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Real quadrics in $\mathbf{C}^n$, complex manifolds and convex polytopes

by

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Dedicated to Alberto Verjovsky on his 60th birthday

Introduction

This work explores the relationships existing between three classes of objects, coming from different domains of mathematics, namely:

(i) *Real algebraic geometry*: the objects here are what we call *links*, that is transverse intersections in $\mathbf{C}^n$ of real quadrics of the form

$$\sum_{i=1}^{n} a_i |z_i|^2 = 0, \quad a_i \in \mathbf{R},$$

with the unit euclidean sphere of $\mathbf{C}^n$.

(ii) *Convex geometry*: the class of simple convex polytopes.

(iii) *Complex geometry*: the class of non-Kähler compact complex manifolds of [30]. They are a generalization by the second author of the manifolds introduced in [27] by S. López de Medrano and A. Verjovsky, and will be called here LV-M manifolds.

The natural connection between these classes goes as follows. First, a link is invariant by the standard action of the real torus $(\mathbf{S}^1)^n$ onto $\mathbf{C}^n$ and the quotient space is easily seen to identify with a simple convex polytope (Lemma 0.12). Secondly, as a direct consequence of the construction of [30], each link (after taking the product with a circle in the odd-dimensional case) can be endowed with a complex structure of an LV-M manifold (Theorem 12.2). Indeed, the links form a large subclass of the class of LV-M manifolds.

The aim of the paper is to describe the topology of the links and to apply the results to address the following question.

Question. How complicated can the topology of the LV-M manifolds be?

This program is achieved by making a reduction to combinatorics of simple convex polytopes: a simple convex polytope encodes the topology of the associated link completely.

As shown by the question, the main motivation comes from complex geometry. Let us explain a little more why we find it important to know the topology of the LV-M manifolds.

Complex geometry is concerned with the study of (compact) complex manifolds. Nevertheless, no general theory exists and only special classes of complex manifolds such as projective or Kähler manifolds or complex manifolds which are at least bimeromorphic to projective or Kähler ones are well understood. Moreover, except for the case of surfaces, there are few explicit examples having none of these properties; explicit meaning that it is possible to work with and to compute things on it. Indeed, the two classical families are the Hopf manifolds (diffeomorphic to $\mathbf{S}^1\times\mathbf{S}^{2n-1}$; see [20]) and the Calabi–Eckmann manifolds (diffeomorphic to $\mathbf{S}^{2p-1}\times\mathbf{S}^{2q-1}$; see [10]).

These classical examples have been developed through a number of papers inspired by the theory of dynamical systems, starting from the construction of deformation spaces of foliations by Girbau–Haefliger–Sundararaman [15] and of deformation spaces of the Hopf and Calabi–Eckmann manifolds by Borcea [5], Haefliger [18] and Loeb–Nicolau [23]. This led to the construction and study of larger and larger classes of new examples, especially in [27], [30] and [6].

In this article, we focus on the class of LV-M manifolds of [30]. It is explicit in the previous sense. Indeed the main complex geometrical properties (algebraic dimension, generic holomorphic submanifolds, local deformation space, etc.) of these objects are established in [30]. Besides, it is proved in [31] that they are small deformations of holomorphic principal bundles over projective toric varieties with a compact complex torus as fiber. In this sense, they constitute a natural generalization of Hopf and Calabi–Eckmann manifolds, which can be deformed into compact complex manifolds fibering in elliptic curves over the complex projective space $\mathbf{P}^{n-1}$ (Hopf case) or over the product of projective spaces $\mathbf{P}^{p-1}\times\mathbf{P}^{q-1}$ (Calabi–Eckmann case). One of the main interests in these manifolds, however, is that they have a richer topology, since it is also proved in [30] that complex structures on certain connected sums of products of spheres can be obtained by this process.

Nevertheless, these examples of connected sums constitute very particular cases of the construction, and the problem of describing the topology in other cases was left wide open in [30]. Of course, due to the lack of examples of non-Kähler and non-Moïshezon compact complex manifolds, the more intricate this topology is, the more interesting is the class of LV-M manifolds. This is the starting point and motivation for this work and leads to the question stated above.

In [30], it was conjectured that they are all diffeomorphic to products of connected sums of sphere products and odd-dimensional spheres.

On the other hand, it follows from the construction that an LV-M manifold N is entirely characterized by a set Λ of m vectors of $\mathbf{C}^n$ (with $n>2m$). Moreover, a homotopy of Λ in $\mathbf{C}^n$ gives rise to a deformation of N as soon as an open condition is fulfilled at each step of the homotopy. If this condition is broken during the homotopy, the diffeomorphism type of the new complex manifold N' is different from that of N. In other words, there is a natural wall-crossing problem, and this leads to the following problem.

Problem. Describe the topological and holomorphic changes occurring after a generic wall-crossing.

This wall-crossing problem is linked with the previous question, since knowing how the topology changes after a wall-crossing, one can expect to describe the most complicated examples. But it has also a holomorphic part, since the initial and final manifolds are complex.

In this article, we address these questions and give a description as complete as we can of the topology of these compact complex manifolds:

• Concerning the question above, the very surprising answer is that the topology of the LV-M manifolds is much more complicated than expected. Indeed, their homology groups can have arbitrary amounts of torsion (Theorem 14.1). Counterexamples are given in §11, as well as a constructive way of obtaining these arbitrary amounts of torsion.

• Concerning the wall-crossing problem, we show that crossing a wall means performing a complex surgery and describe precisely these surgeries from the topological and the holomorphic point of view (Theorems 5.4 and 13.3).

As an easy but nice consequence, we obtain that affine compact complex manifolds (that is compact complex manifolds with an affine atlas) can have arbitrary amount of torsion. It becomes thus quite difficult to classify, up to diffeomorphism, affine compact complex manifolds or manifolds having a holomorphic affine connection in high dimensions ($\geqslant 3$).

It is interesting to compare this result with the Kähler case: it is known that affine Kähler manifolds are covered by complex tori (see [22]), so the difference here is striking. Of course, it is known for a long time that such a statement is false for non-Kähler manifolds (think about the Hopf surfaces). Nevertheless, one could expect a rigidity result, which is definitively not the case. Notice also that a statement similar to Theorem 14.1 is unknown for Kähler manifolds.

The paper is organized as follows. In §0, we collect the basic facts about the links. In particular, we introduce the simple convex polytope associated to a link, as well as a subspace arrangement whose complement has the same homotopy type as the associated link. We recall the previously known cases studied in [25] and [26]. Finally, we prove that links are equivariantly homeomorphic to moment-angle manifolds coming from simple polytopes introduced in [12] and intensively studied in [9].

Parts I and II deal exclusively with the properties of links as smooth manifolds, without any reference to the LV-M manifolds. On the contrary, Part III deals with LV-M manifolds. The connection is made at the beginning of Part III, where it is explained that the links form a large subclass of the LV-M manifolds (but not all of them).

In Part I, we prove that the classes of links, up to equivariant diffeomorphism (equivariant with respect to the action of the real torus) and up to product with circles, are in one-to-one correspondence with the combinatorial classes of simple convex polytopes (Theorem 4.1). This is the first main result of this part. It allows us to translate problems about the differential topology of the links entirely in the world of combinatorics of simple convex polytopes. As a by-product of Theorem 4.1, we prove that there exists a unique smooth structure on a moment-angle manifold compatible with its natural torus action (Corollary 4.7). On the other hand, we recall the notion of flips of simple polytopes of [29] and [38] in §2, and prove some auxiliary results. We define in §3 a set of equivariant elementary surgeries on the links, and prove in §4 (Theorem 4.8) that performing a flip on a simple convex polytope means performing an equivariant surgery on the associated link. Finally, we introduce in §5 the notion of wall-crossing of links and prove the second main theorem of this part, namely the wall-crossing theorem (Theorem 5.4): crossing a wall for a link is equivalent to performing a flip for the associated simple convex polytope, and therefore the wall-crossing can be described in terms of elementary surgeries. As a consequence, we generalize a result of McGavran (see [28]) and describe explicitely the diffeomorphism type of certain families of links in §6.

In Part II, we give a formula for computing the cohomology ring of a link in terms of subsets of the associated simple convex polytope. To do this, we use results of Buchstaber and Panov [9], and Baskakov [2] on the cohomology of moment-angle manifolds. The formula is stated as cohomology theorem (Theorem 10.1) and is proved in §10 after some preliminary material in §8 and §9. Notice that it is also a cohomology formula for the coordinate subspace arrangement mentioned before. Finally, applications and examples are given in §11, and it is proved that the homology groups of a link can have arbitrary torsion (Theorem 11.11).

In Part III, we apply the previous results to the family of LV-M manifolds. In §12, we recall very briefly their construction and prove that an even-dimensional link admits such a complex structure, as well as the product of an odd-dimensional link with a circle. We resolve the holomorphic wall-crossing problem in §13 (Theorem 13.3). Finally, in §14, we obtain, as an easy consequence of Theorem 11.11, that the homology groups of an LV-M manifold can have arbitrary amount of torsion, and, as an easy consequence of the construction, that such a statement is true for affine compact complex manifolds. The article ends with some open questions in §15.

Although the main motivation comes from complex geometry, Part I (especially §6) should also be of interest to readers working on smooth torus actions on manifolds. It can be seen as a continuation of [25], [26] and [28]. On the other hand, the links form an explicit smooth realization of moment-angle manifolds and the surgery results of Part I can be seen as a diffeomorphic version (that is up to equivariant diffeomorphism) of results of [9, §6.4], obtained up to equivariant homeomorphism. The relationship between links and moment-angle complexes gives interesting open questions (see §15). Finally, the cohomology formula of Part II has its own interest as a geometric reformulation of the formula of Baskakov and Buchstaber–Panov, and a nice simplification of the Goresky–MacPherson [16] and De Longueville [24] formulas for a special class of subspace arrangements.

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0. Preliminaries

In this section, we give the basic definitions, notation and lemmas. Some of the results are stated and sometimes proved in [30] or [31], but in different versions; in this case we give the original reference, but at the same time, we give at least some indication about the proof to be self-contained.

In this paper, we denote by $\mathbf{S}^{n-1}$ the unit euclidean sphere of $\mathbf{R}^n$, and by $\mathbf{D}^n$ (respectively, $\overline{\mathbf{D}^n}$) the unit euclidean open (respectively, closed) ball of $\mathbf{R}^n$. We identify $\mathbf{C}^p$ and $\mathbf{R}^{2p}=(\mathbf{R}^2)^p$ via the map sending each complex coordinate onto its real and imaginary parts. Smooth means C^∞. Polytope means convex polytope and a polyhedron is a polytope of dimension 3. Recall that two convex polytopes are combinatorially equivalent if there exists a bijection between their posets of faces which respects the inclusion. Two combinatorially equivalent convex polytopes are PL-homeomorphic, and the classes of convex poytopes up to combinatorial equivalence coincide with the classes of convex polytopes up to PL-homeomorphism. In the sequel, *we make no distinction between a convex polytope and its combinatorial class.* No confusion should arise from this abuse.

Definition 0.1. A *special real quadric* in $\mathbf{C}^n$ is a set of points $z\in\mathbf{C}^n$ satisfying

$$\sum_{i=1}^{n} a_i|z_i|^2=0$$

for some fixed n-tuple $(a_1, ..., a_n)$ in $\mathbf{R}^n$.

We are interested in the topology of the transverse intersection of a finite (but arbitrary) number of special real quadrics in $\mathbf{C}^n$ with the euclidean unit sphere. We call such an intersection the *link* of the system of special real quadrics.

Let $A\in M_{n,p}(\mathbf{R})$, that is A is a real matrix with n columns and p rows. We write A as $(A_1, ..., A_n)$. To A, we may associate p special real quadrics in $\mathbf{C}^n$ and a link, which we denote by X_A. The corresponding system of equations, that is

$$\begin{cases} \sum_{i=1}^n A_i|z_i|^2=0, \\ \sum_{i=1}^n |z_i|^2=1, \end{cases}$$

will be denoted by (S_A).

Notice that we include the special case $p=0$. In this situation, $A=0$ is a matrix of $M_{n,0}(\mathbf{R})$ and X_A is $\mathbf{S}^{2n-1}$.

Definition 0.2. Let $A\in M_{n,p}(\mathbf{R})$. We say that A is *admissible* if it gives rise to a non-empty link X_A whose system (S_A) is non-degenerate at every point of X_A. We denote by $\mathcal{A}$ the set of admissible matrices.

In this paper, we restrict ourselves to the case where A is admissible. A link is thus a smooth compact manifold of dimension $2n-p-1$ without boundary. Moreover, it has trivial normal bundle in $\mathbf{C}^n$, so it is orientable.

We denote by $\mathcal{H}(A)$ the convex hull of the vectors $A_1, ..., A_n$ in $\mathbf{R}^p$.

LEMMA 0.3. (Cf. [31, Lemma 1.1]) *Let $A\in M_{n,p}(\mathbf{R})$. Then A is admissible if and only if it satisfies the following conditions*:

(i) (*Siegel condition*) $0\in\mathcal{H}(A)$;

(ii) (*weak hyperbolicity condition*) $0\in\mathcal{H}((A_i)_{i\in I})\Rightarrow|I|>p$.

Proof. Clearly X_A is non-vacuous if and only if the Siegel condition is satisfied. Let $z\in X_A$ and let

$$I_z=\{i\in\{1, ..., n\}: z_i\neq 0\}=\{i_1, ..., i_q\}, \quad i_1\leqslant ...\leqslant i_q. \tag{1}$$

The system (S_A) is non-degenerate at z if and only if the matrix

$$\tilde{A}_z = \begin{pmatrix} A_{i_1} & \dots & A_{i_q} \\ 1 & \dots & 1 \end{pmatrix}$$

has maximal rank, i.e. rank $p+1$.

Assume the weak hyperbolicity condition. As $z \in X_A$, we have $0 \in \mathcal{H}((A_i)_{i \in I_z})$. By Carathéodory's theorem [17, p. 15], there exists a subset $J=\{j_1, ..., j_{p+1}\} \subset I_z$ such that 0 belongs to $\mathcal{H}((A_i)_{i \in J})$. Moreover, $(A_{j_1}, ..., A_{j_{p+1}})$ has rank p, otherwise, still by Carathéodory's theorem, 0 would be in the convex hull of p of these vectors, contradicting the weak hyperbolicity condition.

As a consequence of these two facts, the vector space of linear relations between $(A_{j_1}, ..., A_{j_{p+1}})$ has dimension 1 and is generated by a solution with all coefficients non-negative. Assume that $\tilde{A}_z$ has rank strictly less than $p+1$. Then, there is a non-trivial linear relation between $(A_{j_1}, ..., A_{j_{p+1}})$, with the additional property that the sum of the coefficients of this relation is zero, yielding a contradiction.

Conversely, assume that the weak hyperbolicity condition is not satisfied. For example, assume that 0 belongs to $\mathcal{H}(A_1, ..., A_p)$ and let $r \in (\mathbf{R}^+)^p$ be such that

$$\sum_{i=1}^{p} r_i A_i = 0 \quad \text{and} \quad \sum_{i=1}^{p} r_i = 1.$$

Then $z = \left(\sqrt{r}_1, ..., \sqrt{r}_p, 0, ..., 0\right)$ belongs to X_A, and $\mathrm{rk}(\tilde{A}_z)$ is at most p, so A is not admissible. □

Note that $\mathcal{A}$ is open in $M_{n,p}(\mathbf{R})$. Let us describe some examples.

Example 0.4. Let $p=1$. Then the A_i's are real numbers. The weak hyperbolicity condition implies that none of the A_i's is zero. Let us say that a of the A_i's are strictly positive, whereas $b=n-a$ of them are strictly negative. The Siegel condition implies that a and b are strictly positive. There is just one special real quadric, which is the equation of a cone over a product of spheres $\mathbf{S}^{2a-1} \times \mathbf{S}^{2b-1}$. As we take the intersection of this quadric with the unit sphere, we finally obtain that X_A is diffeomorphic to $\mathbf{S}^{2a-1} \times \mathbf{S}^{2b-1}$.

Example 0.5. Let $p=2$. Then the A_i's are points in the plane containing 0 in their convex hull (Siegel condition). The weak hyperbolicity condition implies that 0 is not on a segment joining two of the A_i's. Two examples of admissible configurations are illustrated in Figure 1.

Assume that we perform a smooth homotopy $(A^t)_{0 \leqslant t \leqslant 1}$ in $\mathbf{R}^2$ between $A^0=A$ and A^1, such that dA_t/dt is never zero and such that A^t still satisfies the Siegel and

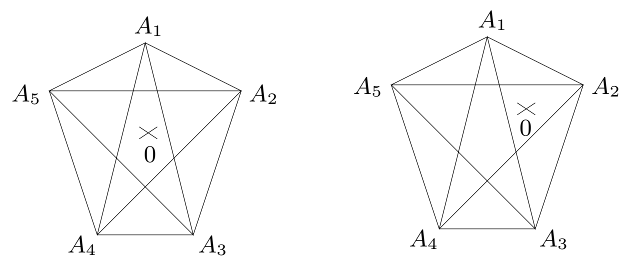

Figure 1.

the weak hyperbolicity conditions for any t. Then the union of the X_{A^t}'s (seen as a smooth submanifold of $\mathbf{C}^n\times\mathbf{R}$) admits a submersion onto $[0,1]$ with compact fibers. Therefore, by Ehresmann's lemma, this submersion is a locally trivial fiber bundle and X_{A^1} is diffeomorphic to $X_{A^0}=X_A$. Using this trick, it can be proven that X_A is diffeomorphic to $X_{A'}$, where A' is a configuration of an odd number $k=2l+1$ of distinct points with weights $n_1,...,n_k$ (see [26]). The result of such a homotopy on the two configurations in Figure 1 is illustrated in Figure 2. The arrows indicate the homotopy, and the numbers appearing on the circles are the weights of the final configuration. These weights encode the topology of the links.

Theorem 0.6. (See [26]) *Let $p=2$ and $A\in\mathcal{A}$. Assume that A is homotopic* (*in the sense given just above*) *to a reduced configuration of $k=2l+1$ distinct points with weights $n_1,...,n_k$.*

(i) *If $l=1$, then X_A is diffeomorphic to* $\mathbf{S}^{2n_1-1}\times\mathbf{S}^{2n_2-1}\times\mathbf{S}^{2n_3-1}$;

(ii) *if $l>1$, then X_A is diffeomorphic to*

$$\mathop{\#}_{i=1}^{k}\mathbf{S}^{2d_i-1}\times\mathbf{S}^{2n-2d_i-2},$$

where # denotes the connected sum and where $d_i=n_i+...+n_{i+l-1}$ (*the indices are taken modulo k*).

In particular, X_A is diffeomorphic to $\mathbf{S}^3\times\mathbf{S}^3\times\mathbf{S}^1$ for the configuration on the right of Figures 1 and 2, and diffeomorphic to $\#(5)\mathbf{S}^3\times\mathbf{S}^4$ (that is the connected sum of five copies of $\mathbf{S}^3\times\mathbf{S}^4$) for the configuration on the left.

Example 0.7. (Products) Let A and B be two admissible configurations of respective dimensions (n,p) and (n',p'). Set

$$C=\begin{pmatrix} & A & & & 0 & \\ -1 & ... & -1 & 1 & ... & 1 \\ & 0 & & & B & \end{pmatrix}.$$

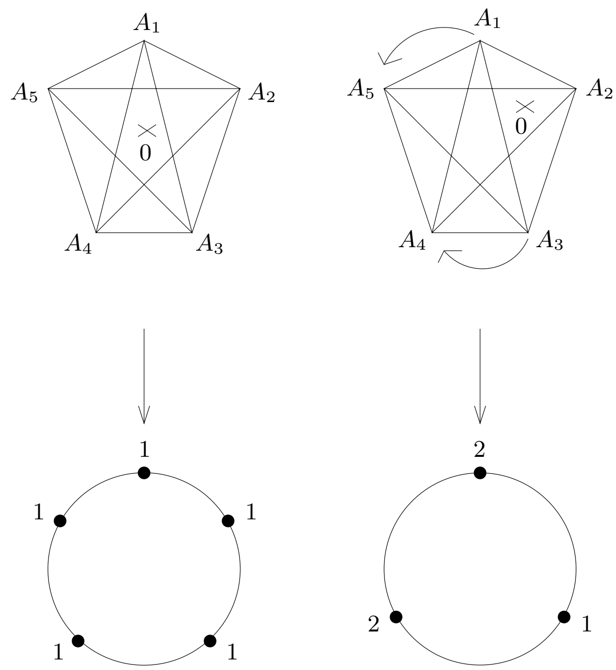

Figure 2.

Then, it is straightforward to check that C is admissible and that X_C is diffeomorphic to the product $X_A \times X_B$. In other words, *the class of links is stable by direct product.* In particular, the product of a link with an odd-dimensional sphere is a link. For example, letting

$$C = \begin{pmatrix} & A & & 0 \\ -1 & \dots & -1 & 1 \end{pmatrix},$$

then X_C is diffeomorphic to $X_A \times \mathbf{S}^1$.

Let $\mathcal{L}_A$ denote the complex coordinate subspace arrangement of $\mathbf{C}^n$ defined by

$$L_I = \{z \in \mathbf{C}^n : z_i = 0 \text{ for } i \in I\} \in \mathcal{L}_A \quad \iff \quad L_I \cap X_A = \varnothing, \tag{2}$$

and let $\mathcal{S}_A$ be its complement in $\mathbf{C}^n$. In other words,

$$\mathcal{S}_A = \{z \in \mathbf{C}^n : 0 \in \mathcal{H}((A_i)_{i \in I_z})\},$$

where I_z is defined as in (1). We have the following lemma.

LEMMA 0.8. *The sets X_A and $\mathcal{S}_A$ have the same homotopy type.*

Proof. This is an argument of foliations and convexity already used in [11], [27], [30] and [31]. We sketch the proof and refer to these articles for more details.

Let $\mathcal{F}$ be the smooth foliation of $\mathcal{S}_A$ given by the action

$$(z,T)\in\mathcal{S}_A\times\mathbf{R}^p\longmapsto\big(z_ie^{\langle A_i,T\rangle}\big)_{i=1}^n\in\mathcal{S}_A.$$

Let $z\in\mathcal{S}_A$ and let F_z be the leaf passing through z. Consider now the map

$$f_z\colon w\in F_z\longmapsto\|w\|^2=\sum_{i=1}^n|w_i|^2.$$

Using the strict convexity of the exponential map, it is easy to check that each critical point of f_z is indeed a local minimum, and that f_z cannot have two local minima and thus cannot have two critical points (see [11] for more details). Now as $z\in\mathcal{S}_A$, then, by definition, 0 is in the convex hull of $(A_i)_{i\in I_z}$. This implies that F_z is a closed leaf and does not accumulate onto $0\in\mathbf{C}^n$ (see [30] and [31, Lemma 2.12] for more details). Therefore, the function f_z has a global minimum, which is unique by the previous argument. Finally, a straightforward computation shows that the minimum of f_z is the point w of F_z such that

$$\sum_{i=1}^n A_i|w_i|^2=0.$$

In particular $w/\|w\|$ belongs to X_A.

As a consequence of all that, the foliation $\mathcal{F}$ is trivial and the space $\mathcal{S}_A$ can be identified with $X_A\times\mathbf{R}^+_*\times\mathbf{R}^p$. More precisely, the map

$$\Phi_A\colon(z,T,r)\in X_A\times\mathbf{R}^p\times\mathbf{R}^+_*\longmapsto r\big(z_ie^{\langle A_i,T\rangle}\big)_{i=1}^n\in\mathcal{S}_A$$

is a global diffeomorphism. □

Let $A\in\mathcal{A}$. The real torus $(\mathbf{S}^1)^n$ acts on $\mathbf{C}^n$ by

$$(u,z)\in(\mathbf{S}^1)^n\times\mathbf{C}^n\longmapsto(u_1z_1,\dots,u_nz_n)\in\mathbf{C}^n. \tag{3}$$

Let X be a subset of $\mathbf{C}^n$, which is invariant by the action (3). We define the *natural torus action* on X as the restriction of (3) to X. In particular, every link X_A for $A\in\mathcal{A}$ is endowed with a natural torus action, as well as $\mathbf{S}^{2n-1}$, $\mathbf{D}^{2n}$ and $\overline{\mathbf{D}^{2n}}$.

Definition 0.9. Let $A,B\in\mathcal{A}$. We say that X_A and X_B are *equivariantly diffeomorphic*, and we write $X_A\underset{\text{eq}}{\simeq}X_B$, if there exists a diffeomorphism between X_A and X_B respecting the natural torus actions on X_A and X_B.

More generally, we say that X_A and $X_B\times(\mathbf{S}^1)^k$ are *equivariantly diffeomorphic*, and write $X_A\underset{\text{eq}}{\simeq}X_B\times(\mathbf{S}^1)^k$, if there exists a diffeomorphism between X_A and $X_B\times(\mathbf{S}^1)^k$ respecting the natural torus actions on X_A and on $X_B\times(\mathbf{S}^1)^k$ (seen as a subset of $\mathbf{C}^n\times\mathbf{C}^k$).

LEMMA 0.10. *There exist* $k\in\mathbf{N}$ *and* $B\in\mathcal{A}$ *such that* X_A *is equivariantly diffeomorphic to* $X_B\times(\mathbf{S}^1)^k$ *and* X_B *is* 2*-connected.*

Proof. Assume that $X_A\cap\{z\in\mathbf{C}^n:z_1=0\}$ is vacuous. Let

$$A_i=\begin{pmatrix} a_i \\ \tilde{A}_i \end{pmatrix}.$$

As A_1 is not zero by the weak hyperbolicity condition, we may assume without loss of generality that $a_1\neq 0$. Then, there exists an equivariant diffeomorphism

$$z\in X_A\longmapsto\left(\frac{z_1}{|z_1|},\frac{z_2}{\sqrt{1-|z_1|^2}},...,\frac{z_n}{\sqrt{1-|z_1|^2}}\right)\in\mathbf{S}^1\times X_B,$$

where B is defined as

$$B=\left(\tilde{A}_2-\tilde{A}_1\frac{a_2}{a_1},...,\tilde{A}_n-\tilde{A}_1\frac{a_n}{a_1}\right).$$

Now, B is admissible since, at each point, the system (S_B) has rank p. We may continue this process until we have $X_A\underset{\text{eq}}{\simeq}X_B\times(\mathbf{S}^1)^k$, where the manifold $X_B\subset\mathbf{C}^{n-k}$ intersects each coordinate hyperplane of $\mathbf{C}^{n-k}$ (note that X_B may be reduced to a point). This means that the subspace arrangement $\mathcal{L}_B$ has complex codimension at least 2 in $\mathbf{C}^n$ and thus, by transversality, $\mathcal{S}_B$ is 2-connected. By Lemma 0.8, this implies that X_B is 2-connected. □

We will denote by $\mathcal{A}_0$ the set of admissible matrices giving rise to a 2-connected link. More generally, let $k\in\mathbf{N}$. We will denote by $\mathcal{A}_k$ the set of admissible matrices giving rise to a link with fundamental group isomorphic to $\mathbf{Z}^k$. Of course, by Lemma 0.10, the set $\mathcal{A}$ is the disjoint union of all of the $\mathcal{A}_k$'s for $k\in\mathbf{N}$. Still from Lemma 0.10, observe that k is exactly the number of coordinate hyperplanes of $\mathbf{C}^n$ lying in $\mathcal{L}_A$.

