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RESEARCH ARTICLE

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Fundamental buckling of circular plates with elastically restrained edges and resting on concentric rigid ring support

Abstract This work presents the existence of buckling mode switching with respect to the radius of concentric rigid ring support. The buckling mode may not be axisymmetric as previously assumed. In general, the plate may buckle in an axisymmetric mode but when the radius of the ring support becomes small, the plate may buckle in an asymmetric mode. The optimum radius of the concentric rigid ring support for maximum buckling load is also determined. Introducing internal rigid ring support, when placed at an optimal position increases the elastic buckling load capacity by 149.39 percent. The numerical results obtained are in good agreement with the previously published data.

Keywords buckling, circular plate, elastically restrained edge, rigid ring support, mode switching

1 Introduction

Buckling of plates is an important topic in many mechanical, civil, marine and aircraft structures. Their buckling load capacities need to be determined in situations where in-plane compressive of forces act on the plates. In fact, the first theoretical examination of buckling of plates was attributed to Bryan [1]. He presented the buckling analysis for rectangular plates under uniform uniaxial compression using the energy criterion of stability. It should be noted that he was not only the first to analyze the stability of plates, but also the first to apply the energy criterion of stability to the buckling problem of plate. The first event that highlighted the importance of plate buckling was related to a series of tests carried out in the labs of University of London on box beams for Britannia Bridge in 1845 [2]. Many researchers [3–6] have studied the elastic axisymmetric buckling problems of circular and annular plates.

Exact analytical solutions are available for circular plates without internal concentric ring support [7,8]. In particular, circular plates with concentric ring supports find applications in aeronautical (instrument mounting bases for space vehicles), rocket launching pads, aircrafts and naval vessels (instrument mounting bases). Based on the Kirchhoff's theory, the elastic buckling of thin circular plates has been extensively studied by many authors after the pioneering work published by Bryan [1]. Since then, there have been extensive studies on the subject covering various aspects such as different materials, boundary and loading conditions. Also the buckling of circular plates was studied by different authors [9,10]. However, these sources only considered axisymmetric case, which may not lead to the correct buckling load. Introducing an internal ring support may increase the elastic buckling capacity of in-plane loaded circular plates significantly. Laura et al. [6] investigated the elastic buckling problem of the aforesaid type of circular plates, who modeled the plate using the classical thin plate theory. In their study only axisymmetric modes are considered. The assumption of axisymmetric buckling, however, does not necessarily lead to the desired lowest critical load. Therefore, the asymmetric buckling problems of circular and annular plates have also been studied [11–13].

Although the circular symmetry of the problem allows for its significant simplification, many difficulties very often arise due to complexity and uncertainty of boundary conditions. This uncertainty could be due to practical applications where the edge of the plate does not fall into the classical boundary conditions. It is an accepted fact that the condition on a periphery often tends to be in between

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the classical boundary conditions (free, clamped and simply supported) and may correspond more closely to some form of elastic restraints, i.e., translational and rotational restraints [14–17]. In a recent study, Wang and Wang [4] showed that when the ring support has a small radius, the buckling mode takes the asymmetric form. But they have studied only the circular plate with ring support and elastically restrained edge against rotation. Bhaskara Rao and Kameswara Rao [17] studied the buckling of circular plates with an internal elastic ring support and elastically restrained edges against translation and rotation. Wang and Wang [14] showed that the axisymmetric mode assumed by the previous authors might not yield the correct buckling load. In certain cases, an asymmetric mode would yield a lower buckling load. But they have studied only the circular plate with ring support and elastically restrained edge against rotation. The purpose of the present work is to complete the results of the buckling of circular plates with an internal rigid ring support and elastically restrained edges against translation and rotation by including the asymmetric modes, thus correctly determining the buckling loads.

2 Definition of the problem

Consider a thin circular plate of radius R, uniform thickness h, Young's modulus E and Poisson's ratio ν and subjected to a uniform in-plane load, N along its boundary, as shown in Fig. 1. The circular plate is also assumed to be made of linearly elastic, homogeneous and isotropic material. The edge of the circular plate is elastically restrained edge against rotation and translation and supported by an internal rigid ring support, as shown in Fig. 1. The problem here is to determine the elastic critical buckling load of a circular plate with concentric rigid ring support and elastically restrained edge against rotation and translation.

