mathbb{A})$ by

$$\mu\longmapsto\eta(\det(\mu))|\det(\mu)|_E^s.$$

Then define the induced representation

$$I(s,\eta)=\mathrm{Ind}_{P(\mathbb{A})}^{U_{m,m}(\mathbb{A})}\eta|\cdot|^s, \tag{10.3}$$

where we denote by $U_{m,m}(\mathbb{A})$ the adelic points of $U(V\oplus V)$ and the induction is normalized. (Note that the modular character $\delta\colon P\to F^\times$ is equal to $\mu\mapsto|\det(\mu)|_E^m$.) Starting with a section $\Phi(\cdot,s)\in I(s,\eta)$ we may form the Eisenstein series

$$E(h,s,\Phi)=\sum_{\gamma\in P(F)\backslash U(V\oplus V)}\Phi(\gamma h,s). \tag{10.4}$$

The group $U(V\oplus V)$ being quasi-split, we may fix a standard maximal compact subgroup $K=U(V\oplus V)(\mathcal{O}_F)$ of $U_{m,m}(\mathbb{A})$. We then say that $\Phi(\cdot,s)\in I(s,\eta)$ is *standard* if it is holomorphic in s and its restriction to K is independent of s. It follows from [77, p. 56] that, if $\Phi(\cdot,s)$ is holomorphic in s, the Siegel Eisenstein series (10.4) converges absolutely for $\mathrm{Re}(s)>\frac{1}{2}m$ and has a meromorphic continuation to the whole s-plane. If $\Phi(\cdot,s)$ is standard, Tan [67] proves that $E(h,s,\Phi)$ is entire in the half-plane $\mathrm{Re}(s)\geqslant 0$ if $\eta\neq\bar\eta^{-1}$. If $\eta=\bar\eta^{-1}$, then $\eta|_{\mathbb{A}^*}=\varepsilon_{E/F}^j$, with $j=0$ or $j=1$, and the poles of $E(h,s,\Phi)$ in the half-plane $\mathrm{Re}(s)\geqslant 0$ are at most simple and occur at the points

$$s\in\left\{\tfrac{1}{2}(m-j),\tfrac{1}{2}(m-j)-1,\dots\right\}.$$

If s_0 is a pole of $E(h,s,\Phi)$ we may consider the Laurent expansion

$$E(h,s,\Phi)=\frac{A(h,\Phi)}{s-s_0}+A_0(h,\Phi)+O(s-s_0),$$

where $A(h,\Phi)=\mathrm{Res}_{s=s_0}E(h,s,\Phi)$. The function $h\mapsto A(h,\Phi)$ defines an automorphic form on $U(V\oplus V)$.

10.5. The global doubling integral

Consider now an irreducible automorphic cuspidal representation $\pi\in\mathcal{A}^c(U(V))$. Denote by $\pi^\vee$ the contragredient representation of π and write η for the character $\eta\circ\det$ of

$U_m(\mathbb{A})$, identified with the adelic points of the subgroup $U(V)^d$ of P. Let H_π (resp. $H_{\pi^\vee}$) be the space of π (resp. $\pi^\vee$). For $f\in H_\pi$ and $f^\vee\in H_{\pi^\vee}$, let

$$\phi(g)=\langle \pi(g)f, f^\vee\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the standard pairing. Given a section $\Phi(\cdot,s)\in I(s,\eta)$, define

$$Z(s,f,f^\vee,\Phi)=\int_{U_m(\mathbb{A})}\phi(g)\Phi(\jmath(g,1),s)\,dg. \tag{10.5}$$

This integral converges absolutely for $\mathrm{Re}(s)>m$.

The basic identity of [20, p. 3] relates (10.5) to some Rankin–Selberg integral

$$Z(s,f,f^\vee,\Phi)=\int_{[U_m\times U_m]}E(\jmath(g_1,g_2),s,\Phi)f(g_1)f^\vee(g_2)\eta^{-1}(g_2)\,dg_1\,dg_2. \tag{10.6}$$

The point here is that:

- If $\Phi(\cdot,s)=\otimes_v\Phi_v(\cdot,s)$, $f=\otimes f_v$ and $f^\vee=\otimes f_v^\vee$ are factorizable, then the zeta integral (10.5) equals the product

$$\prod_v Z(s,f_v,f_v^\vee,\Phi_v),$$

where, letting $\phi_v(g)=\langle\pi_v(g)f_v,f_v^\vee\rangle$,

$$Z(s,f_v,f_v^\vee,\eta_v,\Phi_v)=\int_{U(V\otimes_F F_v)}\phi_v(g)\Phi_v(\jmath(g,1),s)\,dg \tag{10.7}$$

is the local zeta integral.

- The Eisenstein series (10.4) admits a meromorphic continuation to the entire complex plane.

Thus all this leads to the meromorphic continuation of some Euler products. We now study the local zeta integrals (10.7).

10.6. Local zeta integrals

Let S be the finite set of ramified places (those v which are either infinite or finite and either η_v is ramified or π_v is not spherical). Assume that Φ is standard at each finite place $v\notin S$. Let v be a finite place of F outside S and let q be the residue characteristic of F. In this paragraph we deal with the local field F_v but we drop the subscript v everywhere. Thus F is a non-Archimedean local field of characteristic 0, E is a quadratic extension of F, V is an m-dimensional vector space over E endowed with a non-degenerate Hermitian form $(\cdot,\cdot)$ and π is an irreducible unitary representation of the group $G=U(V)$. It may also be that $E=F\oplus F$ is a split extension, then $G\cong \mathrm{GL}(m,F)$ and the computations below still work (see [20]) and are equivalent to the computations made by Godement and Jacquet [25].

Denote by H the double of G so that $\jmath\colon G\times G\hookrightarrow H$. For $s\in\mathbb{C}$ and for any character η of $E^\times$, let $I(s,\eta)$ be the so-called *degenerate principal series* consisting of smooth functions on H which satisfy

$$\Phi(u\mu h,s)=\eta(\det(\mu))|\det(\mu)|_E^{s+m/2}\Phi(h,s), \tag{10.8}$$

where $u\in U$ and $\mu\in M$.

The local zeta integral is defined as follows. For $f\in\pi$ and $f^\vee\in\pi^\vee$ let

$$\phi(g)=\langle \pi(g)f,f^\vee\rangle$$

be the corresponding matrix coefficient. For a section $\Phi(\cdot,s)\in I(s,\eta)$, define

$$Z(s,f,f^\vee,\Phi)=\int_G \phi(g)\Phi(\jmath(g,1),s)\,dg. \tag{10.9}$$

The integral converges for large $\mathrm{Re}(s)$ and defines a non-zero element

$$Z(s)\in \mathrm{Hom}_{G\times G}(I(s,\eta),\pi^\vee\otimes(\eta\pi)). \tag{10.10}$$

We may identify the group H as the subgroup of $\mathrm{GL}(2m,E)$ which preserves the Hermitian form with matrix

$$\begin{pmatrix} 0 & 1_m \\ 1_m & 0\end{pmatrix}.$$

Let then K be the maximal compact subgroup of H obtained by intersecting this group with $\mathrm{GL}(2m,\mathcal{O}_E)$, where $\mathcal{O}_E$ is the ring of integers of E. We let $\Phi^0(\cdot,s)$ be the standard section whose restriction to K is 1. Recall (see [51]) that if G is the split group, π is an unramified principal series, η is unramified, and f and $f^\vee$ are non-trivial and K-fixed, then, up to a non-zero constant, $\phi(g)$ is the zonal spherical constant associated with π and the local zeta integral

$$Z(s,f,f^\vee,\Phi^0)=\frac{L\big(s+\frac{1}{2},\pi\times\eta\big)}{b(s,\eta)}. \tag{10.11}$$

Here $L(s,\pi\times\eta)$ is the usual Langlands Euler factor associated with π, η and the standard representation of the L-group ${}^LG$, and

$$b(s,\eta)=\prod_{j=0}^{m-1} L(2s+m-j,\eta^0\varepsilon_{E/F}^j),$$

where η^0 is the restriction of η to $F^\times$. Note that in $b(s,\eta)$ each L-factor is a local (Tate) factor.

10.7. From now on, we fix f_v, $f_v^\vee$ and $\Phi_v = \Phi_v^0$ as above for each $v \notin S$. We conclude that

$$Z(s, f, f^\vee, \Phi) = \frac{1}{b^S(s, \eta)} L^S\left(s + \frac{1}{2}, \pi \times \eta\right) \prod_{v \in S} Z(s, f_v, f_v^\vee, \Phi_v), \tag{10.12}$$

where $b^S(s, \eta)$ is the product of the local factors $b(s, \eta_v)$ over the set of finite places $v \notin S$.

We remark that the proof of [47, Proposition 7.2.1]—which generalizes immediately to the unitary case—implies that for any point $s \in \mathbb{C}$ there exist choices of f, $f^\vee$ and Φ such that the local zeta integral $Z_v(s, f, f^\vee, \Phi)$ is non-zero.

Now recall that we have

$$b^S(s, \eta) Z(s, f, f^\vee, \Phi) = \int_{[U_m \times U_m]} E^*(\jmath(g_1, g_2), s, \Phi) f(g_1) f^\vee(g_2) \eta^{-1}(g_2)\, dg_1\, dg_2, \tag{10.13}$$

where $E^*(h, s, \Phi) = b^S(s, \eta) E(h, s, \Phi)$ is the *normalized Eisenstein series*.

We conclude from (10.12) and (10.13) that any pole of $L^S\big(s + \frac{1}{2}, \pi \times \eta\big)$ must be a pole of $b^S(s, \eta) Z(s, f, f^\vee, \Phi)$ for a suitable choice of Φ, and hence also a pole of the normalized Eisenstein series $E^*(h, s, \Phi)$. Since $b^S(s, \eta)$ does not vanish in the half-plane $\mathrm{Re}(s) \geqslant 0$, we conclude the following result.

Proposition 10.2. *The following statements hold*:

(1) *If* $\eta \neq \bar{\eta}^{-1}$, *the partial* L-*function* $L^S\big(s + \frac{1}{2}, \pi \times \eta\big)$ *is entire in the half-plane* $\mathrm{Re}(s) \geqslant 0$.

(2) *If* $\eta = \bar{\eta}^{-1}$ *then* $\eta = \varepsilon_{E/F}^j$ *with* $j = 0$ *or* $j = 1$. *Then the partial* L-*function*

$$L^S\big(s + \tfrac{1}{2}, \pi \times \eta\big)$$

has at most simple poles in the half-plane $\mathrm{Re}(s) \geqslant 0$ *and these can only occur for*

$$s \in \big\{ \tfrac{1}{2}(m - j), \tfrac{1}{2}(m - j) - 1, \dots \big\}.$$

We now explain the relation of the doubling method with the Weil representation.

10.8. Doubling the Weil representation

Consider an n-dimensional vector space W over E equipped with a τ-skew-Hermitian $\langle \cdot, \cdot \rangle$, as in §7.1. The space $\mathbb{W} + \mathbb{W} = W \otimes_E (V \oplus V) = \mathbb{W} \oplus \mathbb{W}$ is then endowed with the symplectic form $[\cdot, \cdot] \oplus -[\cdot, \cdot]$. We denote by Ω the Weil representation of $\mathrm{Mp}_{4nm}(\mathbb{A})$—the group of adelic points of $\mathrm{Mp}(\mathbb{W} + \mathbb{W})$—corresponding to the same character ψ of $\mathbb{A}/F$.

The obvious embedding

$$i\colon \mathrm{Sp}_{2nm}(\mathbb{A})\times \mathrm{Sp}_{2nm}(\mathbb{A}) \longrightarrow \mathrm{Sp}_{4nm}(\mathbb{A})$$

leads to a homomorphism

$$\tilde{i}\colon \mathrm{Mp}_{2nm}(\mathbb{A})\times \mathrm{Mp}_{2nm}(\mathbb{A}) \longrightarrow \mathrm{Mp}_{4nm}(\mathbb{A})$$

such that

$$\Omega\circ\tilde{i} = \omega\otimes\omega^\vee,$$

where $\omega^\vee$ is the contragredient representation of ω, and is the same as $\omega_{\bar\psi}$ (the Weil representation associated with $\bar\psi$). Clearly $\mathbb{W}+\mathbb{W}=(\mathbb{X}\oplus\mathbb{X})+(\mathbb{Y}\oplus\mathbb{Y})$ is a complete polarization of $\mathbb{W}+\mathbb{W}$ and Ω can be realized on $\mathcal{S}((\mathbb{X}\oplus\mathbb{X})(\mathbb{A}))$.

The choice of χ also defines a homomorphism

$$U_n(\mathbb{A})\times U_{2m}(\mathbb{A}) \longrightarrow \mathrm{Mp}_{4nm}(\mathbb{A}) \tag{10.14}$$

lifting the natural map

$$U_n(\mathbb{A})\times U_{2m}(\mathbb{A}) \longrightarrow \mathrm{Sp}_{4nm}(\mathbb{A}),$$

and so, we obtain a representation Ω_χ of $U_n(\mathbb{A})\times U_{2m}(\mathbb{A})$ on $\mathcal{S}((\mathbb{X}\oplus\mathbb{X})(\mathbb{A}))$. We will rather work with another model of the Weil representation which we now describe.

10.9. Set

$$\mathbb{W}^d = W\otimes_E V^d \quad \text{and} \quad \mathbb{W}_d = W\otimes_E V_d.$$

Then $\mathbb{W}+\mathbb{W}=\mathbb{W}_d+\mathbb{W}^d$ is another complete polarization of $\mathbb{W}+\mathbb{W}$. Thus Ω can also be realized on $L^2(\mathbb{W}_d(\mathbb{A}))$. We denote by Ω^- this realization. There is an isometry

$$\delta: L^2((\mathbb{X}+\mathbb{X})(\mathbb{A})) \longrightarrow L^2(\mathbb{W}_d(\mathbb{A}))$$

intertwinning the action of $\mathrm{Mp}_{4nm}(\mathbb{A})$ on the two spaces. Explicitly, δ is given as follows. We identify $\mathbb{W}_d$ and $\mathbb{W}$ via the map

$$(w,-w)\longmapsto w,$$

and write $w\in\mathbb{W}$ as $w=(x,y)$, according to the decomposition $\mathbb{W}=\mathbb{X}+\mathbb{Y}$. Then, for $\Psi\in L^2((\mathbb{X}+\mathbb{X})(\mathbb{A}))$, we have

$$\delta(\Psi)(w) = \int_{\mathbb{X}(\mathbb{A})} \psi(2[u,y])\Psi(u+x,u-x)\,du. \tag{10.15}$$

10.10. Ichino sections

It follows from [42] that the action of $M(\mathbb{A})$ on $\varphi \in \mathcal{S}(\mathbb{W}_d(\mathbb{A}))$ is given by

$$(\Omega_\chi^-(\mu)\varphi)(\omega) = \chi_2(\det(\mu))|\det(\mu)|^{n/2}\varphi(\mu^{-1}\omega). \tag{10.16}$$

In particular, setting

$$\Phi(h) = (\Omega_\chi^-(h)\varphi)(0), \tag{10.17}$$

we have

$$\Phi \in I(s_0, \chi_2), \tag{10.18}$$

with $s_0 = \frac{1}{2}(n-m)$. In general a holomorphic section $\Phi(\cdot, s)$ is said to be associated with φ if it is holomorphic in s and $\Phi(h, s_0) = (\Omega_\chi^-(h)\varphi)(0)$. We call *Ichino sections* the sections which are associated with some $\varphi \in \mathcal{S}(\mathbb{W}_d(\mathbb{A}))$.

11. Rallis' inner product formula and the proof of Theorem 10.1

We first want to construct an n-dimensional skew-Hermitian space W over E and a pair of characters $\chi = (\chi_1, \chi_2)$ of $\mathbb{A}_E^\times / E^\times$ (as in §7.1) such that

$$\Theta_{\psi,\chi,V}^W(\pi) \neq 0. \tag{11.1}$$

To prove (11.1) we shall compute the Petersson product

$$\langle \theta_{\psi,\chi,\phi_1}^{f_1}, \theta_{\psi,\chi,\phi_2}^{f_2} \rangle = \int_{[U_n]} \theta_{\psi,\chi,\phi_1}^{f_1}(g') \overline{\theta_{\psi,\chi,\phi_2}^{f_2}(g')}\, dg', \tag{11.2}$$

where $f_1, f_2 \in \mathcal{H}_\pi$, using Rallis' inner product formula.