The action (3) induces the following action of $\mathbf{S}^1$ onto a link X_A:

$$(u,z)\in\mathbf{S}^1\times X_A\longmapsto uz\in X_A. \tag{4}$$

We call this action the *diagonal action* of $\mathbf{S}^1$ onto X_A. We have the following lemma.

LEMMA 0.11. *Let* $A\in\mathcal{A}$. *Then the Euler characteristic of* X_A *is zero.*

Proof. The diagonal action is the restriction to X_A of a free action of $\mathbf{S}^1$ onto $\mathbf{S}^{2n-1}$, so is free. Therefore, we may construct a smooth non-vanishing vector field on X_A from a constant unit vector field on $\mathbf{S}^1$. □

The quotient space of X_A by the natural torus action is given by the positive solutions of the system

$$A\cdot r=0, \quad \sum_{i=1}^{n} r_i=1. \tag{5}$$

By the weak hyperbolicity condition, it has maximal rank. We may thus parametrize its set of solutions by

$$r_i=\langle v_i,u\rangle+\varepsilon_i, \quad u\in\mathbf{R}^{n-p-1},$$

for some $v_i\in\mathbf{R}^{n-p-1}$ and some $\varepsilon_i\in\mathbf{R}$. Projecting onto $\mathbf{R}^{n-p-1}$, this gives an identification of the quotient of X_A by the action (3) as

$$\{u\in\mathbf{R}^{n-p-1}:\langle v_i,u\rangle\geqslant-\varepsilon_i\}. \tag{6}$$

LEMMA 0.12. *Let $A\in\mathcal{A}_k$. The identification of the quotient space of X_A defined in* (6) *is a realization of a* (*full*) *simple convex polytope of dimension $n-p-1$ with $n-k$ facets.*

We denote by P_A the convex polytope corresponding to (6). We call it the *associate polytope* of X_A. We denote by P_A^* the dual of P_A, which is thus a simplicial polytope.

Proof. As this set is the quotient space of the compact manifold X_A by the action of a compact torus, it is a compact subset of $\mathbf{R}^{n-p-1}$.

Using (6), it is a bounded intersection of half-spaces, i.e. a realization of a (full) convex polytope of dimension $n-p-1$.

For every subset I of $\{1,...,n\}$, let

$$Z_I=\{z\in\mathbf{C}^n:z_i=0 \text{ if and only if } i\in I\}.$$

Let $z\in X_A$ and define I_z as in (1). Then, for every z' belonging to the orbit of z, we have $I_z=I_{z'}$, and thus the action respects each set Z_{I_z}. Moreover, the action induces a trivial foliation of $X_A\cap Z_{I_z}$.

It follows from all this that each k-face of P_A corresponds to a set of orbits of points z with fixed I_z, i.e. to a set $X_A\cap Z_{I_z}$. In particular, there is a numbering of the faces of P_A such that each j-face is numbered by the $(n-p-1-j)$-tuple I of the corresponding Z_I. As a first consequence, the number of facets of P_A is exactly equal to the number of coordinate hyperplanes of $\mathbf{C}^n$ whose intersection with X_A is non-vacuous, that is, it is equal to $n-k$ (see the remark just after the proof of Lemma 0.10). As a second consequence of this numbering, each vertex v corresponds to an $(n-p-1)$-tuple I, and each facet having v as vertex corresponds to a singleton of I: each vertex is thus attached to exactly $n-p-1$ facets, i.e. the convex polytope is simple. □

Following the numbering introduced in the proof of the previous lemma, we will see P_A as a poset whose elements are subsets of $\{1, ..., n\}$ satisfying

$$I \in P_A \iff L_I \cap X_A \neq \varnothing \iff Z_I \subset \mathcal{S}_A \iff 0 \in \mathcal{H}((A_i)_{i\in I^c}), \tag{7}$$

where $I^c = \{1, ..., n\} \backslash I$. We equip P_A with the order coming from the inclusion of faces. Of course P_A^* will be seen as the same set, but with the reversed order.

Let $(v_1, ..., v_n)$ be a set of vectors of some $\mathbf{R}^q$. Following [4], a *Gale diagram* of $(v_1, ..., v_n)$ is a set of points $(w_1, ..., w_n)$ in $\mathbf{R}^{n-q-1}$ satisfying, for all proper subsets I of $\{1, ..., n\}$,

$$0 \in \operatorname{Relint}(\mathcal{H}(w_i)_{i\in I}) \iff \mathcal{H}(v_i)_{i\in I^c} \text{ is a proper face of } \mathcal{H}(v_1, ..., v_n), \tag{8}$$

where $\operatorname{Relint}(\cdot)$ denotes the relative interior of a set.

Now, consider (6). Notice that we may assume that the ε_i's are positive, taking as $(\varepsilon_1, ..., \varepsilon_n)$ a particular solution of (5). Under this assumption, let $B_i = v_i/\varepsilon_i$ for i between 1 and n. The convex hull of $(B_1, ..., B_n)$ is a realization of P_A^*. Using (8) and the weak hyperbolicity condition, it is easy to prove the following result.

LEMMA 0.13. (Cf. [30, Lemma VII.2]) *The set* $(A_1, ..., A_n)$ *is a Gale diagram of* $(B_1, ..., B_n)$.

Notice that if $(A_1, ..., A_n)$ is a Gale diagram of two different sets $(B_1, ..., B_n)$ and $(C_1, ..., C_n)$, and if $\mathcal{H}(B_1, ..., B_n)$ is a *simplicial* polytope, then $\mathcal{H}(C_1, ..., C_n)$ is also simplicial and is combinatorially equivalent to $\mathcal{H}(B_1, ..., B_n)$. We now have the following result.

THEOREM 0.14. (Realization theorem; see [30, Theorem 14]) *Let P be a simple convex polytope. Then, for every $k \in \mathbf{N}$, there exists $A(k) \in \mathcal{A}_k$ such that $P_{A(k)} = P$. In particular, every simple convex polytope can be realized as the associate polytope of some* 2-*connected link.*

Proof. Let P be a simple polytope and let P^* be its dual. Realize P^* in $\mathbf{R}^q$ (with $q = \dim P^*$) as the convex hull of its vertices $(v_1, ..., v_n)$.

Let us start with $k=0$. By Lemma 0.13, it is sufficient to find $A(0) \in \mathcal{A}_0$ such that $A(0)$ is a Gale diagram of P^*.

This can be done by taking a Gale transform ([17, p. 84]) of $(v_1, ..., v_n)$, that is by taking the transpose of a basis of the solutions of

$$\begin{cases} \sum_{i=1}^n x_i v_i = 0, \\ \sum_{i=1}^n x_i = 0. \end{cases}$$

We thus obtain n vectors $(A_1, ..., A_n)$ in $\mathbf{R}^{n-q-1}$. Set $A(0)=(A_1, ..., A_n)$ and $p=n-q-1$. We have now to check that $A(0)\in\mathcal{A}_0$. By (8), the Gale transform $(A_1, ..., A_n)$ satisfies the Siegel condition. Assume that 0 belongs to the relative interior of $\mathcal{H}(A_i)_{i\in I}$ for some $I=\{i_1, ..., i_r\}$ with $r\leqslant p$. Then $\mathcal{H}(v_i)_{i\in I^c}$ is a proper face of P^*, so it has dimension less than $q=n-p-1$. But it has $n-r$ vertices. Since $n-r\geqslant n-p$, this face cannot be simplicial, yielding a contradiction. The weak hyperbolicity condition is fulfilled.

Finally, as $P^*=P^*_{A(0)}$ has n vertices, the link $X_{A(0)}$ intersects each coordinate hyperplane of $\mathbf{C}^n$, so it is 2-connected (see Lemma 0.8).

Now, using the construction detailed in Example 0.7, we can find $A(k)\in\mathcal{A}_k$, for every k, such that $P_{A(k)}=P$. □

Note that, when P^* is the n-simplex, the previous construction (for a 2-connected link) yields $p=0$, and the corresponding X_A is the standard sphere of $\mathbf{C}^{n-1}$.

To finish with these preliminaries, we discuss now the relationship between links and moment-angle complexes coming from simple polytopes. These complexes were first introduced in [12]. We follow [9, §6].

Let P be a simple convex polytope with set of facets $\mathcal{F}=\{F_1, ..., F_n\}$. For each facet F_i, denote by T_{F_i} the 1-dimensional coordinate subgroup of the n-torus $T_{\mathcal{F}}\simeq(\mathbf{S}^1)^n$ corresponding to F_i. Then, assign to every face G the coordinate subtorus

$$T_G = \prod_{F_i\supset G} T_{F_i} \subset T_{\mathcal{F}}.$$

For every point $q\in P$, denote by $G(q)$ the unique face containing q in its relative interior.

Then, the moment-angle complex $\mathcal{Z}_P$ is the identification space

$$\mathcal{Z}_P = (T_{\mathcal{F}}\times P)/\sim,$$

where $(t_1, p)\sim(t_2, q)$ if and only if $p=q$ and $t_1t_2^{-1}\in T_{G(q)}$.

The moment-angle complex depends only on the combinatorial type of P and comes naturally equipped with a continuous action of $T_{\mathcal{F}}$ on it, with orbit space P. It is a topological manifold ([9, Lemma 6.2]).

The next lemma follows from this definition and from the previous description of the action of $(\mathbf{S}^1)^n$ onto a link.

Lemma 0.15. *Let X_A be a 2-connected link with associate polytope P. Then X_A is equivariantly homeomorphic to $\mathcal{Z}_P$.*

Moreover, Buchstaber and Panov prove that $\mathcal{Z}_P$ is a smooth manifold such that the natural torus action is smooth. In fact, Buchstaber and Panov give several ways of endowing $\mathcal{Z}_P$ with such a structure. Let us just describe one of them. The moment-angle manifold $\mathcal{Z}_P$ may be equivariantly realized in $(\overline{\mathbf{D}})^{2n}$ as

$$\bigcup_I \{(z_1, ..., z_n) \in (\overline{\mathbf{D}})^{2n} : |z_i| = 1 \text{ for } i \notin I\},$$

where I runs over all proper faces of P (following the numbering of the faces previously defined). This is a sort of manifold with corners, and it is possible to equivariantly "straighten the angles" (compare with [6, Proposition 2.3]). Notice that this smooth structure is compatible with the torus action in the following sense.

Definition 0.16. A smooth structure on $\mathcal{Z}_P$ is *compatible* with the torus action if

(i) the torus action is smooth;

(ii) for every *closed* face F of P, the set $\pi^{-1}(F)$ (where $\pi: \mathcal{Z}_P \to P$ is the natural projection) is a smooth invariant submanifold of $\mathcal{Z}_P$ with trivial invariant normal bundle.

On the other hand, Lemma 0.15 gives also a smooth compatible structure on $\mathcal{Z}_P$: that of a link (point (ii) of Definition 0.16 is checked in Proposition 1.1). Nevertheless, *it is not clear neither that these two smooth manifolds are equivariantly diffeomorphic, nor that the different smooth structures on $\mathcal{Z}_P$ described in* [9] *are the same.* Indeed, Buchstaber and Panov do not touch the following question.

Question 0.17. Does there exist a unique smooth structure on $\mathcal{Z}_P$ compatible with the torus action (up to equivariant diffeomorphism)?

We give an affirmative answer to this question in §4 (Corollary 4.7).

Part I. Elementary surgeries, flips and wall-crossing

1. Submanifolds of X_A given by a face of P_A

Let $A \in \mathcal{A}$ and F be a proper face of P_A numbered by I. Then, we may associate a link with F and A, which we will denote by X_F (by a slight abuse of notation), smoothly embedded in X_A. To do this, just recall by (7) that

$$B = (A_j)_{j \in I^c}$$

is admissible and thus gives rise to a link X_B in $\mathbf{C}^{n-b}$, where b is the cardinality of I. Now, X_B is naturally embedded into X_A as X_F by defining

$$X_F = L_I \cap X_A, \tag{9}$$

where L_I was defined in (2). Moreover, the natural torus action of $(\mathbf{S}^1)^n$ onto X_A gives, by restriction to L_I, the natural torus action of $(\mathbf{S}^1)^{n-b}$ onto $X_F \underset{\text{eq}}{\sim} X_B$.

We have the following result.

Proposition 1.1. *Let* $A\in\mathcal{A}$ *and* F *be a face of* P_A *of codimension* b. *Then,*

(i) X_F *is a smooth submanifold of codimension* $2b$ *of* X_A *which is invariant under the natural torus action*;

(ii) *the quotient space of* X_F *by the natural torus action is* $F\subset P_A$;

(iii) X_F *has trivial invariant tubular neighborhood in* X_A.

Proof. The points (i) and (ii) are direct consequences of the definition (9) of X_F. Let us prove (iii). For $\varepsilon>0$, define

$$L_I^\varepsilon=\{z\in\mathbf{C}^n:\textstyle\sum_{i\in I}|z_i|^2<\varepsilon\}$$

and

$$W_F^\varepsilon=X_A\cap L_I^\varepsilon.$$

For simplicity, assume that $I=\{1,...,b\}$. Set $y_j=z_j$ for $1\leqslant j\leqslant b$, and $w_j=z_{b+j}$ for $1\leqslant j\leqslant n-b$. For $\varepsilon>0$ sufficiently small, the map

$$\pi\colon(y,w)\in W_F^\varepsilon\longmapsto\frac{1}{\sqrt{\varepsilon}}y\in\mathbf{D}^{2b}$$

is a smooth submersion. Indeed, a straightforward computation shows that the previous map is a submersion as soon as W_F^ε does not intersect any of the sets

$$\{w_j=0:b+j\in J\},$$

for J satisfying $F\cap F_J=\varnothing$ (cf. the proof of Lemma 0.3). As this submersion has compact fibers, it is a locally trivial fiber bundle by Ehresmann's lemma. It is even a trivial bundle, since $\mathbf{D}^{2b}$ is contractible. Notice now that the action of $(\mathbf{S}^1)^n$ onto W_F^ε can be decomposed into an action of $(\mathbf{S}^1)^b$ leaving the y-coordinates fixed and an action of $(\mathbf{S}^1)^{n-b}$ leaving the w-coordinates fixed. The fibers of the previous submersion are invariant with respect to the action of $(\mathbf{S}^1)^{n-b}$, whereas the disk $\mathbf{D}^{2b}$ is invariant with respect to the action of $(\mathbf{S}^1)^b$. All this implies that W_F^ε is equivariantly diffeomorphic to $X_F\times\mathbf{D}^{2b}$ endowed with its natural torus action. □

In the case where F is a simplicial face, we can identify precisely X_F.

Proposition 1.2. *Let* $A\in\mathcal{A}_0$. *Then, the following statements are equivalent*:

(i) X_A *is equivariantly diffeomorphic to the unit euclidean sphere* $\mathbf{S}^{2n-1}$ *of* $\mathbf{C}^n$ *equipped with the action induced by the standard action of* $(\mathbf{S}^1)^n$ *on* $\mathbf{C}^n$;

(ii) X_A *is diffeomorphic to* $\mathbf{S}^{2n-1}$;

(iii) X_A *has the homotopy type of* $\mathbf{S}^{2n-1}$;

(iv) P_A *is the* $(n-1)$*-simplex.*

Proof. When $p=0$, the link X_A is the unit euclidean sphere $\mathbf{S}^{2n-1}$ of $\mathbf{C}^n$, and the natural torus action comes from the standard action of $(\mathbf{S}^1)^n$ on $\mathbf{C}^n$. On the other hand, when P_A is the $(n-1)$-simplex, we have $p=0$, since the dimension of P_A is $n-p-1$; in this way, we get an equivalence between (i) and (iv).

Of course, (i) implies (ii) and (ii) implies (iii). So assume now that X_A is a homotopy sphere of dimension $2n-1$. Recall that a polytope with n vertices is k*-neighbourly* if its k-skeleton coincides with the k-skeleton of an $(n-1)$-simplex (cf. [17, Chapter 7]). In particular, an $(n-1)$-simplex is $(n-2)$-neighbourly.

Applying Lemma 1.3 below gives that P_A^* is $(n-2)$-neighbourly. But, its dimension being $n-p-1$, this implies that p equals 0 and that it is the $(n-1)$-simplex. Therefore (iii) implies (iv). □

Lemma 1.3. *Let $A\in\mathcal{A}_0$. Then, the link X_A is $2k$-connected if and only if P_A^* is a $(k-1)$-neighbourly polytope.*

Proof. Assume that P_A^* is $(k-1)$-neighbourly. This means that every subset of $\{1,...,n\}$ of cardinalilty less than k, numbers a face of P_A^*. Using (2) and (7), this means that every coordinate subspace of $\mathcal{L}_A$ has at least complex codimension $k+1$. By transversality, this implies that $\mathcal{S}_A$ is $2k$-connected and thus, by Lemma 0.8, the link X_A is $2k$-connected.

Now, assume moreover that P_A^* is *not* k-neighbourly. Then, there exists a coordinate subspace L_I in $\mathcal{L}_A$ of codimension $k+1$. The unit sphere $\mathbf{S}^{2k+1}$ of the complementary coordinate subspace L_{I^c} lies in $\mathcal{S}_A$ and is not null-homotopic in $\mathcal{S}_A$. Therefore, $\mathcal{S}_A$ and thus X_A are not $(2k+1)$-connected. □

Corollary 1.4. *Let $A\in\mathcal{A}$. Then P_A is the $(n-p-1)$-simplex if and only if X_A is equivariantly diffeomorphic to $\mathbf{S}^{2n-2p-1}\times(\mathbf{S}^1)^p$.*

Proof. Assume that P_A is the $(n-p-1)$-simplex. Since the polytope P_A has $n-p$ facets, we know that $A\in\mathcal{A}_p$. By Lemma 0.10, there exists $B\in\mathcal{A}_0$ such that

$$X_A \underset{\text{eq}}{\simeq} X_B\times(\mathbf{S}^1)^p.$$

Now, this implies that $P_B=P_A$, so that P_B is the $(n-p-1)$-simplex. We conclude by Proposition 1.2.

The converse is obvious by Proposition 1.2. □

Corollary 1.5. *Let F be a simplicial face of P_A of codimension b. Then X_F is equivariantly diffeomorphic to $\mathbf{S}^{2n-2p-2b-1}\times(\mathbf{S}^1)^p$.*

2. Flips of simple polytopes

We will make use of the notion of flips of simple polytopes. This section is deeply inspired by [38, §3] (see also [29]). The main difference is that we only deal with combinatorial types of simple polytopes.

Definition 2.1. Let P and Q be two simple polytopes of the same dimension q. Let W be a simple polytope of dimension $q+1$. We say that W is a *cobordism* between P and Q if P and Q are disjoint facets of W. In addition, if $W\setminus(P\sqcup Q)$ contains no vertex, we say that W is a *trivial* cobordism; if $W\setminus(P\sqcup Q)$ contains a unique vertex, we say that W is an *elementary* cobordism.

In the next section, we will relate this notion of cobordism of polytopes to the classical notion of cobordism of manifolds (here of links) via Theorem 0.14. This will justify the terminology.

Notice that the existence of a trivial cobordism between P and Q implies that $P=Q$; notice also that a cobordism of simple polytopes may be decomposed into a finite number of elementary cobordisms.

Now, let W be an elementary cobordism between P and Q, and let v denote the unique vertex of $W\setminus(P\sqcup Q)$. An edge attached to v has another vertex which may belong to P or Q. Let us say that, among the $q+1$ edges attached to v, a of them join P and b of them join Q.

Definition 2.2. (Cf. [38, §3.1]) We call the *index* of v, or the *index* of the cobordism, the couple of integers (a,b), where a (respectively, b) is the number of edges of W attached to v and joining P (respectively, Q).

Let P and Q be two simple polytopes of the same dimension q. Assume that there exists an elementary cobordism W between them and let (a,b) denote its index. Then we say that Q is obtained from P by performing a *flip* of type (a,b) on P, or that P undergoes a *flip* of type (a,b).

An example of a flip of type $(1,2)$ is illustrated in Figure 3.

Notice that if Q is obtained from P by a flip of type (a,b), then obviously P is obtained from Q by a flip of type (b,a). Note also that we have the obvious relation $a+b=q+1$, with $1\leqslant a\leqslant q$ and $1\leqslant b\leqslant q$.

Lemma 2.3. *Every simple convex q-polytope can be obtained from the q-simplex by a finite number of flips.*

Proof. Let P be a simple convex q-polytope. Consider the product $P\times[0,1]$ and cut off one vertex of $P\times\{1\}$ by a generic hyperplane. The resulting polytope, say W, is

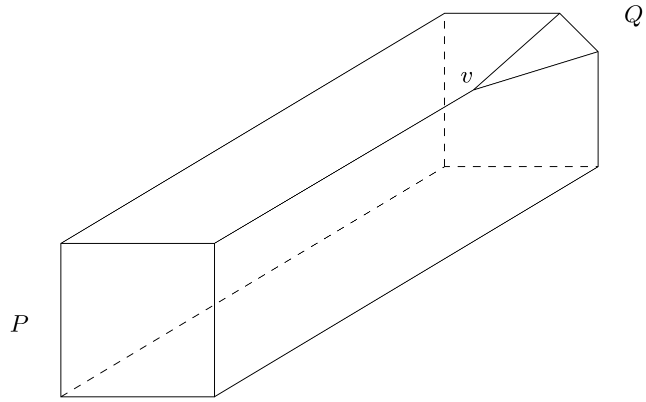

Figure 3.

simple and realizes a cobordism between P (seen as $P\times\{0\}$) and the q-simplex (seen as the simplicial facet created by the cut). As observed after Definition 2.1, this cobordism may be decomposed into a finite number of elementary cobordisms, that is of flips. □

Following [38, §3.2], it is possible to give a more precise description of a flip of type (a,b). We use the same notation as before. Let $F_1, ..., F_{q+1}$ be the facets of W attached to the vertex v. As W is simple, a sufficiently small neighborhood of v in W is PL-isomorphic to the neighborhood of a point in a $(q+1)$-simplex. As a consequence, each facet F_i contains all the edges attached to v but one. Assume that $(F_1, ..., F_b)$ contain all the edges joining P, whereas $(F_{b+1}, ..., F_{q+1})$ contain all the edges joining Q. Let

$$F_P = P\cap F_1\cap ...\cap F_b \quad \text{and} \quad F_Q = Q\cap F_{b+1}\cap ...\cap F_{q+1}.$$

The face $F_1\cap ...\cap F_b$ (respectively, $F_{b+1}\cap ...\cap F_{q+1}$) is a pyramid with base F_P (respectively, F_Q) and apex v. As these faces are simple as convex polytopes, this implies that F_P and F_Q are simplicial. More precisely, if $a=1$ (respectively, $b=1$), then F_P (respectively, F_Q) is a point, and $F_P\cap F_{q+1}=\varnothing$ (respectively, $F_Q\cap F_1=\varnothing$). Otherwise, F_P is a simplicial face of strictly positive dimension $q-b=a-1$ with facets $F_P\cap F_{b+1}, ..., F_P\cap F_{q+1}$ (respectively, F_Q is a simplicial face of strictly positive dimension $b-1$ with facets $F_Q\cap F_1, ..., F_Q\cap F_b$).

In Figure 4, F_P is a point and F_Q is a segment. There are three facets, namely F_1, F_2 and F_3, containing v.

The flip destroys the face F_P and creates the face F_Q in its place. Continuously, the face F_P is homothetically reduced to a point and then this point is inflated to the face F_Q. In a more static way of thinking, a trivial neighborhood of F_P in P is cut off, and a closed trivial neighborhood of F_Q in Q is glued. In particular, the simple polytope

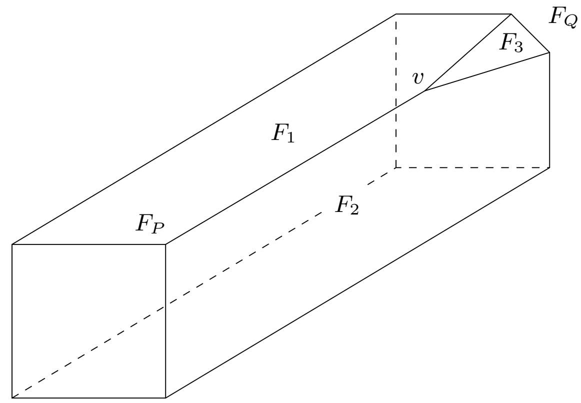

Figure 4.

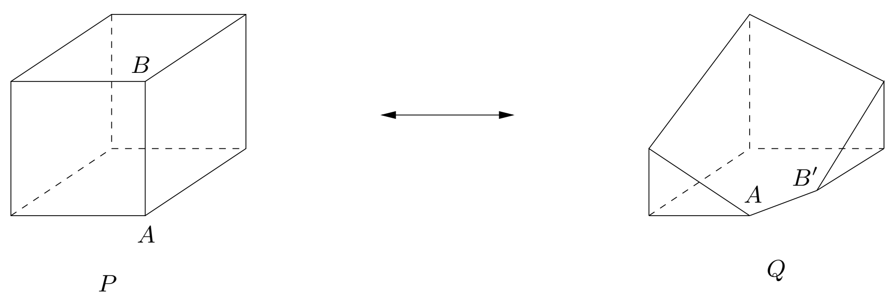

Figure 5.

obtained from P by cutting off a neighborhood of F_P by a hyperplane and the polytope obtained from Q by cutting off a neighborhood of F_Q by a hyperplane are the same. Let us denote this polytope by T.

Definition 2.4. The simple convex polytope T will be called the *transition polytope* of the flip between P and Q.

Remark 2.5. This definition is not the same as that of transition polytope in [38].

Notice that T has just one extra facet (with respect to P and Q), except for the special case of index $(1,1)$. Let us call this extra facet F.

Figure 5 describes a flip of type $(2,2)$. We simply drew the initial state P and the final state Q, and indicated the two edges F_P of vertices A and B, and F_Q of vertices A and B'.

To visualize the 4-dimensional cobordism between P and Q, just perform the following homotopy: move the hyperplane supporting the upper facet of the cube to the bottom, in order to contract the edge AB to its lower vertex A; then, move the hyper-

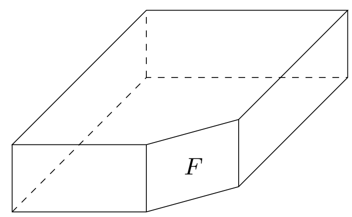

Figure 6.

plane supporting the right facet of the cube to the right, in order to inflate the transverse edge AB', keeping A fixed. The transition polytope T is depicted in Figure 6.