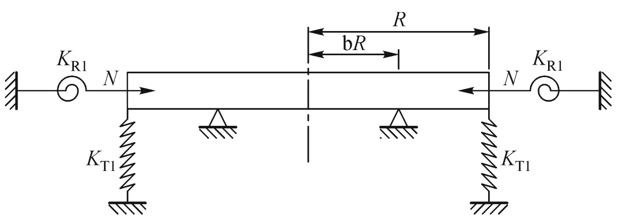

Fig. 1 Buckling of a circular plate with concentric rigid ring support and elastically restrained edge against rotation and translation

3 Formulation of the problem

The plate is elastically restrained against rotation and translation at the edge of radius, R and supported on an internal rigid ring of smaller radius bR as shown in Fig. 1. Let subscript I denote the inner region $0 \leqslant \overline{r} \leqslant b$ and the subscript II denote the outer region $b \leqslant \overline{r} \leqslant 1$. Where r is the normalized radial distance with R. Here, all lengths are normalized by R. Using classical (Kirchhoff's) plate theory, the following fourth order differential equation for buckling in polar coordinates(r,θ).

$$D\nabla^4 w + N\nabla^2 w = 0, \tag{1}$$

where w is the lateral displacement, N is the uniform compressive load at the edge. After normalizing the lengths by the radius of the plate R, Eq. (1) can be written as

$$D\nabla^4 \overline{w} + k^2\nabla^2 \overline{w} = 0, \tag{2}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial \overline{r}^2} + \dfrac{1}{\overline{r}}\dfrac{\partial}{\partial \overline{r}} + \dfrac{1}{\overline{r}^2}\dfrac{\partial^2}{\partial \theta^2}$ is the Laplace operator in the polar coordinates r and θ.

After normalization, the inner and outer radius parameters are b and 1 respectively. $\overline{D} = Eh^3/12(1-\nu^2)$ is the flexural rigidity, $\overline{w} = w/R$, is normalized transverse displacement of the plate. $k^2 = R^2 N/\overline{D}$ is non-dimensional load parameter. Suppose there are n nodal diameters. In polar coordinates (r,θ) set

$$\overline{w}(\overline{r},\theta) = \overline{u}(\overline{r})\cos(n\theta). \tag{3}$$

Considering the boundness at the origin, the general solution [16] for the two regions is

$$\overline{u}_I(r) = C_1 J_n(k\overline{r}) + C_2 Y_n(k\overline{r}) + C_3 \overline{r}^n + C_4 \left\{ \begin{array}{c} \log \overline{r} \\ \overline{r}^{-n} \end{array} \right\}, \tag{4}$$

$$\overline{u}_{II}(r) = C_5 J_n(k\overline{r}) + C_6 \overline{r}^n, \tag{5}$$

where top form of the Eq. (4) is used for $n = 0$ (Axisymmetric) and the bottom form is used for $n \neq 0$ (Asymmetric), C_1, C_2, C_3, C_4, C_5 & C_6 are constants, $J_n(.)$ & $Y_n(.)$ are the Bessel functions of the first and second kind of order, n, respectively. Substituting Eq. (4) into Eq. (3), gives the following

$$\overline{w}_I(\overline{r},\theta) = \left[C_1 J_n(k\overline{r}) + C_2 Y_n(k\overline{r}) + C_3 \overline{r}^n + C_4 \left\{ \begin{array}{c} \log \overline{r} \\ \overline{r}^{-n} \end{array} \right\} \right] \cos(n\theta), \tag{6}$$

$$\overline{w}_{II}(\overline{r},\theta) = [C_5 J_n(k\overline{r}) + C_6 \overline{r}^n]\cos(n\theta). \tag{7}$$