11.1. Rallis' inner product formula

We work with the dual pair $(U(V \oplus V), U(W))$ and the associated Weil representation Ω_χ. The corresponding theta distribution is

$$\Theta(\phi) = \sum_{\xi, \eta \in \mathbb{X}(F)} \phi(\xi, \eta),$$

and it follows from [30, §1] that, as a representation of $U_m(\mathbb{A}) \times U_m(\mathbb{A}) \subset U_{2m}(\mathbb{A})$,

$$\Omega_{\chi \circ \mathrm{J}} = \omega_\chi \otimes \left(\chi_2 \cdot \omega_\chi^\vee\right),$$

where $\jmath$ is the obvious embedding $U_m(\mathbb{A})\times U_m(\mathbb{A})\subset U_{2m}(\mathbb{A})$.

With notation as in (11.2), we have $\phi_1\otimes\bar{\phi}_2\in\mathcal{S}((\mathbb{X}\oplus\mathbb{X})(\mathbb{A}))$, and a simple formal calculation gives the right-hand side of (11.2) as

$$\int_{[U_m\times U_m]} f_1(g_1)\chi_2^{-1}(g_2)\overline{f_2(g_2)}\left(\int_{[U_n]}\Theta_{\psi,\chi,\phi_1\otimes\bar{\phi}_2}(\jmath(g_1,g_2),g')\,dg'\right)dg_1\,dg_2, \tag{11.3}$$

where $\Theta_{\psi,\chi,\phi_1\otimes\bar{\phi}_2}$ is defined by the obvious analogue of (7.3). Unfortunately the inner theta integral

$$\int_{[U_n]}\Theta_{\psi,\chi,\phi_1\otimes\bar{\phi}_2}(\jmath(g_1,g_2),g')\,dg'$$

diverges in general. Following Kudla and Rallis [47] we will regularize this integral. But here again we shall rather work with the model $(\Omega_\chi^-,\mathcal{S}(\mathbb{W}_d(\mathbb{A})))$.

It follows from (10.15) that we have

$$\delta(\phi_1\otimes\bar{\phi}_2)(0)=\langle\phi_1,\phi_2\rangle, \tag{11.4}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in the Hilbert space $L^2(\mathbb{X}(\mathbb{A}))$.

Now on the space $\mathcal{S}(\mathbb{W}_d(\mathbb{A}))$ the theta distribution takes the form

$$\Theta^-(\varphi)=\sum_{\xi\in\mathbb{W}_d(F)}\varphi(\xi).$$

Setting

$$\varphi=\delta(\phi_1\otimes\bar{\phi}_2), \tag{11.5}$$

we therefore see that (11.3) is equal to

$$\int_{[U_m\times U_m]} f_1(g_1)\chi_2^{-1}(g_2)\overline{f_2(g_2)}\left(\int_{[U_n]}\Theta^-_{\psi,\chi,\varphi}(\jmath(g_1,g_2),g')\,dg'\right)dg_1\,dg_2. \tag{11.6}$$

Here again the inner theta integral

$$\int_{[U_n]}\Theta^-_{\psi,\chi,\varphi}(\jmath(g_1,g_2),g')\,dg'$$

diverges in general but we now describe how Kudla and Rallis [47] and Ichino [36] introduce a regularization of it.

11.2. The regularized theta integral

Suppose that $m\geqslant n$. It then follows from [36, §2] that, for a given finite place v of F, there is an element α of the spherical Hecke algebras of $U_{2m}(F_v)$ such that, for every

$\varphi \in \mathcal{S}(\mathbb{W}_d(\mathbb{A}))$ and $h \in U_{2m}(\mathbb{A})$, the theta function $\Theta^-_{\psi,\chi,\Omega^-(\alpha)\varphi}(h,\cdot)$ is rapidly decreasing on $U(W)\backslash U_n(\mathbb{A})$.(10) It follows that the theta integral

$$\int_{[U_n]} \Theta^-_{\psi,\chi,\Omega^-(\alpha)\varphi}(h,g')\,dg'$$

is absolutely convergent. Now if φ is such that the original theta-integral

$$\int_{[U_n]} \Theta^-_{\psi,\chi,\varphi}(h,g')\,dg'$$

is absolutely convergent, Ichino [36, Lemmas 2.3 and 2.2] proves that there exists some non-zero constant c_α, independent of φ, such that

$$\int_{[U_n]} \Theta^-_{\psi,\chi,\Omega^-(\alpha)\varphi}(h,g')\,dg' = c_\alpha \int_{[U_n]} \Theta^-_{\psi,\chi,\varphi}(h,g')\,dg'.$$

The *regularized theta integral* is then defined to be the function

$$B(g,\varphi) = \frac{1}{c_\alpha} \int_{[U_n]} \Theta^-_{\psi,\chi,\Omega^-(\alpha)\varphi}(h,g')\,dg'.$$

It is well defined and independent of the choices of v and α; see [36, §2]. The function $h \mapsto B(h,\varphi)$ defines an automorphic form on $U(V\oplus V)$ and the linear map

$$\begin{aligned} \mathcal{S}(\mathbb{W}_d(\mathbb{A})) &\longrightarrow \mathcal{A}(U(V\oplus V)), \\ \varphi &\longmapsto B(\cdot,\varphi), \end{aligned}$$

is $U_{2m}(\mathbb{A})$-equivariant.

Remark. By the Howe duality [34], [56] the spherical Hecke algebras of $U_{2m}(F_v)$ and $U_n(F_v)$ generate the same algebra of operators on $\mathcal{S}(\mathbb{W}_d(F_v))$ through the Weil representation Ω^-. It follows from [36, §2] that one may associate with α an element α' of the spherical Hecke algebra of $U_n(F_v)$ such that $\Omega^-(\alpha) = \Omega^-(\alpha')$ and

$$\alpha' \cdot \mathbf{1} = c_\alpha \cdot \mathbf{1}, \tag{11.7}$$

where $\mathbf{1}$ is the trivial representation of $U_n(F_v)$.

PROPOSITION 11.1. *Let ψ, χ, ϕ and f be as above. Then we have*

$$\langle \theta^{f_1}_{\psi,\chi,\phi_1}, \theta^{f_2}_{\psi,\chi,\phi_2} \rangle = \int_{[U_m\times U_m]} f_1(g_1)\chi_2^{-1}(g_2)\overline{f_2(g_2)} B(\jmath(g_1,g_2),\varphi)\,dg_1\,dg_2,$$

where φ is given by (11.5).

(10) Note that α acts on φ through the Weil representation.

Proof. We have

$$
\begin{aligned}
&\int_{[U_m\times U_m]} f_1(g_1)\chi_2^{-1}(g_2)\overline{f_2(g_2)}B(\jmath(g_1,g_2),\varphi)\,dg_1\,dg_2\\
&=\frac{1}{c_\alpha}\int_{[U_m\times U_m]} f_1(g_1)\chi_2^{-1}(g_2)\overline{f_2(g_2)}\int_{[U_n]}\Theta^-_{\psi,\chi,\Omega^-(\alpha)\varphi}(\jmath(g_1,g_2),g')\,dg_1\,dg_2\,dg'\\
&=\frac{1}{c_\alpha}\int_{[U_n]}\left(\int_{[U_m\times U_m]} f_1(g_1)\chi_2^{-1}(g_2)\overline{f_2(g_2)}\Theta^-_{\psi,\chi,\Omega^-(\alpha)\varphi}(\jmath(g_1,g_2),g')\,dg_1\,dg_2\right)1(g')\,dg'\\
&=\frac{1}{c_\alpha}\int_{[U_n]}\left(\int_{[U_m\times U_m]} f_1(g_1)\chi_2^{-1}(g_2)\overline{f_2(g_2)}\Theta^-_{\psi,\chi,\varphi}(\jmath(g_1,g_2),g')\,dg_1\,dg_2\right)(\alpha'\cdot 1)(g')\,dg'\\
&=\int_{[U_n]}\left(\int_{[U_m\times U_m]} f_1(g_1)\chi_2^{-1}(g_2)\overline{f_2(g_2)}\Theta^-_{\psi,\chi,\varphi}(\jmath(g_1,g_2),g')\,dg_1\,dg_2\right)dg'\\
&=\int_{[U_n]}\theta^{f_1}_{\psi,\chi,\phi_1}(g')\overline{\theta^{f_2}_{\psi,\chi,\phi_2}(g')}\,dg',
\end{aligned}
$$

where we have used the remark above. □

11.3. Ichino's regularized Siegel–Weil formula

The proof of Theorem 10.1 will follow from the following reformulation of the main result of [36]. We provide the details of the proof in Appendix B.

THEOREM 11.2. *Let s_0 be a positive real number $<\frac{1}{2}m$ and let $\Phi(\cdot,s)$ be a holomorphic section of $I(s,\chi_2)$ such that the Siegel Eisenstein series $E(h,s,\Phi)$ has a simple pole at $s=s_0$ whose residue $A(\cdot,\Phi)$ generates an irreducible automorphic representation. Then there exists a global skew-Hermitian space W over E of dimension $n=m-2s_0$ and a function $\varphi\in\mathcal{S}(\mathbb{W}_d(\mathbb{A}))$ such that*

$$A(h,\Phi)(=\mathrm{Res}_{s=s_0}E(h,s,\Phi))=cB(h,\varphi),$$

where c is a non-zero explicit constant.

Assuming Theorem 11.2 we can now prove Theorem 10.1.

11.4. Proof of Theorem 10.1

Let s_0 be a pole of the partial L-function $L^S\big(s+\frac{1}{2},\pi\times\eta\big)$ where η is some character of $E^\times\backslash\mathbb{A}_E^\times$. It follows from Proposition 10.2 that $\eta=\bar{\eta}^{-1}$ and that s_0 is a half-integer $\leqslant\frac{1}{2}m$. We set $n=m-2s_0$ and suppose that n is positive. Note that we necessarily have $\eta|_{\mathbb{A}^\times}=\varepsilon^n_{E/F}$. We set $\chi_2=\eta$ and fix some arbitrary choice of character χ_1 such that $\chi_1|_{\mathbb{A}^\times}=\varepsilon^m_{E/F}$.

Since s_0 is a pole of the partial L-function $L^S\big(s+\frac{1}{2}, \pi\times\chi_2\big)$ there are choices of $f\in\pi$, $f^\vee\in\pi^\vee$ and of a holomorphic section $\Phi(\cdot, s)$ of $I(s, \chi_2)$ such that

$$\int_{[U_m\times U_m]} A(\jmath(g_1, g_2), \Phi) f(g_1) f^\vee(g_2)\chi_2^{-1}(g_2)\, dg_1\, dg_2 \neq 0. \tag{11.8}$$

In particular the Siegel Eisenstein series $E(h, s, \Phi)$ has a simple pole at $s=s_0$. We may moreover modify Φ so that $A(\cdot, \Phi)$ generates an irreducible automorphic representation and (11.8) is still non-zero. Theorem 11.2 then implies that there exists a global skew-Hermitian space W over E of dimension $n=m-2s_0$ and a function $\varphi\in\mathcal{S}(\mathbb{W}_d(\mathbb{A}))$ such that

$$A(h, \Phi) = cB(h, \varphi), \tag{11.9}$$

where c is a non-zero explicit constant. Set $f_1=f$ and let $f_2\in\mathcal{H}_\pi$ be the element corresponding to the conjugate of $f^\vee\in\mathcal{H}_{\pi^\vee}$. Writing φ as a linear combination of $\delta(\phi_1\otimes\bar{\phi}_2)$, it follows from Proposition 11.1 and equations (11.8) and (11.9) that there exist elements $\phi_1, \phi_2\in\mathcal{S}(\mathbb{X}(\mathbb{A}))$ such that

$$\langle\theta^{f_1}_{\psi,\chi,\phi_1}, \theta^{f_2}_{\psi,\chi,\phi_2}\rangle = \int_{[U_m\times U_m]} f_1(g_1)\chi_2^{-1}(g_2)\overline{f_2(g_2)} B(\jmath(g_1, g_2), \varphi)\, dg_1\, dg_2$$

is non-zero. This proves that $\Theta^W_{\psi,\chi,V}(\pi)\neq 0$.

Lemma 11.3. *The representation $\Theta^W_{\psi,\chi,V}(\pi)$ is a cuspidal automorphic representation of $U(W)$.*

Proof. Let W' be the smallest element of the Witt tower $\{W_r\}_r$ such that

$$\Theta^{W'}_{\psi,\chi,V}(\pi)\neq 0.$$

By the Rallis theta tower property [62], $\Theta^{W'}_{\psi,\chi,V}(\pi)$ is a cuspidal automorphic representation of $U(W)$. Let n' be the dimension of W'. We have $n'\leqslant n$ and, by inverting the above arguments, we get that $L^S\big(s+\frac{1}{2}, \pi\times\eta\big)$ has a pole at $s=\frac{1}{2}(m-n')$. Since by hypothesis $L^S(s, \pi\times\eta)$ is holomorphic in the half-plane $\mathrm{Re}(s)>\frac{1}{2}(a-1)$, where $a=m-n$, we conclude that $n'\geqslant n$. The lemma follows. □

11.5. We can now conclude the proof of Theorem 10.1. The non-vanishing of

$$\langle\theta^{f_1}_{\psi,\chi,\phi_1}, \theta^{f_2}_{\psi,\chi,\phi_2}\rangle$$

implies the non-vanishing of

$$\int_{[U_n]} \overline{\theta^{f_2}_{\psi,\chi,\phi_2}(g')}\bigg(\int_{[U_m]} \theta_{\psi,\chi,\phi_1}(g, g') f_1(g)\, dg\bigg)\, dg' = \int_{[U_n]} \theta^{f_1}_{\psi,\chi,\phi_1}(g')\overline{\theta^{f_2}_{\psi,\chi,\phi_2}(g')}\, dg'. \tag{11.10}$$

But it follows from Lemma 11.3 that $g' \mapsto \overline{\theta^{f_2}_{\psi,\chi,\phi_2}(g')}$ is rapidly decreasing and that we may therefore invert the order of integration in the left-hand side of (11.10). Setting $F(g') = \overline{\theta^{f_2}_{\psi,\chi,\phi_2}(g')}$, we get that

$$\theta^F_{\psi,\chi,\phi_1}(g) = \int_{[U_n]} F(g')\theta_{\psi,\chi,\phi_1}(g,g')\,dg'$$

is non-zero and is not orthogonal to the space of π. Since the image of $\Theta^V_{\psi,\chi,W}$ is an automorphic subrepresentation of $L^2([U_m])$, this proves that π is in the image of the cuspidal ψ-theta correspondence from the group $U(W)$.

12. Weak Arthur theory

In this section we recall a small part of Arthur's work on the endoscopic classification of automorphic representations of classical groups. This will be used in the following section to verify the hypotheses of Theorem 10.1 in our (geometric) cases.

12.1. Notation

Let E be a CM-field with totally real maximal subfield F which is a number field, and let $\mathbb{A}$ be the ring of adeles of F. We will always assume that a specified embedding of E into the algebraic closure of F has been fixed. We set $\Gamma_F = \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Let V be a non-degenerate Hermitian vector space over E with $\dim_E V = m$.

We let G be an inner form of $U_{E/F}(m)$, the quasi-split unitary group over F, whose group of F-points is given by

$$U_{E/F}(m)(F) = \{g \in \mathrm{GL}_m(E) : {}^t\bar{g}Jg = J\}.$$

Here J is the anti-diagonal matrix

$$J = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$$

and $z \mapsto \bar{z}$ is the Galois conjugation of E/F. In this section we simply denote by $U(m)$ the unitary group $U_{E/F}(m)$. We finally denote by $\mathrm{GL}(m)$ the F-algebraic group $\mathrm{Res}_{E/F}(\mathrm{GL}_m|_E)$. We will identify automorphic representations of $\mathrm{GL}(m)$ with automorphic representations of $\mathrm{GL}_m(\mathbb{A}_E)$.

12.2. *L*-groups

The (complex) dual group of $U(m)$ is

$$U(m)^\vee = \mathrm{GL}_m(\mathbb{C}),$$

and the L-group of $U(m)$ is the semi-direct product

$${}^L U(m) = \mathrm{GL}_m(\mathbb{C}) \rtimes \Gamma_F,$$

where the action of Γ_F factors through $\mathrm{Gal}(E/F)$ and the action of the non-trivial element $\sigma \in \mathrm{Gal}(E/F)$ is given by

$$\sigma(g) = \Phi_m\, {}^t g^{-1} \Phi_m^{-1}, \quad g \in \mathrm{GL}_m(\mathbb{C}).$$

Here Φ_m is the anti-diagonal matrix with alternating ± 1 entries:

$$\Phi_m = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ (-1)^{m-1} & & 0 \end{pmatrix}.$$

Note that $\Phi_m^2 = (-1)^{m-1}$ so that σ is of order 2. Moreover, σ fixes the standard splitting of $\mathrm{GL}_m(\mathbb{C})$.