PROPOSITION 2.6. (i) *The extra facet F of T is $F_P \times F_Q$, that is a product of an $(a-1)$-simplex by a $(b-1)$-simplex.*

(ii) *A neighborhood of F_P in P (respectively, F_Q in Q) is $F_P \times \mathcal{C}(F_Q)$ (respectively, $\mathcal{C}(F_P) \times F_Q$), where $\mathcal{C}(F_Q)$ (respectively, $\mathcal{C}(F_P)$) denotes the pyramid with base F_Q (respectively, F_P).*

Proof. Assume that P is a simplex. Cut off a neighborhood of F_P by a hyperplane. The created facet is a product of the simplex F_P by a simplex S of complementary dimension, whereas the cut part is $F_P \times \mathcal{C}(S)$, with the notation introduced in the statement of the proposition. Then both statements follow, since the neighborhood of a simplicial face in a simple convex polytope is PL-homeomorphic to the neighborhood of a face of the same dimension in a simplex. □

In particular, P and Q can be recovered from T (up to exchange of P and Q): the face poset of P is obtained from that of T by identifying two faces $A \times B$ and $A \times B'$ of $F_P \times F_Q$, and the face poset of Q is obtained from that of T by identifying two faces $A \times B$ and $A' \times B$ of $F_P \times F_Q$.

Combining this observation with Proposition 2.6 yields the following result.

COROLLARY 2.7. *Let Q and Q' be obtained from P by a flip of type (a, b) along the same simplicial face F_P. Then $Q = Q'$.*

Given a simple convex polytope T with a facet F which is a product of simplices $S_{a-1} \times S_{b-1}$, we may define two posets from the poset of the faces of T making the identifications explained just before Corollary 2.7. These two posets *may* or *may not* be the face posets of some simple convex polytopes P and Q (see the examples below). In the case they are, we write $P = T/S_{a-1}$ and $Q = T/S_{b-1}$. Of course, in the case of a flip, with the same notation as before, we have $P = T/F_P$ and $Q = T/F_Q$. The next result is a reformulation of Corollary 2.7 which will be useful in the sequel.

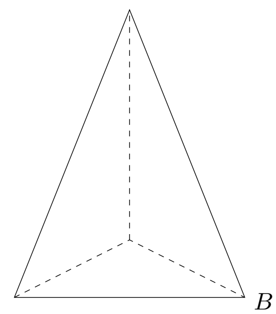

Figure 7.

Corollary 2.8. *Let Q be obtained from P by a flip along F_P and let T be the transition polytope. Let P' and Q' be two simple convex polytopes satisfying $P'=T/F_P$ and $Q'=T/F_Q$. Then $P=P'$ and $Q=Q'$.*

Let us describe another way of visualizing a flip. Let P be a simple polytope and F_P a simplicial face of P of dimension $a-1$. Let Q be a simple polytope and assume that Q is obtained from P by performing a flip on F_P. Cutting off F_P by a hyperplane, one obtains the transition polytope T. Consider now a simplex Δ of the same dimension as P and an $(a-1)$-face F' of Δ. Cutting off F' by a hyperplane, one obtains, with the notation of Proposition 2.6, the polytope $F'\times S$, where S is the maximal simplicial face of Δ without intersection with F'. It follows from Proposition 2.6 and Corollary 2.8 that the polytope Q is the gluing of $T=P\setminus(F_P\times\mathcal{C}(S))$ and $\Delta\setminus(F_P\times\mathcal{C}(S))=F'\times S$.

Finally, from all that precedes, a complete combinatorial characterization of a flip may easily be derived. In the following statement, we consider also flips of type $(q+1,0)$, that is destructions of a q-simplex.

Proposition 2.9. ([38, Theorem 3.4.1]) *Let Q be a simple polytope obtained from P by a flip of type (a,b). Using the same notation as before, the following properties hold*:

(i) *if $a\neq 1$, then the facets $P\cap F_{b+1},...,P\cap F_{q+1}$ undergo flips of index $(a-1,b)$*;

(ii) *the facets $P\cap F_1,...,P\cap F_b$ undergo flips of index $(a,b-1)$*;

(iii) *the other facets keep the same combinatorial type.*

It is however important to remark that the notion of "combinatorial flip" is not well defined in the class of simple polytopes: the result of cutting off a neighborhood of a simplicial face of a simple polytope and gluing the neighborhood of another simplex in its place *may not be* a convex polytope. Let us give three examples of this crucial fact.

Example 2.10. Let P be the 3-simplex (see Figure 7). Then, the result of cutting off an edge AB and gluing a transverse edge in its place (that is the result of a "combinatorial

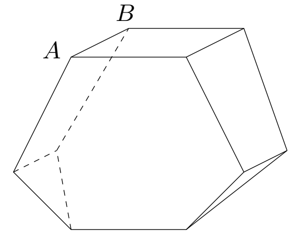

Figure 8.

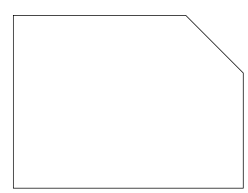

Figure 9.

2-flip") is not the combinatorial type of a 3-polytope.

Example 2.11. More generally, let P be a simple convex polytope and F_P a simplicial face of dimension q, with $q>2$. Then, we cannot perform a flip along a strict face of F_P.

Example 2.12. Consider the polytope illustrated in Figure 8 (the "hexagonal book"). Then, the 2-flip along the edge AB does not exist.

We finish this section with the following result.

PROPOSITION 2.13. *Let P be a simple convex polytope and Q be obtained from P by a flip of type (a,b). Let W be the elementary cobordism between P and Q. Assume that P has d facets. Then W has $d+2$ facets if $a\neq 1$, and $d+3$ facets if $a=1$.*

Proof. In the special case where $a=b=1$, we have that $P=Q$ is the segment and W is the pentagon (see Figure 9). Thus, $d=2$ and W has $d+3$ facets.

Assume that a and b are both different from 1. Then P and Q have the same number d of facets and there is a one-to-one correspondence between the facets of P and the facets of Q: according to Proposition 2.9, each facet of P is transformed through a flip (case (i) or (ii)) or just shifted (case (iii)) to a facet of Q. There are d facets of W which realize the previous trivial and elementary cobordisms. Adding 2 to this number, taking P and Q into account, gives that W has $d+2$ facets.

Now assume that $a=1$ and $b\neq 1$. Then, as before, the d facets of P correspond to d facets of W realizing cobordisms with d facets of Q. But this time Q has $d+1$ facets, and this extra facet belongs to an extra facet of W which does not intersect P. Adding the two facets P and Q gives thus $d+3$ facets for W.

Finally, when $b=1$ and $a\neq 1$, the polytope Q has $d-1$ facets; switching the roles of P and Q in the previous case yields that W has $(d-1)+3=d+2$ facets. □

3. Elementary surgeries

In this section, we translate the notions of cobordism and flip of simple polytopes at the level of the links, by introducing elementary surgeries on links. Notice that "equivariant surgeries" of moment-angle complexes (up to equivariant homeomorphism) were considered in [9, §§6.23–6.25] in connection with the so-called bistellar moves of simplicial complexes. Bistellar moves are dual operations to flips of simple polytopes.

We will make use several times of the following result.

Theorem 3.1. (Extension of equivariant isotopies) *Let M and V be smooth compact manifolds endowed with a smooth torus action. Let $f: V\times[0,1]\to M$ be an equivariant isotopy. Then f can be extended to an equivariant diffeotopy $F: M\times[0,1]\to M$ such that $F_t|_V\equiv f_t$ for $0\leqslant t\leqslant 1$.*

A proof of this fact *in the non-equivariant case* can be found in [19, Chapter 8]. Now, we may assume that the diffeotopy extending an equivariant isotopy is also equivariant (see [7, §VI.3]), so that this theorem holds in the equivariant setting.

Let $A\in\mathcal{A}$ and F be a *simplicial* face of P_A of codimension b. As explained in §1, it gives rise to an invariant submanifold X_F of X_A (see definition (9)) with trivial invariant tubular neighborhood.

By Corollary 1.5, as F is simplicial of codimension b, we have that X_F is equivariantly diffeomorphic to $\mathbf{S}^{2a-1}\times(\mathbf{S}^1)^p$ (where $a=n-p-b$).

But now, we can perform an equivariant surgery on X_A as follows: choose a closed invariant tubular neighborhood

$$\nu: X_F\times\overline{\mathbf{D}^{2b}}\longrightarrow\overline{W}_F,$$

where $W_F\subset X_A$ is an open (invariant) neighborhood of X_F. Then fix an equivariant identification

$$\xi: \mathbf{S}^{2a-1}\times(\mathbf{S}^1)^p\longrightarrow X_F.$$

Finally, set

$$\phi\equiv\nu\circ(\xi,\mathrm{id}): \mathbf{S}^{2a-1}\times(\mathbf{S}^1)^p\times\overline{\mathbf{D}^{2b}}\longrightarrow\overline{W}_F.$$

We call ϕ a *standard product neighborhood* of X_F.

Then, remove W_F and glue $\overline{\mathbf{D}^{2a}}\times(\mathbf{S}^1)^p\times\mathbf{S}^{2b-1}$ by ϕ along the boundary. We thus obtain a topological manifold Y. Since the natural torus actions on $\overline{\mathbf{D}^{2a}}\times(\mathbf{S}^1)^p\times\mathbf{S}^{2b-1}$ and on $\mathbf{S}^{2a-1}\times(\mathbf{S}^1)^p\times\overline{\mathbf{D}^{2b}}$ coincide on their common boundary, this topological manifold supports a continuous action of $(\mathbf{S}^1)^n$ which extends the natural torus action on $X_A\backslash W_F$. Using invariant collars for the boundary of $X_A\backslash W_F$ and for the boundary of $\overline{\mathbf{D}^{2a}}\times(\mathbf{S}^1)^p\times\mathbf{S}^{2b-1}$, we may smooth Y as well as the action in such a way that the natural inclusions of $X_A\backslash W_F$ and $\overline{\mathbf{D}^{2a}}\times(\mathbf{S}^1)^p\times\mathbf{S}^{2b-1}$ in it are equivariant embeddings. As a consequence of Theorem 3.1, it can be proven that, up to equivariant diffeomorphism, there are no other differentiable structure and smooth action on Y satisfying this property (see [19, Chapter 8] for the non-equivariant case). The manifold Y endowed with such a differentiable structure and such a smooth torus action, is the result of our surgery.

Here is a combinatorial description of this surgery. Recall that P_A identifies with the quotient of X_A by the natural torus action. The neighborhood W_F then corresponds to a neighborhood of F in P_A. Consider now a simplex Δ of the same dimension as P_A and a face F' of Δ of the same dimension as F. By Corollary 1.4, the link X_Δ corresponding to Δ is equivariantly diffeomorphic to $\mathbf{S}^{2n-2p-1}\times(\mathbf{S}^1)^p$, and a neighborhood $W_{F'}$ of $X_{F'}$ (coming from a neighborhood of F' in Δ) is equivariantly diffeomorphic to W_F. The complement $X_\Delta\backslash W_{F'}$ is equivariantly diffeomorphic to

$$(\mathbf{S}^{2n-2p-1}\backslash(\mathbf{S}^{2a-1}\times\mathbf{D}^{2b}))\times(\mathbf{S}^1)^p=\overline{\mathbf{D}^{2a}}\times\mathbf{S}^{2b-1}\times(\mathbf{S}^1)^p.$$

The surgery consists of removing W_F from X_A and $W_{F'}\underset{\text{eq}}{\sim}W_F$ from X_Δ, and of gluing the resulting manifolds along their boundary:

$$(X_A\backslash W_F)\cup_\psi(X_\Delta\backslash W_{F'}). \tag{10}$$

The map ψ may be written as $\phi\circ(\phi')^{-1}$, where ϕ (respectively, ϕ') is a standard product neighborhood of X_F in X_A (respectively, of $X_{F'}$ in X_Δ).

We conclude from this description and from Corollary 2.8 that, at the level of the associate polytope, this surgery coincides exactly with a flip.

Definition 3.2. Let $A\in\mathcal{A}$. Let (a,b) be a couple of positive integers with $a+b=n-p$. Let F be a simplicial face of P_A of codimension b. The equivariant transformation

$$(X_A\backslash(\mathbf{S}^{2a-1}\times(\mathbf{S}^1)^p\times\mathbf{D}^{2b}))\cup_\phi(\overline{\mathbf{D}^{2a}}\times(\mathbf{S}^1)^p\times\mathbf{S}^{2b-1})$$

of X_A is called *elementary surgery* of type (a,b) along X_F. Here, $\mathbf{S}^{2a-1}\times(\mathbf{S}^1)^p\times\mathbf{D}^{2b}$ is embedded in X_A by means of a standard product neighborhood ϕ, and the gluing is made along the common boundary by the restriction of ϕ to this boundary.

In the particular case where $a=1$, we restrict the definition of elementary surgery to the case where X_A is equivariantly diffeomorphic to $X_B\times\mathbf{S}^1$ and where the surgery is made as follows:

$$((X_B\setminus((\mathbf{S}^1)^p\times\mathbf{D}^{2b}))\times\mathbf{S}^1)\cup_\phi((\mathbf{S}^1)^p\times\mathbf{S}^{2b-1}\times\overline{\mathbf{D}^2}).$$

These surgeries depend *a priori* on the choice of ϕ. But, in fact, we have the following lemma.

LEMMA 3.3. *The result of an elementary surgery is independent of the choice of ϕ. In other words, given two standard product neighborhoods ϕ and ϕ', the manifolds*

$$X_\phi=(X_A\setminus(\mathbf{S}^{2a-1}\times(\mathbf{S}^1)^p\times\mathbf{D}^{2b}))\cup_\phi(\overline{\mathbf{D}^{2a}}\times(\mathbf{S}^1)^p\times\mathbf{S}^{2b-1})$$

and

$$X_{\phi'}=(X_A\setminus(\mathbf{S}^{2a-1}\times(\mathbf{S}^1)^p\times\mathbf{D}^{2b}))\cup_{\phi'}(\overline{\mathbf{D}^{2a}}\times(\mathbf{S}^1)^p\times\mathbf{S}^{2b-1})$$

are equivariantly diffeomorphic.

Proof. It is enough to prove that ϕ and ϕ' are equivariantly isotopic. As in the non-equivariant case, the uniqueness of gluing for isotopic diffeomorphisms is a direct consequence of Theorem 3.1.

Now, any two invariant tubular neighborhoods of X_F are equivariantly isotopic by [7, §VI.2]. Thus, we may assume that

$$\phi(\mathbf{S}^{2a-1}\times(\mathbf{S}^1)^p\times\overline{\mathbf{D}^{2b}})=\phi'(\mathbf{S}^{2a-1}\times(\mathbf{S}^1)^p\times\overline{\mathbf{D}^{2b}})$$

and that the map $f=\phi'\circ\phi^{-1}$ is of the form

$$(z,e^{it},w)\in\mathbf{S}^{2a-1}\times(\mathbf{S}^1)^p\times\overline{\mathbf{D}^{2b}}\longmapsto(f_1(z,e^{it}),f_2(z,e^{it}),A(z,e^{it})\cdot w),$$

where A is a smooth invariant map from $\mathbf{S}^{2a-1}\times(\mathbf{S}^1)^p$ to the group of matrices $\mathrm{SO}(2b)$, and i, in this proof, stands for the imaginary unit. Moreover, the equivariance of f implies that each matrix $A(z,e^{it})$ is of the form

$$\begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_b}\end{pmatrix}.$$

We may thus easily equivariantly isotope f to

$$(z,e^{it},w)\in\mathbf{S}^{2a-1}\times(\mathbf{S}^1)^p\times\overline{\mathbf{D}^{2b}}\longmapsto(f_1(z,e^{it}),f_2(z,e^{it}),w),$$

and it is enough to prove that the equivariant diffeomorphism $\tilde{f}=(f_1,f_2)$ of $\mathbf{S}^{2a-1}\times(\mathbf{S}^1)^p$ is equivariantly isotopic to the identity.

Still by equivariance, we have

$$\tilde{f}(z,e^{it})=e^{it}\tilde{f}(z,1),$$

so we may equivariantly isotope $\tilde{f}$ to a map of the form

$$(z,e^{it})\in\mathbf{S}^{2a-1}\times(\mathbf{S}^1)^p\longmapsto(h(z),e^{it})\in\mathbf{S}^{2a-1}\times(\mathbf{S}^1)^p,$$

where h is an equivariant diffeomorphism of $\mathbf{S}^{2a-1}$. Finally, using Lemma 3.4 (stated and proved below), h and thus f are equivariantly isotopic to the identity. This is enough to show the result. □

LEMMA 3.4. *Let h be an equivariant diffeomorphism of the sphere $\mathbf{S}^{2a-1}$. Then f is equivariantly isotopic to the identity.*

Proof. We proceed by induction on a. For $a=1$, the map h is a translation so the result is clear. Assume the result true for some $a\geqslant 1$, and let h be an equivariant diffeomorphism of $\mathbf{S}^{2a+1}$. By equivariance, the submanifold

$$X=\{z\in\mathbf{S}^{2a+1}:z_{a+1}=0\}\underset{\text{eq}}{\sim}\mathbf{S}^{2a-1}$$

is invariant by h. We shall construct two invariant tubular neighborhoods of X. First, consider, for $0<\varepsilon<1$,

$$X_\varepsilon=\{z\in\mathbf{S}^{2a+1}:|z_{a+1}|^2\leqslant\varepsilon\}\underset{\text{eq}}{\sim}\mathbf{S}^{2a-1}\times\overline{\mathbf{D}^2},$$

and the equivariant bundle map

$$z\in X_\varepsilon\overset{\xi}{\longmapsto}\frac{1}{\sqrt{1-|z_{a+1}|^2}}(z_1,...,z_a,0)\in X.$$

Secondly, let f be the restriction of h^{-1} to X. Set $\widetilde{X}_\varepsilon=f^*X_\varepsilon$ (pull-back bundle by f), and let $\tilde{f}$ denote the natural map between $\widetilde{X}_\varepsilon$ and X_ε. The map $h\circ\tilde{f}$ defines the second tubular neighborhood of X in $\mathbf{S}^{2a+1}$.

By [7, §VI.3], there exists an equivariant isotopy of tubular neighborhoods

$$H: X_\varepsilon\times[0,1]\longrightarrow\mathbf{S}^{2a+1},$$

with $H_0\equiv\text{id}$ and $H_1(X_\varepsilon)\equiv h\circ\tilde{f}(\widetilde{X}_\varepsilon)\equiv h(X_\varepsilon)$. In particular, H_1 differs from h by an equivalence of equivariant bundles:

$$\begin{array}{ccc} X_\varepsilon & \xrightarrow{h^{-1}\circ H_1} & X_\varepsilon \\ \downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle\xi} \\ X & \xrightarrow{f} & X. \end{array}$$

Since $X \underset{\text{eq}}{\sim} \mathbf{S}^{2a-1}$, by induction, the map f is equivariantly isotopic to the identity and it is easy to lift this isotopy to an isotopy G between H_1 and h.

Combining H and G, we obtain an equivariant isotopy

$$F\colon [0,1]\times X_\varepsilon \longrightarrow \mathbf{S}^{2a+1}$$

such that F_0 is the natural inclusion map and $F_1 \equiv h|_{X_\varepsilon}$.

By Theorem 3.1, F extends to an equivariant diffeotopy between some map g, with $g|_{X_\varepsilon} \equiv h$, and the identity. As this construction can be achieved for any choice of $0<\varepsilon<1$, we may assume that $g \equiv h$ on the whole sphere. □

We note that the result of such a surgery *may* or *may not* be a link. Indeed, in Examples 2.10, 2.11 and 2.12, we may perform elementary surgeries, but the quotient space of the new manifold by the action of the real torus cannot be identified with a simple polytope, therefore the new manifold is not a link.

Consider now the following more subtle case. Let X_A be a link and let Q be the simple convex polytope obtained from P_A by performing a flip of type (a,b) along some simplicial face F. Then, call Y the manifold obtained from X_A by performing an elementary surgery of type (a,b) along X_F. As the surgery is equivariant, the manifold Y is endowed with a smooth action of the real torus on it. It follows from Corollary 2.8 that the quotient space of Y by this action can be identified with Q. This means that this quotient space is in bijection with Q, that the orbit over a point in the interior of Q is $(\mathbf{S}^1)^n$, whereas the orbit over a point in the interior of a facet of Q is $(\mathbf{S}^1)^{n-1}$, and so on. The resulting polytope is still called associate polytope. Finally, each closed face of Q corresponds to an invariant submanifold of Y with trivial invariant tubular neighborhood. In fact, every such face S is obtained from a face R of P_A by a certain flip, as precised in Proposition 2.9. The corresponding invariant submanifold Y_S is thus obtained from X_R by performing the corresponding elementary surgery. More precisely, if we write

$$Y = (X_A \setminus W_F) \cup_\psi (X_\Delta \setminus W_{F'})$$

as in (10), then we have

$$Y_S = (X_R \setminus (W_F \cap X_R)) \cup_\psi (X_{R'} \setminus (W_{F'} \cap X_{R'}))$$

for some well-chosen face R' of Δ. Let

$$\nu\colon X_R \times \mathbf{D}^{2b'} \longrightarrow W_R \subset X_A$$

be a trivial invariant tubular neighborhood of X_R (we denote the codimension of X_R in X_A by b'). We assume that W_R is small enough to have

$$\nu^{-1}(W_R \cap W_F) = (X_R \cap W_F) \times \mathbf{D}^{2b'}.$$

Then the composition

$$(X_{R'}\cap W_{F'})\times \mathbf{D}^{2b'} \xmapsto{(\psi,\mathrm{id})} (X_R\cap W_F)\times \mathbf{D}^{2b'} \xmapsto{\psi^{-1}} W_{F'}$$

can be extended to a (trivial) invariant tubular neighborhood

$$\nu'\colon X_{R'}\times \mathbf{D}^{2b'} \longrightarrow W_{R'}\subset X_\Delta,$$

since $\psi^{-1}\circ\nu$ maps $X_R\cap W_F$ onto $X_{R'}\cap W_{F'}$. Finally, set $\nu_S\equiv\nu\cup_\psi\nu'$. Then ν_S maps

$$((X_R\setminus W_F)\times \mathbf{D}^{2b'})\cup_{(\psi,\mathrm{id})}((X_{R'}\setminus W_{F'})\times \mathbf{D}^{2b'}) = Y_S\times \mathbf{D}^{2b'}$$

to $(W_R\setminus W_F)\cup_\psi(W_{R'}\setminus W_{F'})$, that is ν_S is a trivial invariant tubular neighborhood of Y_S.

Assume that Y_S is equivariantly diffeomorphic to some $\mathbf{S}^{2a'-1}\times(\mathbf{S}^1)^{p'}$. Then we may perform an elementary surgery corresponding to this choice of Y_S. In particular, we may perform an elementary surgery corresponding to any choice of a flip of Q, as soon as the corresponding invariant submanifold of Y is equivariantly diffeomorphic to some $\mathbf{S}^{2a'-1}\times(\mathbf{S}^1)^{p'}$. In this case, we say that the flip is *good*.

We may then repeat this process and construct manifolds obtained from a link by a finite number of elementary surgeries corresponding to good flips of the associate polytope.

Nevertheless, it is not clear a priori that Y, as well as the manifolds obtained from Y, are equivariantly diffeomorphic to a link, that is to a transverse intersection of special real quadrics.

Definition 3.5. A *pseudolink* is a manifold obtained from a link by a finite number of elementary surgeries corresponding to good flips of the associate polytopes.

We will now see that every flip is good.

PROPOSITION 3.6. *Let X be a pseudolink such that its associate polytope P is a d-simplex. Then X is, up to product with circles, equivariantly diffeomorphic to the unit euclidean sphere $\mathbf{S}^{2d+1}$ of $\mathbf{C}^{d+1}$ endowed with the natural action of $(\mathbf{S}^1)^{d+1}$ on it.*

Proof. The proof is by induction on d. If $d=0$, then X is obviously a product of circles, and the proposition is satisfied.

Assume now that the proposition is true for simplices of dimension at most d and consider a pseudolink X whose associate polytope P is a $(d+1)$-simplex. Then P can be seen as a pyramid with a d-simplex P' as base, and can be decomposed into a closed neighborhood of P' glued along the common boundary with a closed neighborhood of a 0-simplex v (a point). This means that X is equivariantly diffeomorphic to the gluing of an invariant closed neighborhood of X'_P with an invariant closed neighborhood of X_v by the identity along the common boundary. We may assume that these neighborhoods are tubular and thus trivial. Using the induction hypothesis and standard product neighborhoods, we may write

$$X \underset{\text{eq}}{\simeq} (\mathbf{S}^{2d+1} \times (\mathbf{S}^1)^p \times \overline{\mathbf{D}^2}) \cup_\phi (\overline{\mathbf{D}^{2b}} \times (\mathbf{S}^1)^p \times \mathbf{S}^1)$$

for some $p \geqslant 0$ and some equivariant diffeomorphism ϕ of $\mathbf{S}^{2d+1} \times (\mathbf{S}^1)^{p+1}$. By Lemma 3.4, we may assume that ϕ is the identity. Therefore X is, up to product with circles, equivariantly diffeomorphic to the unit euclidean sphere $\mathbf{S}^{2d+3}$ of $\mathbf{C}^{d+2}$ endowed with the natural action of $(\mathbf{S}^1)^{d+2}$ on it. □

COROLLARY 3.7. *Every flip of the associate polytope of a pseudolink is good.*

We finish this section with a proposition which will be useful in the sequel.

PROPOSITION 3.8. *Let $A \in \mathcal{A}_k$ and $B \in \mathcal{A}_l$. Assume that X_B is obtained from X_A by performing an elementary surgery of type (a, b) corresponding to a flip.*

(i) *If $1 < a < n$ or $a = b = 1$, then $k = l$;*

(ii) *if $a = 1 \neq b$, then $k = l + 1$;*

(iii) *if $a = n \neq 1$, then $k = l - 1$.*

Proof. As the links X_A and X_B have the same dimensions, as well as P_A and P_B, the numbers n and p are the same for both links. This implies that k (respectively, l) is equal to n minus the number of facets of P_A (respectively, P_B) (see Lemma 0.12). Now, the result easily follows from the fact that a flip of type (a, b) does not create nor destroy any facet if $1 < a < n$ or $a = b = 1$ (see Figure 9 on page 75), creates a facet if $a = 1 \neq b$ and destroys a facet if $a = n \neq 1$ (see Proposition 2.9). □

4. The rigidity theorem

We are now in position to prove the following result.

THEOREM 4.1. (Rigidity theorem) (i) *Every pseudolink is a link.*

(ii) *Let $A, B \in \mathcal{A}_k$ for some k. Then $X_A \underset{\text{eq}}{\simeq} X_B$ if and only if $P_A = P_B$.*

Remark 4.2. Statement (ii) is easily deduced from Lemma 0.15 if we replace equivariant *diffeomorphism* by equivariant *homeomorphism*. The difficulty is related to the possibility that there exist several smooth structures on $\mathcal{Z}_P$ compatible with the torus action (cf. Lemma 0.15 and the subsequent discussion). We will prove that this is not the case, but as a *consequence* of Theorem 4.1.