The boundary conditions at outer region of the circular plate in terms of rotational stiffness (K_{R1}) and translational stiffness (K_{T1}) are given by the following expressions

$$M_r(\overline{r},\theta) = K_{R1}\frac{\partial \overline{w}_I(\overline{r},\theta)}{\partial \overline{r}}, \tag{8}$$

$$V_r(\bar{r},\theta) = -K_{T1}\overline{w}_I(\bar{r},\theta). \tag{9}$$

From Eqs. (8) and (9) it can be easily seen that as K_{R1} and K_{T1} become equal to infinity, the slope and deflection become zero and hence corresponds to the case of a clamped end conditions. The translational spring stiffness K_{T1} becomes very important in simulating practical cases such as a guided end by making K_{R1} to be equal to infinity and K_{T1} equal to zero.

The radial moment and the radial Kirchhoff shear at outer edge are defined as follows

$$M_r(\bar{r},\theta) = -\frac{D}{R^3}\left[\frac{\partial^2\overline{w}_I(\bar{r},\theta)}{\partial\bar{r}^2} + \nu\left(\frac{1}{\bar{r}}\frac{\partial\overline{w}_I(\bar{r},\theta)}{\partial\bar{r}} + \frac{1}{\bar{r}^2}\frac{\partial^2\overline{w}_I(\bar{r},\theta)}{\partial\theta^2}\right)\right], \tag{10}$$

$$V_r(\bar{r},\theta) = -\frac{D}{R^3}\left[\frac{\partial}{\partial\bar{r}}\nabla^2\overline{w}_I(\bar{r},\theta) + (1-\nu)\frac{1}{\bar{r}}\cdot\frac{\partial}{\partial\theta}\left(\frac{1}{\bar{r}}\frac{\partial^2\overline{w}_I(\bar{r},\theta)}{\partial\bar{r}\partial\theta} - \frac{1}{\bar{r}^2}\frac{\partial\overline{w}_I(\bar{r},\theta)}{\partial\theta}\right)\right]. \tag{11}$$

Equations (8) and (10) yield the following

$$\left[\frac{\partial^2\overline{w}_I(\bar{r},\theta)}{\partial\bar{r}^2} + \nu\left(\frac{1}{\bar{r}}\frac{\partial\overline{w}_I(\bar{r},\theta)}{\partial\bar{r}} + \frac{1}{\bar{r}^2}\frac{\partial^2\overline{w}_I(\bar{r},\theta)}{\partial\theta^2}\right)\right] = -R_{11}\frac{\partial\overline{w}_I(\bar{r},\theta)}{\partial\bar{r}}. \tag{12}$$

From Eqs. (9) and (11) we get the following

$$\left[\frac{\partial}{\partial\bar{r}}\nabla^2\overline{w}_I(\bar{r},\theta)+(1-\nu)\frac{1}{\bar{r}}\frac{\partial}{\partial\theta}\left(\frac{1}{\bar{r}}\frac{\partial^2\overline{w}_I(\bar{r},\theta)}{\partial\bar{r}\partial\theta} - \frac{1}{\bar{r}^2}\frac{\partial\overline{w}_I(\bar{r},\theta)}{\partial\theta}\right)\right] = T_{11}\overline{w}_I(\bar{r},\theta), \tag{13}$$

where $R_{11} = \dfrac{K_{R1}R^2}{D}$ and $T_{11} = \dfrac{K_{T1}R^3}{D}$.

Apart from the elastically restrained edge against rotation and translation, there is an internal rigid ring support constraint and the continuity requirements of slope and curvature at the support, i.e., at $\bar{r} = b$

$$\overline{w}_I(b,\theta) = 0, \tag{14}$$

$$\overline{w}_{II}(b,\theta) = 0, \tag{15}$$

$$\overline{w}_I'(b,\theta) = \overline{w}_{II}'(b,\theta), \tag{16}$$

$$\overline{w}_I''(b,\theta) = \overline{w}_{II}''(b,\theta). \tag{17}$$

The non-trivial solutions to Eq. (12), (13), (14)–(17) are sought. The lowest value of k is the square root of the normalized buckling load. From Eqs. (4), (5), (12), (13) and (14)–(17) yields the following equations.