Now the (complex) dual group of $\mathrm{GL}(m,E)$, seen as a group over F, is

$$\mathrm{GL}(m)^\vee = \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C}),$$

and the L-group of $\mathrm{GL}(m)$ is the semi-direct product

$${}^L\mathrm{GL}(m) = (\mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})) \rtimes \Gamma_F,$$

where now σ acts by

$$\sigma(g, g') = (g', g), \quad g, g' \in \mathrm{GL}_m(\mathbb{C}).$$

12.3. Representations induced from square integrable automorphic representations

By [57], one may parameterize the discrete automorphic spectrum of $\mathrm{GL}(m)$ by a set of formal tensor products

$$\Psi = \mu \boxtimes R,$$

where μ is an irreducible, unitary, cuspidal automorphic representation of $\mathrm{GL}(d)$ and R is an irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension n, for positive integers d and n such that $m=dn$. For any such Ψ, we form the induced representation

$$\mathrm{ind}(\mu|\cdot|_{\mathbb{A}_E}^{(n-1)/2}, \mu|\cdot|_{\mathbb{A}_E}^{(n-3)/2}, ..., \mu|\cdot|_{\mathbb{A}_E}^{(1-n)/2})$$

(normalized induction from the standard parabolic subgroup of type $(d, ..., d)$). We then write Π_Ψ for the unique irreducible quotient of this representation.

We may more generally associate a representation Π_Ψ of $\mathrm{GL}(m)$, induced from square integrable automorphic representations, to a formal sum of formal tensor products

$$\Psi = (\mu_1 \boxtimes R_{n_1}) \boxplus ... \boxplus (\mu_r \boxtimes R_{n_r}), \tag{12.1}$$

where μ_j is an irreducible, unitary, cuspidal automorphic representation of $\mathrm{GL}(d_j)/F$, R_{n_k} is an irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension n_k and $m=n_1d_1+...+n_rd_r$. With each $\mu_j \boxtimes R_{n_k}$ we associate a square integrable automorphic form Π_j of $\mathrm{GL}(n_kd_k)$. We then define Π_Ψ as the induced representation

$$\mathrm{ind}(\Pi_1 \otimes ... \otimes \Pi_r)$$

(normalized induction from the standard parabolic subgroup of type $(n_1d_1, ..., n_rd_r)$). This is an irreducible representation of $\mathrm{GL}_m(\mathbb{A}_E)$, since it was proved by Tadic [66] and Vogan [70] that a representation induced by a unitary irreducible one is irreducible.

12.4. Unramified base change

Let $\pi = \otimes'_v \pi_v$ be an automorphic representation of $G(\mathbb{A})$ occuring in the discrete spectrum.[11] For almost every finite place v of F both $G(F_v)$ and π_v are unramified. Let F_v be the local field associated with such a place and let W'_{F_v} be its Weil–Deligne group. Then by the Satake isomorphism π_v is associated with an L-parameter $\varphi_v: W'_{F_v} \to {}^LU(m)$; see e.g. [54].

The (fixed) embedding of E into the algebraic closure of F specifies $E_v = E \otimes_F F_v$. This realizes W'_{E_v} as a subgroup of W'_{F_v}. Restricting φ_v to W'_{E_v} one gets an L-parameter for an unramified irreducible representation Π_v of $\mathrm{GL}_m(E_v)$—the *principal base change* of π_v.

Remark. There is another base change.[12] We fix a unitary character

$$\chi: \mathbb{A}_E^\times / E^\times \longrightarrow \mathbb{C}^\times$$

---

(11) Note that it is always the case if G is anisotropic.

(12) It is sometimes called *unstable base change* but we will avoid this confusing terminology.

whose restriction to $\mathbb{A}^\times$ is the character $\varepsilon_{E/F}$ associated, by classfield theory, with the quadratic extension E/F. Outside some finite set S of places of F the character χ is unramified. Let $v\notin S$, the *non-principal base change* for unramified representations of $G(F_v)$ is the one described above but twisted by the character χ_v. The definition depends on the choice of the character but so does the transfer between functions; see e.g. [59]. In the statement below we will solely consider the principal base change, note however that we have to consider both base changes in the proofs, and we will not make this very explicit. The definition depends on the choice of the character but only up to a twist: two different choices differ by a character of $E_v^\times$ trivial on $F_v^\times$. The group of such characters of $E_v^\times$ is exactly the group of characters of $U_{E_v/F_v}(m)$ by the following correspondence: let ω be a character of $E_v^\times$ trivial on $F_v^\times$ and denote by ω^1 the character of the subgroup of $U_{E_v/F_v}(m)$ defined by $\omega^1(g)=\omega(z)$, where z is any element of $E_v^\times$ such that $\det(g)=z/\bar{z}$. The principal and the non-principal base change commute with the twist by $\omega\circ\det$ on the $\mathrm{GL}(m,E_v)$ side and the twist by ω^1 on the $U_{E_v/F_v}(m)$ side.

12.5. Weak base change

The following proposition is essentially due to Arthur [3, Corollary 3.4.3] though it is not stated for unitary groups; see [59, Corollary 4.3.8] for a statement in the latter case when G is quasi-split. The reduction from the general case to the quasi-split case follows the same lines;([13]) we provide some details in Appendix A. (When Kaletha, Minguez, Shin and White have finished their three announced papers, this will be included.)

PROPOSITION 12.1. *Let π be an irreducible automorphic representation of $G(\mathbb{A})$ which occurs* (*discretely*) *as an irreducible subspace of $L^2(G(F)\backslash G(\mathbb{A}))$. Then, there exists a* (*unique*) *global representation $\Pi=\Pi_\Psi$ of $\mathrm{GL}(m,\mathbb{A}_E)$, induced from square integrable automorphic representations, associated with a parameter Ψ as in* (12.1) *and a finite set S of places of F containing all Archimedean ones such that, for all $v\notin S$, the representations π_v and Π_v are both unramified and Π_v is the principal base change of π_v.*

We will refer to Π as the *weak base change* of π.

Remark. Proposition 12.1 puts serious limitations on the kind of non-tempered representations that can occur discretely: e.g. an automorphic representation π of $G(\mathbb{A})$ which occurs discretely in $L^2(G(F)\backslash G(\mathbb{A}))$ and which is non-tempered at one place $v\notin S$ is non-tempered at all places outside S.

([13]) The use of the stable twisted trace formula being replaced by the (untwisted) stable trace formula.

The above remark explains how Arthur's theory will be used in our proof. We now want to get a global control on the automorphic representations with a prescribed type at infinity.

12.6. Standard representations and characters

We first recall the results of local harmonic analysis that we will need. We therefore fix a place v of F and, until further notice, let G denote the group of F_v-points of the unitary group. Denote by $\mathcal{H}(G)$ the Hecke algebra of locally constant functions of compact support on G.([14])

By Langlands' classification, any admissible representation π of G can be realized as the Langlands subquotient of some standard representation. Recall that the latter can be identified with a G-orbit of a couple $\varrho=(M,\sigma)$ where $M\subset G$ is a Levi subgroup and σ is an irreducible representation of M that is tempered modulo the center.

Both π and ϱ determine real linear forms Λ_π and Λ_ϱ—the *exponents*—on $\mathfrak{a}_M$; they measure the failure of the representation to be tempered.

Following Arthur we denote by $\varrho_\pi=(M_\pi,\sigma_\pi)$ the standard representation corresponding to π. We furthermore recall that the distribution character of π has a decomposition

$$\operatorname{trace}\,\pi(f)=\sum_{\varrho} n(\pi,\varrho)\operatorname{trace}\varrho(f),\quad f\in\mathcal{H}(G),$$

into standard characters ϱ, where the coefficients $n(\pi,\varrho)$ are uniquely determined integers such that all but finitely many of them are equal to 0 and $n(\pi,\varrho_\pi)=1$. If $n(\pi,\varrho)\neq 0$, then $\Lambda_\varrho\leqslant\Lambda_\pi$ in the usual sense that $\Lambda_\pi-\Lambda_\varrho$ is a nonnegative integral combination of simple roots of the root system associated with the inducing parabolic,([15]) with equality $\Lambda_\pi=\Lambda_\varrho$ if and only if $\varrho=\varrho_\pi$.

12.7. Archimedean packets

Suppose now that v is a Archimedean place. Local principal base change associates with the standard module ϱ_π a standard module of $\mathrm{GL}_m(\mathbb{C})$ that we denote by $\mathrm{St}(\pi)$. We now describe this module in terms of L-parameters. Recall that L-packets are in one-to-one correspondence with admissible L-parameters $\varphi\colon W'_{F_v}\to {}^LU(m)$.

([14]) We will mainly deal with the situation where v is Archimedean, so that $\mathcal{H}(G)=C_c^\infty(G)$, except in Appendix A where we have to deal with all places.

([15]) In loose terms: the Langlands subquotient of ϱ is more tempered than π.

Restricting φ to W'_{E_v} one gets an L-parameter which defines a standard module of $\mathrm{GL}_m(\mathbb{C})$; this representation is precisely $\mathrm{St}(\pi)$.

Now there is a local analogue to §12.3. A standard representation of $\mathrm{GL}_m(\mathbb{C})$ can be parametrized by a formal sum of formal tensor product (12.1), where each μ_j is now a tempered irreducible representation of $\mathrm{GL}(d_j, \mathbb{C})$. The other components R_{n_j} remain irreducible representations of $\mathrm{SL}_2(\mathbb{C})$. With each $\mu_j \boxtimes R_{n_j}$, we associate the unique irreducible quotient Π_j of

$$\mathrm{ind}(\mu_j |\cdot|^{(n_j-1)/2}, \mu_j |\cdot|^{(n_j-3)/2}, ..., \mu_j |\cdot|^{(1-n_j)/2})$$

(normalized induction from the standard parabolic subgroup of type $(d_j, ..., d_j)$). We then define Π_Ψ as the induced representation

$$\mathrm{ind}(\Pi_1 \otimes ... \otimes \Pi_r)$$

(normalized induction from the standard parabolic subgroup of type $(n_1 d_1, ..., n_r d_r)$). The representation Π_Ψ is irreducible and unitary. We will abusively denote by Ψ the standard representation associated with Π_Ψ. Now by the local Langlands correspondence, the standard module Ψ can be represented as a homomorphism

$$\Psi: W'_{E_v} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_m(\mathbb{C}). \tag{12.2}$$

Arthur associates with such a parameter the L-parameter $\varphi_\Psi: W'_{E_v} \to \mathrm{GL}_m(\mathbb{C})$ given by

$$\varphi_\Psi(w) = \Psi\left(w, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}\right).$$

And $\mathrm{St}(\pi) = \Psi$ if and only if $\varphi|_{W'_{E_v}} = \varphi_\Psi$.

12.8. Weak classification

We now come back to the global situation. Let π be an irreducible automorphic representation of $G(\mathbb{A})$ which occurs (discretely) as an irreducible subspace of $L^2(G(F) \backslash G(\mathbb{A}))$. It follows from Proposition 12.1 that π determines an irreducible automorphic representation $\Pi = \Pi_\Psi$ of $\mathrm{GL}_m(\mathbb{A}_E)$. Given an Archimedean place v, it is not true in general that $\mathrm{St}(\pi_v) = \Psi_v$. Our main technical result in this third part of the paper is the following theorem, whose proof—a slight refinement of the proof of Proposition 12.1—is postponed to Appendix A.

THEOREM 12.2. *Let π be an irreducible automorphic representation of $G(\mathbb{A})$ which occurs (discretely) as an irreducible subspace of $L^2(G(F)\backslash G(\mathbb{A}))$. Assume that for every Archimedean place v the representation π_v has regular infinitesimal character. Let Π_Ψ be the automorphic representation of $\mathrm{GL}(m, \mathbb{A}_E)$ associated with π by weak base change. Then, for every Archimedean place v, the (unique) irreducible quotient of the standard $\mathrm{GL}_m(\mathbb{C})$-module $\mathrm{St}(\pi_v)$ occurs as an (irreducible) subquotient of the local standard representation Ψ_v.*

Remark. Theorem 12.2 follows from Proposition 12.1 and the description of the cohomological Arthur packets recently obtained by Arancibia, Moeglin and Renard [1]. We give a direct self-contained proof in Appendix A.

13. Applications

In this section we derive the corollaries to Theorem 12.2 that are used in the paper.

13.1. Application to L-functions

Let π be an irreducible automorphic representation of $G(\mathbb{A})$ which occurs (discretely) as an irreducible subspace of $L^2(G(F)\backslash G(\mathbb{A}))$ and let $\Pi=\Pi_\Psi$ be the automorphic representation of $\mathrm{GL}(m)$ associated with π by weak base change. Write

$$\Psi = (\mu_1 \boxtimes R_{n_1}) \boxplus \ldots \boxplus (\mu_r \boxtimes R_{n_r}).$$

We factor each $\mu_j = \otimes_v \mu_{j,v}$, where v runs over all places of F. Let S be a finite set of places of F containing the set S of Proposition 12.1, and all v for which some $\mu_{j,v}$ or π_v is ramified. We can then define the formal Euler product

$$L^S(s, \Pi_\Psi) = \prod_{j=1}^{r} \prod_{v \notin S} L_v\left(s - \frac{n_j - 1}{2}, \mu_{j,v}\right) L_v\left(s - \frac{n_j - 3}{2}, \mu_{j,v}\right) \ldots L_v\left(s - \frac{1 - n_j}{2}, \mu_{j,v}\right).$$

Note that $L^S(s, \Pi_\Psi)$ is the partial L-function of a very special automorphic representation of $\mathrm{GL}(m)$; it is the product of partial L-functions of the square integrable automorphic representations associated with the parameters $\mu_j \boxtimes R_{n_j}$. According to Jacquet and Shalika [37], $L^S(s, \Pi_\Psi)$, which is an absolutely convergent product for $\mathrm{Re}(s) \gg 0$, extends to a meromorphic function of s. Moreover, it follows from Proposition 12.1 and the definition of $L^S(s, \pi)$ that

$$L^S(s, \pi) = L^S(s, \Pi_\Psi).$$

13.2. Infinitesimal character

It follows from Theorem 12.2 that for each infinite place v the representations π_v and Π_v both have the same infinitesimal character. It is computed in the following way. Let v_0 be an Archimedean place of F. We may associate with Ψ the parameter $\varphi_{\Psi_{v_0}}:\mathbb{C}^*\to G^\vee\subset \mathrm{GL}_m(\mathbb{C})$ given by

$$\varphi_{\Psi_{v_0}}(z)=\Psi_{v_0}\left(z, \begin{pmatrix} (z\bar{z})^{1/2} & \\ & (z\bar{z})^{-1/2}\end{pmatrix}\right).$$

Being semisimple, it is conjugate into the diagonal torus $\{\mathrm{diag}(x_1, ..., x_m)\}$. We may therefore write $\varphi_{\Psi_{v_0}}=(\eta_1, ..., \eta_m)$, where each η_j is a character $z\mapsto z^{P_j}\bar{z}^{Q_j}$. One easily checks that the vector

$$\nu_\Psi=(P_1, ..., P_m)\in\mathbb{C}^m\cong\mathrm{Lie}(T)\otimes\mathbb{C}$$

is uniquely defined modulo the action of the Weyl group $W=\mathfrak{S}_m$ of $G(F_{v_0})$. The following proposition is detailed in [6].

Proposition 13.1. *The infinitesimal character of π_{v_0} is the image of ν_Ψ in $\mathbb{C}^m/\mathfrak{S}_m$.*

From now on we assume that π has a regular([16]) and integral infinitesimal character at every infinite place. This forces the Archimedean localizations of the cuspidal automorphic representations μ_j to be induced of unitary characters of type $(z/\bar{z})^{p/2}$, where $p\in\mathbb{Z}$. Moreover, we have

$$\tfrac{1}{2}p+\tfrac{1}{2}(n_j-1)-\tfrac{1}{2}(m-1)\in\mathbb{Z}.$$

We can now relate Arthur's theory to Theorem 10.1.