Remark 4.3. Let $p=0$ and $n\geqslant 2$. Then X_A is the unit euclidean sphere $\mathbf{S}^{2n-1}$ of $\mathbf{C}^n$. We may perform an equivariant surgery as follows:

$$(X_A\setminus(\mathbf{S}^1\times\mathbf{D}^{2n-2}))\cup(\overline{\mathbf{D}^2}\times\mathbf{S}^{2n-3})=(\overline{\mathbf{D}^2}\times\mathbf{S}^{2n-3})\cup(\overline{\mathbf{D}^2}\times\mathbf{S}^{2n-3})=\mathbf{S}^2\times\mathbf{S}^{2n-3}.$$

This surgery looks like an elementary surgery of type $(1,n-1)$. In particular, it is easy to check that the quotient space of $\mathbf{S}^2\times\mathbf{S}^{2n-1}$ by the induced torus action can be identified with the prism with an $(n-2)$-simplex as base, i.e. the simple convex polytope obtained from the $(n-1)$-simplex P_A by a flip of type $(1,n-1)$. Nevertheless, this is not an elementary surgery by Definition 3.2 (X_A is simply-connected) and the resulting manifold is not a link by Theorem 4.1, but a quotient of a link by an action of $\mathbf{S}^1$. The simply-connected link corresponding to the prism with an $(n-2)$-simplex as base is

$$\begin{aligned}((\mathbf{S}^{2n-1}\setminus(\mathbf{S}^1\times\mathbf{D}^{2n-2}))\times\mathbf{S}^1)\cup(\mathbf{S}^1\times\mathbf{S}^{2n-3}\times\overline{\mathbf{D}^2})&\\ =(\overline{\mathbf{D}^2}\times\mathbf{S}^1\times\mathbf{S}^{2n-3})\cup(\mathbf{S}^1\times\overline{\mathbf{D}^2}\times\mathbf{S}^{2n-3})&=\mathbf{S}^3\times\mathbf{S}^{2n-3}.\end{aligned}$$

Proof of Theorem 4.1. Let P be a convex simple polytope. Call the *length* of P the minimal number of flips necessary to pass from the simplex (of the same dimension as P) to P. This number exists by Lemma 2.3.

The proof is by induction on the length of the associate polytope. More precisely, the induction hypothesis (at order l) is that statements (i) and (ii) are true for links and pseudolinks with associate polytopes of length less than or equal to l. This hypothesis is satisfied at order 0 by Propositions 1.2 and 3.6.

Assume the hypothesis at order l, and consider a pseudolink X with associate polytope P of length $l+1$. Then, if P undergoes some well-chosen flip, we obtain a simple convex polytope Q with length l. As usual, let (a,b) denote the type of flip and F the simplicial face along which the flip is made. Remark that this implies that P is obtained from Q by performing a flip of type (b,a) along some simplicial face F'. Perform an elementary surgery of type (a,b) along the submanifold of X corresponding to F. We recover a pseudolink Y whose associate polytope is Q. By induction, Y is a link X_A for A belonging to some $\mathcal{A}_k$. Define

$$k'=\begin{cases}k, & \text{if } 1<a<n \text{ or } a=b=1,\\ k+1, & \text{if } a=1\neq b,\\ k-1, & \text{otherwise.}\end{cases}$$

In the last case, notice that $k-1$ is positive: X is obtained from X_A by an elementary surgery of type $(1,n)$, so, by Definition 3.2, the link X_A is not simply-connected. By Theorem 0.14, there exists $B\in\mathcal{A}_{k'}$ such that P_B is equal to P. Perform an elementary surgery of type (a,b) along the submanifold of X_B corresponding to F. By induction, the result of this surgery is a link $X_{A'}$. Due to the choice of k', we have $A'\in\mathcal{A}_k$ by Proposition 3.8. Therefore, the second statement of the induction hypothesis implies that $X_{A'}\underset{\text{eq}}{\simeq}X_A$.

The conclusion is that both X_B and X are obtained from the same link $X_{A'}\underset{\text{eq}}{\simeq}X_A$ by performing an elementary surgery of type (b,a) along the same invariant submanifold (the submanifold corresponding to F' in Q). Therefore, X_B and X are equivariantly diffeomorphic and X is a link. This proves the first statement for associate polytopes of length $l+1$. Moreover, if one now considers any link X_C with $P_C=P$ and $C\in\mathcal{A}_{k'}$, then the same proof implies that $X_B\underset{\text{eq}}{\simeq}X_C$. As these considerations do not depend on the value of k', this proves one implication of statement (ii). But the converse is easy: two equivariantly diffeomorphic links have the same combinatorics of orbits, that is they have the same associate polytope. The statements are thus valid for length $l+1$. □

COROLLARY 4.4. *Let $A\in\mathcal{A}_k$ and $B\in\mathcal{A}_0$. Then $X_A\underset{\text{eq}}{\simeq}X_B\times(\mathbf{S}^1)^k$ if and only if $P_A=P_B$.*

Proof. By Lemma 0.10, there exists $A'\in\mathcal{A}_0$ such that the link X_A is equivariantly diffeomorphic to $X_{A'}\times(\mathbf{S}^1)^k$. In particular, this implies that $P_{A'}=P_A$. The statement then follows by applying Theorem 4.1. □

COROLLARY 4.5. *Let $\Phi:[0,1]\to\mathcal{A}\cap M_{n,p}(\mathbf{R})$ be a continuous path of admissible matrices of the same dimensions. Set $A_t=\Phi(t)$ for all $t\in[0,1]$. Then X_{A_0} is equivariantly diffeomorphic to X_{A_1}.*

Proof. Let $I\subset\{1,...,n\}$ be such that 0 belongs to the convex hull of $(((A_0)_i)_{i\in I})$. Then 0 belongs to the convex hull of $(((A_t)_i)_{i\in I})$ for all $t\in[0,1]$, otherwise there would be a time t_0 at which the weak hyperbolicity condition would be broken, and the path Φ would not be a path of admissible matrices. As a consequence of Lemma 0.13 and condition (8), the associate polytopes P_{A_t} are all equal. Moreover this implies that all the X_{A_t}'s belong to the same $\mathcal{A}_k$. We may thus conclude, by Theorem 4.1, that X_{A_0} and X_{A_1} are equivariantly diffeomorphic. □

COROLLARY 4.6. *Let $A,B,C\in\mathcal{A}$. Then $X_C\underset{\text{eq}}{\simeq}X_A\times X_B$ (up to product with circles) if and only if $P_C=P_A\times P_B$.*

Proof. It is an immediate consequence of Example 0.7 and Theorem 4.1, noting that, in Example 0.7, we have $P_C=P_A\times P_B$. □

Finally, we give a positive answer to Question 0.17.

COROLLARY 4.7. *There exists a unique smooth compatible structure on the moment-angle manifold $\mathcal{Z}_P$ (in the sense of Definition 0.16): that of the corresponding link.*

Proof. First, notice that the structure of link is compatible with the torus action by Proposition 1.1.

Now, put a compatible smooth structure on $\mathcal{Z}_P$. By performing a finite number of equivariant surgeries corresponding to well-chosen flips of P on this smooth manifold (this is possible by Definition 0.16 (ii)), we obtain a smooth compatible structure on $\mathcal{Z}_\Delta$, where Δ is the simplex of the same dimension as P. Remark that the proof of Proposition 3.5 works in this case, so that $\mathcal{Z}_\Delta$, endowed with this smooth structure, is equivariantly diffeomorphic to a sphere. This implies that $\mathcal{Z}_P$ is a pseudo-link. So it is a link by Theorem 4.1. □

The second statement of Theorem 4.1 is definitely false if we replace equivariant diffeomorphism by diffeomorphism. A counterexample is given in [26, p. 242]. We will see other interesting counterexamples in §6 (see Example 6.2).

We may now merge the two previous sections in the following theorem, which is a direct consequence of the description of flips given in §2, of the description of elementary surgeries given in §3 and of Theorem 4.1.

THEOREM 4.8. *Let $A, B \in \mathcal{A}$ have the same dimensions n and p. Assume that P_B is obtained from P_A by performing a flip of type (a, b) along some simplicial face F. Then X_B is obtained (up to equivariant diffeomorphism) from X_A by performing an elementary surgery of type (a, b) along some X_F.*

As noted above, the converse of Theorem 4.8 is false. Indeed, in Examples 2.10, 2.11 and 2.12, we may perform elementary surgeries which will not correspond to flips. In other words, *the class of links (up to equivariant diffeomorphism) is not stable under elementary surgeries.*

COROLLARY 4.9. *Let $A \in \mathcal{A}$. Then X_A is obtained (up to equivariant diffeomorphism) from $\mathbf{S}^{2n-2p-1} \times (\mathbf{S}^1)^p$ by performing a finite number of elementary surgeries.*

Proof. Let W be the simple polytope obtained from the product $P_A \times [0, 1]$ by cutting off a neighborhood of a vertex of $P_A \times \{1\}$ by a hyperplane (cf. Lemma 2.3). Then W is a cobordism between P_A and the simplex of dimension $n-p-1$. If it is trivial, then P_A is the $(n-p-1)$-simplex, otherwise it can be decomposed into a finite number of elementary cobordisms. Now apply Theorem 4.8 for each elementary cobordism, and conclude in both cases by Corollary 1.4. □

COROLLARY 4.10. *Let $A, B \in \mathcal{A}$ have the same dimensions n and p. Assume that X_B is obtained from X_A by an elementary surgery. Then there exists an equivariant cobordism between $X_A \times (\mathbf{S}^1)^2$ and $X_B \times (\mathbf{S}^1)^2$.*

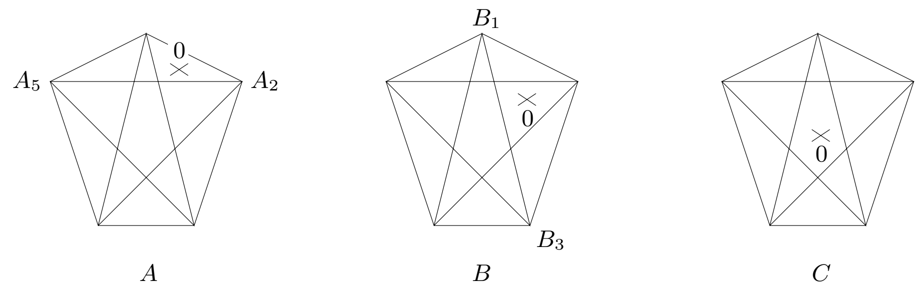

Figure 10.

Proof. Let $k \in \mathbf{N}$ be such that $A \in \mathcal{A}_k$. Let (a, b) be the type of the elementary surgery transforming X_A into X_B. Let W be the corresponding elementary cobordism between P_A and P_B. We define

$$l = \begin{cases} k-1, & \text{if } a=1, \\ k, & \text{otherwise} \end{cases}$$

(note that if $a=1$, then $k>0$ by Definition 3.2). By use of Theorem 0.14, there exists a link X_C such that $P_C = W$ and $C \in \mathcal{A}_l$. By Lemma 0.12 and Proposition 2.13, we know that P_C has $n-l+2$ facets. As it has dimension $n-p$, C is a configuration of $n+2$ points in $\mathbf{R}^{p+1}$, so X_C has dimension $2n-p+2$. Using the fact that P_A and P_B are disjoint facets of P_C, and that X_A and X_B have dimension $2n-p-1$, we may embed, by Proposition 1.1, the link $X_A \times \mathbf{S}^1$ (respectively, $X_B \times \mathbf{S}^1$) as a smooth submanifold of X_C of codimension 2 with trivial normal bundle. The manifold obtained from X_C by removing an open trivial tubular neighborhood of each of these submanifolds is an equivariant cobordism between $X_A \times (\mathbf{S}^1)^2$ and $X_B \times (\mathbf{S}^1)^2$. □

5. Wall-crossing

We will now use the previous results to resolve the wall-crossing problem (compare with [6, §4]). Let us start with an example to make the next explanations clearer.

Example 5.1. Consider the links related to the three admissible configurations illustrated in Figure 10 (the vertices of each configuration are numbered clockwise).

Here $n=5$ and $p=2$. Note that B and C are translations of A in $\mathbf{R}^2$.

Nevertheless, the corresponding links are very different. From [25] (see Example 0.5) or [28], we can conclude that

$$X_A \underset{\text{eq}}{\simeq} \mathbf{S}^5 \times \mathbf{S}^1 \times \mathbf{S}^1,$$
$$X_B \underset{\text{eq}}{\simeq} \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1,$$
$$X_C \underset{\text{eq}}{\simeq} \#(5)\mathbf{S}^3 \times \mathbf{S}^4,$$

where $\#(5)\mathbf{S}^3 \times \mathbf{S}^4$ denotes the connected sum of five copies of $\mathbf{S}^3 \times \mathbf{S}^4$. By Corollary 4.5, as long as we move the configuration A smoothly without breaking the weak hyperbolic condition, i.e. *without crossing a wall*, the link X_A remains unchanged. But to go from A to B we have to cross the wall A_2A_5, and to go from B to C we have to cross the wall B_1B_3; finally notice that we cannot pass directly from A to C with a single wall-crossing. The best we can do is to perform two wall-crossings.

Definition 5.2. Let $A \in \mathcal{A}$. A *wall* of A is a hyperplane of $\mathbf{R}^p$ passing through p vectors of A and no more than p (the data of the hyperplane is thus equivalent to the data of the p vectors) and which does not support a facet of $\mathcal{H}(A)$.

From the definition, the intersection of the set $\{A_1, ..., A_n\}$ with each open half-space defined by the wall is not vacuous.

Definition 5.3. Let $A, B \in \mathcal{A}$ have the same dimensions n and p. Let W be a wall of A. We say that B is obtained from A by *crossing the wall* W if

(i) the configuration B is a translate of A by some vector v of $\mathbf{R}^p$;

(ii) the configuration $A+tv$ is admissible for every $t \in [0,1]$, *except for one value* $t_0 \in]0,1[$;

(iii) at t_0, the point $0 \in \mathbf{R}^p$ belongs to the translate of W by $t_0 v$ and does not belong to any other wall.

In other words, the point 0 "moves" continuously in the direction $-v$ and crosses the wall W, hence the terminology.

Let $A \in \mathcal{A}$ and W be a wall of A. Then W splits $\mathbf{R}^p$ into two open half-spaces containing the $n-p$ vectors of A not belonging to W. More precisely, one of the two open half-spaces, let us denote it by W^+, contains 0 and a vectors of A, whereas the other, that we call W^-, contains b vectors of A. We say that the wall W is of *type* (a,b). We have $a+b=n-p$ with $1 \leqslant a \leqslant n-p-1$ and $1 \leqslant b \leqslant n-p-1$.

Now, let B be obtained from A by crossing W. If, by abuse of notation, we still call W^+ and W^- the open half-spaces of $\mathbf{R}^p$ separated by the translate of W, then W^+ still contains a vectors of B (which are exactly the translates of the a vectors of A lying in W^+) and W^- contains b vectors of B, but now 0 lies in W^-. In particular, before the wall-crossing, 0 belongs to the convex hull of the set consisting of the p vectors of the wall W and any vector of W^+; after crossing the wall, 0 belongs to the convex hull of the set consisting of the p vectors of the wall W and any vector of W^-.

THEOREM 5.4. (Wall-crossing theorem) *Let $A, B \in \mathcal{A}$ with the same dimensions n and p. Assume that $p>0$. Then, the following conditions are equivalent*:

(i) *the convex polytope P_B is obtained from P_A by a flip of type (a,b) along the simplicial face F_J*;

(ii) *there exists $X_{B'} \underset{\text{eq}}{\simeq} X_B$ and $X_{A'} \underset{\text{eq}}{\simeq} X_A$ such that $X_{B'}$ is obtained from $X_{A'}$ by a single wall-crossing of A', which is of type (a,b).*

In the particular case where $p=0$, the notion of wall is meaningless. This explains the restriction $p>0$ in the statement of Theorem 5.4.

Combining this result with Theorem 4.8 yields the following corollary.

COROLLARY 5.5. *Under the same hypotheses, X_B is obtained from X_A by an elementary surgery of type (a,b) along X_{F_J}.*

In other words, the class of links (up to equivariant diffeomorphism) is not stable under elementary surgeries but *is stable under elementary surgeries coming from wall-crossings.*

Proof of Theorem 5.4. The argument is purely combinatorial. Assume (i). Then we can form the simple convex polytope P_C having P_A and P_B as separated facets and one single extra vertex of index (a,b). Let $k \in \mathbf{N}$ be such that $A \in \mathcal{A}_k$. We define an integer l as in the proof of Corollary 4.10:

$$l = \begin{cases} k-1, & \text{if } a=1, \\ k, & \text{otherwise} \end{cases}$$

(the assumption $p>0$ excludes the case $a=b=1$). Note that P_C has dimension $n-p$ and it has $n+2-l$ facets by Proposition 2.13. By Theorem 0.14, there exists a link X_C corresponding to P_C with $C \in \mathcal{A}_l$. We know that C is a configuration of $n+2$ vectors of $\mathbf{R}^{p+1}$, say $C=(C_0, ..., C_{n+1})$. We may assume that $C_+ = C \setminus \{C_0\}$ satisfies $X_{C_+} \underset{\text{eq}}{\simeq} X_A \times \mathbf{S}^1$ and that $C_- = C \setminus \{C_{n+1}\}$ satisfies $X_{C_-} \underset{\text{eq}}{\simeq} X_B \times \mathbf{S}^1$ (see Corollary 4.10). Moreover, as $P_A \cap P_B$ is vacuous (as a face of P_C), we have that $C \setminus \{C_0, C_{n+1}\}$ is not admissible. We say that $\{C_0, C_{n+1}\}$ is *indispensable*. In particular, this means that there exists a hyperplane of $\mathbf{R}^{p+1}$ passing through 0 which strictly separates $\{C_0, C_{n+1}\}$ from $\overline{C} = C \setminus \{C_0, C_{n+1}\}$. Scaling each vector of $\overline{C}$ by a strictly positive real number if necessary, we may assume that $\overline{C}$ lies in an affine hyperplane H of $\mathbf{R}^{p+1}$ without changing the equivariant diffeomorphism type of X_C (see Corollary 4.5).

Under this assumption, the convex hull of C_+ is a pyramid with base $\overline{C}$, apex C_{n+1} and containing 0. In particular, C_{n+1} is indispensable. This implies that, if we

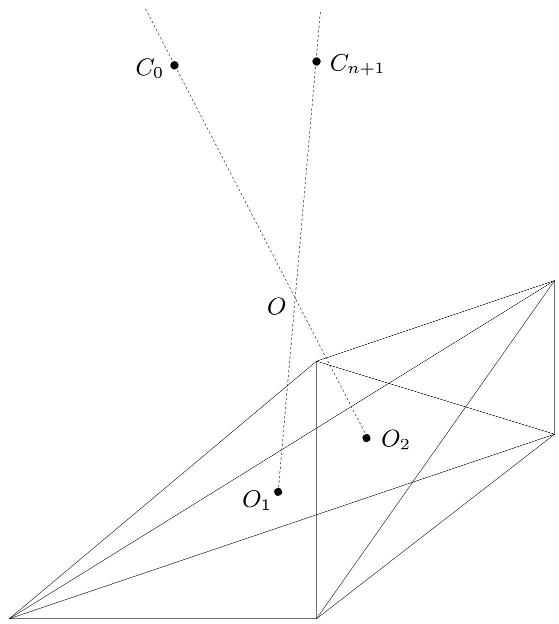

Figure 11.

project 0 onto the hyperplane H by letting $\bar{0}=H\cap(0C_{n+1})$ (where $(0C_{n+1})$ denotes the line passing through the origin and the point C_{n+1}), then identifying H with $\mathbf{R}^p$ and $\bar{0}$ with the zero of $\mathbf{R}^p$ yields an admissible configuration A' of n vectors in $\mathbf{R}^p$ satisfying $X_{A'} \underset{\text{eq}}{\sim} X_A$ (cf. Lemma 0.10).

Performing the same transformation on the convex hull of C_-, viewed as a cone over $\overline{C}$ with apex C_0, we obtain an admissible configuration B' of n vectors in $\mathbf{R}^p$ satisfying $X_{B'} \underset{\text{eq}}{\sim} X_B$ and such that B' is obtained from A' by a translation.

Figure 11 should illustrate this construction. Taking $\bar{0}$ as O_1 (respectively, O_2) gives the configuration A' (respectively, B').

From the construction, there is a translation sending the configuration A' to B'. Let us now prove that this translation induces exactly one wall-crossing and characterizes it.

LEMMA 5.6. *Let $I\subset\{1,...,n\}$ be of cardinality p. Assume that $\{A'_i:i\in I\}$ defines a wall W of A'. Then W is crossed when changing from A' to B' if and only if* 0 *is in the convex hull of* $\{C_0,C_{n+1}\}\cup\{C_i:i\in I\}$.

Proof. The proof is direct. Let W be a wall of A' defined by I. The hyperplane passing through W and 0, let us call it H_1, separates $\mathbf{R}^{p+1}$ into two open half-spaces. Clearly, W is crossed when changing from A' to B' if and only if C_0 and C_{n+1} do not belong to the same open half-space. If this is the case, then H_1 cuts the segment $[C_0,C_{n+1}]$

in one point C_{t_0}, and 0 belongs to the convex hull of $\{C_{t_0}\}\cup\{C_i : i\in I\}$. Therefore, 0 is in Δ, the convex hull of $\{C_0, C_{n+1}\}\cup\{C_i : i\in I\}$.

Conversely, assume that C_0 and C_{n+1} belong to the same open half-space defined by H_1. Then, the intersection of Δ and H_1 is included in W, and thus it does not contain 0. □

Now, by Lemma 0.13 and condition (8), a set of $p+2$ vertices of C including C_0 and C_{n+1} and containing 0 in its convex hull, corresponds to a vertex of P_C which neither belongs to P_A nor to P_B. As the flip transforming P_A into P_B is elementary, there exists only one such simplex, and thus B' is obtained from A' by a single wall-crossing along the wall W_J corresponding to the extra vertex of P_C. Let us determine the type of the wall.

Let I be the set of indices defining W. As before, let W^+ (respectively, W^-) be the open half-space containing $\bar{0}$ (respectively, not containing $\bar{0}$) before performing the wall-crossing. A point A'_i belongs to W^+ if and only if the convex hull of $\{A'_i\}\cup\{A'_j : j\in I\}$ in $\mathbf{R}^p$ contains $\bar{0}$. Since 0 belongs to the segment $[\bar{0}, C_{n+1}]$, this is the case if and only if the convex hull of $\{C_{n+1}\}\cup\{C_i\}\cup\{C_j : j\in I\}$ contains 0 in $\mathbf{R}^{p+1}$. Through condition (8), this determines a vertex v of $P_A\subset P_C$. Moreover, since 0 belongs to the convex hull of $\{C_0, C_{n+1}\}\cup\{C_j : j\in I\}$, by Lemma 5.6, and to the convex hull of $\{C_0, C_{n+1}\}\cup\{C_i\}\cup\{C_j : j\in I\}$, we know, still by (8), that there is an edge from v to the extra vertex of P_C (that is the vertex of $P_C\setminus(P_A\sqcup P_B)$). As this vertex has index (a, b), the wall W separates A' into a vectors belonging to W^+ and b vectors belonging to W^-.

Conversely, assume (ii). Let us define a new admissible configuration as follows. Let

$$C_i = \begin{pmatrix} A'_i \\ -1 \end{pmatrix} \in \mathbf{R}^{p+1}, \quad 1\leqslant i\leqslant n,$$

and let $\bar{0}=(0,-1)\in\mathbf{R}^p\times\mathbf{R}$. Consider the hyperplane $H=\mathbf{R}^p\times\{1\}\subset\mathbf{R}^{p+1}$. Let C_0 be the intersection of H with the line $(0\bar{0})$. We may now move $\bar{0}$ inside $\mathbf{R}^p\times\{-1\}$ *without moving the points* C_i to realize the wall-crossing from A' to B'. Define C_{n+1} as the intersection of H with $0\bar{0}$ after the translation of $\bar{0}$. Then C is obviously an admissible configuration. We obtain exactly the same picture as before.

Moreover, $C\setminus\{C_{n+1}\}$ is an admissible configuration which is a pyramid with base $\overline{C}=(C_1, ..., C_n)$ and apex C_0, thus

$$X_{C\setminus\{C_{n+1}\}} = X_C\cap\{z : z_{n+1}=0\} \underset{\text{eq}}{\simeq} X_{A'}\times\mathbf{S}^1.$$

In the same way,

$$X_{C\setminus\{C_0\}} = X_C\cap\{z : z_0=0\} \underset{\text{eq}}{\simeq} X_{B'}\times\mathbf{S}^1.$$

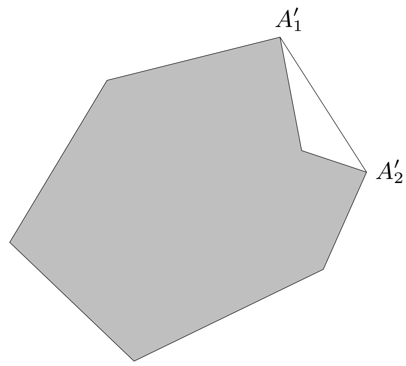

Figure 12.

From the construction, we obviously have $X_{\bar{C}}=\varnothing$. Therefore P_C is a cobordism between $P_{A'}$ and $P_{B'}$. But, as above, using Lemmas 0.13, 5.6 and condition (8), it is straightforward to check that P_C has a single extra vertex which is of index (a,b), and that P_C is an elementary cobordism between P_A and P_B along some simplicial face F_J. □

COROLLARY 5.7. *Let $A\in\mathcal{A}$. Then there exists $A'\in\mathcal{A}$ such that*

(i) *the link X_A is equivariantly diffeomorphic to $X_{A'}$;*

(ii) *the configuration A' is obtained by wall-crossings from a configuration A'' satisfying $X_{A''}\underset{\text{eq}}{\simeq}\mathbf{S}^{2n-2p-1}\times(\mathbf{S}^1)^p$.*

Proof. Let A' be a generic perturbation of A, that is a small perturbation of A whose convex hull is simplicial. In this situation, a hyperplane of $\mathbf{R}^p$ contains at most p vertices of A'. By Corollary 4.5, we may assume that $X_{A'}\underset{\text{eq}}{\simeq}X_A$. For simplicity, assume that the convex hull of $(A_1',...,A_p')$ is a facet of $\mathcal{H}(A_1',...,A_n')$. Consider the region $\mathcal{R}$ of $\mathbf{R}^p$ defined as follows: $\mathcal{R}$ is the union of the simplices whose vertices are constituted by $p-1$ points among $(A_1',...,A_p')$ and two points among $(A_{p+1}',...,A_n')$.

The shaded region in Figure 12 is an example of such an $\mathcal{R}$.

Notice that a point of $\mathcal{H}(A_1',...,A_n')$ which is sufficiently close to the center of $\mathcal{H}(A_1',...,A_p')$ does not belong to $\mathcal{R}$. Define A'' as an admissible configuration obtained as a translate of A' such that 0 does not belong to the corresponding translate of $\mathcal{R}$. In particular, A'' is obtained from A' by wall-crossings. Then A_1'', ..., A_p'' are indispensable points of A'', so, by Lemma 0.10, we have that $A''\in\mathcal{A}_k$ for $k\geqslant p$. This implies that $P_{A''}$ has dimension $n-p-1$ and has at most $n-p$ facets. Therefore, $k=p$ and P_A is the $(n-p-1)$-simplex. Thus, by Corollary 1.4, we have $X_{A''}\underset{\text{eq}}{\simeq}\mathbf{S}^{2n-2p-1}\times(\mathbf{S}^1)^p$. □

Remark 5.8. Generically, we may take $A'=A$.