$$\begin{aligned}&\left[\frac{k^2}{4}P_2 + \frac{k}{2}(\nu + R_{11})P_1 - \left(\frac{k^2}{2} + \nu n^2\right)J_n(k)\right]C_1\\&+\left[\frac{k^2}{4}Q_2 + \frac{k}{2}(\nu + R_{11})Q_1 - \left(\frac{k^2}{2} + \nu n^2\right)Y_n(k)\right]C_2\\&+[n((n-1)(1-\nu) + R_{11})]C_3\\&+\left\{\begin{array}{l}(\nu + R_{11}) - 1\\ n((n+1)(1-\nu) - R_{11})\end{array}\right\}C_4 = 0,\end{aligned} \tag{18}$$

$$\begin{aligned}&\left[\frac{k^3}{8}P_3 + \frac{k^2}{4}P_2 - \frac{k}{2}\left(\frac{3}{4}k^2 + n^2(2-\nu) + 1\right)P_1\right.\\&\left.+\left(n^2(3-\nu) - \frac{k^2}{2} - T_{11}\right)J_n(k)\right]C_1\\&+\left[\frac{k^3}{8}Q_3 + \frac{k^2}{4}Q_2 - \frac{k}{2}\left(\frac{3}{4}k^2 + n^2(2-\nu) + 1\right)Q_1\right.\\&\left.+\left(n^2(3-\nu) - \frac{k^2}{2} - T_{11}\right)Y_n(k)\right]C_2\\&+\left[n^2(n-1)\nu - n^3 - T_{11}\right]C_3\\&-\left\{\begin{array}{l}n^2(2-\nu)\\ -n^2(n+1)\nu + n^3 - T_{11}\end{array}\right\}C_4 = 0,\end{aligned} \tag{19}$$

where

$$P_1 = J_{n-1}(k) - J_{n+1}(k);\ P_2 = J_{n-2}(k) + J_{n+2}(k);$$

$$P_3 = J_{n-3}(k) - J_{n+3}(k);\ Q_1 = Y_{n-1}(k) - Y_{n+1}(k);$$

$$Q_2 = Y_{n-2}(k) + Y_{n+2}(k);\ Q_3 = Y_{n-3}(k) - Y_{n+3}(k);$$

$$J_n(kb)C_1 + Y_n(kb)C_2 + b^nC_3 + \left\{\begin{array}{l}\log b\\ b^{-n}\end{array}\right\}C_4 = 0, \tag{20}$$

$$J_n(kb)C_5 + b^nC_6 = 0, \tag{21}$$

$$\begin{aligned}&\frac{k}{2}P_1'\,C_1 + \frac{k}{2}Q_1'\,C_2 + nb^{n-1}C_3\\&+\left\{\begin{array}{l}\frac{1}{b}\\ -nb^{-n-1}\end{array}\right\}C_4 - \frac{k}{2}P_1'\,C_5 - nb^{n-1}C_6 = 0,\end{aligned} \tag{22}$$

$$\begin{aligned}&\frac{k^2}{4}(P_2' - 2J_n(kb))C_1 + \frac{k^2}{4}(Q_2' - 2Y_n(kb))C_2\\&+n(n-1)b^{n-2}C_3 - \left\{\begin{array}{l}\frac{1}{b^2}\\ n(n+1)b^{-n-2}\end{array}\right\}C_4\\&-\frac{k^2}{4}(P_2' - 2J_n(kb))C_5 - n(n-1)b^{n-2}C_6 = 0,\end{aligned} \tag{23}$$

where

$$P_1' = J_{n-1}(kb) - J_{n+1}(kb);\ P_2' = J_{n-2}(kb) + J_{n+2}(kb);$$

$$Q_1' = Y_{n-1}(kb) - Y_{n+1}(kb);\ Q_2' = Y_{n-2}(kb) + Y_{n+2}(kb);$$

The coefficient of C_4 has two values corresponding to region I in Eqs. (18)–(23). Therefore, the top form of Eqs. (18)–(23) corresponding to region, I is used for $n = 0$ (axisymmetric buckling) and the bottom form is used for $n \neq 0$ (asymmetric buckling). Therefore, the top form of Eqs. (18)–(23) are used for $n = 0$ (axisymmetric buckling) and the bottom form is used for $n \neq 0$ (asymmetric buckling).