Proposition 13.2. *Assume that for some Archimedean place v_0 of F the local representation π_{v_0} is a Langlands' quotient of a standard representation with an exponent $(z/\bar{z})^{p/2}(z\bar{z})^{(a-1)/2}$. Then, the following statements hold*:

(1) *In the parameter Ψ some factor $\mu_j\boxtimes R_{n_j}$ is such that $n_j\geqslant a$. In particular, if $a>\frac{1}{2}m$, the representation μ_j is a character.*

(2) *If we assume that π has trivial infinitesimal character and that $3a>m+|p|$. Then in the parameter Ψ some factor $\mu_j\boxtimes R_{n_j}$ is such that $n_j\geqslant a$, and the representation μ_j is a character.*

Proof. It follows from Theorem 12.2 that, if π_{v_0} is a Langlands' quotient of a standard representation with an exponent $(z/\bar{z})^{p/2}(z\bar{z})^{(a-1)/2}$, then the associated standard representation Ψ_{v_0} (of $\mathrm{GL}_m(\mathbb{C})$) contains a character of absolute value $\geqslant\frac{1}{2}(a-1)$. This

([16]) Equivalently, the P_j's are all distinct.

forces one of the factors $\mu_j \boxtimes R_{n_j}$ in Ψ to be such that $n_j \geqslant a$. Since $\sum_j n_j = m$, there can, if $a > \frac{1}{2}m$, only be factors $\mu_j \boxtimes R_{n_j}$ in Ψ, where $n_j \geqslant a$ and μ_j is a character. This proves the first part of the proposition.

We now assume that π has trivial infinitesimal character. It follows, in particular, that p and $m-a$ have the same parity. Suppose that $3a > m+|p|$. Note that if $|p| \geqslant a$ then $a > \frac{1}{2}m$ and the result follows from the first case; we will therefore assume that $|p| \leqslant a$.

Now we have $a > \frac{1}{2}m$ and, as above, this forces one of the factors $\mu_j \boxtimes R_{n_j}$ in Ψ to be such that $n_j \geqslant a$ and either μ_j is a character or μ_j is 2-dimensional. We only have to deal with the latter case. Set $n = n_j$. The localization μ_j in v_0 is induced from two characters $(z/\bar{z})^{p_j/2}$, $j=1,2$, and Theorem 12.2 implies that the set

$$\left\{ \tfrac{1}{2}(a+p-1), \tfrac{1}{2}(a+p-3), ..., \tfrac{1}{2}(p+1-a) \right\} \tag{13.1}$$

is contained in one of the two sets

$$I_k = \left\{ \tfrac{1}{2}(n+p_k-1), \tfrac{1}{2}(n+p_k-3), ..., \tfrac{1}{2}(p_k+1-n) \right\}, \quad k=1,2,$$

say $k=1$. The infinitesimal character of π_{v_0} is regular and integral, it is therefore a collection of m distinct half-integers. Let m_+ be the number of positive entries and m_- be the number of negative entries. The entries must include the (necessary disjoint) sets I_k, $k=1,2$. Now, since $|p| < a$ (by the reduction already made), the set (13.1) contains either 0 or $\pm\frac{1}{2}$ and consequently so does I_1. This forces I_2 to be totally positive or negative. And since (13.1) contains $\left[\frac{1}{2}(a+p)\right]$ positive elements and $\left[\frac{1}{2}(a-p)\right]$ negative elements, we conclude that

$$\max\{m_-, m_+\} \geqslant n + \left[\tfrac{1}{2}(a-|p|)\right] \geqslant a + \left[\tfrac{1}{2}(a-|p|)\right].$$

On the other hand $\max\{m_-, m_+\} = \left[\frac{1}{2}m\right] + |m_+ - m_-|$ and since $a-|p|$ and m have the same parity, we end up with

$$3a - |p| \leqslant m + 2|m_+ - m_-|.$$

Since $m_+ = m_-$ (π_{v_0} has trivial infinitesimal character) we get a contradiction. □

Remark. It follows from the proof that we can replace the assumption that the infinitesimal character is that of the trivial representation by the more general assumption that its entries are all integral, resp. half-integral but non-integral, and that it is *balanced*, i.e. that it contains as many positive and negative entries.

PROPOSITION 13.3. *Let π be as in Proposition 13.2, with either $a>\frac{1}{2}m$ or with trivial infinitesimal character and $3a>1+|p|$. Then there exists a character η of $\mathbb{A}_E^\times/E^\times$ and an integer $b\geqslant a$ such that the partial L-function $L^S(s,\eta\times\pi)$, where S is a finite set of places which contains all the Archimedean places and all the places where π ramifies, is holomorphic in the half-plane $\mathrm{Re}(s)>\frac{1}{2}(b+1)$ and has a simple pole at $s=\frac{1}{2}(b+1)$.*

Proof. Proposition 13.2 provides a character $\eta=\mu_j$ of $\mathbb{A}_E$ and an integer $b=n_j\geqslant a$. Writing $L^S(s,\eta\times\pi)$ explicitly on a right half-plane of absolute convergence using the description of §13.1; we get a product of $L^S\big(s-\frac{1}{2}(b-1),\eta\times\eta\big)$ some terms of the form

$$L^S\big(s-\tfrac{1}{2}(b'-1),\eta\times\varrho\big).$$

Our hypothesis on a forces $b'<a\leqslant b$; so such a factor is non-zero at $s=\frac{1}{2}(b+1)$. The conclusion of the proposition follows. □

13.3. Langlands' parameters of cohomological representations

The restriction to $\mathbb{C}^\times$ of the Langlands' parameter of a cohomological representation $A_\mathfrak{q}$ is explicitly described in [5, §5.3]. We use here the following alternate description given by Cossutta [11].

Recall that we have $\mathfrak{q}=\mathfrak{q}(X)$, with $X=(t_1,...,t_{p+q})\in\mathbb{R}^{p+q}$ such that $t_1\geqslant...\geqslant t_p$ and $t_{p+q}\geqslant...\geqslant t_{p+1}$. Since $A_\mathfrak{q}$ only depends on the intersection $\mathfrak{u}\cap\mathfrak{p}$, we may furthermore choose X such that the Levi subgroup L associated with $\mathfrak{l}$ has no compact (non-abelian) simple factor.

We associate with these data a parameter

$$\Psi\colon W_\mathbb{R}\times\mathrm{SL}_2(\mathbb{C})\longrightarrow {}^LU(m)$$

such that

(1) Ψ factors through ${}^LL$, that is

$$\Psi\colon W_\mathbb{R}\times\mathrm{SL}_2(\mathbb{C})\xrightarrow{\Psi_L}{}^LL\longrightarrow {}^LU(m),$$

where the last map is the canonical extension [64, Proposition 1.3.5] of the injection $L^\vee\subset U(m)^\vee$;

(2) φ_{Ψ_L} is the L-parameter of the trivial representation of L.

The restriction of the parameter Ψ to $\mathrm{SL}_2(\mathbb{C})$ therefore maps $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ to a principal unipotent element in $L^\vee\subset U(m)^\vee$. The restriction $\varphi\colon\mathbb{C}^\times\to\mathrm{GL}_m(\mathbb{C})$ of the Langlands' parameter of $A_\mathfrak{q}$ to $\mathbb{C}^\times$ is given by

$$\varphi(z)=\Psi\left(z,\begin{pmatrix}(z\bar z)^{1/2}&\\&(z\bar z)^{-1/2}\end{pmatrix}\right).$$

More explicitly, let $z_1, ..., z_r$ be the different values of the $\{t_j\}_{j=1}^m$ and let $\{p_k\}_{k=1}^r$ and $\{q_k\}_{k=1}^r$ be the integers such that

$$(t_1, ..., t_p) = (\underbrace{z_1, ..., z_1}_{p_1 \text{ times}}, ..., \underbrace{z_r, ..., z_r}_{p_r \text{ times}})$$

and

$$(t_{p+q}, ..., t_{p+1}) = (\underbrace{z_1, ..., z_1}_{q_1 \text{ times}}, ..., \underbrace{z_r, ..., z_r}_{q_r \text{ times}}).$$

We then have

$$L = \prod_{j=1}^r U(p_j, q_j),$$

with $\sum_j p_j = p$ and $\sum_j q_j = q$. Moreover, if $p_j q_j = 0$, then either p_j or q_j is equal to 1. We let $m_j = p_j + q_j$, $j = 0, ..., r$, and set

$$k_j = -m_1 - ... - m_{j-1} + m_{j+1} + ... + m_r.$$

The canonical extension ${}^L L \to {}^L U(m)$ of the block diagonal map

$$\mathrm{GL}_{m_1}(\mathbb{C}) \times ... \times \mathrm{GL}_{m_r}(\mathbb{C}) \longrightarrow \mathrm{GL}_m(\mathbb{C})$$

then maps $z \in \mathbb{C}^\times \subset W_\mathbb{R}$ to

$$\begin{pmatrix} (z/\bar{z})^{k_1/2} I_{m_1} & & \\ & \ddots & \\ & & (z/\bar{z})^{k_r/2} I_{m_r} \end{pmatrix}.$$

Now the parameter Ψ_L maps $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ to a principal unipotent element in each factor of $L^\vee \subset U(m)^\vee$. The parameter Ψ_L therefore contains an $\mathrm{SL}_2(\mathbb{C})$ factor of the maximal dimension in each factor of $L^\vee$. These factors consist of $\mathrm{GL}_{m_j}(\mathbb{C})$, $j = 1, ..., r$. The biggest possible $\mathrm{SL}_2(\mathbb{C})$ representation in the jth factor is R_{m_j}. We conclude that Ψ decomposes as

$$(\mu_1 \boxtimes R_{m_1} \boxplus \mu_1^{-1} \boxtimes R_{m_1}) \boxplus ... \boxplus (\mu_r \boxtimes R_{m_r} \boxplus \mu_r^{-1} \boxtimes R_{m_r}), \tag{13.2}$$

where μ_j is the unitary characters of $\mathbb{C}^\times$ given by $\mu_j(z) = (z/\bar{z})^{k_j/2}$. Denoting the character $z \mapsto z/\bar{z}$ by μ, we conclude that

$$\mathrm{St}(A_\mathfrak{q}) = (\mu^{k_1/2} \boxtimes R_{m_1}) \boxplus ... \boxplus (\mu^{k_r/2} \boxtimes R_{m_r}). \tag{13.3}$$

13.4. In particular, it follows from (13.3) that

$$\mathrm{St}(A(b\times q,a\times q))=\mu^{(m-1)/2}\boxplus\mu^{(m-3)/2}\boxplus\ldots\boxplus\mu^{(m-2b+1)/2}$$
$$\boxplus(\mu^{(a-b)/2}\boxtimes R_{p+q-a-b})\boxplus\mu^{(2a-1-m)/2}\boxplus\ldots\boxplus\mu^{(1-m)/2}.$$

Therefore from Theorem 10.1, Proposition 13.2 and the paragraph following it, we deduce the following result.

PROPOSITION 13.4. *Let $\pi\in\mathcal{A}^c(U(V))$ and let v be an infinite place of F such that $U(V)(F_v)\cong U(p,q)$. Assume that π_v is (isomorphic to) the cohomological representation $A(b\times q,a\times q)$ of $U(p,q)$ with $3(a+b)+|a-b|<2m$. Then, there exists some $(a+b)$-dimensional skew-Hermitian space W over E such that π is in the image of the cuspidal ψ-theta correspondence from the group $U(W)$.*

Proof. We begin by translating the notation of §13.1 to the notation of this section. The $|p|$ and b of §13.1 are here $|a-b|$ and $p+q-a-b=m-(a+b)$, respectively. The hypothesis of this proposition is the same as the hypothesis of §13.1. One deduces the fact that the partial L-function as in §13.1 has a pole at a point s_0 with $s_0\geqslant\frac{1}{2}(m-(a+b)+1)$. Using Theorem 10.1, we obtain the fact that π is in the image of the ψ-theta correspondence with a skew-Hermitian space W of dimension $m-2s_0-1\geqslant m-(m-(a+b))=(a+b)$. But it is easy to see that a strict inequality is impossible at the infinite place v, and we obtain the equality as in the statement of the proposition. Moreover, the representation of $U(W)$ in this correspondence is a discrete series at the place v and is, therefore, necessarily a cuspidal representation. □

13.5. Proof of Theorem 6.1(2)

It suffices to prove that, if $\pi_f^\sigma\in\mathrm{Coh}_f^{b',a'}$ for some $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and some integers b' and a' such that $a+b=a'+b'$, then either $(a',b')=(a,b)$ or $(a',b')=(b,a)$. To do so, recall that, corresponding to π_f, there is a parameter Ψ given by Proposition 12.1. Now fix an unramified finite prime v and let $\omega_{\pi_v}:\mathcal{H}_v\to\mathbb{C}$ be the unramified character of the Hecke algebra associated with the local (unramified) representation π_v by Satake transform [54]. Recall that ω_{π_v} is associated with some unramified character χ_{π_v} of a maximal torus of $G(\mathbb{Q}_v)$ (considered up to the Weyl group action). Since the Hecke algebra and its action on $H^\bullet(S(K),\mathbb{C})$ admits a definition over $\mathbb{Q}$, the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the characters of the Hecke algebra associated with representations $\pi_f\in\mathrm{Coh}_f$, and therefore on χ_{π_v}, so that $\chi_{\pi_v^\sigma}=\chi_{\pi_v}^\sigma$. Note that the Galois action preserves the norm. Now, if $\pi_f\in\mathrm{Coh}_f^{a,b}$ with $3(a+b)+|a-b|<2m$, the global parameter Ψ contains a unique factor $\mu\boxtimes R_{m-a-b}$, where μ is a unitary Hecke character, and all the other factors are associated with smaller $\mathrm{SL}_2(\mathbb{C})$-representations—this follows from Proposition 13.4 and (the proof of) Proposition 13.2. Localizing this parameter Ψ at the finite unramified place v, we conclude that the character χ_{π_v} has a constituent which is a Hecke character of "large" norm, corresponding to the $\mathrm{SL}_2(\mathbb{C})$-representation of dimension $m-a-b$. This singularizes the character μ_v so that the Galois action on χ_{π_v} yields the usual action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on μ. Now, if μ is $(z/\bar{z})^{(b-a)/2}$ at infinity, then for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the character μ^σ is either $(z/\bar{z})^{(a-b)/2}$ or $(z/\bar{z})^{(b-a)/2}$ at infinity. And we conclude from Proposition 13.4 that the contribution of π_f^σ to $\mathrm{SH}^{(a+b)q}(S(K),\mathbb{C})$ can only occur in

$$H^{a\times q,b\times q}(S(K),\mathbb{C})\oplus H^{b\times q,a\times q}(S(K),\mathbb{C}).$$

Appendices

Appendix A. Proof of Theorem 12.2

A.1. The quasi-split case

Suppose that G is quasi-split. We want to relate the discrete automorphic spectra of G and that of the (disconnected) group $\widetilde{G}$ which is equal to $\mathrm{GL}(m)$ twisted by the exterior automorphism $\theta: g \mapsto {}^t\bar{g}^{-1}$. Following Arthur, this goes through the use of the stable trace formula. The notation is as in the preceding paragraph except that we will use $\widetilde{G}$ to denote the twisted linear group as in [59].

A.1.1. Discrete parts of trace formulae

Let $f=\bigotimes_v f_v$ be a decomposable smooth function with compact support in $G(\mathbb{A})$. We are essentially interested in

$$\operatorname{trace} R^G_{\mathrm{dis}}(f) = \operatorname{trace} f|_{L^2_{\mathrm{dis}}(G(F)\backslash G(\mathbb{A}))}, \tag{A.1}$$

where L^2_{dis} is the discrete part of the space of automorphic forms on $G(F)\backslash G(\mathbb{A})$. We fix a positive real number t and (following Arthur) we will only compare sums, relative to G and $\mathrm{GL}(m)$, over representations whose infinitesimal character has norm $\leqslant t$.

For a fixed t, Arthur defines a distribution $f \mapsto I^G_{\mathrm{disc},t}(f)$ as a sum of the part of (A.1) relative to t and of terms associated with some representations induced from Levi subgroups; see [59, formula (4.1.1)]. There is an analogous distribution $f \mapsto \tilde{I}^m_{\mathrm{disc},t}(f)$ in the twisted case; see [59, formula (4.1.3)]. Proposition 12.1 is based on the comparison of these two distributions via their stabilized versions.

A.1.2. Stable distribution, endoscopic groups and transfer

Fix a place v and let $G=G(F_v)$. Langlands and Shelstad [50] have conjectured the existence of a remarkable family of maps—or rather of correspondences—$f\rightsquigarrow f^H$ which transfer functions on G to functions on so-called endoscopic groups H, certain quasi-split groups of dimension smaller than $\dim G$. These maps are an analytic counterpart to the fact that non-conjugate elements in G can be conjugate over the algebraic closure $G(\overline{F}_v)$. This goes through the study of orbital integrals.