6. Elementary surgery of type $(1,n)$

Let X_A be a link. Assume that P_A is obtained from the simplex (of the same dimension) by uniquely performing flips of type $(1,n)$. Then, in this case, we may explicitely describe the diffeomorphism type of the link. First, we have the following lemma.

LEMMA 6.1. *Let $A\in\mathcal{A}_k$ with $k>1$. Let X_B be obtained from X_A by performing an elementary surgery of type $(1,n)$ along some invariant submanifold corresponding to a vertex. Then the diffeomorphism type of X_B is independent of the choice of the vertex on which the flip occurs.*

Proof. Let v and v' be two vertices of P_A. We want to prove that, if X_B and $X_{B'}$ denote the links obtained from X_A by performing an elementary surgery of type $(1,n)$ along X_v (respectively, $X_{v'}$), then these two links are diffeomorphic. It is enough to show this in the case where v and v' belong to the same edge E. Let us describe X_E. By Corollary 1.5, the link X_E is diffeomorphic to $\mathbf{S}^3\times(\mathbf{S}^1)^p$. The real torus $(\mathbf{S}^1)^{p+2}=\mathbf{S}^1\times\mathbf{S}^1\times T$ acts on X_E in the following manner: decompose $\mathbf{S}^3$ as the union of two solid tori $(\mathbf{S}^1\times\mathbf{D}^2)\cup(\mathbf{D}^2\times\mathbf{S}^1)$. Then $\mathbf{S}^1\times\mathbf{S}^1$ acts on each solid torus in the natural way (that is the first factor by translations on $\mathbf{S}^1$ and the second factor tangentially to each circle on $\mathbf{D}^2$) and this describes the induced action on $\mathbf{S}^3$; finally, T acts by translations on $(\mathbf{S}^1)^p$. Therefore, X_v is exactly given as $(\mathbf{S}^1\times\{0\})\times(\mathbf{S}^1)^p$, that is as the core circle of the first solid torus product with $(\mathbf{S}^1)^p$; similarly, $X_{v'}$ is exactly given as $(\{0\}\times\mathbf{S}^1)\times(\mathbf{S}^1)^p$, that is as the core circle of the second solid torus product with $(\mathbf{S}^1)^p$. There exists an isotopy in $\mathbf{S}^3$ which sends $\mathbf{S}^1\times\{0\}$ to $\{0\}\times\mathbf{S}^1$, and this isotopy can be extended by the identity on $(\mathbf{S}^1)^p$ to obtain an isotopy in X_E sending X_v to $X_{v'}$. Moreover, as it is the identity on $(\mathbf{S}^1)^p$, it maps the circle which will be filled by a 2-disk in the surgery giving X_B, to the circle which will be filled by a 2-disk in the surgery giving $X_{B'}$. Therefore, the two elementary surgeries give the same result, that is X_B is diffeomorphic to $X_{B'}$. □

Of course, in the previous lemma, the class of X_B *modulo equivariant diffeomorphisms* depends on the vertex on which the surgery occurs: generally, the corresponding flips give different polytopes so, by Theorem 4.1, different equivariant smooth classes of links. Here is such an example.

Example 6.2. Consider the polyhedron shown in Figure 13 (the "hexagonal book"). Let X_A be the corresponding link, with $A\in\mathcal{A}_1$. Then, we may perform an elementary surgery of type $(1,3)$ on X_A in three ways, corresponding to the three vertices A, B and C indicated in the picture. By Lemma 6.1, the resulting manifolds are all diffeomorphic; however, by Theorem 4.1, no two of them are equivariantly diffeomorphic. In particular,

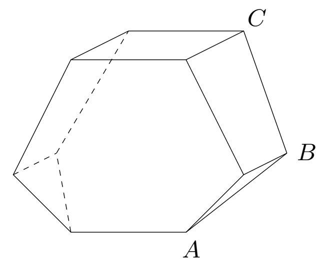

Figure 13.

this gives an example of a manifold which admits three different "structures of link".

We may now describe explicitely the links corresponding to polytopes obtained from the simplex (of the same dimension) by cutting off vertices.

THEOREM 6.3. (See [28]) *Let X_A be a simply-connected link such that P_A is obtained from the q-simplex* (*of the same dimension*) *by l flips of type* $(1,n)$ (*we assume that* $l>0$). *Then X_A is diffeomorphic to the following connected sum of products of spheres*:

$$X_A \simeq \mathop{\#}_{j=1}^{l} j\binom{l+1}{j+1}\mathbf{S}^{2+j}\times\mathbf{S}^{2q+l-j-1}.$$

The proof of Theorem 6.3 is done for polygons in [28, Theorem 3.4], but the proof of this generalization is the same. Notice that Theorem 6.3 shows that, for any dimension of the associate polytope and for any value of p, there exist infinite families which are connected sums of products of spheres, as in Example 0.5.

Going back to Example 6.2, we see that the manifold

$$\#(10)\mathbf{S}^3\times\mathbf{S}^8\,\#(20)\mathbf{S}^4\times\mathbf{S}^7\,\#(19)\mathbf{S}^5\times\mathbf{S}^6$$

admits three different actions of $(\mathbf{S}^1)^8$ with a convex polyhedron as quotient.

This example can be easily generalized as follows.

Example 6.4. Consider the l-gonal book P_l for $l>3$. It is obtained from the tetrahedron by $l-3$ flips of type $(1,3)$. By Theorem 6.3, it thus gives rise to a 2-connected link diffeomorphic to

$$X_l = \mathop{\#}_{j=1}^{l-3} j\binom{l-2}{j+1}\mathbf{S}^{2+j}\times\mathbf{S}^{2+l-j}$$

Consider an l-gonal facet of P_l. Number its vertices as indicated in Figure 14.

The simple convex polyhedra obtained from X_{l-1} by cutting off a vertex v_i are all different when i ranges from 1 to $\lfloor l/2\rfloor$ (where $\lfloor\cdot\rfloor$ denotes the integer part). One of

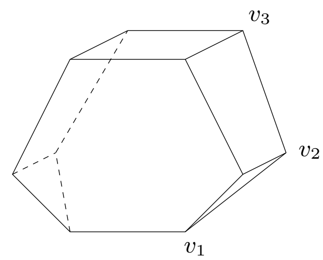

Figure 14.

these polyhedra being the l-gonal book, we have, by Lemma 6.1, that the corresponding links are all diffeomorphic to X_l.

In other words, the manifold X_l admits at least $\lfloor l/2 \rfloor$ link structures. Therefore, the number of link structures that X_l has, tends to infinity when l tends to infinity. Notice that the dimension of X_l is $l+4$.

Part II. The cohomology ring of a link

Thanks to Theorems 0.14 and 4.1, there is exactly one 2-connected link (up to equivariant diffeomorphism) associated with any simple convex polytope (recall that we always consider a convex polytope only up to combinatorial equivalence). In this part, we give an explicit formula for the cohomology ring of a 2-connected link in terms of its associate polytope. We use this formula to show that the cohomology of a link can have arbitrary amount of torsion.

7. Notation and auxiliary results

We denote by P a simple convex polytope and by X the associated 2-connected link, that is we drop the subscript A referring to the choice of a matrix.

Furthermore, we denote by

- d the dimension of P;
- n the number of facets of P;
- ∂P the boundary of P (we consider it as a cell complex);
- P_b the barycentric subdivision of ∂P (in the same way, the barycentric subdivision of a simplicial complex Γ will be denoted by Γ_b; if a set I numbers a simplex σ of Γ, then we number the center of σ in Γ_b by the same set I, that is we identify a simplex of Γ and its center in Γ_b);

- $\mathcal{F}$ the set of the facets of P (we identify it with the set $\{1,...,n\}$, i.e. we see it as an ordered set);
- $\mathcal{I}$ a subset of $\mathcal{F}$ (we also see it as an ordered set);
- $|\mathcal{I}|$ the cardinality of $\mathcal{I}$;
- $\bar{\mathcal{I}}$ the complement of $\mathcal{I}$ in $\mathcal{F}$;
- $F_{\mathcal{I}}$ the intersection of the facets of P that are in $\mathcal{I}$ (it is either empty or a face of P);
- $P_{\mathcal{I}}$ the union of the facets of P that are in $\mathcal{I}$;
- IJ the set $I\cup J$ (with I and J two *disjoint* subsets of $\mathcal{F}$) endowed with the following order: every element of I is less than every element of J;
- ε_{IJ} the sign of the permutation sending IJ onto $I\cup J$;
- P^* the dual polytope of P (we then consider $\mathcal{F}$ as its vertex set);
- $P^*_{\mathcal{I}}$ the maximal simplicial subcomplex of P^* with vertex set $\mathcal{I}$;
- Δ_I either the simplicial face of P^* with vertex set $I\subset\mathcal{F}$ or the empty set, that is

$$\Delta_I = \begin{cases} P^*_I, & \text{if } P^*_I \text{ is a simplex,} \\ \varnothing, & \text{otherwise;} \end{cases}$$

- δ_i^j the Kronecker symbol;
- $H_i(A,\mathbf{Z})$ (respectively, $\widetilde{H}_i(A,\mathbf{Z})$) the ith homology group (respectively, reduced homology group) of a manifold or a simplicial complex A with coefficients in $\mathbf{Z}$ (by convention, we set $\widetilde{H}_{-1}(\varnothing,\mathbf{Z})=\mathbf{Z}$);
- $H^i(A,\mathbf{Z})$ (respectively, $\widetilde{H}^i(A,\mathbf{Z})$) the ith cohomology group (respectively, reduced cohomology group) of a manifold or a simplicial complex A with coefficients in $\mathbf{Z}$.

Definition 7.1. For a non-empty face F of P, the vector space underlying the affine space in which F has non-empty interior will be called the (*vector*) *space of* F. By abuse of notation, we will still denote the space of F by F. No confusion should arise from this.

Definition 7.2. A proper face of P will be called an $\mathcal{I}$*-face* (respectively, an $\bar{\mathcal{I}}$*-face*) if every facet of P containing it is in $\mathcal{I}$ (respectively, in $\bar{\mathcal{I}}$).

We now prove some preliminary results on simple polytopes.

LEMMA 7.3. *Let P be a simple polytope and let $\mathcal{I}\subset\mathcal{F}$. Then, a non-empty intersection of elements of $\mathcal{I}$ is an $\mathcal{I}$-face.*

Proof. This comes directly from the fact that the neighborhood of a face in a simple polytope is the product of this face by a simplex. Hence, for every face F of P, there is a unique subset $\mathcal{I}$ such that $F_{\mathcal{I}}=F$ and F is an $\mathcal{I}$-face. □

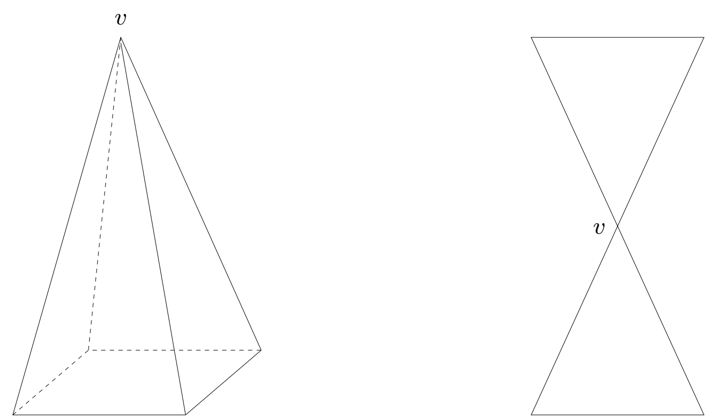

Figure 15.

Lemma 7.3 is false for non-simple polytopes. In Figure 15, the polytope is a pyramid with rectangular base and apex v, whereas the set $\mathcal{I}$ consists of two faces whose intersection is v. Nevertheless, v is not an $\mathcal{I}$-face.

Lemma 7.4. *Let P be a simple polytope. Consider a subset $\mathcal{I}$ of $\mathcal{F}$. Then,*
(i) *the complex $(P^*_{\mathcal{I}})_b$ is a deformation retract of $P_{\mathcal{I}}$;*
(ii) *the set $P_{\mathcal{I}}$ has the same homotopy type as its interior in ∂P.*

Proof. The barycentric subdivision of ∂P is a simplicial complex whose vertices are all the (non-empty) faces of P. By Lemma 7.3, the complex $(P^*_{\mathcal{I}})_b$ is isomorphic to the subcomplex of this subdivision associated to $\mathcal{I}$-faces. Each point M of $P_{\mathcal{I}}$ belongs to a *unique* minimal simplex of P_b, and this simplex has at least one vertex belonging to $(P^*_{\mathcal{I}})_b$ (the center of the minimal face which contains it). Take the barycentric coordinates of M in this simplex. We may then construct a retraction of $P_{\mathcal{I}}$ on $(P^*_{\mathcal{I}})_b$ by cancelling the bad barycentric coordinates (i.e. coordinates associated with vertices which do not belong to $(P^*_{\mathcal{I}})_b$).

To prove (ii), just remark that the previous construction also yields a retraction of the interior of $P_{\mathcal{I}}$ onto $(P^*_{\mathcal{I}})_b$. □

Corollary 7.5. *The set $P^*_{\mathcal{I}}$ is a deformation retract of $\partial P^* \setminus P^*_{\mathcal{I}}$.*

Proof. Following the proof of Lemma 7.4, there exists a retraction of $P_{\mathcal{I}}=(P_{\mathcal{I}})_b$ onto $(P^*_{\mathcal{I}})_b$, hence of $P^*_b \setminus (P^*_{\mathcal{I}})_b$ onto $P_b \setminus (P_{\mathcal{I}})_b$. But this last set is exactly the interior of $P_{\bar{\mathcal{I}}}$. The conclusion follows then by Lemma 7.4. □

8. Orientation

In this section, we fix some conventions of orientation. They are necessary to obtain a cohomology formula with sign. Let us start with the orientation of ∂P. We consider P as being realized in $\mathbf{R}^d$. We orient $\mathbf{R}^d$ and thus obtain an orientation of P.

8.1. Orientation of a facet and of a boundary

Recall that if we consider an oriented polytope, there is a canonical orientation of its boundary by stating that for any facet F of this polytope, a basis consisting of the normal outward pointing vector followed by a positively oriented basis of the space of the facet is a positively oriented basis of the space of the polytope.

8.2. Orientation of a face of P

Consider an ordered set $(H_1, ..., H_k)$ of facets of P with non-empty intersection. Then $F_{(H_1,\ldots,H_k)}$ denotes the intersection of these facets endowed with the following orientation: taking a basis $(v_1, ..., v_k, \mathcal{B})$ of the space of P, where v_j denotes the normal outward pointing vector of H_j and $\mathcal{B}$ is a basis of the space of our face, we state that both bases have the same orientation. Remark that even a 0-dimensional face has two “orientations”.

Remark 8.1. To orient a face of P is equivalent to order the set of facets containing it. In particular, given an orientation of a convex polytope, *there is no canonical orientation of the faces which are not facets.*

Definition 8.2. A d-tuple $(H_1, ..., H_d)$ of facets of P with non-empty intersection will be called *direct* if $(v_1, ..., v_d)$ is a positively oriented basis. It will be called *undirect* otherwise.

8.3. Orientation of an intersection

Consider an n-dimensional oriented vector space E and two oriented subspaces F and F', of strictly positive dimensions d and d', respectively, and whose sum is E. Then the vector space $F\cap F'$ is oriented with the convention that if $\mathcal{B}=(v_1, ..., v_{d+d'-n})$ is a basis of $F\cap F'$, $(w_1, ..., w_{n-d'}, v_1, ..., v_{d+d'-n})$ is a positive basis of F and $(v_1, ..., v_{d+d'-n}, w'_1, ..., w'_{n-d})$ a positive basis of F', then the basis $\mathcal{B}$ of $F\cap F'$ and the basis

$$(w_1, ..., w_{n-d'}, v_1, ..., v_{d+d'-n}, w'_1, ..., w'_{n-d})$$

have the same sign. In the special case where $F\cap F'$ is reduced to $\{0\}$, we state that $F\cap F'$ is *positively oriented* if $(w'_1,...,w'_{n-d},w_1,...,w_{n-d'})$ is a positive basis of $\mathbf{R}^d$. This convention is taken to guarantee the statement of Lemma 8.4 (below) in this special case.

Remark 8.3. With this definition, the orientations of $F\cap F'$ and $F'\cap F$ may differ.

The previous convention is a generalization of the convention of orientation of a face, since we have the following result.

LEMMA 8.4. *With the orientation conventions above, F_{IJ} is equal to $F_I\cap F_J$ as an oriented face.*

Proof. Let v_i (respectively, v'_i) denote the normal outward pointing vector of the ith facet of I (respectively, J). We may assume that F_I and F_J are orthogonal. Let $\mathcal{B}$ be a basis of $F_I\cap F_J$. Then $(v_1,...,v_k,v'_1,...,v'_{k'},\mathcal{B})$ is a positive basis of $\mathbf{R}^d$ if and only if $(v'_1,...,v'_{k'},\mathcal{B})$ is a positive basis of F_I, whereas $(v'_1,...,v'_{k'},\mathcal{B},v_1,...,v_k)$ is a positive basis of $\mathbf{R}^d$ if and only if $(\mathcal{B},v_1,...,v_k)$ is a positive basis of F_J. The claim follows then easily. □

LEMMA 8.5. *Let P be an oriented polytope. Let F be a face of P. Fix an orientation of F. With the orientation conventions above, the oriented boundary of F is given by*

$$\partial F=\sum_{\substack{H\in\mathcal{F}\\ F\cap H\neq F,\varnothing}} F\cap H,$$

where F is considered as an oriented polytope and H is endowed with the canonical orientation of ∂P.

Proof. We may find $\langle I\rangle=(i_1,...,i_k)$ such that $F_{\langle I\rangle}=F$ as oriented faces (the angles mean that the order on I may be different from its natural order). Now, set $\mathcal{F}=(i_1,...,i_n)$ (as ordered sets) up to an even permutation. For $k<j\leqslant n$, the oriented face $F_{\langle I\rangle\{i_j\}}$ is a facet of $F_{\langle I\rangle}$ (if non-empty) which is easily seen to be positively oriented with respect to the convention about the orientation of a facet. Therefore,

$$\partial F=\sum_{k<j\leqslant n} F_{\langle I\rangle\{i_j\}}.$$

The result follows now by Lemma 8.4. □

8.4. Orientation of a simplicial face of P^*

We orient Δ_I for $I\subset\mathcal{F}$ by stating that if $i_0<...<i_k$ are the ordered elements of I, then the basis $\overrightarrow{e_0e_1},...,\overrightarrow{e_0e_k}$ is a positively oriented basis of the space of Δ_E (where the e_i's are the vertices of $\Delta_I\subset P_I^*\subset\mathbf{R}^d$).

Remark 8.6. We recall that, in the sequel, a subset I of $\mathcal{F}$ will always be considered as an ordered set, with the order induced from the order of $\mathcal{F}$. In particular, the simplex Δ_I is thus an oriented simplex (if non-empty), as well as the face F_I (if non-empty).

9. Alexander duals

To prove the cohomology theorem (Theorem 10.1), we need to compute *explicit* Alexander duals of simplicial cycles. We make use of [1, vol. 3, Chapter XIII]. We first recall this construction in our context.

Let $I=\{i_0, ..., i_k\}\subset\mathcal{F}$. The star dual F_I^* of F_I is defined as the maximal subcomplex of the barycentric subdivision $P_b=P_b^*$ of ∂P whose vertices are the centers of the faces of P containing F_I (see [1, vol. 1, pp. 143–144]). An orientation is fixed on F_I^* by demanding that the intersection number of F_I with F_I^* is $+1$ ([1, vol. 3, pp. 11–17]).

LEMMA 9.1. *The star dual of F_I is the barycentric subdivision of the oriented simplex* $(-1)^{kd}\Delta_I$.

Proof. The set of proper faces of P containing F_I is the set $\{F_J : J\subset I\}$, so, using the duality between P and P^*, the star dual of F_I is, up to sign, the maximal complex of P_b^* with vertex set $\{J : J\subset I\}$, that is the barycentric subdivision of Δ_I.

To compute the sign, let $\mathcal{B}$ be a positive basis of F_I. By §8, this means that the set $\mathcal{B}_0=(v_{i_0}, ..., v_{i_k}, \mathcal{B})$ is a positive basis of $\mathbf{R}^d$. On the other hand, assuming that 0 belongs to P and multiplying each normal vector v_i by a positive scalar if necessary, we may realize P^* as the convex hull of the points $(v_1, ..., v_n)$. We see, following the conventions of §8, that a positive basis of Δ_I is given by $\mathcal{B}'=(\overrightarrow{v_{i_0}v_{i_1}}, ..., \overrightarrow{v_{i_0}v_{i_k}})$. By [1, vol. 1, pp. 143–144], the barycentric subdivision of Δ_I is the star dual of F_I if and only if the set $(v_{i_0}, \mathcal{B}, \mathcal{B}')$ is a positive basis of $\mathbf{R}^d$, which is equivalent to asking for $(v_{i_0}, \mathcal{B}, v_{i_1}, ..., v_{i_k})$ to be a positive basis. Comparing it to $\mathcal{B}_0$, we have that its sign is $(-1)^{k(d-k-1)}$. Hence the statement holds. □

Let c be a (cellular) k-cycle of $P_{\mathcal{I}}$. Assume that $k<d-1$. In ∂P, the cycle c is thus a boundary. Indeed, it can be written

$$c=\sum_{|I|=d-k-1} a_I\partial F_I=\partial\left(\sum_{|I|=d-k-1} a_I F_I\right)=\partial C,\quad a_I\in\mathbf{Z}.$$

Since we are interested in the homology class $[c]\in\widetilde{H}_k(P_{\mathcal{I}}, \mathbf{Z})$, we may assume that a_I is zero if F_I is in $P_{\mathcal{I}}$, that is if $I\cap\mathcal{I}$ is non-empty. This means that we may assume that a_I is non-zero only if $I\subset\bar{\mathcal{I}}$.

Consider the star dual of C, that is the $(d-k-2)$-cochain

$$C^* = \sum_{\substack{|I|=d-k-1\\ I\subset\bar{\mathcal{I}}}} a_I F_I^*.$$

From now on, we see C^* as a *cochain*, that is we identify each simplex Δ_I with the cochain taking value 1 on Δ_I. Then C^* is a cocycle in $\partial P^*\setminus P^*_{\mathcal{I}}$. The cohomology class of C^* in $\widetilde{H}^{d-k-2}(\partial P^*\setminus P^*_{\mathcal{I}}, \mathbf{Z})$ is the Alexander dual of $[c]$.

Using Lemma 9.1, Corollary 7.5 and the fact that C^* is geometrically realized in $P^*_{\bar{\mathcal{I}}}$, we conclude the following result.

Lemma 9.2. *The Alexander dual of the class* $\left[\sum a_I \partial F_I\right] \in \widetilde{H}_k(P_{\mathcal{I}}, \mathbf{Z})$ *is the class* $(-1)^{d(k+1)}\left[\sum a_I \Delta_I\right] \in \widetilde{H}^{d-k-2}(P^*_{\bar{\mathcal{I}}}, \mathbf{Z})$.

10. The cohomology theorem

We may now state the cohomology theorem.

Theorem 10.1. (Cohomology theorem) *For any i, we have an isomorphism*

$$H^i(X, \mathbf{Z}) \simeq \bigoplus_{\mathcal{I}\subset\mathcal{F}} \widetilde{H}_{d+|\bar{\mathcal{I}}|-i-1}(P_{\mathcal{I}}, \mathbf{Z}).$$

We denote by $\psi([c])$ the preimage by this isomorphism of a class $[c]$ in any factor of the right-hand side. Moreover, consider two classes $[c]\in\widetilde{H}_k(P_{\mathcal{I}}, \mathbf{Z})$ and $[c']\in\widetilde{H}_{k'}(P_{\mathcal{J}}, \mathbf{Z})$, and denote by $[c]\cap[c']$ their intersection class in $\widetilde{H}_{k+k'-d+1}(P_{\mathcal{I}\cap\mathcal{J}}, \mathbf{Z})$. Then, the cup product of their images by ψ is given by

$$\psi([c]) \smile \psi([c']) = \begin{cases} \varepsilon\psi([c]\cap[c']), & \text{if } \mathcal{I}\cup\mathcal{J}=\mathcal{F},\\ 0, & \text{otherwise,}\end{cases}$$

where

$$\varepsilon = \begin{cases} 1, & \text{if } \mathcal{I}=\mathcal{F} \text{ or } \mathcal{J}=\mathcal{F},\\ \varepsilon_{IJ}\varepsilon_{\bar{\mathcal{I}}\setminus I, \bar{\mathcal{J}}\setminus J}, & \text{otherwise.}\end{cases}$$

Remark 10.2. The following formula for the homology groups of X in terms of P^* also holds:

$$H_i(X, \mathbf{Z}) \simeq \bigoplus_{\mathcal{I}\subset\mathcal{F}} \widetilde{H}_{i-|\mathcal{I}|-1}(P^*_{\mathcal{I}}, \mathbf{Z}).$$

In some cases, this formula is easier to use to compute the homology groups. We will prove this formula at the same time as the formula of Theorem 10.1.

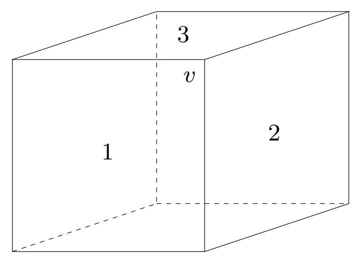

Figure 16.

Remark 10.3. If $\mathcal{I}$ and $\mathcal{J}$ are complementary in $\mathcal{F}$ and we take classes $[c]\in\widetilde{H}_k(P_{\mathcal{I}},\mathbf{Z})$ and $[c']\in\widetilde{H}_{k'}(P_{\mathcal{J}},\mathbf{Z})$ with $k+k'=d-2$, then their intersection class in $\widetilde{H}_{-1}(\varnothing,\mathbf{Z})\simeq\mathbf{Z}$ is their linking number. In particular, *Poincaré duality on X is given by Alexander duality on ∂P.*

Example 10.4. Let P be the cube. Number its facets in the following way: 1, 2 and 3 denote three faces adjacent to a vertex v (as in Figure 16), whereas $1'$ (respectively, $2'$ and $3'$) is the opposite face to 1 (respectively, 2 and 3). The order on $\mathcal{F}$ is given by

$$1<2<3<1'<2'<3'.$$

The sets $P_{\{1,2,1',2'\}}$, $P_{\{1,3,1',3'\}}$ and $P_{\{2,3,2',3'\}}$ have the homotopy type of a circle. Set $c_{12}=\partial F_3$, $c_{13}=\partial F_2$ and $c_{23}=\partial F_1$. Then $[c_{12}]$ (respectively, $[c_{13}]$ and $[c_{23}]$) is a generator of $\widetilde{H}_1(P_{\{1,2,1',2'\}},\mathbf{Z})$ (respectively, $\widetilde{H}_1(P_{\{1,3,1',3'\}},\mathbf{Z})$ and $\widetilde{H}_1(P_{\{2,3,2',3'\}},\mathbf{Z})$).