4 Solution

For the given values of n,ν,R_{11},T_{11}, and b the above set of equations, gives an exact characteristic equation for non-trivial solutions of the coefficients C_1,C_2,C_3,C_4,C_5, and C_6. Non-trivial solution, the determinant of $[C]_{6x6}$ vanishes**.** The value of k, calculated from the characteristic equation by a simple root search method. Using Maple, computer software with symbolic capabilities, solves this problem. Here, all equations have been written in the format of the programming, which are compatible with the maple computer software.

5 Results and discussion

Poisson's ratio used in the calculations is 0.3. Buckling load parameters for axisymmetric and asymmetric modes and for various values of rotational and translational spring stiffness parameters are determined. Figures 2–5 show the variations of buckling load parameter k, with respect to the internal rigid ring support radius b, for various values of rotational and translational spring stiffness parameters. It is observed from Figs. 2–5, that for a given value of rotational and translational spring stiffness parameters, the curve is composed of two segments. This is due to the switching of buckling modes. For a smaller internal rigid ring support radius b, the plate buckles in an asymmetric mode (i.e., $n \neq 0$). In this segment (as shown by dotted lines in Figs. 2–5) the buckling load decreases as b decreases in value. For a larger internal rigid ring support radius b, the plate buckles in an axisymmetric mode (i.e., $n = 0$). In this segment (as shown by continuous lines in Figs. 2–5) the buckling load increases as b decreases up to a peak point corresponds to maximum buckling load and thereafter decreases as b decreases in value as shown in Figs. 2–5. The cross over radius i.e., mode switching for

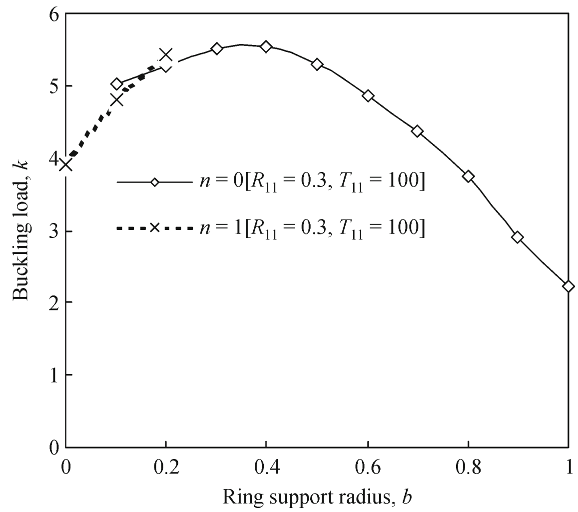

Fig. 2 Buckling load parameter k, versus internal rigid ring support radius b, for $R_{11} = 0.3$ and $T_{11} = 100$

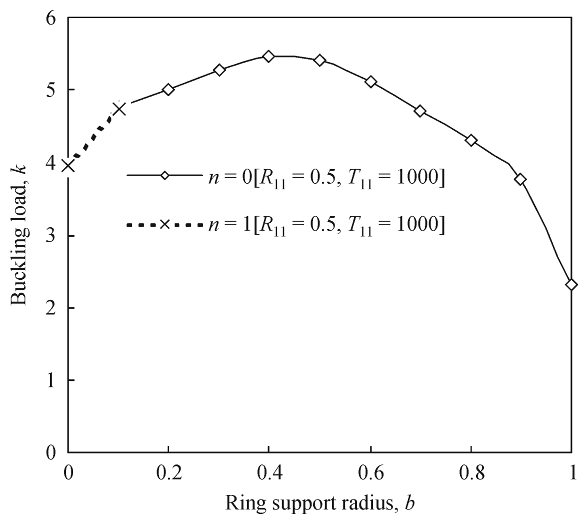

Fig. 3 Buckling load parameter k, versus internal rigid ring support radius b, for $R_{11} = 0.5$ and $T_{11} = 1000$

various values of rotational and translational spring stiffness parameters are presented in Table 1. Here, 10^{16} has been considered as infinity (∞). As R_{11} & $T_{11} \to \infty$, the edge of the plate becomes clamped and as $b \to 1$, buckling solution for axisymmetric case is 3.83163, which is in good agreement with that of Wang and Wang [14].