Two functions in $\mathcal{H}(G)$ are *equivalent*, resp. *stably equivalent*, if they have the same orbital integrals, resp. stable orbital integrals (see [2]). We denote by $\mathcal{I}(G)$, resp. $\mathcal{SI}(G)$, the corresponding orbit space. It is known that invariant distributions on G annihilate any function $f\in\mathcal{H}(G)$ such that all the orbital integrals of f vanish. An invariant distribution is a *stable distribution* on G if it annihilates any function $f\in\mathcal{H}(G)$ such that all the stable orbital integrals of f vanish. Equivalently, it is an invariant distribution which lies in the closed linear span of the stable orbital integral; see e.g. [50]. The theory of endoscopy describes invariant distributions on G in terms of stable distributions on certain groups of dimension less than or equal to G—the endoscopic groups.

Due to the recent proofs by Ngô [60] of the fundamental lemma and Waldspurger's and Shelstad's work, the Langlands–Shelstad conjecture is now a theorem. We need the more general version which includes the disconnected group $\widetilde{G}$, and which is also known due to the same authors [72], [65], [75]. In both cases endoscopic groups are described in [73]; see also [59]. The group G appears as a (principal) endoscopic subgroup of $\widetilde{G}$; this is the key point to relate representations of G and $\mathrm{GL}(m)$. In what follows, we denote by $\tilde{\mathcal{I}}(m)$ and $\widetilde{\mathcal{SI}}(m)$ the quotients of $\widetilde{\mathcal{H}}(m)$ defined as above.

A.1.3. Local stabilization

Let $\mathcal{I}_{\mathrm{cusp}}(G)$ be the image in $\mathcal{I}(G)$ of the *cuspidal* functions on $\mathcal{H}(G)$, i.e. the functions whose orbital integrals associated with semi-simple elements contained in a proper parabolic subgroup all vanish. We similarly define $\tilde{\mathcal{I}}_{\mathrm{cusp}}(m)$ (see [74]).

Arthur [2] then stabilizes $\mathcal{I}_{\mathrm{cusp}}(G)$. In particular he defines the stable part $\mathcal{SI}_{\mathrm{cusp}}(G)$ of $\mathcal{I}_{\mathrm{cusp}}(G)$, and similarly for all the endoscopic groups (or rather endoscopic data). The transfer maps $f\mapsto\oplus_H f^H$ induce a linear isomorphism

$$\mathcal{I}_{\mathrm{cusp}}(G)\cong\bigoplus_H \mathcal{SI}_{\mathrm{cusp}}(H)^{\mathrm{Out}_G(H)}, \tag{A.2}$$

where we sum over the endoscopic groups (or rather endoscopic data) not forgetting G

itself. The twisted analogue of (A.2) is

$$\tilde{\mathcal{I}}_{\mathrm{cusp}}(m) \cong \bigoplus_{H} \mathcal{SI}_{\mathrm{cusp}}(H)^{\mathrm{Out}_{\widetilde{G}}(H)}; \tag{A.3}$$

see [72], where Waldspurger deals with a much more general situation.

We say that a virtual representation π—in the Grothendieck ring (with complex coefficients) of the representations of G—is *stable* if $f \mapsto \operatorname{trace} \pi(f)$ is a stable distribution. Assume that π is a finite linear combination of elliptic representations of G, then, by [2], π is stable if and only if trace $\pi(f)=0$ for any $f \in \mathcal{I}_{\mathrm{cusp}}(G)$ in the kernel of the projection of $\mathcal{I}_{\mathrm{cusp}}(G)$ onto $\mathcal{SI}_{\mathrm{cusp}}(G)$ in the above decomposition. Moreover, the map $f \mapsto (\pi \mapsto \operatorname{trace} \pi(f))$ induces an isomorphism from $\mathcal{SI}_{\mathrm{cusp}}(G)$ to the space of linear forms with finite support on the elliptic and stable virtual representations.

Now let π be any virtual representation of G that is a finite combination of elliptic representations of G. Its distribution character defines a distribution on $\mathcal{I}_{\mathrm{cusp}}(G)$ and it follows from (A.2) that, for every endoscopic data H, we may associate with π a virtual representation π_{st}^H such that for every $f \in \mathcal{I}_{\mathrm{cusp}}(G)$ we have

$$\operatorname{trace} \pi(f) = \sum_{H} \operatorname{trace} \pi_{\mathrm{st}}^H (f^H). \tag{A.4}$$

We denote by π_{st} the virtual representation π_{st}^G. This is a stable, elliptic representation.

In the Archimedean case, it follows from [64] that if π is a discrete series representation of G, then the virtual representation π_{st} is the sum of the representations in the L-packet of π. Moreover, if π is only elliptic but not discrete, then $\pi_{\mathrm{st}}=0$.

A.1.4. Stable standard modules

If $\varrho=(M,\sigma)$ is a standard module with σ elliptic, we define the associated *stable standard module* to be the virtual representation obtained as the induced module $\operatorname{ind}_M \sigma_{\mathrm{st}}$. If π is any irreducible admissible representation of G, we denote by π_{st} the stable standard module associated with ϱ_π. This is a virtual representation induced from a Langlands' packet of discrete series. The next proposition again follows from [2].

Proposition A.1. *The set of stable standard modules is a basis of the vector space of stable distributions that are supported on a finite set of characters of G.*

Let H be an endoscopic data in $\widetilde{G}$. Given an admissible irreducible representation π of H, as H is a product of unitary groups, we have associated with it a stable standard module π_{st}. We may transfer π_{st} to a standard module for $\widetilde{G}$ and therefore for $\mathrm{GL}_m(E_v)$. This transfer is the standard module of $\mathrm{GL}_m(E_v)$ associated with any standard module occuring in π_{st} by local principal base change, and we denote it by $\mathrm{St}(\pi)$.

We now come back to the global situation.

A.1.5. Stabilization of trace formulae

The stabilization of the distributions $I^G_{\mathrm{disc},t}$ refers to a decomposition

$$I^G_{\mathrm{disc},t}(f)=\sum_{H\in\mathcal{E}_{\mathrm{ell}}(G)}\iota(G,H)S^H_{\mathrm{disc},t}(f^H),\quad f\in\mathcal{H}(G); \tag{A.5}$$

see e.g. [59, formula (4.2.1)]. Here we sum over a set of representatives of endoscopic data in G, we denote by f^H the Langlands–Kottwitz–Shelstad transfer of f to H and, for every H, $S^H_{\mathrm{disc},t}$ is a stable distribution. The coefficients $\iota(G,H)$ are positive rational numbers.

Similarly, the stabilization of the distributions $\tilde{I}^m_{\mathrm{disc},t}$ refers to a decomposition

$$\tilde{I}^m_{\mathrm{disc},t}(\tilde{f})=\sum_{H\in\tilde{\mathcal{E}}_{\mathrm{ell}}(m)}\tilde{\iota}(m,H)S^H_{\mathrm{disc},t}(\tilde{f}^H),\quad \tilde{f}\in\widetilde{\mathcal{H}}(m). \tag{A.6}$$

Now we fix a finite set S of places of F which contains all the Archimidean places of F. We moreover assume that S contains all the ramification places of G. If $v\notin S$, $G\times F_v$ is isomorphic to the (quasi-split) group $G(F_v)$ and splits over a finite unramified extension of F_v; in particular $G\times F_v$ contains a hyperspecial compact subgroup K_v (see [69, §1.10.2]). Let $\mathcal{H}_v$ be the corresponding (spherical) Hecke algebra and let $\mathcal{H}^S=\prod_{v\notin S}\mathcal{H}_v$.

We may decompose (A.5) according to characters of $\mathcal{H}^S$. Namely, for $f\in\mathcal{H}^S$, we have a decomposition (see [59, formula (4.3.1)])

$$I^G_{\mathrm{disc},t}(f)=\sum_{c^S}I^G_{\mathrm{disc},c^S,t}(f),$$

where $c^S=(c_v)_{v\in S}$ runs over a family of compatible Satake parameters—called Hecke eigenfamilies in [3], [59]—consisting of those families that arise from automorphic representation of $G(\mathbb{A})$, and $I^G_{\mathrm{disc},c^S,t}$ is the c^S eigencomponent of $I^G_{\mathrm{disc},t}$. It then follows from [3, Lemma 3.3.1] or [59, Lemma 4.3.2] that

$$I^G_{\mathrm{disc},c^S,t}(f)=\sum_{H\in\mathcal{E}_{\mathrm{ell}}(G)}\iota(G,H)S^H_{\mathrm{disc},c^S,t}(f^H), \tag{A.7}$$

where on the right-hand side c^S is rather the Hecke eigenfamily for H which corresponds to c^S under the L-embedding ${}^LH\to{}^LG$ that is part of the endoscopic data, and $S^H_{\mathrm{disc},c^S,t}$ is the stable part of the trace formula for H restricted to the representations which are unramified outside S and belong to the c^S eigencomponent.

Similarly, in the twisted case we have

$$\tilde{I}^m_{\mathrm{disc},c^S,t}(\tilde{f})=\sum_{H\in\tilde{\mathcal{E}}_{\mathrm{ell}}(m)}\tilde{\iota}(m,H)S^H_{\mathrm{disc},c^S,t}(\tilde{f}^H). \tag{A.8}$$

A Hecke eigenfamily c^S determines at most one irreducible automorphic representation $\Pi=\Pi_\Psi$ of $\mathrm{GL}_m(\mathbb{A}_E)$ such that, for every $v\notin S$, the unramified representation Π_v corresponds to the Satake parameter c_v. We write $\Pi=0$ or $\Psi=0$ if Π does not exist. Note that Proposition 12.1 states that if $\Pi=0$ then the c^S eigencomponent of the discrete part of $L^2(G(F)\backslash G(\mathbb{A}))$ is trivial. In fact Arthur (and Mok) prove that, if $\Pi=0$, then for all $f\in\mathcal{H}^S$ we have

$$I^G_{\mathrm{disc},c^S,t}(f)=0=S^G_{\mathrm{disc},c^S,t}(f).$$

The proof goes by induction on m; see [3, Proposition 3.4.1] and [59, Proposition 4.3.4]. Following their proof, we now prove the following refined version of Proposition 12.1.

Proposition A.2. *Let π be an irreducible automorphic representation of $G(\mathbb{A})$ which occurs (discretely) as an irreducible subspace of $L^2(G(F)\backslash G(\mathbb{A}))$. Assume that for every Archimedean place v the representation π_v has regular infinitesimal character. Let S be a finite set of places—including all the Archimedean ones—such that π is unramified outside S and belongs to the c^S eigencomponent of the discrete part of $L^2(G(F)\backslash G(\mathbb{A}))$, and let $\Pi=\Pi_\Psi$ be the associated automorphic representation. Then, for every Archimedean place v, the (unique) irreducible quotient of the standard $\mathrm{GL}_m(\mathbb{C})$-module $\mathrm{St}(\pi_v)$ occurs as an (irreducible) subquotient of Ψ_v.*(17)

Remark. If $\Psi=0$ it follows in particular from the proposition that π cannot exist as proved in [59, Proposition 4.3.4].

Proof. Mok [59] has proved that $I^H_{\mathrm{disc},c^S,t}$ is of finite length and we will freely use that fact to simplify the proof. Let $v\in S$ be an Archimedean place and let $\pi_S\in I^H_{\mathrm{disc},c^S,t}$ be an irreducible representation. Denote by γ_v the collection of m characters of $\mathbb{C}^*$ obtained as the restriction of the Langlands parameter of π_v to $\mathbb{C}^*$; considering this collection as a Langlands parameter for a representation of $\mathrm{GL}(m,\mathbb{C})$, we obtain the Langlands parameter of the representation $\mathrm{St}(\pi_v)$ of $\mathrm{GL}(m,\mathbb{C})$. We recall that Ψ_v is the local component of the representation of $\mathrm{GL}_m(\mathbb{A})$ defined by c^S. The proposition is a corollary of Proposition 12.1 and the following lemma. □

Lemma A.3. *The Langlands quotient of $\mathrm{St}(\pi_v)$ is a subquotient of the standard module associated with Ψ_v.*

Proof. We call an irreducible representation included in $I^H_{\mathrm{disc},c^S,t}$ *maximal* if this representation does not appear as a subquotient of the standard module (which at each place is the product of the local standard modules) of another representation entering $S^H_{\mathrm{disc},c^S,t}$. We will prove the lemma for a maximal representation. Denote by π'_S such a representation.

(17) Here Ψ_v is considered as a standard module; see §12.7.

We first prove that if the lemma holds for any maximal element of $I^H_{\mathrm{disc},c^S,t}$, then it holds for any element of $I^H_{\mathrm{disc},c^S,t}$. Take any element π_S of $I^H_{\mathrm{disc},c^S,t}$, and assume that its standard module occurs in the decomposition of the standard module of the maximal element π'_S of $I^H_{\mathrm{disc},c^S,t}$ and that we know the lemma for π'_S. Here we use the deep result of Salamanca–Riba [63]: if π'_v is unitary and has an infinitesimal character which is integral and regular, then π'_v has cohomology. So π'_v is of the form $A_{\mathfrak{q}'}(\lambda')$ and Johnson has in his thesis [38] computed the decomposition of such a module in terms of standard modules. We provide details below (with explicit parameters).

Denote by γ'_v the analogue of γ_v (as defined before the lemma) for π'. We recall that the normalizer of $\mathfrak{q}'$ in H is a product of unitary groups $\prod_{i=1}^{\ell} U(p_i,q_i)$ and that λ' gives a character of this group, that is a set of half integers r_i for $i{=}[1,\ell]$. It is not necessary to know exactly what γ'_v is. In fact it is enough to know that it is a collection of characters $z^{x_{i,j_i}}\bar{z}^{x'_{i,j_i}}$, with $i{\in}[1,\ell]$ and $j_i{\in}[1,m_i]$, where $m_i{:=}(p_i{+}q_i)$, satisfying

$$\{x_{i,j_i}:j_i\in[1,m_i]\}=\bigl\{r_i+k:k\in\bigl[\tfrac12(m_i-1),-\tfrac12(m_i-1)\bigr]\bigr\} \tag{A.9}$$

$$\{x_{i,j_i}:j_i\in[1,m_i]\}=\bigl\{-r_i+k:k\in\bigl[\tfrac12(m_i-1),-\tfrac12(m_i-1)\bigr]\bigr\}. \tag{A.10}$$

We also know that Ψ_v is an induced representation of unitary characters. Thus we can decompose $m{=}\sum_{t=1}^{\ell'} m_t$ and for any t we have a unitary character of $\mathbb{C}^*$, $(z/\bar{z})^{n_t}$. The subquotients of the standard module Ψ_v are exactly the representations whose Langlands' parameters are collections of m characters of $\mathbb{C}^*$ that can be partitioned in ℓ' subsets such that in each subset, indexed by t, the characters are of the form $z^x\bar{z}^{x'}$ with

$$x\in n_t+\bigl[\tfrac12(m_t-1),-\tfrac12(m_t-1)\bigr]\quad\text{and}\quad x'\in -n_t+\bigl[\tfrac12(m_t-1),-\tfrac12(m_t-1)\bigr].$$

Recall that by hypothesis the lemma holds for π'_v. Now, for each $i{\in}[1,\ell]$, there exists $t{\in}[1,\ell']$ such that

$$\bigl\{r_i+k:k\in\bigl[\tfrac12(m_i-1),-\tfrac12(m_i-1)\bigr]\bigr\}\subset n_t+\bigl[\tfrac12(m_t-1),-\tfrac12(m_t-1)\bigr],$$

and, by symmetry,

$$\bigl\{-r_i+k:k\in\bigl[\tfrac12(m_i-1),-\tfrac12(m_i-1)\bigr]\bigr\}\subset -n_t+\bigl[\tfrac12(m_t-1),-\tfrac12(m_t-1)\bigr].$$

To prove the lemma for π_v, we therefore only have to prove that the Langlands' parameter γ_v of π_v is a collection of m characters of $\mathbb{C}^*$ which can be decomposed in ℓ subsets satisfying exactly the property (A.9) and (A.10) above with the same numbers. But this follows from Johnson's thesis: in the exact sequence [38, assertion (3), p. 378], the standard modules which appear all satisfy $\Delta^+{\supset}\Delta(\bar{\mathfrak{u}})$ where $\bar{\mathfrak{u}}$ is the nilradical of the opposite parabolic subalgebra of $\mathfrak{q}$. Our assertion follows (we can permute inside the blocks but not between two different blocks).