The sets $P_{\{1,1'\}}$, $P_{\{2,2'\}}$ and $P_{\{3,3'\}}$ have the homotopy type of a pair of points. Set $c_1=\partial F_{32}$, $c_2=\partial F_{31}$ and $c_3=\partial F_{21}$. Then $[c_1]$ (respectively, $[c_2]$ and $[c_3]$) is a generator of $\widetilde{H}_0(P_{\{1,1'\}},\mathbf{Z})$ (respectively, $\widetilde{H}_0(P_{\{2,2'\}},\mathbf{Z})$ and $\widetilde{H}_0(P_{\{3,3'\}},\mathbf{Z})$).

Finally, set $c=\partial F_{132}$ and consider the generator $[c]$ of $\widetilde{H}_{-1}(\varnothing,\mathbf{Z})$.

Theorem 10.1 gives the cohomology groups of X, which are as follows:

i	$\widetilde{H}^i(X,\mathbf{Z})$
$1,2,4,5,7,8$	$\{0\}$
3	$\mathbf{Z}\cdot\psi([c_{12}])\oplus\mathbf{Z}\cdot\psi([c_{13}])\oplus\mathbf{Z}\cdot\psi([c_{23}])$
6	$\mathbf{Z}\cdot\psi([c_1])\oplus\mathbf{Z}\cdot\psi([c_2])\oplus\mathbf{Z}\cdot\psi([c_3])$
9	$\mathbf{Z}\cdot[c]$

and the only non-zero cup products are

$$\begin{aligned}
\psi([c_{12}])\smile\psi([c_3]) &= -\psi([c_{13}])\smile\psi([c_2]) = \psi([c_{23}])\smile\psi([c_1]) = [c],\\
\psi([c_{12}])\smile\psi([c_{13}]) &= \psi([c_1]),\\
\psi([c_{12}])\smile\psi([c_{23}]) &= \psi([c_2]),\\
\psi([c_{13}])\smile\psi([c_{23}]) &= \psi([c_3]).
\end{aligned}$$

By Corollary 4.6 and Example 0.7, we know that X is the product of spheres $\mathbf{S}^3\times\mathbf{S}^3\times\mathbf{S}^3$. We here recover its cohomology ring.

Proof of Theorem 10.1 *and Remark* 10.2. We use the fact that X has the same homotopy type as the moment-angle complex $\mathcal{Z}_P$ (Lemma 0.15). Buchstaber and Panov proved ([9, Theorems 7.6 and 7.7] for the case over a field, [3] for the general case; see also [35, Theorem 4.7]) that there exists an isomorphism of bigraded algebras

$$H^*(X,\mathbf{Z})\simeq H\Big[\bigwedge[u_1,...,u_n]\otimes\mathbf{Z}[P^*],d\Big],$$

where $\mathbf{Z}[P^*]$ is the Stanley–Reisner ring of P^* [36, Chapter II], that is the polynomial ring $\mathbf{Z}[v_1,...,v_n]$ modulo the ideal generated by $v_{i_1}...v_{i_k}\notin P^*$ (where the v_i's are considered as abstract variables in the polynomial ring and as vertices of P^*). The bigrading on the left-hand side is explained in [9, §7]. On the right-hand side, we have

$$\operatorname{bideg} u_i=(-1,2),\quad \operatorname{bideg} v_i=(0,2),\quad du_i=v_i\quad\text{and}\quad dv_i=0.$$

From this isomorphism, Buchstaber and Panov proved ([9, Proposition 8.18]) that

$$H_i(X,\mathbf{Z})\simeq\bigoplus_{\mathcal{I}\subset\mathcal{F}}\widetilde{H}_{i-|\mathcal{I}|-1}(P^*_{\mathcal{I}},\mathbf{Z}),$$

which is exactly the formula of Remark 10.2.([1])

By Poincaré duality, we have

$$H^i(X,\mathbf{Z})\simeq H_{n+d-i}(X,\mathbf{Z})\simeq\bigoplus_{\mathcal{I}\subset\mathcal{F}}\widetilde{H}_{d+|\bar{\mathcal{I}}|-i-1}(P^*_{\mathcal{I}},\mathbf{Z})\simeq\bigoplus_{\mathcal{I}\subset\mathcal{F}}\widetilde{H}_{d+|\bar{\mathcal{I}}|-i-1}(P_{\mathcal{I}},\mathbf{Z}),$$

since $P_{\mathcal{I}}$ and $P^*_{\mathcal{I}}$ are homotopically equivalent, by Lemma 7.4.

To obtain a more geometric formulation of Theorem 10.1, define isomorphisms

$$\begin{aligned}
[\partial F_I]\in\widetilde{H}_k(P_{\mathcal{I}},\mathbf{Z}) &\longmapsto (-1)^{d(k+1)}[\Delta_I]\in\widetilde{H}^{d-k-2}(P^*_{\bar{\mathcal{I}}},\mathbf{Z})\\
&\longmapsto u_{\bar{\mathcal{I}}\setminus I}v_I\in H^{|\bar{\mathcal{I}}|+d-k-1}\Big[\bigwedge[u_1,...,u_n]\otimes\mathbf{Z}[P^*],d\Big]
\end{aligned}$$

([1]) To be more precise, this formula is proved *over a field* in [9, Proposition 8.18], since at that time the previous isomorphism of bigraded algebras was known only over a field. The same proof works over the integers, once one has the isomorphism over the integers.

and

$$[\partial P] \in \widetilde{H}_{d-1}(P, \mathbf{Z}) \longmapsto 1 \in H^0\Big[\bigwedge[u_1, ..., u_n] \otimes \mathbf{Z}[P^*], d\Big],$$

where v_I stands for $v_{i_1} \dots v_{i_{d-k-1}}$ and the same convention is used on u. The first arrow is the Alexander duality described in Lemma 9.2 and the second arrow is a well-defined isomorphism by [2, Theorem 1] (see also [35, Lemma 4.5 and Theorem 5.1]).

Set $R = \bigwedge[u_1, ..., u_n] \otimes \mathbf{Z}[P^*]$. From these isomorphisms we may transfer the wedge product on R to a product on the homology classes of the $P_{\mathcal{I}}$'s. We just handle the non-trivial cases. Let $\mathcal{I}$ and $\mathcal{J}$ be two proper subsets of $\mathcal{F}$ whose union is $\mathcal{F}$. Let k and k' be less than $d-1$. Let $I \subset \bar{\mathcal{I}}$ with $|I| = d-k-1$ and $J \subset \bar{\mathcal{J}}$ with $|J| = d-k'-1$. Note that

$$\varepsilon \cdot \varepsilon_{IJ} \cdot u_{(\bar{\mathcal{I}} \cup \bar{\mathcal{J}}) \setminus (I \cup J)} v_{IJ} = u_{\bar{\mathcal{I}} \setminus I, \bar{\mathcal{J}} \setminus J} v_{IJ},$$

where ε is defined in the statement of Theorem 10.1. We then obtain the following commutative diagram:

$$\begin{array}{ccc} [\partial F_I] \otimes [\partial F_J] \in \widetilde{H}_k(P_{\mathcal{I}}) \otimes \widetilde{H}_{k'}(P_{\mathcal{J}}) & \longrightarrow & \varepsilon [\partial F_{IJ}] \in \widetilde{H}_{k+k'-d+1}(P_{\mathcal{I} \cap \mathcal{J}}) \\ \downarrow & & \downarrow \\ [u_{\bar{\mathcal{I}} \setminus I} v_I] \otimes [u_{\bar{\mathcal{J}} \setminus J} v_J] \in H^p(R) \otimes H^q(R) & \longrightarrow & [u_{\bar{\mathcal{I}} \setminus I, \bar{\mathcal{J}} \setminus J} v_{IJ}] \in H^{p+q}(R), \end{array}$$

where $p = |\bar{\mathcal{I}}| + d - k - 1$ and $q = |\bar{\mathcal{J}}| + d - k' - 1$. Now, let

$$c = \sum_{\substack{|I| = d-k-1 \\ I \subset \bar{\mathcal{I}}}} a_I \partial F_I, \quad a_I \in \mathbf{Z},$$

represent a class of $\widetilde{H}_k(P_{\mathcal{I}}, \mathbf{Z})$ and

$$c' = \sum_{\substack{|J| = d-k'-1 \\ J \subset \bar{\mathcal{J}}}} b_J \partial F_J, \quad b_J \in \mathbf{Z},$$

represent a class of $\widetilde{H}_{k'}(P_{\mathcal{J}}, \mathbf{Z})$.

We want to compute the intersection class of $[c]$ and $[c']$. These two classes are naturally realized in the boundaries of $P_{\mathcal{I}}$ and $P_{\mathcal{J}}$, but do not meet transversely. We can nevertheless "push" them in the interior of these sets so that they do.

Definition 10.5. For each facet H, consider an affine function l_H on the space of P which is zero on H and positive on $P \setminus H$. For $\varepsilon > 0$, set $H_\varepsilon = l_H^{-1}(\varepsilon) \cap P$, and for a face F of P, set $F_\varepsilon = \bigcap_{H \supset F} H_\varepsilon$.

The following lemma is clear.

LEMMA 10.6. *Let F and F' be two faces of P that are not contained in a common facet and have non-empty intersection. Then, if ε is small enough, ∂F_ε and $\partial F'_\varepsilon$ meet transversely and their intersection is $\partial(F\cap F')_\varepsilon$. Moreover, this works for oriented faces.*

We now can compute the homology class of the intersection of our two cycles. For this, consider for every facet H of P an affine function on the space of P which is zero on H and positive on $P\setminus H$.

Take $\varepsilon>0$ small enough. Define $[c_\varepsilon]$ as follows: for an element I of $\bar{\mathcal{I}}$ and a facet H of $\mathcal{I}$ meeting F_I, call $(F_I\cap H)_{H,\varepsilon}$ the set $(F_I\cap H)_\varepsilon$ when we consider H as a simple polytope and restrict the affine functions of the facets meeting H to the facets of H. Just remark now that

$$[c]=\Big[\sum a_I(F_I\cap H)_{H,\varepsilon}\Big],$$

where the sum runs over $H\in\mathcal{I}$ and $I\in\bar{\mathcal{I}}$ such that $F_I\cap H\neq\varnothing$, since the cycle in the brackets above is homotopic to $\sum a_IF_I\cap H$.

Of course, the same is true for $[c']$. But these cycles meet transversely and, thanks to Lemma 10.6, their intersection can be written as

$$[c]\cap[c']=\Big[\sum a_Ib_J(F_I\cap F_J\cap H)_{H,\varepsilon}\Big],$$

where the sum runs over every $H\in\mathcal{I}$, $I\in\bar{\mathcal{I}}$ and $J\in\overline{\mathcal{J}}$ such that $F_I\cap F_J\cap H\neq\varnothing$; thus, by Lemmas 8.4 and 8.5, we get

$$[c]\cap[c']=\sum a_Ib_J[\partial F_{IJ}],$$

and this completes the proof. □

Remark 10.7. Theorem 10.1 can also be obtained, by use of Alexander duality, directly from Baskakov's formula [2, Theorem 1]. However, it is not possible to obtain a formula *with sign* using only Baskakov's result as stated in [2]. On the other hand, in a previous version of this paper, Theorem 10.1 was obtained from Goresky–MacPherson's [16] and de Longueville's [24] formulas about the cohomology ring of the subspace arrangement $\mathcal{S}$ defined in §0 (see also [9, §8.2]). The proof was based on Alexander duality too, but was much more involved.

11. Applications to the topology of the links

In this section we make use of the previous results on the cohomology ring of a 2-connected link X to investigate how complicated the topology of a link can be. We will see that the complexity increases when the dimension d of the associate polytope P increases, and that the topology of a link may finally be "arbitrarily complicated".

For $d=0$ the unique 2-connected link is a point, for $d=1$ it is $\mathbf{S}^3$ (this is the case $p=0$ and $n=2$). For the polygons, the situation is not so easy, and the links are products of odd-dimensional spheres or connected sums of products of spheres: this case was completely described in [28] (cf. Theorem 6.3). In higher dimensions, the only known case is the special case where $p=2$ ([25], [26]), where the same type of manifolds is obtained (cf. Example 0.5). On the other hand, the generalization of McGavran's results stated as Theorem 6.3 shows that, for any value of d, there is an infinite number of examples where the link is a connected sum of products of spheres. This leads naturally to the following question, whose positive answer was stated as a conjecture in [30].

Question A. Is it true that any 2-connected link may be decomposed into a product of odd-dimensional spheres and connected sums of products of spheres?

A weaker version of this question is the following.

Question A′. At least, is it true that the cohomology ring of a 2-connected link is isomorphic to the cohomology ring of a product of odd-dimensional spheres and connected sums of products of spheres?

This supposes to resolve first the following (easier?) question.

Question A″. Is it true that the homology of a 2-connected link is always without any torsion?

An immediate application of Theorem 10.1 is that the answer is positive if $d\leqslant 4$.

COROLLARY 11.1. *If the polytope P has dimension at most 4, then the homology of the associated manifold is torsion-free.*

Proof. In this case, every homology group of the form $\widetilde{H}_k(P_{\mathcal{I}},\mathbf{Z})$ is torsion-free, as $P_{\mathcal{I}}$ lies in ∂P, which is a sphere of dimension $\leqslant 3$ (see [1, vol. 3, Chapter XIII, §4.12]). So is a direct sum of such groups as are the cohomology groups of X by Theorem 10.1. □

We emphasize that this result, obtained as an easy consequence of Theorem 10.1, cannot be easily deduced from the classical form of the Goresky–MacPherson formula (for example in the version of [24]) applied to the complement of the subspace arrangement $\mathcal{S}$, since the dimension of the complex Δ, on which the homology computations have to be done, can be much greater than 3. Therefore, this corollary illustrates all the interest in having a formula in terms of subsets of the associate polytope.

We will now prove that, even in dimension 3, the answer to Questions A and A′ is negative. To see this, we will first compute how the cohomology of a link X changes when performing an elementary surgery of type $(1,n)$ on $X\times\mathbf{S}^1$, that is when performing a vertex cutting on P. Recall that, by Lemma 6.1, the diffeomorphism type of the new link X' is independent of the choice of the vertex to be cut off.

PROPOSITION 11.2. *Let X and X' be as above. Assume that $d\geqslant 2$. Then*

$$\begin{aligned} &H^0(X',\mathbf{Z})\simeq H^{n+d+1}(X',\mathbf{Z})\simeq\mathbf{Z},\\ &H^1(X',\mathbf{Z})\simeq H^2(X',\mathbf{Z})\simeq H^{n+d-1}(X',\mathbf{Z})\simeq H^{n+d}(X',\mathbf{Z})\simeq 0,\\ &H^i(X',\mathbf{Z})\simeq H^i(X,\mathbf{Z})\oplus H^{i-1}(X,\mathbf{Z})\oplus\mathbf{Z}^{\binom{n-d}{i-2d+1}}\oplus\mathbf{Z}^{\binom{n-d}{i-2}},\quad \textit{for } 3\leqslant i\leqslant n+d-4, \end{aligned}$$

where $\binom{l}{k}=0$ if $k<0$ or $k>l$. Moreover, the product is given by the following rules considering two cohomology classes $[c]$ and $[c']$ of X'.

Rule 1: *If one of $[c]$ and $[c']$ is in $H^0(X',\mathbf{Z})$ or $H^{n+d+1}(X',\mathbf{Z})$, then the product is the obvious one. Assume that this is not the case. Then denote by $S_{i,j}$, for $3\leqslant i\leqslant n+d-2$ and $1\leqslant j\leqslant 4$, the terms of the sum above, when they exist, that is*

$$H^i(X',\mathbf{Z})=S_{i,1}\oplus S_{i,2}\oplus S_{i,3}\oplus S_{i,4}.$$

For $j=1,2$, decompose $S_{i,j}$ as $\bigoplus_{\mathcal{I}\subset\mathcal{F}}S_{\mathcal{I},j}$, as in Theorem 10.1. Finally, denote by S_j, for $1\leqslant j\leqslant 4$, the sum of the $S_{i,j}$'s when i varies. We assume that $[c]$ is in $S_{\mathcal{I},j}$ and $[c']$ in $S_{\mathcal{J},j'}$.

Rule 2: *If $\{j,j'\}\neq\{1\},\{1,2\},\{3,4\}$ then $[c]\smile[c']=0$. Call φ_1 and φ_2 the applications of $H^i(X,\mathbf{Z})$ in $S_{i,1}$ and $S_{i+1,2}$.*

Rule 3: *If $j=j'=1$, then we may assume that $[c]=\varphi_1([c_1])$ and $[c']=\varphi_1([c_1'])$. Then $[c]\smile[c']=\varphi_1([c_1]\smile[c_1'])$.*

Rule 4: *If $j=1$ and $j'=2$, then we may assume that $[c]=\varphi_1([c_1])$ and $[c']=\varphi_2([c_2'])$. Then $[c]\smile[c']=\varphi_2([c_1]\smile[c_2'])$.*

Rule 5: *The cup product from $S_3\times S_4$ to $H^{n+d+1}(X,\mathbf{Z})\simeq\mathbf{Z}$ is a unimodular bilinear form, which is diagonal in the canonical bases (when these bases are suitably ordered). Note that the product vanishes when the dimensions do not correspond.*

In particular, if the cohomology of X has no torsion, then neither has the cohomology of X'.

Remark 11.3. The isomorphisms are not completely canonical. Some judicious choices have to be made to obtain the desired rules about the cup product.

Proof of Proposition 11.2. Let v be the cut vertex, $\mathcal{F}_v$ be the set of the facets of P that contain v, and F be the "new" facet (we will not distinguish a facet of P, even in $\mathcal{F}_v$, from the "same" facet of P').

Notation 11.4. For a subset $\mathcal{I}$ of $\mathcal{F}$, we will denote by $\mathcal{I}_2$ the subset of the facets of P' having the same elements as $\mathcal{I}$. Then, set $\mathcal{I}_1=\mathcal{I}_2\cup\{F\}$.

Let $\mathcal{I}\subset\mathcal{F}$ be such that the intersection of $\mathcal{I}$ with $\mathcal{F}_v$ is proper and non-empty; then v belongs to the topological boundary of $P_{\mathcal{I}}$, and both $P'_{\mathcal{I}_1}$ and $P'_{\mathcal{I}_2}$ are homotopy equivalent to $P_{\mathcal{I}}$. Therefore, the three sets have the same reduced homology groups.

Consider now a subset $\mathcal{I}$ of $\mathcal{F}$ that contains $\mathcal{F}_v$. Then $P'_{\mathcal{I}_1}$ is homotopy equivalent to $P_{\mathcal{I}}$, hence has the same reduced homology groups, and $P'_{\mathcal{I}_2}$ is homotopy equivalent to $P_{\mathcal{I}}$ minus a point. Therefore, if $\mathcal{I}\neq\mathcal{F}$, then the reduced homology groups of $P'_{\mathcal{I}_2}$ are isomorphic to the ones of $P_{\mathcal{I}}$, except for $\widetilde{H}_{d-2}(P'_{\mathcal{I}_2},\mathbf{Z})$ which is isomorphic to $\widetilde{H}_{d-2}(P_{\mathcal{I}},\mathbf{Z})\oplus\mathbf{Z}$. If $\mathcal{I}=\mathcal{F}$, then $P'_{\mathcal{I}_2}$ is contractible, hence has no reduced homology.

Consider now a subset $\mathcal{I}$ of $\mathcal{F}$ that is disjoint from $\mathcal{F}_v$. Then $P'_{\mathcal{I}_2}$ is homotopy equivalent to $P_{\mathcal{I}}$, hence has the same reduced homology groups, and $P'_{\mathcal{I}_1}$ is homotopy equivalent to the disjoint union of $P_{\mathcal{I}}$ with a point. Therefore, if $\mathcal{I}\neq\varnothing$, then the reduced homology groups of $P'_{\mathcal{I}_1}$ are isomorphic to the ones of $P_{\mathcal{I}}$, except for $\widetilde{H}_0(P'_{\mathcal{I}_2},\mathbf{Z})$ which is isomorphic to $\widetilde{H}_0(P_{\mathcal{I}},\mathbf{Z})\oplus\mathbf{Z}$. If $\mathcal{I}=\varnothing$, then $P'_{\{F\}}=F$ is contractible and has no reduced homology.

Let i be an integer. Then, the above results allow us to compute $H^i(X',\mathbf{Z})$:

$$\begin{aligned} H^i(X',\mathbf{Z}) &\simeq \bigoplus_{\mathcal{I}\subset\mathcal{F}} \widetilde{H}_{d+|\bar{\mathcal{I}}_1|-i-1}(P'_{\mathcal{I}_1},\mathbf{Z}) \oplus \bigoplus_{\mathcal{I}\subset\mathcal{F}} \widetilde{H}_{d+|\bar{\mathcal{I}}_2|-i-1}(P'_{\mathcal{I}_2},\mathbf{Z}) \\ &\simeq \bigoplus_{\mathcal{I}\subset\mathcal{F}} \widetilde{H}_{d+|\bar{\mathcal{I}}|-i-1}(P'_{\mathcal{I}_1},\mathbf{Z}) \oplus \bigoplus_{\mathcal{I}\subset\mathcal{F}} \widetilde{H}_{d+|\bar{\mathcal{I}}|-i}(P'_{\mathcal{I}_2},\mathbf{Z}), \end{aligned}$$

which is isomorphic to

$$\begin{aligned} &\bigoplus_{\substack{\mathcal{I}\subset\mathcal{F}\\ \mathcal{I}\cap\mathcal{F}_v\neq\varnothing}} \widetilde{H}_{d+|\bar{\mathcal{I}}|-i-1}(P_{\mathcal{I}},\mathbf{Z}) \oplus \bigoplus_{\substack{\mathcal{I}\subset\mathcal{F}\\ \mathcal{I}\cap\mathcal{F}_v=\varnothing\\ \mathcal{I}\neq\varnothing}} \big(\widetilde{H}_{d+|\bar{\mathcal{I}}|-i-1}(P_{\mathcal{I}},\mathbf{Z})\oplus\mathbf{Z}^{\delta_{i+1}^{d+|\bar{\mathcal{I}}|}}\big) \\ &\qquad \oplus \bigoplus_{\substack{\mathcal{I}\subset\mathcal{F}\\ \mathcal{I}\not\supset\mathcal{F}_v}} \widetilde{H}_{d+|\bar{\mathcal{I}}|-i}(P_{\mathcal{I}},\mathbf{Z}) \oplus \bigoplus_{\substack{\mathcal{I}\subset\mathcal{F}\\ \mathcal{I}\supset\mathcal{F}_v\\ \mathcal{I}\neq\mathcal{F}}} \big(\widetilde{H}_{d+|\bar{\mathcal{I}}|-i}(P_{\mathcal{I}},\mathbf{Z})\oplus\mathbf{Z}^{\delta_{d-2}^{d+|\bar{\mathcal{I}}|-i}}\big) \end{aligned}$$

and finally to

$$\bigoplus_{\substack{\mathcal{I}\subset\mathcal{F}\\ \mathcal{I}\neq\varnothing}} \widetilde{H}_{d+|\bar{\mathcal{I}}|-i-1}(P_{\mathcal{I}},\mathbf{Z}) \oplus \bigoplus_{\substack{\mathcal{I}\subset\mathcal{F}\\ \mathcal{I}\neq\mathcal{F}}} \widetilde{H}_{d+|\bar{\mathcal{I}}|-i}(P_{\mathcal{I}},\mathbf{Z}) \oplus \bigoplus_{\substack{\mathcal{I}\subset\mathcal{F}\\ \mathcal{I}\cap\mathcal{F}_v=\varnothing\\ \mathcal{I}\neq\varnothing}} \mathbf{Z}^{\delta_{i-d+1}^{|\bar{\mathcal{I}}|}} \oplus \bigoplus_{\substack{\mathcal{I}\subset\mathcal{F}\\ \mathcal{I}\supset\mathcal{F}_v\\ \mathcal{I}\neq\mathcal{F}}} \mathbf{Z}^{\delta_{i-2}^{|\bar{\mathcal{I}}|}}.$$

The first sum

$$\bigoplus_{\substack{\mathcal{I}\subset\mathcal{F}\\ \mathcal{I}\neq\varnothing}} \widetilde{H}_{d+|\bar{\mathcal{I}}|-i-1}(P_{\mathcal{I}},\mathbf{Z})$$

is isomorphic to $H^i(X,\mathbf{Z})$, except when $d+n-i-1=-1$, that is $i=d+n$.

Also, the second sum

$$\bigoplus_{\substack{\mathcal{I}\subset\mathcal{F}\\ \mathcal{I}\neq\mathcal{F}}} \widetilde{H}_{d+|\bar{\mathcal{I}}|-i}(P_{\mathcal{I}},\mathbf{Z})$$

is isomorphic to $H^{i-1}(X,\mathbf{Z})$, except when $d-i=d-1$, that is $i=1$.

On the other hand,

$$\sum_{\substack{\mathcal{I}\subset\mathcal{F}\\ \mathcal{I}\cap\mathcal{F}_v=\varnothing\\ \mathcal{I}\neq\varnothing}} \delta^{|\bar{\mathcal{I}}|}_{i-d+1}$$

is the number of non-empty subsets of $\mathcal{F}\setminus\mathcal{F}_v$ having $n-i+d-1$ elements. It is $\binom{n-d}{n-i+d-1}$, except when $n-i+d-1=0$, that is $i=n+d-1$, in which case this sum is zero.

We also have that

$$\sum_{\substack{\mathcal{F}_v\subset\mathcal{I}\subset\mathcal{F}\\ \mathcal{I}\neq\mathcal{F}}} \delta^{|\bar{\mathcal{I}}|}_{i-2} = \sum_{\substack{\bar{\mathcal{I}}\subset\mathcal{F}\\ \bar{\mathcal{I}}\cap\mathcal{F}_v=\varnothing\\ \bar{\mathcal{I}}\neq\varnothing}} \delta^{|\bar{\mathcal{I}}|}_{i-2}$$

is the number of non-empty subsets of $\mathcal{F}\setminus\mathcal{F}_v$ having $i-2$ elements. It is $\binom{n-d}{i-2}$, except when $i-2=0$, that is $i=2$, in which case this sum is zero.

Putting all these results together, being $(n-d)-(n-i+d-1)=i-2d+1$, we get the isomorphisms of the proposition.

Let us now describe the cup product. Rule 1 is obvious.

To continue, we have to clearly define the sums S_j, because they derive from isomorphims which are not, as we shall see right now, canonical.