The optimum location of the ring support and the corresponding buckling load parameters for various rotational spring stiffness parameters ($R_{11} = 0.350$ & ∞)

Table 1 The Cross over radius (switching of. buckling mode) of the rigid ring support for various values of rotational, R_{11} and translational stiffness, T_{11} parameters when $\nu = 0.3$

	$R_{11} = 0.3,\ T_{11} = 100$	$R_{11} = 0.5,\ T_{11} = 1000$	$R_{11} = 50,\ T_{11} = 1000$	$R_{11} = T_{11} = \infty$
b	0.1589	0.1102	0.1557	0.1523

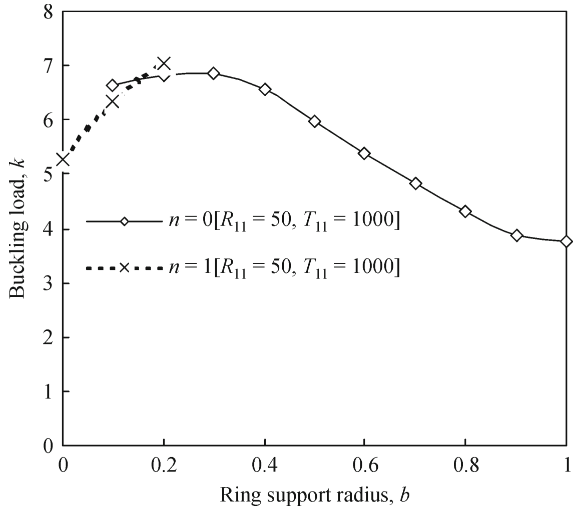

Fig. 4 Buckling load parameter k, versus internal rigid ring support radius b, for $R_{11} = 50$ and $T_{11} = 1000$

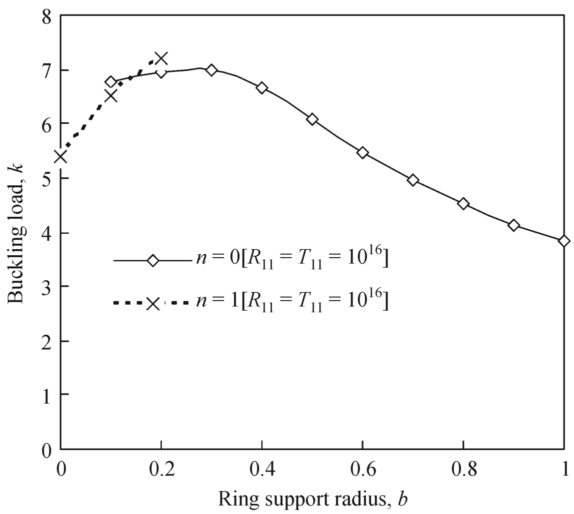

Fig. 5 Buckling load parameter k, versus internal rigid ring support radius b, for $R_{11} = T_{11} = 10^{16}$

and translational spring stiffness parameters ($T_{11} = 1001000$ & ∞) are presented in the Table 2. The optimum location is the location of the ring support at which the buckling load parameter k becomes maximum. It is observed that the optimal rigid ring support radius parameter decrease with increase in rotational spring stiffness parameter and also the optimal buckling load capacity increases with rotational spring stiffness parameter. Introducing internal rigid ring support, when placed at an optimal position increases the elastic buckling capacity significantly, and the percentage of increase in buckling loads is presented in Table 2. The percentage increase in buckling load is calculated by comparing the buckling load parameter value obtained when the circular plate is having the ring support with that of circular plate without having the ring support**.** It is observed that the percentage increase in buckling load parameter decreases with increase in R_{11}. This is due to the amount of increase in buckling load without ring support with R_{11} is more than that of increase in buckling load with rigid ring support with R_{11}.