Remark. This is not mysterious, at least in our case. It is just the fact that the induced representation of $\mathrm{GL}_2(\mathbb{C})$ of two characters $z^x\bar{z}^{x'}$ and $z^y\bar{z}^{y'}$ is irreducible if $x>y$ but $x'<y'$ or the symmetric relations and this is precisely what occurs if the two characters are in subsets indexed by j and j', with $j\neq j'$.

We now prove the lemma for maximal representations. When we decompose $I^H_{\mathrm{disc},c^S,t}$ in terms of standard modules, we are sure that the standard module associated with any maximal representation π'_S occurs with the same coefficient as the representation itself. We now look at the coefficient with which the stable standard module of π'_S occurs in $S^H_{\mathrm{disc},c^S,t}$. Up to a positive global constant $i(H)$, it is either the same coefficient as in $I^H_{\mathrm{disc},c^S,t}$ or π'_S comes from endoscopy. In this latter case we argue by induction because endoscopic groups are product of smaller unitary groups. So we assume that the standard module of π'_S occurs in $S^H_{\mathrm{disc},c^S,t}$ with the coefficient $i(H)$ times the multiplicity of π in $I^H_{\mathrm{disc},c^S,t}$. We now look at the sum of all H and decompose in the Grothendieck group; the standard module of π'_S can be canceled by a representation occurring in the decomposition of the standard module of $\pi_S^{H'}$, where $\pi_S^{H'}$ occurs in $S^{H'}_{\mathrm{disc},c^S,t}$. If this does not occur, then by transfer the standard module of π'_S appears in the decomposition of Ψ_S, and we are done for π'_S. (In fact in that case what we get is stronger than the statement of the lemma.)

In the case where we have a simplification, by positivity, $\pi_S^{H'}$ is not maximal for H' but, by an easy induction, we know the lemma for $\pi_S^{H'}$. We argue explicitly as above that this also proves the lemma for π_S, we leave the details to the reader especially as the deep results of Mok ultimately yield that this cannot happen: the representation of $\mathrm{GL}_m(\mathbb{A})$ determined by c^S is the transfer of a unique endoscopic group. $\square$

A.2. The general (non-quasi-split) case

The notation is as in the preceding two paragraphs. We do not assume G to be quasi-split anymore.

Let π be an irreducible automorphic representation of $G(\mathbb{A})$ which occurs (discretely) as an irreducible subspace of $L^2(G(F)\backslash G(\mathbb{A}))$. Let S be a finite set of places—including all the Archimedean ones—such that both G and π are unramified outside S. It still makes sense to consider a Hecke eigenfamily (outside S) c^S. Again it determines at most one irreducible automorphic representation $\Pi=\Pi_\Psi$ of $\mathrm{GL}_m(\mathbb{A}_E)$ such that for every $v\notin S$ the unramified representation Π_v corresponds to the Satake parameter c_v, and we write $\Pi=0$, or $\Psi=0$, if Π_Ψ does not exist.

We may restate Theorem 12.2 in the following way.

Theorem A.4. *Let π be an irreducible automorphic representation of $G(\mathbb{A})$ which* *occurs (discretely) as an irreducible subspace of* $L^2(G(F)\backslash G(\mathbb{A}))$. *Assume that for every Archimedean place* v *the representation* π_v *has regular infinitesimal character. Let* S *be a finite set of places—including all the Archimedean ones—such that both* G *and* π *are unramified outside* S. *Let* c^S *be the Hecke eigenfamily associated with* π *and let* Π_Ψ *be the associated automorphic representation of* $\mathrm{GL}(m)$. *Then, for every Archimedean place* v, *the (unique) irreducible quotient of the standard* $\mathrm{GL}_m(\mathbb{C})$*-module* $\mathrm{St}(\pi_v)^{\mathrm{BC}}$ *occurs as an (irreducible) subquotient of the local standard representation* Ψ_v.

Proof. Here we use the stable trace formula (A.7) for the group G. We write the left-hand side of (A.7) as a linear combination of standard modules. The contribution of the standard module associated with π_v might be zero but then there must exist π', an irreducible automorphic representation of $G(\mathbb{A})$ which occurs (discretely) as an irreducible subspace of $L^2(G(F)\backslash G(\mathbb{A}))$, such that π'_v and π_v share the same infinitesimal character and π_v is a subquotient of the standard module associated with π'_v. It is then enough to prove the theorem for π'. From now on we will therefore assume that the standard module associated with π_v contributes to the left-hand side of (A.7).

We now write the right-hand side as a sum of stable standard modules. At least one of these has an L-parameter whose restriction to $\mathbb{C}^\times$ is the parameter of the standard module of π_v. At this point it is not clear that this stable standard module is associated with a square integrable automorphic representation of an endoscopic group. Using the same induction as above one may however assume this is the case. Now since endoscopic groups are (products of) quasi-split unitary groups the theorem follows from the quasi-split case (see the proposition above). □

Appendix B. Proof of Theorem 11.2

We fix s_0 and a character $\chi=\chi_2$ of $\mathbb{A}_E^\times/E^\times$ as in Theorem 11.2. Let $n=m-2s_0$ and $n'=m+2s_0$, so that

$$-s_0=\tfrac{1}{2}(m-n').$$

Note that we have

$$m<n'<2m \quad \text{and} \quad \chi|_{\mathbb{A}_F^\times}=\varepsilon_{E/F}^{n}=\varepsilon_{E/F}^{n'}.$$

B.1. Representations associated with skew-Hermitian spaces

In this paragraph we fix a prime and omit it from notation as in §10.6. Recall that the isometry class of a τ-skew-Hermitian space W of dimension n over E is determined by the Hasse invariant $\varepsilon=\varepsilon(W)=\pm 1$. We will write W_1 and W_2 for the two distinct τ-skew-Hermitian spaces of dimension n over E.

We fix an arbitrary choice of character χ_1 of $E^\times$ such that $\chi_1|_{F^\times}=\varepsilon_{E/F}^m$. We may consider the local analogue of the Weil representation Ω_χ^- of the preceding paragraph; see (10.16). This yields a representation of $U_{2m}(F)\times U_n(F)$ on $\mathcal{S}(W^m)$. The group $U_n(F)$ acts via a twist χ_1 of its linear action on $\mathcal{S}(W^m)$. Let $R(W,\chi)$ be the maximal quotient of $\mathcal{S}(W^m)$ on which $U_n(F)$ acts by χ_1. Kudla and Sweet [48] show that $R(W,\chi)$ is an admissible representation of $U_{2m}(F)$ of finite length and with a unique irreducible quotient. Moreover, since $1\leqslant n\leqslant m$, it follows from [41, Theorem 1.2] that $R(W_1,\chi)$ and $R(W_2,\chi)$ are irreducible inequivalent representations of $U_{2m}(F)$.

B.2. Automorphic representations associated with skew-Hermitian spaces

We now return to the global situation. Assume that $1\leqslant n<m$ and let $\mathcal{C}=\{W_v\}_v$ be a collection of local skew-Hermitian spaces of dimension n such that W_v is unramified outside of a finite set of places of F, then—following Kudla–Rallis [47, §3]—we may define a global irreducible representation

$$\Pi(\mathcal{C})=\bigotimes_v R(W_v,\chi_v) \tag{B.1}$$

of $U_{2m}(\mathbb{A})$. Such representations are of two types: those for which the W_v's are the localizations of some (unique) global skew-Hermitian space W over E—in this case we write $\Pi(W)$ for $\Pi(\mathcal{C})$—and those for which no such global space exists. Given a collection $\mathcal{C}=\{W_v\}_v$, the obstruction to the existence of a global space is just the requirement that

$$\prod_v \varepsilon(W_v)=1.$$

Let $\mathcal{A}(U(V\oplus V))$ be the set of irreducible automorphic representations of $U_{2m}(\mathbb{A})$. The following proposition is the analogue of [47, Theorem 3.1] in the unitary case.

PROPOSITION B.1. *Let $\mathcal{C}=\{W_v\}_v$ be a collection of local skew-Hermitian spaces of dimension n such that W_v is unramified outside of a finite set of places of F and*

$$\dim \operatorname{Hom}_{U_{2m}(\mathbb{A})}(\Pi(\mathcal{C}),\mathcal{A}(U(V\oplus V)))\neq 0.$$

Then, there exists a global skew-Hermitian space W over E such that the W_v's are the localizations of W.

Proof. The proof makes use of ideas of Howe. We consider the Fourier coefficients of automorphic representations with respect to the unipotent radical of the Siegel parabolic subgroup $P=MN$ of the quasi-split unitary group $U(V\oplus V)$. Identifying the latter with the subgroup of $\mathrm{GL}(2m,E)$, which preserves the Hermitian form with matrix $\left(\begin{smallmatrix} 0 & 1_m \\ -1_m & 0 \end{smallmatrix}\right)$, we have

$$N=\left\{\begin{pmatrix} 1_m & b \\ 0 & 1_m \end{pmatrix} : b\in \mathrm{Her}_m(F)\right\},$$

where

$$\mathrm{Her}_m(F)=\{b\in M_m(F): b={}^t\bar{b}\}.$$

For $\beta\in\mathrm{Her}_m(F)$ we define the character ψ_β of $N(\mathbb{A})$ by

$$n(b)=\begin{pmatrix} 1_m & b \\ 0 & 1_m \end{pmatrix}\longmapsto \psi(\mathrm{trace}(b\beta)),$$

and the βth Fourier coefficient of $f\in\mathcal{A}(U(V\oplus V))$ by

$$W_\beta(f)(g)=\int_{N(F)\backslash N(\mathbb{A})} f(n(b)g)\psi(-\mathrm{trace}(\beta b))\, db. \tag{B.2}$$

The latter defines a linear functional

$$\begin{aligned} W_\beta: \mathcal{A}(U(V\oplus V)) &\longrightarrow \mathbb{C}, \\ f &\longmapsto W_\beta(f)(e). \end{aligned}$$

Now, if $A\in\mathrm{Hom}_{U_{2m}(\mathbb{A})}(\Pi(\mathcal{C}),\mathcal{A}(U(V\oplus V)))$, then $A_\beta=W_\beta\circ A$ defines a linear functional on $\Pi(\mathcal{C})$ such that

$$\begin{aligned} A_\beta(\pi(n)f) &= \psi_\beta(n)A_\beta(f), && \text{for all } n\in N(\mathbb{A}_f), \\ A_\beta(\pi(X)f) &= d\psi_\beta(X)A_\beta(f), && \text{for all } X\in \mathrm{Lie}(N_\infty), \end{aligned}$$

and A_β has continuous extension to the smooth vectors of $\Pi(\mathcal{C})$.

Let

$$\Omega_\beta=\{w\in W^m : \langle w,w\rangle=\beta\}.$$

It follows from [36, Lemmas 5.1 and 5.2] on the local functionals that, if $\Omega_\beta=\varnothing$, then $A_\beta=0$. In particular, if $\mathrm{rank}(\beta)>n$, then $A_\beta=0$. Now, if $\Omega_\beta\neq\varnothing$ and $\mathrm{rank}(\beta)=n$, we must have equality of Hasse invariants $\varepsilon_v(\beta)=\varepsilon_v(W_v)$. So that either $\mathcal{C}$ correspond to a global skew-Hermitian space, or $A_\beta=0$ for all β with $\mathrm{rank}(\beta)\geqslant n$. The proposition therefore follows from the next lemma. $\square$

Lemma B.2. *Let* $A\in\mathrm{Hom}_{U_{2m}(\mathbb{A})}(\Pi(\mathcal{C}),\mathcal{A}(U(V\oplus V)))$ *be such that* $A_\beta=0$ *for all* β *with* $\mathrm{rank}(\beta)\geqslant n$. *Then* $A=0$.

Proof. Fix a non-Archimedean place v and let ϕ be a compactly supported function on N_v whose Fourier transform vanishes on the set of $\beta_v\in\mathrm{Her}_m(F_v)$ with $\mathrm{rank}(\beta_v)<n$, and is non-zero on the set of β_v such that $\Omega_{\beta_v}\neq\varnothing$. Then [36, Lemma 5.1] shows that ϕ does not act by zero in $R(W_v,\chi_v)$ or $\Pi(\mathcal{C})$. On the other hand, by hypothesis, ϕ acts by zero in the image of A. Thus $A=0$ by irreducibility of $\Pi(\mathcal{C})$. ☐

B.3. Proof of Theorem 11.2

If Φ is a section of $I(s_0,\chi)$, we may extend it to a holomorphic section $\Phi(\cdot,s)$ of $I(s,\chi)$ and consider the residue $A(\cdot,\Phi)$ in $s=s_0$ of the Siegel Eisenstein series $E(h,s,\Phi)$. This residue does not depend on the holomorphic extension. We therefore get a $U_{2m}(\mathbb{A})$-intertwining map $A\colon I(s_0,\chi)\to\mathcal{A}(U(V\oplus V))$. Now it follows from [36, Lemma 6.1] that this map factors through the quotient

$$I(s_0,\chi)_\infty\otimes\bigg(\bigoplus_{\mathcal{C}}\Pi_f(\mathcal{C})\bigg),$$

where $\mathcal{C}$ runs over all collections of local skew-Hermitian spaces, as above, of dimension n. The proposition above therefore associates with any irreducible residue of a Siegel Eisenstein series a global space W of dimension n over E. Theorem 11.2 then follows from [36, Theorem 4.1].(18)

Appendix C. The local product formula

In this appendix we show that the cocycles $\psi_{nq,nq}$ of this paper (when transformed into cocycles with values in the appropriate Schrödinger model) are equal to the cocycles $\varphi_{nq,nq}$ introduced by Kudla–Millson in [44]; see also [46]. We stated this relation without proof in Proposition 5.17 (5). It is almost proved in [46]: on the fifth line of page 158 the cocycle $\varphi_{q,q}$ is defined by (see also (C.5) below)

$$\varphi_{q,q}=\frac{1}{2^{2q}}D^+\overline{D}^+\varphi_0.$$

Then on the next page, Theorem 5.2 (ii), it is stated that

$$\varphi_{nq,nq}=\varphi_{q,q}\wedge\ldots\wedge\varphi_{q,q}. \tag{C.1}$$

(18) Beware that our W is Ichino's V'.

The previous equation reduces the problem to proving the equality of cocycles for the case $n=1$. This is because the intertwiner $B_{V_+\otimes W}\otimes 1$ below commutes with the outer exterior product and the cocycle $\psi_{nq,nq}$ factors as above by definition. In what follows we will be using the symplectic vector space obtained from the tensor product of Hermitian spaces $V\otimes W$ where $(V,(\cdot\,,\cdot)_V)$ is a Hermitian space of signature (p,q) and $(W,(\cdot\,,\cdot)_W)$ is a Hermitian space of signature $(1,1)$. Since the analysis in what follows will be controlled by W, with V essentially a parameter space, we will first describe the required structures on W alone.

C.1. The Schrödinger and Fock models of the Weil representation for $U(W)$

We consider a Hermitian space $(W,(\cdot\,,\cdot)_W)$ of signature $(1,1)$. We let $W_{\mathbb{R}}$ denote the real vector space underlying W. Then $W_{\mathbb{R}}$ has the standard integrable almost complex structure J_W induced by multiplication by i. It is also equipped with the symplectic form

$$\langle\cdot\,,\cdot\rangle_W = -\operatorname{Im}(\cdot\,,\cdot)_W.$$

Finally, recall that we denote by θ_W the Cartan involution of W and let

$$J_0 = J_W\circ\theta_W = \theta_W\circ J_W.$$

Then J_0 is positive definite with respect to $\langle\cdot\,,\cdot\rangle_W$.

We now describe two bases for W. Let $\{\varepsilon_1,\varepsilon_2\}$ be the orthogonal complex basis of W such that

$$(\varepsilon_1,\varepsilon_1)=1 \quad\text{and}\quad (\varepsilon_2,\varepsilon_2)=-1.$$

Set

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{2}}(\varepsilon_1 - i\varepsilon_2), & e_2 &= \frac{1}{\sqrt{2}}(i\varepsilon_1+\varepsilon_2) = J(e_1),\\ f_1 &= \frac{1}{\sqrt{2}}(i\varepsilon_1-\varepsilon_2) = J_0(e_1), & f_2 &= \frac{1}{\sqrt{2}}(-\varepsilon_1 - i\varepsilon_2) = J_0(e_2). \end{aligned}$$

Then $\{e_1,e_2,f_1,f_2\}$ is a (real) symplectic basis of the underlying real vector space $W_{\mathbb{R}}$.