Look first at the isomorphism $\widetilde{H}_0(P'_{\mathcal{I}_1},\mathbf{Z})\simeq\widetilde{H}_0(P_{\mathcal{I}},\mathbf{Z})\oplus\mathbf{Z}$, where $\mathcal{I}$ is non-empty and does not meet $\mathcal{F}_v$. This isomorphism is canonical when (non-reduced) homology is concerned, but the cycles that are added (multiples of the singleton $\langle v\rangle$) are not cycles in reduced homology. Look now at the isomorphism $\widetilde{H}_{d-2}(P'_{\mathcal{I}_2},\mathbf{Z})\simeq\widetilde{H}_{d-2}(P_{\mathcal{I}},\mathbf{Z})\oplus\mathbf{Z}$, where $\mathcal{I}\neq\mathcal{F}$ and $\mathcal{I}$ contains $\mathcal{F}_v$. The projection of $\widetilde{H}_{d-2}(P'_{\mathcal{I}_2},\mathbf{Z})$ over $\widetilde{H}_{d-2}(P_{\mathcal{I}},\mathbf{Z})$ is canonical (hence so is its kernel, which is the factor $\mathbf{Z}$), but the inclusion of $\widetilde{H}_{d-2}(P_{\mathcal{I}},\mathbf{Z})$ in $\widetilde{H}_{d-2}(P'_{\mathcal{I}_2},\mathbf{Z})$ is not.

Consider a non-empty subset $\mathcal{I}$ of $\mathcal{F}$ disjoint from $\mathcal{F}_v$. Choose any reduced homology class in $\widetilde{H}_0(P'_{\mathcal{I}_1},\mathbf{Z})$, whose value on the connected component F of $P'_{\mathcal{I}_1}$ is equal to 1, and call $[c_{\mathcal{I}}]$ this class. It is clear that the groups $\mathbf{Z}\cdot[c_{\mathcal{I}}]$ and $\widetilde{H}_0(P_{\mathcal{I}},\mathbf{Z})$, whose inclusion in $\widetilde{H}_0(P'_{\mathcal{I}_1},\mathbf{Z})$ follows from the inclusion $P_{\mathcal{I}}\subset P'_{\mathcal{I}_1}$, give the desired isomorphism. Doing this for every $\mathcal{I}$, we thus have

$$S_3=\bigoplus_{\substack{\mathcal{I}\subset\mathcal{F}\\ \mathcal{I}\cap\mathcal{F}_v=\varnothing\\ \mathcal{I}\neq\varnothing}} \mathbf{Z}\cdot[c_{\mathcal{I}}].$$

Consider now $\bar{\mathcal{I}}$. It is a proper subset of $\mathcal{F}$ which contains $\mathcal{F}_v$. The linking operation on $\widetilde{H}_0(P'_{\mathcal{I}_1},\mathbf{Z})\times\widetilde{H}_{d-2}(P'_{\bar{\mathcal{I}}_1},\mathbf{Z})$ is well defined, and the subgroup of the homology classes that are not linked with $[c_{\mathcal{I}}]$ is isomorphic to $\widetilde{H}_{d-2}(P_{\bar{\mathcal{I}}},\mathbf{Z})$. As a consequence, $\widetilde{H}_{d-2}(P'_{\bar{\mathcal{I}}_1},\mathbf{Z})$ is the direct sum of this subgroup with the group generated by the class $[c'_{\bar{\mathcal{I}}}]$ of a sphere that surrounds F (this group is also the kernel of the projection coming from the inclusion $P'_{\bar{\mathcal{I}}_1}\subset P_{\bar{\mathcal{I}}}$). We thus obtain

$$S_4=\bigoplus_{\substack{\mathcal{I}\subset\mathcal{F}\\ \mathcal{I}\cap\mathcal{F}_v=\varnothing\\ \mathcal{I}\neq\varnothing}}\mathbf{Z}\cdot[c'_{\bar{\mathcal{I}}}].$$

Rule 5 is now clear. More precisely, if we take $[c_{\mathcal{I}}]$ and $[c'_{\overline{\mathcal{J}}}]$ as explained above, the cup product of the corresponding cohomology classes is zero if $\mathcal{I}\neq\mathcal{J}$. Indeed, if $\mathcal{I}\neq\mathcal{J}$, then $\mathcal{I}\cup\overline{\mathcal{J}}\neq\mathcal{F}$ or $\mathcal{I}\cap\overline{\mathcal{J}}\neq\varnothing$. By Theorem 10.1, the cup product is automatically zero in the first case; in the second case, it lies in $\widetilde{H}_{-1}(\mathcal{I}\cap\overline{\mathcal{J}},\mathbf{Z})$. As this group is reduced to zero, the cup product is zero too. On the other hand, the cup product of the classes associated to $[c_{\mathcal{I}}]$ and $[c'_{\bar{\mathcal{I}}}]$ is, up to sign, the top class of X' (more precise choices allow to exactly obtain the top class). This gives Rule 5.

For Rule 2, remark first that if both $[c]$ and $[c']$ are in S_j, with $j\neq1$, then the union of the corresponding subsets of $\mathcal{F}\cup\{F\}$ is not all of $\mathcal{F}\cup\{F\}$ (indeed, F is not in this union if $j=2$ or $j=4$, and $\mathcal{F}_v$ does not intersect the union if $j=3$). We then just have to see that $[c]\smile[c']$ vanishes if $j\leqslant2$ and $j'\geqslant3$.

Consider first a class $[c'_{\bar{\mathcal{I}}}]$ in S_4. It is realized by a $(d-2)$-sphere which surrounds F. Remark that every (reduced) homology class in a $P_{\mathcal{I}}$ can be realized by a cycle which is far away from v (except if $\mathcal{I}=\mathcal{F}$, but then the corresponding class is in $H^0(X',\mathbf{Z})$ and Rule 1 applies). As F (and thus the sphere realizing $[c'_{\bar{\mathcal{I}}}]$) can be thought of very close to v, they do not intersect (neither are they linked). Hence, if $[c']$ is in S_4 and $[c]$ is in $S_{j'}$, with $j'\leqslant2$, then $[c]\smile[c']=0$.

Consider now a class $[c_{\mathcal{I}}]$ in S_3. Let $\mathcal{J}\neq\mathcal{F}$ and let $[a_{\mathcal{J}}]$ be a class of $\widetilde{H}_k(P'_{\mathcal{J}_2},\mathbf{Z})$. By arguments similar to those used in the proof of Rule 5, we have that the intersection class $[c_{\mathcal{I}}]\cap[a_{\mathcal{J}}]$ corresponds to a non-trivial cohomology class of X' if and only if $[a_{\mathcal{J}}]$ is a multiple of $[c'_{\overline{\mathcal{J}}}]$. But such a class is not in S_2, and thus the cup product of a class of S_2 with a class of S_3 is always zero.

Rules 3 and 4 derive from our Theorem 10.1. Assume that F is the greatest element for the order that we consider on $\mathcal{F}\cup\{F\}$.

Given a proper non-empty subset $\mathcal{I}$ of $\mathcal{F}$, and $[a]\in\widetilde{H}_k(P_{\mathcal{I}},\mathbf{Z})$, recall that $\psi([a])$ is its image in $H^{|\bar{\mathcal{I}}|+d-k-1}(X,\mathbf{Z})$. Let $\psi_i([a])=\varphi_i(\psi([a]))$ for $i=1,2$. Via our isomorphisms, $[a]$ is identified with some classes $[a_j]\in\widetilde{H}_k(P'_{\mathcal{I}_j},\mathbf{Z})$ for $j=1,2$. Denoting by ψ' the application on X' which is equivalent to ψ on X, we have $\psi_j([a])=\psi'([a_j])$ for $j=1,2$.

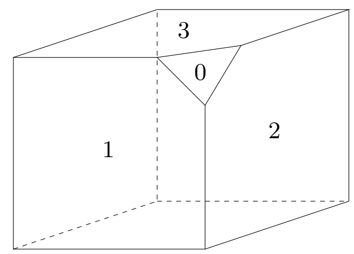

Figure 17.

Consider now $[a]\in\widetilde{H}_k(P_{\mathcal{I}},\mathbf{Z})$ and $[b]\in\widetilde{H}_{k'}(P_{\mathcal{J}},\mathbf{Z})$ ($\mathcal{I}$ and $\mathcal{J}$ proper and non-empty) with $a=\partial F_I$ and $b=\partial F_J$. Assume moreover that $\mathcal{I}\cup\mathcal{J}=\mathcal{F}$ (otherwise, cup products are zero). Remark that $[a_1]\cap[b_j]=([a]\cap[b])_j$ for $j=1,2$. We then compute

$$\psi_1([a])\smile\psi_1([b])=\psi'([a_1])\smile\psi'([b_1])=\varepsilon_{IJ}\varepsilon_{\bar{\mathcal{I}}_1\setminus I,\bar{\mathcal{J}}_1\setminus J}\psi'([a_1]\cap[b_1]),$$

that is, using the definitions $\mathcal{I}_1=\mathcal{I}\cup\{F\}$ and $\mathcal{J}_1=\mathcal{J}\cup\{F\}$,

$$\begin{aligned}\psi_1([a])\smile\psi_1([b])&=\varepsilon_{IJ}\varepsilon_{\bar{\mathcal{I}}\setminus I,\bar{\mathcal{J}}\setminus J}\psi'(([a]\cap[b])_1)\\&=\varphi_1(\varepsilon_{IJ}\varepsilon_{\bar{\mathcal{I}}\setminus I,\bar{\mathcal{J}}\setminus J}\psi([a]\cap[b]))\\&=\varphi_1(\psi([a])\smile\psi([b])).\end{aligned}$$

Rule 3 follows from this and Rule 4 by similar computations. □

Example 11.5. Consider the cube as a simple polytope. By Corollary 4.6, the associated manifold is the product of three 3-spheres (cf. Example 10.4). Cut now a vertex. The resulting simple polytope has dimension 3 and seven facets, hence the associated manifold X has dimension 10. Note also a $\mathfrak{S}_3$-symmetry. Let us compute its cohomology ring as an application of Proposition 11.2.

Number the "cut face" by 0, the faces adjacent to 0 by 1, 2 and 3 (see Figure 17), and the faces "opposite" to 1, 2, and 3 by $1'$, $2'$ and $3'$, respectively.

The cohomology groups of X are free, and the corresponding Betti numbers are given in the following table:

i	0	1	2	3	4	5	6	7	8	9	10
$b_i(X)$	1	0	0	6	6	2	6	6	0	0	1

Denote by λ_i, for $1\leqslant i\leqslant 3$, the cohomology classes which generate $H^3(\mathbf{S}^3\times\mathbf{S}^3\times\mathbf{S}^3,\mathbf{Z})$, and by λ_{ij} the cup product $\lambda_i\smile\lambda_j$. For $l=1,2$ let $\lambda_{i,l}$ (respectively, $\lambda_{ij,l}$) be $\varphi_l(\lambda_i)$ (respectively, $\varphi_l(\lambda_{ij})$). The expression $e_{\mathcal{I}}$, for some $\mathcal{I}\subset\{0,1,2,3,1',2',3'\}$, denotes the generator of a cohomology class of $P_{\mathcal{I}}$, and will only be used when $P_{\mathcal{I}}$ has only one non-zero reduced homology group and when this group is isomorphic to $\mathbf{Z}$ (e.g. $P_{\mathcal{I}}$ has the homotopy type of a circle). Finally, by σ we denote a permutation of the set $\{1,2,3\}$. Letting σ vary among the permutations of $\{1,2,3\}$, we have

- $H^3(X,\mathbf{Z})$ is generated by $\lambda_{\sigma(1),1}$ and $e_{123\sigma(1)'\sigma(2)'}$;
- $H^4(X,\mathbf{Z})$ is generated by $\lambda_{\sigma(1),2}$ and $e_{123\sigma(1)'}$;
- $H^5(X,\mathbf{Z})$ is generated by e_{123} and $e_{01'2'3'}$;
- $H^6(X,\mathbf{Z})$ is generated by $\lambda_{\sigma(1)\sigma(2),1}$ and $e_{0\sigma(1)'\sigma(2)'}$;
- $H^7(X,\mathbf{Z})$ is generated by $\lambda_{\sigma(1)\sigma(2),2}$ and $e_{0\sigma(1)'}$.

The products of these generators are zero except for the following products:

(i) $\lambda_{\sigma(1),1}\smile\lambda_{\sigma(2),1}=\lambda_{\sigma(1)\sigma(2),1}$;

(ii) $\lambda_{\sigma(1),1}\smile\lambda_{\sigma(2),2}=\lambda_{\sigma(1)\sigma(2),2}$ and $\lambda_{\sigma(2),1}\smile\lambda_{\sigma(1),2}=\lambda_{\sigma(1)\sigma(2),2}$;

(iii) the products which give the top class, i.e. $\lambda_{\sigma(1),1}\smile\lambda_{\sigma(2)\sigma(3),2}$, $e_{\mathcal{I}}\smile e_{\bar{\mathcal{I}}}$ and $\lambda_{\sigma(1)\sigma(2),1}\smile\lambda_{\sigma(3),2}$.

It is easy to check that, in the previous example, the cohomology ring of the associated link is isomorphic neither to that of a sphere, nor to that of a connected sum of sphere products, nor to that of the product of such manifolds. The answer to Questions A and A′ is thus negative even in dimension 3. Notice that the exact diffeomorphism type of the link of the previous example is not clear. We may ask the following question.

Question. Describe this manifold more precisely: for instance, can it be decomposed into a connected sum of manifolds?

In dimension 3, we may in fact characterize precisely which simple polytopes give rise to connected sums of sphere products as links, and which manifolds appear in this way. We have the following result.

Proposition 11.6. *Let P be a simple polyhedron. Then, the following statements are equivalent*:

(i) *the cohomology ring of the associated link X is isomorphic to that of a connected sum of sphere products*;

(ii) *the link X is diffeomorphic to a connected sum of sphere products*;

(iii) *there exists $l>0$ such that X is diffeomorphic to*

$$\mathop{\#}_{j=1}^{l} j\binom{l+1}{j+1}\mathbf{S}^{2+j}\times\mathbf{S}^{6+l-j-1};$$

(iv) *there exists $l>0$ such that P is obtained from the 3-simplex by cutting off l well-chosen vertices.*

Proof. By application of Theorem 6.3, we know that (iv) implies (iii), and of course (iii) implies (ii), and (ii) implies (i), so it is sufficient to prove that (i) implies (iv). We assume thus that the cohomology ring of the associated link X is isomorphic to that of a connected sum of sphere products.

Definition 11.7. Let $\mathcal{I}$ be a subset of $\mathcal{F}$. We say that $\mathcal{I}$ is a 1*-cycle of facets* of P if $K_{\mathcal{I}}$ is a cycle (i.e. a connected graph all of whose vertices are bivalent).

A 1-cycle of facets can also be viewed as the data of an integer $k\geqslant 3$ and an injective map from $\mathbf{Z}_k$ into $\mathcal{I}$, such that the images of two elements meet if and only if the two elements are equal or consecutive in $\mathbf{Z}_k$, and if moreover any triple of facets do not have any common vertex. The integer k is then called the *length* of the 1-cycle of facets.

Claim. Let F and F' be any two disjoint facets of P. Then $\mathcal{F}\setminus\{F,F'\}$ contains a 1-cycle of facets.

To see this, consider the set $\mathcal{I}_F$ of facets that meet F (except for F itself). Consider the maps ϕ from $\mathbf{Z}_k$ into $\mathcal{I}_F$ having the following properties:

(i) for all $i\in\mathbf{Z}_k$, $\phi(i)$ meets $\phi(i+1)$;

(ii) for all $i\in\mathbf{Z}_k$, consider the segment on $\phi(i)$ joining the centers of the edges $\phi(i)\cap\phi(i-1)$ and $\phi(i)\cap\phi(i+1)$. We require the polygon obtained by concatenation of all these segments to be non-trivial in the homology of $P_\partial\setminus(F\cup F')$.

There exist such maps: order $\mathcal{I}_F$ such that the bijective order-preserving map from $\mathbf{Z}_{|\mathcal{I}_F|}$ to $\mathcal{I}_F$ satisfies (i). Then this map also satisfies (ii), since the polygon obtained from it is homotopic to the boundary of F. Moreover, let us prove that a minimal subset of $\mathcal{I}_F$ fulfilling these conditions is a 1-cycle of facets.

First, such a minimal subset cannot contain exactly three globally meeting facets, as in this case the polygon considered in the point (ii) would be contained in a contractible subset (the union of the three facets) of $P_\partial\setminus(F\cup F')$, which is not allowed.

Assume now that in this minimal subset $\{C_1,...,C_k\}$, the facet C_1 meets C_j, for some $2<j<k$. Then $\{C_1,...,C_j\}$ and $\{C_1,C_j,C_{j+1},...,C_k\}$ satisfy (i), and one of them satisfies (ii), as the polygon of $C_1,...,C_k$ is homologically the sum of the polygons of these two subsets. This gives a contradiction, and the proof of the claim is completed.

We denote by $(*)$ the property, for a simple 3-dimensional polytope, that all its 1-cycles of facets have length 3.

Assume that P does not satisfy $(*)$. Then we can take a 1-cycle of facets $\mathcal{I}$ of P of length $k\geqslant 4$. In particular, I_1 and I_3 are disjoint. The complement of $P_{\mathcal{I}}$ in P has two connected components $\mathcal{X}$ and $\mathcal{Y}$.

The group $H_1(P_{\mathcal{I}},\mathbf{Z})$ is isomorphic to $\mathbf{Z}$, generated by the class of the "polygon" T whose vertices are the centers of the intersections of facets of $\mathcal{I}$.

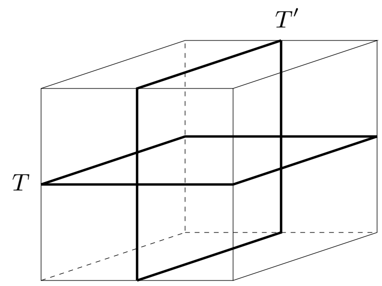

Figure 18.

Consider now $\mathcal{J}=\{I_1,I_3\}\cup(\mathcal{F}\backslash\mathcal{I})$. The group $H_1(P_{\mathcal{J}},\mathbf{Z})$ is isomorphic to $\mathbf{Z}$ too, generated by the class of a cycle T' which is decomposed as follows: for $i=1$ or $i=3$, let x_i (respectively, y_i) be in the intersection of I_i with $\mathcal{X}$ (respectively, $\mathcal{Y}$). Consider a segment in I_i joining x_i to y_i, and a path in the interior of $\mathcal{X}$ (respectively, $\mathcal{Y}$) joining x_1 to x_3 (respectively, y_1 to y_3). The cycle T' is obtained by the concatenation of these four paths.

Figure 18 illustrates such a situation. Here, P is the cube with the same numbering of facets as in Example 10.4. The 1-cycle of facets is $\mathcal{I}=\{1,2,1',2'\}$, so $\mathcal{J}=\{1,3,1',3'\}$, whereas $\mathcal{X}=3$ and $\mathcal{Y}=3'$.

Now $\mathcal{I}\cup\mathcal{J}=\mathcal{F}$ and $\mathcal{I}\cap\mathcal{J}=\{I_1,I_3\}$. On I_3 and I_1, the intersection of T and T' is exactly one point. In particular, the intersection class of these two cycles in $H_0(I_1\cup I_3,\mathbf{Z})$ cannot be zero. By Theorem 10.1, the class $\psi([T])$ (respectively, $\psi([T'])$) is non-trivial of dimension $|\bar{\mathcal{I}}|+1$ (respectively, $|\bar{\mathcal{J}}|+1$). Still by Theorem 10.1, the cup product $\psi([T])\smile\psi([T'])$ is a non-trivial cohomology class.

This class does not belong to the top-dimensional cohomology group of X, since the top class corresponds to the generator of $\widetilde{H}_{-1}(\varnothing,\mathbf{Z})$. This means that the cohomology ring of X is not isomorphic to that of a connected sum of sphere products, yielding a contradiction. The polytope P has only 1-cycles of facets of length 3.

We now have to show the converse, i.e. if P satisfies $(*)$, then P is obtained from the tetrahedron by vertex cutting. Remark that a polyhedron which is obtained from the tetrahedron by vertex cutting has (at least) two disjoint triangular facets (except if it is the tetrahedron itself).

Assume that P has a triangular face. Then, if P is not itself the tetrahedron, we can perform a flip of type $(3,1)$ along this face so that it disappears. The resulting polytope Q satisfies $(*)$ too, as we cannot have created new 1-cycles of facets. It has one face less than P, and P is obtained from Q by vertex cutting.

Hence, by induction on the number of facets, we just have to show that a polytope having the property $(*)$ has necessarily a triangular face. Consider a polytope P fulfilling $(*)$. If P is not a tetrahedron, it has two disjoint facets and, according to the claim, a 1-cycle of facets (F_1, F_2, F_3) of length 3. Now, the plane H passing through the centers of the intersections $F_i \cap F_j$ intersects no other facet. The intersections P^+ and P^- of P with the two half-spaces delimited by H are simple convex polytopes satisfying $(*)$ and with a triangular face $H \cap P$. If P^+ is P itself, then P has a triangular face. Otherwise, P^+ has strictly less faces than P and, by induction, is obtained from the tetrahedron by vertex cutting. As it cannot be the tetrahedron (because $F_1 \cap F_2 \cap F_3$ is empty), it has two disjoint triangular facets, and in particular one which is disjoint from $H \cap P$. This facet is also a triangular facet of P, which completes the proof. □

In higher dimension, the simple polytopes obtained from the simplex (of the same dimension) by cutting off vertices, still give rise to links whose cohomology ring is isomorphic to that of a connected sum of products of spheres, by Theorem 6.3. Nevertheless, they are not the only ones, and a nice characterization of all the polytopes having this property does not seem to exist. In particular, the results of [26] recalled in Example 0.5 give examples of connected sums of products of spheres which cannot be obtained by Theorem 6.3. We use the notation of Example 0.5. Let $n=10$ and $n_1=...=n_5=2$. Then, the associated link X is diffeomorphic to $\#(5)\mathbf{S}^7 \times \mathbf{S}^{10}$. Since X is 6-connected, it is not diffeomorphic to one of the links obtained by Theorem 6.3: none of them is 3-connected. Moreover, we may construct other examples. To do that, recall that an (even-dimensional) polytope is called *neighbourly* if every subset of cardinality $d/2$ determines a face, and that such a polytope is simplicial (see §2 and [17]). A polytope whose dual is neighbourly is therefore simple and is called a *dual neighbourly* polytope. Here, we will only consider the even-dimensional case.

Proposition 11.8. *Assume that P is dual neighbourly and of even dimension. Then the cohomology ring of X is isomorphic to the one of a connected sum of sphere products.*

Proof. We try to compute the reduced homology groups of $P_{\mathcal{I}}$, for $\mathcal{I}$ proper and nonempty. Recall that this set is homotopy equivalent to the subcomplex of P^* corresponding to the *maximal* subcomplex whose vertices are those related to the facets of $\mathcal{I}$. For $k<d/2-1$, the $(k+1)$-skeleton of $P^*_{\mathcal{I}}$ is complete, by definition of neighbourlyness, hence $P_{\mathcal{I}}$ has trivial reduced k-(co)homology.

The torsion part of $\widetilde{H}_{d/2-1}(P_{\mathcal{I}}, \mathbf{Z})$ is isomorphic to the torsion part of the group $\widetilde{H}^{d/2}(P_{\bar{\mathcal{I}}}, \mathbf{Z})$. By Lemma 7.4 and the Alexander–Pontryagin duality (see [1, vol. 3, p. 53]), it is also isomorphic to the torsion part of the group $\widetilde{H}_{d/2-2}(P_{\bar{\mathcal{I}}}, \mathbf{Z})$, and hence is trivial. In the same way, for $k \geqslant d/2$, the group $\widetilde{H}_k(P_{\mathcal{I}}, \mathbf{Z})$ is isomorphic to the direct sum of the free part of $\widetilde{H}_{d-k-2}(P_{\bar{\mathcal{I}}}, \mathbf{Z})$ and of the torsion part of $\widetilde{H}_{d-k-3}(P_{\bar{\mathcal{I}}}, \mathbf{Z})$, both being trivial.

To sum up, the reduced homology groups of $P_{\mathcal{I}}$ vanish except in dimension $d/2-1$, in which case it is free.

Furthermore, if the homology intersection of two such classes is non-zero, then it must lie in the reduced homology group of dimension -1 of some subset of $\mathcal{F}$, which must be the empty set. Finally, to conclude, we just have to see that the linking number is a unimodular bilinear form on $\widetilde{H}_{d/2-1}(P_{\mathcal{I}}, \mathbf{Z}) \times \widetilde{H}_{d/2-1}(P_{\bar{\mathcal{I}}}, \mathbf{Z})$, which follows from the "little Pontryagin duality" (see [1, vol. 3, p. 91]). This proves the proposition. □

Example 11.9. The (even-dimensional) cyclic polytopes ([17, §4.7]) are examples of neighbourly polytopes. For any d and any $v \geqslant d+1$, there exists a unique cyclic polytope $C(d,v)$ of dimension d with v vertices. Let us take $d=4$. Then $C(4,5)$ is the 4-simplex, while $C(4,6)$ is dual to the product of two triangles. Using the Dehn–Sommerville equations ([17, Chapter 9]), it is easy to check that $C(4,7)$ has 28 faces of dimension 2, and that $C(4,8)$ has 40 such faces. Comparing these numbers with the number of 2-faces of the 6-simplex and of the 7-simplex, this means that in $C(4,7)$ there exist 7 subsets $\mathcal{I}$ such that $P^*_{\mathcal{I}}$ is not contractible but homotopic to a circle, and in $C(4,8)$ there exist 16 such subsets. Using the homology formula of Remark 10.2, Proposition 11.8 and Lemma 0.11, we easily get the following table:

v	5	6	7	8
X	$\mathbf{S}^9$	$\mathbf{S}^5 \times \mathbf{S}^5$	$\#(7)\mathbf{S}^5 \times \mathbf{S}^6$	$\#(16)\mathbf{S}^5 \times \mathbf{S}^7 \#(15)\mathbf{S}^6 \times \mathbf{S}^6$

In the first three cases, the table gives the diffeomorphism type of X; in the third case, this follows from the fact that the same example can be obtained from Example 0.5 (take $n=k=7$ and use Lemma 1.3). On the contrary, it guarantees only the cohomology ring of X in the last case. Notice that this last case can be obtained neither from Theorem 6.3, nor from Example 0.5.

This leads to the following conjecture.

Conjecture. If P is dual neighbourly, then X is actually the connected sum of sphere products (if not a sphere).

Remark 11.10. One difficult step in proving the conjecture is to prove that if P is dual neighbourly, then X has the homotopy type of a connected sum of sphere products. Related to this is the following more general question.

Question. Let X and X' be two links. Assume that they have isomorphic cohomology rings. Are they homotopy equivalent?

We will go back to this question in Part III.

To finish with this part, we have to answer Question A″. Indeed, a link may not only have torsion in (co)homology, but arbitrary torsion.