The results of this kind were scarce in the literature. However, the results are compared with the following cases. Table 3, presents the buckling load parameters k, for a circular plate with an internal rigid ring support and elastically restrained edge against rotation, against those obtained by Wang et al. [4], Laura et al. [6] and Wang et al. [18]. Reference [4] is considered because the plate can be considered as thin for $\tau = h/R = 0.001$. It shows that the present results are in good agreement. The buckling load parameters k, for clamped and simply supported edges are compared with those obtained by Wang et al. [4] and Wang et al. [18] as shown in Tables 3 and 4 respectively. The buckling load parameters for rotational stiffness parameter $R_{11} = \infty$ are shown in Table 5. Also the optimum location of the rigid ring support and the corresponding buckling load parameters for rotational spring stiffness parameter 0.3, 50 and ∞ are calculated by substituting $T_{11} = \infty$ and it is in good agreement with earlier results obtained by Wang [13].

6 Conclusions

The fundamental buckling of thin circular plates with an internal rigid ring support and elastically restrained edge against rotation and translation is presented in this paper**.** It

Table 2 Optimum location of the rigid ring support; the corresponding buckling load parameters and percentage increase in buckling load parameters

R_{11}	0.3	50	∞
T_{11}	100	1000	∞
b_{opt}	0.3211	0.27022	0.2600
k_{opt}	5.53449	6.87832	7.01485
age increase in buckling load/%	149.39	83.11	83.07

Table 3 Comparison of buckling load parameter k, with Laura et al. [6] and Wang et al. [18] for simply supported edge for rotational stiffness parameter $R_{11} = 0$ & $\nu = 0.3$

Rigid ring support radius, b	Wang et al. [4]	Laura et al. [6]	Wang et al. [18]	Present
0.1	–	4.5244	4.5235	4.52341
0.2	4.7703	4.7718	4.7702	4.77018
0.3	5.0711	5.0725	5.0710	5.07091
0.4	5.3297	5.3301	5.3296	5.32964
0.5	5.3667	5.3666	5.3666	5.36659
0.6	5.1264	5.1284	5.1261	5.12606
0.7	4.7730	4.7789	4.7727	4.77266
0.8	4.4219	4.4249	4.4215	4.42141
0.9	4.1069	4.1122	4.1063	4.10629

Table 4 Comparison of buckling load parameter k, with Wang et al. [4] and Wang et al. [18] for clamped edge for rotational stiffness parameter

Rigid ring support radius, b	Wang et al. [4]	Wang et al. [18]	Present
0.1	–	6.5009*	6.50095*
0.2	6.9559	6.9558	6.95582
0.3	6.9948	6.9947	6.99475
0.4	6.6627	6.6625	6.66248
0.5	6.0749	6.0745	6.07454
0.6	5.4760	5.4755	5.4755
0.7	4.9532	4.9526	4.95263
0.8	4.5134	4.5127	4.51266
0.9	4.1448	4.1436	4.14357
0.99	3.8667	3.8604	3.86061

* Asymmetric buckling load parameters

Table 5 Comparison of buckling load parameter k, with Laura et al. [6] for rotational stiffness parameter

Rigid ring support radius, b	Laura et al. [6]	Present
0.1	6.7720	6.50105
0.2	6.9649	6.95592
0.3	6.9964	6.99485
0.4	6.6693	6.66257
0.5	6.0852	6.07454
0.6	5.4845	5.47550
0.7	4.9588	4.95263
0.8	4.5277	4.51266
0.9	4.1509	4.14357

is observed that the buckling mode switches from asymmetric mode to *axisymmetric* mode at a particular rigid ring support radius. The cross over radius (switching of mode) is determined for different values of rotational and translational constraints. The optimal location of the internal rigid ring support for maximum buckling load is also found. The optimal location of internal rigid ring support is affected by the rotational and translational stiffness parameters. Also, it is observed that for $T_{11} = 0$, the symmetric buckling mode is independent of the internal elastic rigid ring support and gives a constant buckling load. The percentage of increase in buckling load capacity by introducing concentric rigid ring support, when it is placed at an optimal position is also determined. These exact solutions can be used to check numerical or approximate results.