Let $E=\operatorname{span}_{\mathbb{R}}(e_1,e_2)$ and $F=J_0(E)=\operatorname{span}_{\mathbb{R}}(f_1,f_2)$. Then

$$W_{\mathbb{R}}=E+F \tag{C.2}$$

is a Lagrangian splitting of the symplectic vector space, and we obtain a Schrödinger model $\mathcal{S}(E)$ of the Weil representation of $U(W)$ realized in the Schwartz space $\mathcal{S}(E)$. We let x and y be the coordinates of E associated with the basis $\{e_1,e_2\}$, and set $z=x+iy$. We let $\mathcal{P}(E)$ denote the subspace of $\mathcal{S}(E)$ given by products of complex-valued polynomials in x and y, with the Gaussian $\varphi_0=\exp(-\pi(x^2+y^2))$, or equivalently by products of complex-valued polynomials in z and $\bar{z}$, with the Gaussian $\varphi_0=\exp(-\pi z\bar{z})$.

We next give two sets of coordinates for the space $W'^{\,0}$ associated with the positive complex structure J_0. We first note that

$$e_1'^{\,0}=\frac{1}{\sqrt{2}}(\varepsilon_1'^{\,0}+i\varepsilon_2'^{\,0})\quad\text{and}\quad e_2'^{\,0}=\frac{i}{\sqrt{2}}(\varepsilon_1'^{\,0}-i\varepsilon_2'^{\,0}).$$

Hence, the vectors $e_1'^{\,0}$ and $e_2'^{\,0}$ are independent (over $\mathbb{C}$) and we have two bases for $W'^{\,0}$, the basis $\{e_1'^{\,0},e_2'^{\,0}\}$ and the basis $\{\varepsilon_1'^{\,0},\varepsilon_2'^{\,0}\}$. We let s and t be the (complex) coordinates for $W'^{\,0}$ relative to the first basis. We will call these coordinates *split* coordinates. We let a' and b'' be the coordinates for $W'^{\,0}$ relative to the second basis $\varepsilon_1'^{\,0}$ and $\varepsilon_2'^{\,0}$. We will call a' and b'' *product* coordinates. In order to understand the superscripts attached to these coordinates and the terminology, note that $\varepsilon_1'^{\,0}=\varepsilon_1'$ and $\varepsilon_2'^{\,0}=\varepsilon_2''$, and hence

$$W'^{\,0}=W'_+\oplus W''_-=\mathbb{C}\varepsilon_1'^{\,0}\oplus\mathbb{C}\varepsilon_2'^{\,0}$$

and

$$\operatorname{Pol}(W'^{\,0})=\operatorname{Pol}(W'_+)\otimes\operatorname{Pol}(W''_-)\cong\mathbb{C}[a']\otimes\mathbb{C}[b''].$$

We next note that the split coordinates and the product coordinates are related by the following result.

Lemma C.1. *We have*

$$a'=\frac{1}{\sqrt{2}}(s+it)\quad\textit{and}\quad b''=\frac{i}{\sqrt{2}}(s-it).$$

We now have the following result.

Lemma C.2. *There is a* $\mathfrak{u}(W)$*-equivariant mapping* $B_W\colon\operatorname{Pol}(W'^{\,0})\to\mathcal{P}(E)$ *satisfying the following conditions*:

(1)

$$B_W(1)=\varphi_0;$$

(2)
$$B_W \circ s \circ B_W^{-1} = x - \frac{1}{\pi}\frac{\partial}{\partial x};$$

(3)
$$B_W \circ t \circ B_W^{-1} = y - \frac{1}{\pi}\frac{\partial}{\partial y};$$

(4)
$$B_W \circ \frac{1}{\pi}\frac{\partial}{\partial s} \circ B_W^{-1} = x + \frac{1}{\pi}\frac{\partial}{\partial x};$$

(5)
$$B_W \circ \frac{1}{\pi}\frac{\partial}{\partial t} \circ B_W^{-1} = y + \frac{1}{\pi}\frac{\partial}{\partial y}.$$

Hence, by Lemma C.1, we have the following.

LEMMA C.3. *For B_W, a' and b'' as above, we have*

(1)
$$B_W \circ a' \circ B_W^{-1} = \frac{1}{\sqrt{2}}\left(z - \frac{1}{\pi}\frac{\partial}{\partial \bar{z}}\right);$$

(2)
$$B_W \circ b'' \circ B_W^{-1} = \frac{1}{\sqrt{2}}\left(\bar{z} - \frac{1}{\pi}\frac{\partial}{\partial z}\right).$$

C.2. The Schrödinger and Fock models for $U(V)\times U(W)$

We now describe and compare two different realizations of the infinitesimal Weil representation associated with the pair $U(V)\times U(W)$, the split Schrödinger model (there is another Schrödinger model for Hermitian spaces, the real points in $V\otimes W$, that we will not use here, but it is the one used in [9, Chapter VIII]) and the Fock model (with two sets of coordinates). So we now need to bring the space V into the picture. We will be brief since all the essential ideas are contained in the previous section.

C.2.1. The split Schrödinger model for $U(V)\times U(W)$

The split Schrödinger model is realized in the Schwartz space $\mathcal{S}((V\otimes E)_{\mathbb{R}})$ using the polarization

$$(V\otimes W)_{\mathbb{R}} = (V\otimes E)_{\mathbb{R}} + (V\otimes F)_{\mathbb{R}}$$

inherited from that of W in equation (C.2). Here the tensor product is over $\mathbb{C}$.

Recall that throughout the paper we have used a basis $\{v_j\}_{j=1}^{p+q}$ for V. Hence, we have a basis $\{v_j\otimes e_1\}_{j=1}^{p+q}$ for the complex vector space $V\otimes E$ and noting that $i(v_j\otimes e_1)=v_j\otimes e_2$, we obtain a basis $\{v_j\otimes e_1, v_j\otimes e_2\}_{j=1}^{p+q}$ for the underlying real vector space $(V\otimes E)_{\mathbb{R}}$. We let $\{x_j, y_j\}_{j=1}^{p+q}$ be the corresponding coordinates. We will also use the complex coordinates $z_j=x_j+iy_j$, $1\leqslant j\leqslant p+q$. We will regard a function in $\mathcal{S}((V\otimes E)_{\mathbb{R}})$ as a function of the z_j's (and their complex conjugates). Once again we have the space $\mathcal{P}(V\otimes E)$ consisting of the product of complex-valued polynomials in x_j and y_j, $1\leqslant j\leqslant p+q$, with the Gaussian $\varphi_0=\exp\left(-\pi\left(\sum_{j=1}^{p+q} x_j^2+y_j^2\right)\right)$, or equivalently the product of polynomials in z_j and $\bar{z}_j$, $1\leqslant j\leqslant p+q$,, with the Gaussian $\varphi_0=\exp\left(-\pi\left(\sum_{j=1}^{p+q} z_j\bar{z}_j\right)\right)$.

This is the model where the cocycles of Kudla–Millson $\varphi_{q,q}$ were originally defined; see [44, Proposition 5.2] or [46, p. 148]. To state their formula we need more notation. In what follows $A(\xi'_{\alpha,\mu})$, resp. $A(\xi''_{\alpha,\mu})$, will denote the operation of left exterior multiplication by $\xi'_{\alpha,\mu}$, resp. $\xi''_{\alpha,\mu}$.

For μ with $p+1\leqslant\mu\leqslant p+q$ define

$$D_\mu^+=\sum_{\alpha=1}^{p}\left(\left(\bar{z}_\alpha-\frac{1}{\pi}\frac{\partial}{\partial z_\alpha}\right)\otimes A(\xi'_{\alpha,\mu})\right) \tag{C.3}$$

and

$$\bar{D}_\mu^+=\sum_{\alpha=1}^{p}\left(\left(z_\alpha-\frac{1}{\pi}\frac{\partial}{\partial \bar{z}_\alpha}\right)\otimes A(\xi''_{\alpha,\mu})\right). \tag{C.4}$$

The formula of Kudla and Millson is then

$$\varphi_{q,q}=\frac{1}{2^{2q}}\left(\left(\prod_{\mu=p+1}^{p+q} D_\mu^+\right)\circ\left(\prod_{\nu=p+1}^{p+q}\bar{D}_\nu^+\right)\right)\varphi_0. \tag{C.5}$$

C.2.2. The Fock model for $U(V)\times U(W)$

The second realization of the Weil representation is the polynomial Fock model

$$\mathrm{Pol}((V\otimes W)'_0)$$

considered in this paper. In what follows we will not need the entire space $(V\otimes W)'_0$ but only the subspace $(V_+\otimes W)'_0$. We will give two bases for $(V_+\otimes W)'_0$. Our computations are then simplified by the following lemma.

Lemma C.4. *We have*

$$p'_{0\,V\otimes W}|_{(V_+\otimes W)\otimes_{\mathbb{R}}\mathbb{C}}=I_{V_+}\otimes p'_{0\,W}$$

or, *in more concise form*,

$$(v\otimes w)'_0=v\otimes w'_0 \quad \textit{for } v\in V_+.$$

Proof. By definition,

$$p'_{V\otimes W}{}^{0} = \tfrac{1}{2}(I_V \otimes I_W \otimes 1 - \theta_V \otimes (J_W \circ \theta_W) \otimes i)$$

and hence

$$\begin{aligned} p'_{V\otimes W}{}^{0} | ((V_+ \otimes W)_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) &= \tfrac{1}{2}(I_{V_+} \otimes I_W \otimes 1 - I_{V_+} \otimes (J_W \circ \theta_W) \otimes i) \\ &= I_{V_+} \otimes (I_W \otimes 1 - J_{W,0} \otimes i). \end{aligned}$$ □

Remark. Lemma C.4 implies that we can carry over the computations of the previous section to the ones we need by simply "tensoring with the standard basis for V_+".

Now recall that

$$\mathrm{Pol}((V_+ \otimes W)'^{0}) = \mathrm{Pol}(V'_+ \otimes W'_+) \otimes \mathrm{Pol}(V''_+ \otimes W''_-). \tag{C.6}$$

The first basis for $(V_+ \otimes W)'^{0}$ is adapted to the split Schrödinger model. It is given by

$$\{v_\alpha \otimes e'^{0}_1, v_\alpha \otimes e'^{0}_2\}_{\alpha=1}^{p}.$$

We define $\{s_\alpha, t_\alpha\}_{\alpha=1}^{p}$ to be the coordinates associated with this basis. We will again call these coordinates split coordinates. The second basis for $(V_+ \otimes W)'^{0}$ is given by

$$\{v_\alpha \otimes \varepsilon'^{0}_1, v_\alpha \otimes \varepsilon'^{0}_2\}_{\alpha=1}^{p}.$$

We let $\{a'_\alpha, b''_\alpha\}_{\alpha=1}^{p}$ be the corresponding coordinates with the same explanation for the name and the superscripts as before. Thus, the tensor product in equation (C.6) corresponds to the tensor product decomposition

$$\mathrm{Pol}(V_+ \otimes W'^{0}) \cong \mathbb{C}[a'_1, \ldots, a'_p] \otimes \mathbb{C}[b''_1, \ldots, b''_p]. \tag{C.7}$$

We next note that the split coordinates and the product coordinates are related by the following lemma.

Lemma C.5. *We have*

$$a'_\alpha = \frac{1}{\sqrt{2}}(s_\alpha + i t_\alpha) \quad \textit{and} \quad b''_\alpha = \frac{i}{\sqrt{2}}(s_\alpha - i t_\alpha).$$

As before we have the following result.

Lemma C.6. *There is a* $\mathfrak{u}(V)\times\mathfrak{u}(W)$*-equivariant embedding* (*not onto*)

$$B_{V_+\otimes W}\colon \mathrm{Pol}(V_+\otimes W'^{0})\longrightarrow \mathcal{P}(V\otimes E)$$

satisfying the following conditions:

(1)

$$B_{V_+\otimes W}(1\otimes 1)=\varphi_0;$$

(2)

$$B_{V_+\otimes W}\circ s_\alpha\circ B^{-1}_{V_+\otimes W}=x_\alpha-\frac{1}{\pi}\frac{\partial}{\partial x_\alpha};$$

(3)

$$B_{V_+\otimes W}\circ t_\alpha\circ B^{-1}_{V_+\otimes W}=y_\alpha-\frac{1}{\pi}\frac{\partial}{\partial y_\alpha};$$

(4)

$$B_{V_+\otimes W}\circ\frac{1}{\pi}\frac{\partial}{\partial s_\alpha}\circ B^{-1}_{V_+\otimes W}=x_\alpha+\frac{1}{\pi}\frac{\partial}{\partial x_\alpha};$$

(5)

$$B_{V_+\otimes W}\circ\frac{1}{\pi}\frac{\partial}{\partial t_\alpha}\circ B^{-1}_{V_+\otimes W}=y_\alpha+\frac{1}{\pi}\frac{\partial}{\partial y_\alpha}.$$

Hence, by Lemma C.5, we have the following lemma.

Lemma C.7. *With the notation above, we have* (1)

$$B_{V_+\otimes W}\circ a'_\alpha\circ B^{-1}_{V_+\otimes W}=\frac{1}{\sqrt{2}}\left(z_\alpha-\frac{1}{\pi}\frac{\partial}{\partial \overline{z_\alpha}}\right);$$

(2)

$$B_{V_+\otimes W}\circ b''_\alpha\circ B^{-1}_{V_+\otimes W}=\frac{1}{\sqrt{2}}\left(\bar z_\alpha-\frac{1}{\pi}\frac{\partial}{\partial z_\alpha}\right).$$

We now define operators C^+_μ and $\overline{C}^+_\mu$, $p+1\leqslant\mu\leqslant p+q$, by

(1) $C^+_\mu=\sum_{\alpha=1}^{p} a'_\alpha\otimes A(\xi''_{\alpha,\mu})$;

(2) $\overline{C}^+_\mu=\sum_{\alpha=1}^{p} b''_\alpha\otimes A(\xi'_{\alpha,\mu})$.

We leave the proof of the next lemma to the reader; see Lemma 5.5 for (1) and Lemma 5.11 for (2).

Lemma C.8. *We have*

(1)

$$\left(\prod_{\mu=p+1}^{p+q} C^+_\mu\right)(1)=\psi_{0,q};$$

(2)
$$\Big(\prod_{\mu=p+1}^{p+q} \overline{C}_{\mu}^{+}\Big)(1)=\psi_{q,0};$$

(3)
$$\Big(\prod_{\mu=p+1}^{p+q} C_{\mu}^{+}\Big)\circ\Big(\prod_{\mu=p+1}^{p+q} \overline{C}_{\mu}^{+}\Big)(1)=\psi_{0,q}\wedge\psi_{q,0}.$$

Again we leave the proof of the next lemma to the reader.

LEMMA C.9. *We have*

(1)
$$B_{V_+\otimes W}\circ\Big(\prod_{\mu=p+1}^{p+q} C_{\mu}^{+}\Big)\circ B_{V_+\otimes W}^{-1}=\frac{1}{2^{q/2}}\Big(\prod_{\mu=p+1}^{p+q} D_{\mu}^{+}\Big);$$

(2)
$$B_{V_+\otimes W}\circ\Big(\prod_{\mu=p+1}^{p+q} \overline{C}_{\mu}^{+}\Big)\circ B_{V_+\otimes W}^{-1}=\frac{1}{2^{q/2}}\Big(\prod_{\mu=p+1}^{p+q} \overline{D}_{\mu}^{+}\Big).$$

We can now prove the local product formula.

