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RESEARCH ARTICLE

Giuseppe CARBONE, Marco CECCARELLI

Comparison of indices for stiffness performance evaluation

Abstract This paper addresses the problem of a numerical evaluation of the stiffness performance for multibody robotic systems. An overview is presented with basic formulation concerning indices that are proposed in literature. New indices are also outlined. Stiffness indices are computed and compared for a case study. Results are used for comparing the effectiveness of the stiffness indices. The main goal is to propose a performance index describing synthetically the elastostatic response of a multibody robotic system and also for design purposes.

Keywords robotics, stiffness, performance indices

1 Introduction

Stiffness performance can significantly affect the accuracy, payload, and dynamic characteristics of a multibody robotic system, as pointed out for example in Refs. [1–20]. In fact, inadequate stiffness of links and/or joints may cause large compliant displacements of an end-effector when external forces and moments are acting on the end-effector itself. These compliant displacements detrimentally affect both accuracy and payload performances. Moreover, insufficient stiffness of links and joints may lead to low natural frequencies, yielding longer stabilization (settling) times and reduced dynamic performance as well as larger link excursions and vibration during transient periods. Therefore, an unsuitable stiffness performance can compromise also the dynamic performance of a multibody robotic system especially if inertia forces are high and operation frequencies are close to natural frequencies, as pointed out for example in Ref. [21]. Therefore, stiffness performance should be carefully taken into account at the design stage, as proposed for example in Refs. [1–7,10–12].

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Giuseppe CARBONE (✉), Marco CECCARELLI
LARM, DiMSAT, University of Cassino, Cassino, Italy
E-mail: carbone@unicas.it

In the literature, attention has been addressed to stiffness analysis and numerical evaluation of stiffness performance with different approaches both from the general viewpoint, like for example in Refs. [1–10], and for specific kinematic architectures, like for example in Refs. [7,11,12]. Experimental determinations and evaluations of stiffness performance are presented as necessary tools for validating the mechanical design of a multibody robotic system in Refs. [14–16]. Experimental tests of stiffness performance are also prescribed in standard codes for robotic manipulators having serial architecture, as reported in Refs. [15,16]. Nevertheless, it is still an open issue to formulate a standard “stiffness performance index” that can be computationally efficient, giving direct engineering insight of the design parameter influence, and that can be translated into experimental determinations for validating a design process.

This paper addresses the above-mentioned issue by providing an overview of the existing indices for stiffness performance evaluation. Then, the effectiveness and computational efficiency of each index is compared by means of a suitable case of study to propose a global performance index for the elastostatic response in a multibody robotics system.

2 Cartesian stiffness matrix

Usually the purpose of a stiffness analysis is the definition of the stiffness of the overall system through the derivation of a Cartesian stiffness matrix $\boldsymbol{K}$. This stiffness matrix $\boldsymbol{K}$ expresses the relationship between the compliant displacements $\Delta\boldsymbol{S}$ occurring to a frame fixed at the end of the kinematic chain when a static wrench $\boldsymbol{W}$ acts upon it and $\boldsymbol{W}$ itself. Considering Cartesian reference frames, 6×1 vectors can be defined for compliant displacements $\boldsymbol{S}$ and external wrench $\boldsymbol{W}$ as

$$\Delta\boldsymbol{S} = (\Delta x,\Delta y,\Delta z,\Delta\alpha,\Delta\gamma,\Delta\delta)^{\mathrm{T}},$$

$$\boldsymbol{W} = (F_x,F_y,F_z,T_x,T_y,T_z)^{\mathrm{T}}, \tag{1}$$

where Δx, Δy, and Δz are the variations of Cartesian coordinates for the origin of a moving frame on manipulator extremity and Δx, Δy, and Δz are the variations of angular coordinates for a moving frame; F_x, F_y, and F_z are the force components acting upon a point on the moving frame along the x, y, and z directions, respectively; and T_x, T_y, and T_z are the torque components acting upon the same point on the movable plate about the x, y, and z directions, respectively. Provided that the assumption of small compliant displacements holds, the relationship between the vectors $\Delta \boldsymbol{S}$ and $\boldsymbol{W}$ can be written as

$$\boldsymbol{W} = \boldsymbol{K}\Delta \boldsymbol{S}, \tag{2}$$

where $\boldsymbol{K}$ is the so-called 6×6 Cartesian or spatial stiffness matrix. It is worth noting that according to the definition in Eq. (2) the stiffness matrix $\boldsymbol{K}$ is in general posture dependent. Moreover, the stiffness matrix $\boldsymbol{K}$ is generally non-symmetric and its entries depends on choice of reference frame because it is not reference frame invariant, as demonstrated for example in Refs. [2,8,17,18].