Nomenclature

$w(r,\theta)$	Transverse deflection of the plate
h	Thickness of a plate
R	Radius of a plate
b	Non-dimensional radius of ring support
ν	Poisson's ratio
E	Young's modulus of a material
D	Flexural rigidity of a material
K_{R1}	Rotational spring stiffness
K_{T1}	Translational spring stiffness
T_{11}	Non-dimensional translational Flexibility parameter
R_{11}	Non-dimensional rotational Flexibility parameter
N	Uniform in-plane compressive load
k	Non-dimensional Buckling Load Parameter

References

1. Bryan G H. On the stability of a plane plate under thrust in its own plane with application to the buckling of the side of a ship. Proceedings of the London Mathematical Society, 1891, 22(1): 54–67
2. Walker A C. A Brief Review of Plate Buckling Research In Behaviour of Thin-Walled Structures, ed. by Rhodes J, Spence J, Elsevier: London, 1984, 375–397
3. Raju K K, Rao G V. Stability of moderately thick annular plates subjected to a uniform radial compressive load at the outer edge. Computers & Structures, 1989, 33(2): 477–482
4. Wang C M, Xiang Y, Kitipornchai S, Liew K M. Axisymmetric buckling of circular mindlin plates with ring supports. Journal of Structural Engineering, 1993, 119(3): 782–793
5. Liew K M, Xiang Y, Kitipornchai S, Wang C M. Buckling and vibration of annular Mindlin plates with internal concentric ring supports subject to in-plane radial pressure. Journal of Sound and Vibration, 1994, 177(5): 689–707
6. Laura P A A, Gutierrez R H, Sanzi H C, Elvira G. Buckling of circular, solid and annular plates with an intermediate circular support. Ocean Engineering, 2000, 27(7): 749–755
7. Timoshenko S P, Gere J M. Theory of Elastic Stability. New York: McGraw Hill, 1961
8. Szilard R. Theory and Analysis of Plates. NJ: Prentice-Hall, Englewood Cliffs, 1974
9. Wolkowisky J H. Buckling of the circular plate embedded in elastic springs, an application to geophysics. Communications on Pure and Applied Mathematics, 1969, 22(5): 367–667
10. Brushes D O, Almroth B O. Buckling of Bars, Plates and Shells, New York: McGraw-Hill, 1975
11. Gerard C P. Asymmetric vibration and stability of circular plates. Computers & Structures, 1978, 9(1): 89–95
12. Thevendran V, Wang C M. Buckling of annular plates elastically restrained against rotation along edges. Thin-walled Structures, 1996, 25(3): 231–246
13. Wang C Y. Buckling of a circular plate with internal elastic ring support. Mechanics Based Design of Structures and Machines, 2003, 31(1): 93–102
14. Wang C Y, Wang C M. Buckling of circular plates with an internal ring support and elastically restrained edges. Thin-walled Structures, 2001, 39(9): 821–825
15. Kim C S, Dickinson S M. The flexural vibration of the isotropic and polar orthotropic annular and circular plates with elastically restrained peripheries. Journal of Sound and Vibration, 1990, 143 (1): 171–179
16. Yamaki N. Buckling of a Thin annular plate under uniform compression. Journal of Applied Mechanics, 1958, 25: 267–273
17. Bhaskara Rao L, Kameswara Rao C. Buckling analysis of circular plates with elastically restrained edges and resting on internal elastic ring support. Mechanics Based Design of Structures and Machines, 2010, 38(4): 440–452
18. Wang C M, Aung T M. Tun Myint Aung. Buckling of circular mindlin plates with an internal ring support and elastically restrained edge. Journal of Engineering Mechanics, 2005, 131(4): 359–366