PROPOSITION C.10. *With the notation above, we have*

$$(B_{V_+\otimes W}\otimes 1)(\psi_{0,q}\wedge\psi_{q,0})=2^q\varphi_{q,q}.$$

Proof. We have

$$\begin{aligned}
&(B_{V_+\otimes W}\otimes 1)(\psi_{0,q}\otimes\psi_{q,0})\\
&\quad=(B_{V_+\otimes W}\otimes 1)\Big(\Big(\prod_{\mu=p+1}^{p+q} C_{\mu}^{+}\Big)\circ\Big(\prod_{\mu=p+1}^{p+q} \overline{C}_{\mu}^{+}\Big)(1)\Big)\\
&\quad=\Big((B_{V_+\otimes W}\otimes 1)\circ\Big(\prod_{\mu=p+1}^{p+q} C_{\mu}^{+}\Big)\circ\Big(\prod_{\mu=p+1}^{p+q} \overline{C}_{\mu}^{+}\Big)\circ(B_{V_+\otimes W}^{-1}\otimes 1)\Big)(B_{V_+\otimes W}\otimes 1)(1)\\
&\quad=\frac{1}{2^q}\Big(\prod_{\mu=p+1}^{p+q} D_{\mu}^{+}\Big)\circ\Big(\prod_{\mu=p+1}^{p+q} \overline{D}_{\mu}^{+}\Big)(\varphi_0)\\
&\quad=2^q\varphi_{q,q}.
\end{aligned}$$
□

Remark. We warn the reader that the KM-cocycle $\varphi_{q,q}$ does not factor in the split Schrödinger model. The cocycles $\varphi_{q,0}=(B_{V_+\otimes W}\otimes 1)(\psi_{q,0})$ and $\varphi_{0,q}=(B_{V_+\otimes W}\otimes 1)(\psi_{0,q})$ both exist in the split Schrödinger model (of course), but it is not true that we have

$\varphi_{q,0}\wedge\varphi_{0,q}=\varphi_{q,q}$. The problem is that the product in $\mathcal{P}(V\otimes E)$ is induced by the internal product, i.e the usual multiplication of functions, whereas the product in

$$\mathrm{Pol}(V'_+\otimes W'_+)\otimes\mathrm{Pol}(V''_+\otimes W''_-)$$

is external, i.e. the tensor product multiplication. For example, when $q=1$, we have

$$\begin{aligned}(B_{V_+\otimes W}\otimes 1)(\psi_{1,0}) &= (B_{V_+\otimes W}\otimes 1)\left(\sum_{\alpha=1}^{p}\bar z_\alpha\otimes\xi'_{\alpha,p+1}\right)=\sqrt{2}\varphi_0\left(\sum_{\alpha=1}^{p}\bar z_\alpha\otimes\xi'_{\alpha,p+1}\right),\\ (B_{V_+\otimes W}\otimes 1)(\psi_{0,1}) &= (B_{V_+\otimes W}\otimes 1)\left(\sum_{\alpha=1}^{p} z_\beta\otimes\xi'_{\beta,p+1}\right)=\sqrt{2}\varphi_0\left(\sum_{\beta=1}^{p} z_\beta\otimes\xi''_{\beta,p+1}\right),\end{aligned}$$

and hence

$$(B_{V_+\otimes W}\otimes 1)(\psi_{1,0})\wedge(B_{V_+\otimes W}\otimes 1)(\psi_{0,1})=2(\varphi_0)^2\left(\sum_{\alpha,\beta=1}^{p}\bar z_\alpha z_\beta\otimes\xi'_{\alpha,p+1}\wedge\xi''_{\beta,p+1}\right),$$

whereas

$$\varphi_{1,1}=\varphi_0\left(\sum_{\alpha,\beta=1}^{p}\bar z_\alpha z_\beta\otimes\xi'_{\alpha,p+1}\wedge\xi''_{\beta,p+1}-\frac{1}{2\pi}\sum_{\alpha=1}^{p}\xi'_{\alpha,p+1}\wedge\xi''_{\alpha,p+1}\right).$$

References

[1] Arancibia, N., Moeglin, C. & Renard, D., Paquets d'Arthur des groupes classiques et unitaires. Preprint, 2015. arXiv:1507.01432 [RT].

[2] Arthur, J., On local character relations. *Selecta Math.*, 2 (1996), 501–579.

[3] — An introduction to the trace formula, in *Harmonic Analysis, the Trace Formula, and Shimura Varieties*, Clay Math. Proc., 4, pp. 1–263. Amer. Math. Soc., Providence, RI, 2005.

[4] Bergeron, N., Restriction de la cohomologie d'une variété de Shimura à ses sous-variétés. *Transform. Groups*, 14 (2009), 41–86.

[5] Bergeron, N. & Clozel, L., Spectre automorphe des variétés hyperboliques et applications topologiques. *Astérisque*, 303 (2005), 218 pp.

[6] — Quelques conséquences des travaux d'Arthur pour le spectre et la topologie des variétés hyperboliques. *Invent. Math.*, 192 (2013), 505–532.

[7] Bergeron, N., Millson, J. & Moeglin, C., Hodge type theorems for arithmetic manifolds associated to orthogonal groups. To appear in *Int. Math. Res. Not.* arXiv:1110.3049 [NT].

[8] Blasius, D. & Rogawski, J., Cohomology of congruence subgroups of $\mathrm{SU}(2,1)^p$ and Hodge cycles on some special complex hyperbolic surfaces, in *Regulators in Analysis, Geometry and Number Theory*, Progr. Math., 171, pp. 1–15. Birkhäuser, Boston, MA, 2000.

[9] Borel, A. & Wallach, N., *Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups*. Mathematical Surveys and Monographs, 67. Amer. Math. Soc., Providence, RI, 2000.

[10] Chern, S.-S., On a generalization of Kähler geometry, in *Algebraic Geometry and Topology*, pp. 103–121. Princeton Univ. Press, Princeton, NJ, 1957.

[11] Cossutta, M., Asymptotique des nombres de Betti des variétés arithmétiques. *Duke Math. J.*, 150 (2009), 443–488.

[12] Deligne, P., Travaux de Shimura, in *Séminaire Bourbaki* (1970/71), Exp. No. 389, Lecture Notes in Mathematics, 244, pp. 123–165. Springer, Berlin–Heidelberg, 1971.

[13] Faltings, G., p-adic Hodge theory. *J. Amer. Math. Soc.*, 1 (1988), 255–299.

[14] Flath, D., Decomposition of representations into tensor products, in *Automorphic Forms, Representations and L-Functions* (Corvallis, OR, 1977), Part 1, Proc. Sympos. Pure Math., 33, pp. 179–183. Amer. Math. Soc., Providence, RI, 1979.

[15] Folland, G. B., *Harmonic Analysis in Phase Space*. Annals of Mathematics Studies, 122. Princeton Univ. Press, Princeton, NJ, 1989.

[16] Fulton, W., *Young Tableaux*. London Mathematical Society Student Texts, 35. Cambridge Univ. Press, Cambridge, 1997.

[17] Fulton, W. & Harris, J., *Representation Theory*. Graduate Texts in Mathematics, 129. Springer, New York, 1991.

[18] Gan, W. T. & Takeda, S., On the regularized Siegel–Weil formula (the second term identity) and non-vanishing of theta lifts from orthogonal groups. *J. Reine Angew. Math.*, 659 (2011), 175–244.

[19] Gelbart, S., Examples of dual reductive pairs, in *Automorphic Forms, Representations and L-Functions* (Corvallis, OR, 1977), Part 1, Proc. Sympos. Pure Math., 33, pp. 287–296. Amer. Math. Soc., Providence, RI, 1979.

[20] Gelbart, S., Piatetski-Shapiro, I. & Rallis, S., *Explicit Constructions of Automorphic L-Functions*. Lecture Notes in Mathematics, 1254. Springer, Berlin–Heidelberg, 1987.

[21] Gelbart, S., Rogawski, J. & Soudry, D., On periods of cusp forms and algebraic cycles for U(3). *Israel J. Math.*, 83 (1993), 213–252.

[22] — Periods of cusp forms and L-packets. *C. R. Acad. Sci. Paris Sér. I Math.*, 317 (1993), 717–722.

[23] — Endoscopy, theta-liftings, and period integrals for the unitary group in three variables. *Ann. of Math.*, 145 (1997), 419–476.

[24] Ginzburg, D., Jiang, D. & Soudry, D., Poles of L-functions and theta liftings for orthogonal groups. *J. Inst. Math. Jussieu*, 8 (2009), 693–741.

[25] Godement, R. & Jacquet, H., *Zeta Functions of Simple Algebras*. Lecture Notes in Mathematics, 260. Springer, Berlin–Heidelberg, 1972.

[26] Gong, Z. & Grenié, L., An inequality for local unitary theta correspondence. *Ann. Fac. Sci. Toulouse Math.*, 20 (2011), 167–202.

[27] Griffiths, P. & Harris, J., *Principles of Algebraic Geometry*. Pure and Applied Mathematics. Wiley-Interscience, New York, 1978.

[28] Grothendieck, A., Hodge's general conjecture is false for trivial reasons. *Topology*, 8 (1969), 299–303.

[29] Harris, M., L-functions of 2×2 unitary groups and factorization of periods of Hilbert modular forms. *J. Amer. Math. Soc.*, 6 (1993), 637–719.

[30] Harris, M., Kudla, S. S. & Sweet, W. J., Theta dichotomy for unitary groups. *J. Amer. Math. Soc.*, 9 (1996), 941–1004.

[31] Harris, M. & Li, J.-S., A Lefschetz property for subvarieties of Shimura varieties. *J. Algebraic Geom.*, 7 (1998), 77–122.

[32] He, H. & Hoffman, J. W., Picard groups of Siegel modular 3-folds and θ-liftings. *J. Lie Theory*, 22 (2012), 769–801.

[33] Hirzebruch, F. & Zagier, D., Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus. *Invent. Math.*, 36 (1976), 57–113.

[34] Howe, R., θ-series and invariant theory, in *Automorphic Forms, Representations and L-Functions* (Corvallis, OR, 1977), Part 1, Proc. Sympos. Pure Math., 33, pp. 275–285. Amer. Math. Soc., Providence, RI, 1979.

[35] — Transcending classical invariant theory. *J. Amer. Math. Soc.*, 2 (1989), 535–552.

[36] Ichino, A., A regularized Siegel–Weil formula for unitary groups. *Math. Z.*, 247 (2004), 241–277.

[37] Jacquet, H. & Shalika, J. A., A non-vanishing theorem for zeta functions of GL_n. *Invent. Math.*, 38 (1976/77), 1–16.

[38] Johnson, J. F., Lie algebra cohomology and the resolution of certain Harish-Chandra modules. *Math. Ann.*, 267 (1984), 377–393.

[39] Kashiwara, M. & Vergne, M., On the Segal–Shale–Weil representations and harmonic polynomials. *Invent. Math.*, 44 (1978), 1–47.

[40] Kottwitz, R. E., Points on some Shimura varieties over finite fields. *J. Amer. Math. Soc.*, 5 (1992), 373–444.

[41] Kottwitz, R. E. & Shelstad, D., Foundations of twisted endoscopy. *Astérisque*, 255 (1999), 190 pp.

[42] Kudla, S. S., Splitting metaplectic covers of dual reductive pairs. *Israel J. Math.*, 87 (1994), 361–401.

[43] — Algebraic cycles on Shimura varieties of orthogonal type. *Duke Math. J.*, 86 (1997), 39–78.

[44] Kudla, S. S. & Millson, J. J., The theta correspondence and harmonic forms. I. *Math. Ann.*, 274 (1986), 353–378.

[45] — The theta correspondence and harmonic forms. II. *Math. Ann.*, 277 (1987), 267–314.

[46] — Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables. *Inst. Hautes Études Sci. Publ. Math.*, 71 (1990), 121–172.

[47] Kudla, S. S. & Rallis, S., A regularized Siegel–Weil formula: the first term identity. *Ann. of Math.*, 140 (1994), 1–80.

[48] Kudla, S. S. & Sweet, W. J., Jr., Degenerate principal series representations for $\mathrm{U}(n,n)$. *Israel J. Math.*, 98 (1997), 253–306.

[49] Landherr, W., Äquivalenz Hermitescher Formen über einem beliebigen algebraischen Zahlkörper. *Abh. Math. Sem. Univ. Hamburg*, 11 (1935), 245–248.

[50] Langlands, R. P. & Shelstad, D., On the definition of transfer factors. *Math. Ann.*, 278 (1987), 219–271.

[51] Li, J.-S., Nonvanishing theorems for the cohomology of certain arithmetic quotients. *J. Reine Angew. Math.*, 428 (1992), 177–217.

[52] Matsushima, Y., A formula for the Betti numbers of compact locally symmetric Riemannian manifolds. *J. Differential Geom.*, 1 (1967), 99–109.

[53] Milne, J. S., Canonical models of (mixed) Shimura varieties and automorphic vector bundles, in *Automorphic Forms, Shimura Varieties, and L-functions*, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., 10, pp. 283–414. Academic Press, Boston, MA, 1990.

[54] Mínguez, A., Unramified representations of unitary groups, in *On the Stabilization of the Trace Formula*, Stab. Trace Formula Shimura Var. Arith. Appl., 1, pp. 389–410. Int. Press, Somerville, MA, 2011.

[55] Moeglin, C., Non nullité de certains relêvements par séries théta. *J. Lie Theory*, 7 (1997), 201–229.

[56] Moeglin, C., Vignéras, M.-F. & Waldspurger, J.-L., *Correspondances de Howe sur un corps p-adique*. Lecture Notes in Mathematics, 1291. Springer, Berlin–Heidelberg, 1987.

[57] Moeglin, C. & Waldspurger, J.-L., Le spectre résiduel de GL(n). *Ann. Sci. École Norm. Sup.*, 22 (1989), 605–674.

[58] — Stabilisation de la formule des traces tordue X: stabilisation spectrale. Preprint, 2014. arXiv:1412.2981 [RT].

[59] Mok, C. P., Endoscopic classification of representations of quasi-split unitary groups. *Mem. Amer. Math. Soc.*, 235 (2015), 248 pp.

[60] Ngô, B. C., Le lemme fondamental pour les algèbres de Lie. *Publ. Math. Inst. Hautes Études Sci.*, 111 (2010), 1–169.

[61] Paul, A., Howe correspondence for real unitary groups. *J. Funct. Anal.*, 159 (1998), 384–431.

[62] Rallis, S., On the Howe duality conjecture. *Compos. Math.*, 51 (1984), 333–399.

[63] Salamanca-Riba, S. A., On the unitary dual of real reductive Lie groups and the $A_g(\lambda)$ modules: the strongly regular case. *Duke Math. J.*, 96 (1999), 521–546.

[64] Shelstad, D., Embeddings of L-groups. *Canad. J. Math.*, 33 (1981), 513–558.

[65] — On geometric transfer in real twisted endoscopy. *Ann. of Math.*, 176 (2012), 1919–1985.

[66] Tadić, M., Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case). *Ann. Sci. École Norm. Sup.*, 19 (1986), 335–382.

[67] Tan, V., Poles of Siegel Eisenstein series on U(n, n). *Canad. J. Math.*, 51 (1999), 164–175.

[68] Tate, J., Conjectures on algebraic cycles in l-adic cohomology, in *Motives* (Seattle, WA, 1991), Proc. Sympos. Pure Math., 55, pp. 71–83. Amer. Math. Soc., Providence, RI, 1994.

[69] Tits, J., Reductive groups over local fields, in *Automorphic Forms*, *Representations and L-Functions* (Corvallis, OR, 1977), Part 1, Proc. Sympos. Pure Math., 33, pp. 29–69. Amer. Math. Soc., Providence, RI, 1979.

[70] Vogan, D. A., Jr., The unitary dual of GL(n) over an Archimedean field. *Invent. Math.*, 83 (1986), 449–505.

[71] Vogan, D. A., Jr. & Zuckerman, G. J., Unitary representations with nonzero cohomology. *Compos. Math.*, 53 (1984), 51–90.

[72] Waldspurger, J.-L., L'endoscopie tordue n'est pas si tordue. *Mem. Amer. Math. Soc.*, 194 (2008), 261 pp.

[73] — Les facteurs de transfert pour les groupes classiques: un formulaire. *Manuscripta Math.*, 133 (2010), 41–82.

[74] — La formule des traces locale tordue. To appear in *Mem. Amer. Math. Soc.* arXiv:1205.1100 [RT].

[75] — Stabilisation de la formule des traces tordue III: intégrales orbitales et endoscopie sur un corps local non-archimédien; réductions et preuves. Preprint, 2013. arXiv:1402.2753 [RT].

[76] Weil, A., Sur certains groupes d'opérateurs unitaires. *Acta Math.*, 111 (1964), 143–211.

[77] — Sur la formule de Siegel dans la théorie des groupes classiques. *Acta Math.*, 113 (1965), 1–87.

Nicolas Bergeron
Sorbonne Universités
UPMC Université Paris 06
Institut de Mathématiques
de Jussieu–Paris Rive Gauche, UMR 7586
CNRS
Université Paris Diderot
Sorbonne Paris Cité
FR-75005 Paris
France
nicolas.bergeron@imj-prg.fr

John Millson
Department of Mathematics
University of Maryland
College Park, MD 20742
U.S.A.
jjm@math.umd.edu

Colette Moeglin
CNRS
Institut de Mathématiques
de Jussieu-Paris Rive Gauche, UMR 7586
Sorbonne Universités
UPMC Université Paris 06
Université Paris Diderot
Sorbonne Paris Cité
FR-75005 Paris
France
colette.moeglin@imj-prg.fr

Received July 29, 2014