Theorem 11.11. (Torsion theorem) *The* (*co*)*homology groups of a* 2*-connected link may have arbitrary amount of torsion. More precisely, let* K *be any finite simplicial complex. Let* N *be the number of vertices of* K. *Then, there exists a* 2*-connected link* X *such that* $H^{i+N+1}(X,\mathbf{Z})$ *contains* $\widetilde{H}_i(K,\mathbf{Z})$ *as a direct summand* (*that is* $H^{i+N+1}(X,\mathbf{Z})=\widetilde{H}_i(K,\mathbf{Z})\oplus\ldots$) *for all* $0\leqslant i\leqslant\dim K$.

This is a very surprising result (at least for the authors), since the links are transverse intersections of quadrics with very special properties.(2)

Proof. Let K be a finite simplicial complex. Let $\{1,...,l\}$ be the vertex set of K. Consider the $(l-1)$-simplex and let its set of facets be $\{1,...,l\}$. For every simplex $I=(i_1,...,i_p)$ of maximal dimension of K, cut off the face of the $(l-1)$-simplex numbered $\{1,...,l\}\setminus I$ by a generic hyperplane. We thus obtain a simple convex polytope P. Notice that its number of facets n is the sum of l plus the number f of facets of K. Set $\mathcal{F}=\{1,...,l,l+1,...,l+f\}$. Finally, consider the associated link X.

The crucial remark is stated in Theorem 11.12 below. It describes an explicit realization of an arbitrary finite simplicial complex as a *maximal* subcomplex of a simplicial convex polytope. Since it has its own interest, we state it separately.

Theorem 11.11 then follows from Theorem 11.12 by application of Remark 10.2. □

Theorem 11.12. *Let* K *be an arbitrary finite simplicial complex with vertex set* $\{1,...,l\}$. *Let* P *be the simple polytope with facet set* $\{1,...,l+f\}$ *defined as above. Then,* K *embeds in* P^* *as the maximal subcomplex* $P^*_{\{1,...,l\}}$.

Proof. The complex K embeds naturally in the $(l-1)$-simplex, but not as the *maximal* subcomplex with vertex set $\{1,...,l\}$. Now, cutting off the face of the $(l-1)$-simplex numbered $\{1,...,l\}\setminus I$ by a generic hyperplane is equivalent, through the duality of polytopes, to perform a barycentric subdivision on the face I of its dual. Hence P^* is the simplicial polytope obtained from the $(l-1)$-simplex Δ by performing a barycentric subdivision of all the faces of $\Delta\setminus K$. We thus obtain a simplicial polytope P^* such that K is the *maximal* simplicial subcomplex of P^* with vertex set $\{1,...,l\}$. □

The proof of Theorem 11.11 is constructive. Here is an example.

Example 11.13. (Compare with [21]) Consider the minimal triangulation of the projective plane $\mathbf{P}^2(\mathbf{R})$ illustrated in Figure 19.

(2) On the other hand, this should not be a surprise to readers working on moment-angle complexes.

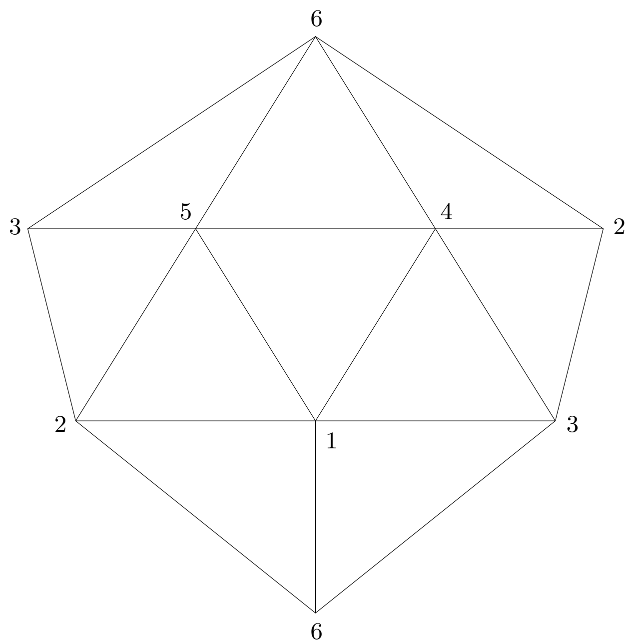

Figure 19.

The simplices of maximal dimensions are

$$\{(356),(456),(246),(235),(145),(125),(134),(234),(126),(136)\}.$$

Consider the 5-simplex and number its facets $\{1,...,6\}$. Cut off the faces of this simplex numbered

$$\{(123),(124),(135),(146),(156),(236),(245),(256),(345),(346)\}$$

by generic hyperplanes. We thus obtain a simple 5-polytope with 16 facets giving rise to a 2-connected link X of dimension 21. Set $\mathcal{F}=\{1,...,16\}$. The complex $P^*_{\{1,...,6\}}$ is homotopic to the projective plane by Theorem 11.12. Then, Remark 10.2 implies that

$$H_8(X,\mathbf{Z})\simeq\bigoplus_{\mathcal{I}\subset\{1,...,16\}}\widetilde{H}_{7-|\bar{\mathcal{I}}|}(P^*_{\mathcal{I}},\mathbf{Z})\simeq\widetilde{H}_1(P^*_{\{1,...,6\}},\mathbf{Z})\oplus...\simeq\widetilde{H}_1(\mathbf{P}^2(\mathbf{R}),\mathbf{Z})\oplus...\simeq\mathbf{Z}_2\oplus...\,.$$

Therefore, not all the homology groups of X are free.

Notice that, due to Corollary 11.1, the dimension of this counterexample is sharp.

Part III. Applications to compact complex manifolds

12. LV-M manifolds and links

We recall very briefly the construction of the LV-M manifolds (see [30] and [31] for more details; this is a generalization of the construction presented in [27]). Let $m>0$ and $n>2m$ be two integers. We identify $\mathbf{C}^m$ and $\mathbf{R}^{2m}$ via the map

$$i: (x_1+iy_1, ..., x_m+iy_m) \in \mathbf{C}^m \longmapsto (x_1, ..., x_m, y_1, ..., y_m) \in \mathbf{R}^{2m},$$

where the i's inside the parentheses stand for the imaginary unit. Let $\Lambda=(\Lambda_1, ..., \Lambda_n)$ be a set of n vectors of $\mathbf{C}^m$ such that

$$A = (A_1, ..., A_n) = (i(\Lambda_1), ..., i(\Lambda_n))$$

satisfies the Siegel condition and the weak hyperbolicity condition, that is such that A is admissible (see Definition 0.2 and Lemma 0.3). Consider the holomorphic foliation $\mathcal{F}$ of the projective space $\mathbf{P}^{n-1}$ given by the following action

$$(T, [z]) \in \mathbf{C}^m \times \mathbf{P}^{n-1} \longmapsto \left[z_1 e^{\langle \Lambda_1, T\rangle}, ..., z_n e^{\langle \Lambda_n, T\rangle}\right] \in \mathbf{P}^{n-1}, \tag{11}$$

where the brackets denote the homogeneous coordinates in $\mathbf{P}^{n-1}$ and where $\langle \cdot, \cdot \rangle$ is the *inner* product of $\mathbf{C}^n$. Define

$$V = \{[z] \in \mathbf{P}^{n-1} : 0 \in \mathcal{H}((A_i)_{i\in I_z})\}, \tag{12}$$

where I_z was defined in (1). We notice that the set I_z is independent of the choice of the representant z of the class $[z]$. Finally, define

$$\widetilde{X}_A = \{[z] \in \mathbf{P}^{n-1} : \textstyle\sum_{i=1}^n A_i |z_i|^2 = 0\}, \tag{13}$$

which is a smooth manifold, due to the weak hyperbolicity condition (see Lemma 0.3).

Then, the following facts are proven in [30] (see also [31]):

(i) the restriction of $\mathcal{F}$ to V is a regular foliation of dimension m;

(ii) the compact smooth submanifold $\widetilde{X}_A$ is a global transversal to $\mathcal{F}$ restricted to V, that is, it cuts every leaf transversally in a unique point.

Therefore, $\widetilde{X}_A$ can be identified with the quotient space of $\mathcal{F}$ restricted to V, and thus inherits a complex structure. We will denote by N_Λ the compact complex manifold obtained in this way. A complex manifold N_Λ for some Λ will be called an *LV-M manifold*. Notice that it has (complex) dimension $n-m-1$.

The main complex properties of these manifolds are investigated in [30], whereas a particularly nice connection with projective toric varieties is explained in [31]. We will not need these results, but we will use the following lemma. Recall that Λ_i (or equivalently A_i) is an *indispensable point* if 0 is not in the convex hull of $(A_j)_{j\neq i}$.

LEMMA 12.1. *Let N_Λ be an LV-M manifold. Assume that Λ has at least $m+1$ indispensable points. Then the complex structure of N_Λ is affine* (*and even linear*), *that is, it may be defined by a holomorphic atlas such that the changes of charts are affine* (*and even linear*) *automorphisms of $\mathbf{C}^{n-m-1}$.*

Proof. Assume that $\Lambda_1, ..., \Lambda_{m+1}$ are indispensable. By (12), this implies that

$$[z] \in V \quad \Longrightarrow \quad z_1 \dots z_{m+1} \neq 0.$$

By construction of N_Λ, we just need to construct a foliated atlas of $(V, \mathcal{F})$ with linear transverse changes of charts. Look at the map $\Phi_z : \mathbf{C}^m \times \mathbf{C}^{n-m-1} \to V$ defined by

$$(T, w) \xrightarrow{\Phi_z} \left[z_1 e^{\langle \Lambda_1, T\rangle}, ..., z_{m+1} e^{\langle \Lambda_{m+1}, T\rangle}, w_1 e^{\langle \Lambda_{m+2}, T\rangle}, ..., w_{n-m-1} e^{\langle \Lambda_n, T\rangle}\right],$$

for a fixed $z=(z_1, ..., z_{m+1}) \in (\mathbf{C}^*)^{m+1}$. Using the weak hyperbolicity condition, it can be shown that the set $(\Lambda_2 - \Lambda_1, ..., \Lambda_{m+1} - \Lambda_1)$ has rank m. As a consequence,

$$\Phi_z(T, w) = \Phi_{z'}(T', w') \quad \Longleftrightarrow \quad w'_i = w_i e^{\langle \Lambda_{m+1+i}, T-T'\rangle} \text{ for all } 1 \leqslant i \leqslant n-m-1,$$

and $T-T'$ belongs to a fixed lattice in $\mathbf{C}^m$. Therefore, Φ_z is a local homeomorphism and can be used as a local foliated chart for every point $(z_1, ..., z_{m+1}, w)$. Since the first $m+1$ homogeneous coordinates of every point in V are non-zero, V can be covered by such charts. Moreover, the previous computation proves that the changes of charts are uniquely determined by translations along a lattice $T \mapsto T+a$, so that the transverse changes of charts have the form

$$w \in \mathbf{C}^{n-m-1} \longmapsto \left(w_1 e^{\langle \Lambda_{m+2}, a\rangle}, ..., w_{n-m-1} e^{\langle \Lambda_n, a\rangle}\right),$$

that is, they are linear. □

To avoid particular cases in the sequel, we add the special case $m=0$: then, there is no action at all, and N is by definition the projective space $\mathbf{P}^{n-1}$.

Let $A \in \mathcal{A}$. The quotient space of X_A by the diagonal action (4) can be identified with $\widetilde{X}_A$. In particular, if X_A is not simply-connected, then, by Lemma 0.10, it is equivariantly diffeomorphic to $X_B \times \mathbf{S}^1$ for some $B \in \mathcal{A}$. It is then easy to check that X_B and $\widetilde{X}_A$ are equivariantly diffeomorphic. On the contrary, when $A \in \mathcal{A}_0$, the manifold $\widetilde{X}_A$ is not a link: for example, think about the case where X_A is diffeomorphic to $\mathbf{S}^3 \times \mathbf{S}^3$ (Example 0.4). Notice that, by (13), every LV-M manifold is diffeomorphic to some $\widetilde{X}_A$, that is to the quotient of an odd-dimensional link by the diagonal action (4); and every LV-M manifold with at least one indispensable point is diffeomorphic to a link.

The following theorem is the motivation for the previous study of the links.

Theorem 12.2. *Let $A\in\mathcal{A}$ with dimensions p and n.*

(i) *If p is odd, that is if X_A is even-dimensional, then X_A admits a complex structure as an LV-M manifold.*

(ii) *If p is even, that is if X_A is odd-dimensional, then $\widetilde{X}_A$ and $X_A\times\mathbf{S}^1$ admit a complex structure as LV-M manifolds.*

Proof. Assume that X_A is odd-dimensional, i.e. that p is even, say $p=2m$. Let Λ denote the preimage $i^{-1}(A)$. Then, by construction, $\widetilde{X}_A$ and N_Λ are diffeomorphic. Therefore, $\widetilde{X}_A$ inherits a complex structure.

If p is odd, define the following matrix with $n+1$ columns and $p+1$ rows:

$$B=\begin{pmatrix} A & 0 \\ 1\,...\,1 & -1 \end{pmatrix}.$$

This is obviously an admissible configuration and, by Lemma 0.10, the links X_B and $X_A\times\mathbf{S}^1$ are equivariantly diffeomorphic. As noticed before, this means that $\widetilde{X}_B$ is diffeomorphic to X_A and we are in the previous case.

Finally, if p is even, consider the following matrix with dimensions $n+2$ and $p+2$:

$$C=\begin{pmatrix} A & 0 & 0 \\ 1\,...\,1 & -1 & 0 \\ 1\,...\,1 & 0 & -1 \end{pmatrix}.$$

Then X_C is equivariantly diffeomorphic to $X_A\times\mathbf{S}^1\times\mathbf{S}^1$, and $\widetilde{X}_C \underset{\text{eq}}{\simeq} X_A\times\mathbf{S}^1$ has a complex structure as an LV-M manifold by what precedes. □

Corollary 12.3. *The product of two links admits a complex structure as an LV-M manifold as soon as it has even dimension.*

Proof. Use Example 0.7 and Theorem 12.2 (i). □

Remark 12.4. Let $A\in\mathcal{A}$ and let $A'\in\mathcal{A}$ be obtained from A by a homotopy which does not break the weak hyperbolicity condition. Then, by Corollary 4.5, the links X_A and $X_{A'}$ are equivariantly diffeomorphic. Nevertheless, the complex structures of X_A and $X_{A'}$ (if p is odd), or of $\widetilde{X}_A$ and $\widetilde{X}_{A'}$ (if p is even), given by Theorem 12.2, are in general not the same; in this way a link X_A, or its diagonal quotient $\widetilde{X}_A$, comes equipped not only with a complex structure, but with a deformation space of complex structures (see [30], where this space is studied).

13. Holomorphic wall-crossing

Let N_Λ be an LV-M manifold. Identifying $\mathbf{C}^m$ with $\mathbf{R}^{2m}$ and Λ with an element of $\mathcal{A}$ via the map i, we may talk of a wall W of Λ (see Definition 5.2) and of a configuration Λ' obtained from Λ by crossing the wall W (Definition 5.3). Up to equivariant diffeomorphism, $N_{\Lambda'}$ is obtained from N_Λ by performing an equivariant smooth surgery described in Theorem 5.4. Indeed, to be more precise, $X_{A'}$ is obtained from X_A by an elementary surgery along some submanifold X_F, and, since everything is equivariant, it is straightforward to check that $\widetilde{X}_{A'}$ is obtained from $\widetilde{X}_A$ by an equivariant surgery of the same type along $\widetilde{X}_F$. Nevertheless, N_Λ and $N_{\Lambda'}$ being complex manifolds, it is natural to ask which *holomorphic* transformation occurs when performing the wall-crossing. This is what we call the *holomorphic wall-crossing problem*.

Remark 13.1. Let $B\in\mathbf{C}^m$ be such that $\Lambda'=\Lambda+B$, that is $\Lambda'=(\Lambda_1+B,...,\Lambda_n+B)$. By Definition 5.3, the configuration $\Lambda+tB$ is admissible for every $t\in[0,1]$, except for one special value t_0. It follows from (11) that N_Λ and $N_{\Lambda+tB}$ are biholomorphic for every $0\leqslant t<t_0$, and that $N_{\Lambda'}$ and $N_{\Lambda+tB}$ are biholomorphic for every $t_0<t\leqslant 1$ (compare with the general case of Remark 12.4). Therefore, the complex structures of the induced links are fixed before and after crossing the wall.

In this section, we will give a complete solution to the holomorphic wall-crossing problem by showing that, in this case, the smooth equivariant surgeries occuring during the wall-crossing are in fact holomorphic surgeries. Let us first recall the following definition.

Definition 13.2. (See [34, p. 15]) Let M be a complex manifold and let S be a holomorphic submanifold of M. Let W be a neighborhood of S. Finally, let $S^*\subset W^*$ be a pair (holomorphic submanifold, complex manifold) such that W^* is a neighborhood of S^*. Given a biholomorphism $f: W\setminus S\to W^*\setminus S^*$, we may construct the well-defined complex manifold M^* by cutting S and pasting S^* by use of f. We say that M^* is obtained from M by a *holomorphic surgery* along (S, W, S^*, W^*, f).

Notice that if f' is smoothly isotopic to f, the result of performing a holomorphic surgery along (S, f') is diffeomorphic but in general not biholomorphic to M^*.

THEOREM 13.3. (Holomorphic wall-crossing theorem) *Let N_Λ be an LV-M manifold. Let $N_{\Lambda'}$ be an LV-M manifold obtained from N_Λ by crossing a wall. Then $N_{\Lambda'}$ is obtained from N_Λ by a holomorphic surgery.*

Proof. Let $\widetilde{X}_F$ be the smooth submanifold of N_Λ along which the elementary surgery occurs. Using §1 and the standard identification of $\mathbf{R}^{2m}$ and $\mathbf{C}^m$, we have that $\widetilde{X}_F$ is the quotient space of the foliation $\mathcal{F}$ restricted to

$$V\cap\{z : z_i=0 \text{ for } i\in I\},$$

for the subset $I\subset\{1, ..., n\}$ numbering F (see (7)). Therefore, it is a holomorphic submanifold of N_Λ corresponding to the admissible subconfiguration $(\Lambda_i)_{i\in I^c}$. By abuse of notation, we still call this complex manifold $\widetilde{X}_F$. On the other hand, we have $V'=V$ and the submanifold $\widetilde{X}'_{F'}$ is the quotient space of $\mathcal{F}'$ restricted to the same $V\cap\{z:z_i=0 \text{ for } i\in I\}$. Define $W=V\setminus\{z:z_i=0 \text{ for } i\in I\}$. As Λ and Λ' differ only by a translation factor, the open complex manifolds $W/\mathcal{F}=N_\Lambda\setminus\widetilde{X}_F$ and $W/\mathcal{F}'=N_{\Lambda'}\setminus\widetilde{X}'_{F'}$ are biholomorphic. More precisely, the identity map of W descends to a biholomorphism f between these two complex manifolds. As a consequence, $N_{\Lambda'}$ is obtained from N_Λ by a holomorphic surgery along $(\widetilde{X}_F, N_\Lambda, \widetilde{X}'_{F'}, N_{\Lambda'}, f)$. □

Remark 13.4. The holomorphic surgery described in the proof of Theorem 13.3 is a very particular case of Definition 13.2, since the neighborhood W of the submanifold $\widetilde{X}_F$ is in fact the whole manifold N_Λ. It is thus a global holomorphic transformation, whereas Definition 13.2 has a local flavour. It is perhaps better to say that N_Λ and $N_{\Lambda'}$ are holomorphic compactifications of the same open complex manifold $N_\Lambda\setminus\widetilde{X}_F=N_{\Lambda'}\setminus\widetilde{X}'_{F'}$.

14. Topology of LV-M manifolds

As an application of Theorem 11.11, we have the following result.

THEOREM 14.1. *The (co)homology groups of a 2-connected LV-M manifold may have arbitrary amount of torsion. More precisely, let K be any finite simplicial complex. Let q be its number of vertices. Then, there exists a 2-connected LV-M manifold N_Λ such that $H^{i+q+1}(N_\Lambda, \mathbf{Z})$ contains $\widetilde{H}_i(K, \mathbf{Z})$ as a direct summand (that is, $H^{i+q+1}(N_\Lambda, \mathbf{Z})=\widetilde{H}_i(K, \mathbf{Z})\oplus...$) for all $0\leqslant i\leqslant \dim K$.*

Proof. Apply Theorem 11.11 to obtain a 2-connected link X with this property. If X is even-dimensional, then we may conclude by Theorem 12.2. Otherwise, we perform a surgery of type $(1, n)$ on $X\times\mathbf{S}^1$. By Proposition 11.2, the resulting 2-connected link X' still has the property. But now X' is even-dimensional and we may conclude again by Theorem 12.2. □

Remark 14.2. As a consequence of a result of [37], every finitely presented group may appear as the fundamental group of a compact complex non-Kählerian 3-fold. The previous theorem is a sort of (much) weaker version of this result for higher dimensional homology groups. Notice that it is not known if a similar statement is true for Kähler manifolds.

Before drawing an interesting consequence of this theorem, we want to go back to the question asked in Remark 11.10. The "holomorphic" version of this question is the following.

Question. Let N and N' be two LV-M manifolds. Assume that they have isomorphic cohomology rings. Are they homotopically equivalent?

In the case of two *simply-connected Kähler* manifolds, there is a partial positive answer to this question: two simply-connected Kähler manifolds with isomorphic cohomology rings have indeed the same *rational* homotopy type (see [13]). Nevertheless, the answer is negative in general. Counterexamples exist even in dimension 2. Consider the open manifold

$$W=\{(w_1,w_2,w_3)\in \mathbf{C}^3\setminus\{(0,0,0)\}: w_1^2+w_2^3+w_3^5=0\}.$$

The quotient space of W by the group generated by a well-chosen weighted homothety is a compact complex surface which is diffeomorphic to $\Sigma\times S^1$, where Σ is the Poincaré sphere (see [8] and [32]). Thinking about the Hopf surfaces, this means that both $S^3\times S^1$ and $\Sigma\times S^1$ admit complex structures. Now, they have isomorphic cohomology rings but different homotopy type (since the Poincaré sphere is not simply-connected).

It seems plausible that the techniques of [13] can be applied to the non-Kähler class of LV-M manifolds and would bring a partial positive answer to the question.(3)

Going back to Theorem 14.1, we easily obtain the following surprising corollary.

Corollary 14.3. *The* (*co*)*homology groups of a* 2*-connected compact complex* affine *manifold may have arbitrary amount of torsion* (*in the sense of Theorem* 14.1).

Proof. By use of Theorem 14.1 and Lemma 12.1, it is enough to prove that, given an LV-M manifold N_Λ of dimensions (m,n), there exists an LV-M manifold $N_{\Lambda'}$ of dimensions (m',n') such that

(i) the manifold $N_{\Lambda'}$ is diffeomorphic to a product of N_Λ with circles;

(ii) the number of indispensable points of $N_{\Lambda'}$ is $m'+1$.

Let L_l be the matrix with $n+2l$ columns

$$\begin{pmatrix} \Lambda_1 & \dots & \Lambda_n & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ -1-i & \dots & -1-i & 1 & i & 0 & 0 & \dots & 0 & 0 \\ -1-i & \dots & -1-i & 0 & 0 & 1 & i & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1-i & \dots & -1-i & 0 & 0 & 0 & 0 & \dots & 1 & i \end{pmatrix},$$

(3) In [14], a preprint which appeared on the `arXiv` after this work was submitted, it is proven that the LV-M manifolds are not formal (see also [35]), so that in fact the techniques of [13] cannot be applied. However, this does not allow one to answer the previous question.

where i stands for the imaginary unit. It is straightforward to check that L_l is admissible, i.e. it has $2l$ indispensable points, and that N_{L_l} is diffeomorphic to $N_\Lambda \times (\mathbf{S}^1)^{2l}$ (see Example 0.7). The equality $m'+1=2l$ is achieved for $l=m+1$. □

This means that it could be quite complicated to classify affine complex manifolds or complex manifolds having a holomorphic affine connection up to diffeomorphism. Notice that an affine compact *Kähler* manifold is covered by a compact complex torus (see [22]).

15. Concluding remarks

We finish this article with some open problems which we find of some interest.

First, the relationship between links and moment-angle complexes coming from simple polytopes leads naturally to the problem of knowing to what extent the results of this paper can be generalized to other classes of moment-angle complexes. The general construction in [9] starts with any finite simplicial complex K. If K is dual to a simple convex polytope P, then the corresponding moment-angle complex $\mathcal{Z}_K$ is exactly $\mathcal{Z}_P$. If the complex is a triangulation of a sphere, it is proven that $\mathcal{Z}_K$ still admits a structure of a smooth manifold. Now, this class is much larger than that of the links, since there exist a lot of sphere triangulations which are not polytopal. Is there a nice realization of these manifolds as intersection of real quadrics? Does Corollary 4.7 remain valid? Is there a generalization of Theorem 4.8? (Recall that Buchstaber and Panov defined a homeomorphic version of equivariant surgeries in the general case.) Finally, do these manifolds admit a complex structure? It should be noted that the class of LV-M manifolds was generalized in [6], using a more combinatorial construction. Does this generalization provide complex structures on non-polytopal moment-angle manifolds?

Secondly, the proof of Corollary 14.3 suggests one to ask the following question.

Question. Let M be a compact complex manifold. Under which assumptions on M does the smooth manifold $M \times (\mathbf{S}^1)^{2N}$ admit a complex affine structure for N sufficiently large? Is it enough to assume that the total real Pontryagin class of M is equal to 1?

We emphasize that the searched complex affine structure on $M \times (\mathbf{S}^1)^{2N}$ does not need to respect M, that is we do not require that M may be embedded as a holomorphic submanifold of $M \times (\mathbf{S}^1)^{2N}$ endowed with its affine complex structure.

Every compact Riemann surface satisfies the conditions of the second part of the question. Since only the elliptic curves admit affine complex structures, the question is interesting and non-trivial even in dimension 1. Every compact complex surface which is spin and has signature zero satisfies the conditions of the second part of the question. Other examples are given by complex manifolds with stably trivial smooth tangent bundle (i.e. such that the Whitney sum of the smooth tangent bundle with a trivial bundle of sufficiently large rank is trivial). Indeed, this is exactly the case for a link X_A endowed with a structure of an LV-M manifold, since it is smoothly embedded in $\mathbf{C}^n$ with trivial normal bundle, so that

$$TX_A \oplus E^{p+1} = T\mathbf{R}^{2n},$$

where TM denotes the tangent bundle of a smooth manifold M, and where E^k denotes the trivial bundle over X_A with fibre $\mathbf{R}^k$.

Notice that the condition on the characteristic classes is necessary. For, if $M \times (\mathbf{S}^1)^{2N}$ admits a complex affine structure, then the total real Chern class of this structure is 1 (see [22]), which implies the same property for the total real Pontryagin class of $M \times (\mathbf{S}^1)^{2N}$. But this class coincide with the total real Pontryagin class of M. In particular, for any $n>1$ and $N \geqslant 0$, the smooth manifold $\mathbf{P}^n \times (\mathbf{S}^1)^{2N}$ does not admit any complex affine structure by computation of its Pontryagin classes (see [33, Example 15.6]).

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