The computation of stiffness matrix $\boldsymbol{K}$ can be achieved with different approaches such as the finite element methods (FEMs) or methods based on models with lumped parameters (MLP). FEM can be used for a stiffness analysis of multibody robotic systems, although with very difficult numerical implementation. In fact, even if FEM methods can be more accurate than MLP methods, they are time consuming and require a complete recalculation at each configuration under analysis. Therefore, the stiffness analysis of multibody robotic systems is usually carried out by means of MLP methods that are based on using lumped stiffness parameters for considering the stiffness properties of links and joints with configuration dependent relationships. Therefore, the main advantages of MLP methods can be understood in reduced computational efforts and possibility to use the same stiffness model for the analysis of several different configurations. These aspects give the possibility to investigate the stiffness performance through the whole workspace of a multibody robotic system in a reasonable amount of computation time. Moreover, MLP methods can be conveniently used for developing parametric models within optimal design procedures while FEM methods are not suitable for this purpose.

The lumped stiffness parameters can be graphically represented as linear or torsion springs, as shown for example in Fig. 1. In particular, Fig. 1(a) shows a model of a 2R serial manipulator with two linear springs and two torsion springs representing the lumped stiffness parameters of links (k_1 and k_2) and motors (k_{T1} and k_{T2}), respectively. Figure 1(b) shows a model of a parallel manipulator with three linear springs representing the lumped stiffness parameters (k_1, k_2, and k_3) of links and linear actuators on each limb. It is worth noting that in Fig. 1 the bold lines show the deformed models after applying a wrench $\boldsymbol{W}$; the dotted curves show the motion of reference points due to compliant displacements.

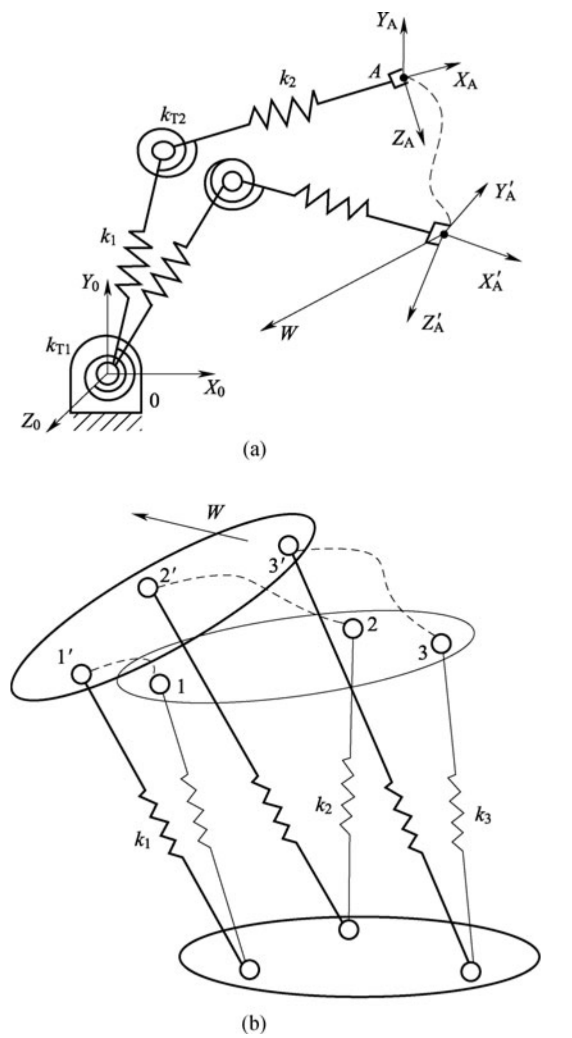

Fig. 1 Schemes of elastically compliant multibody robotic systems. (a) 2R serial manipulator; (b) parallel manipulator with three SPS legs

In general, one can define the relationship between the external wrench $\boldsymbol{W}$ and the forces acting on each single lumped parameter. For example, this relationship can be written for the model in Fig. 1(b) as

$$\boldsymbol{W} = f_1\boldsymbol{s}_1 + f_2\boldsymbol{s}_2 + f_3\boldsymbol{s}_3, \tag{3}$$

where f_i (i = 1,2,3) are the modules of the force acting on each limb; $\boldsymbol{s}_i$ (i = 1,2,3) are the vectors expressing the directions of the limbs. (A similar approach is possible for the model of Fig. 1(a)). In addition, one can write

$$f_i = k_i \Delta \ell_i \quad \text{with} \quad i = 1,2,3, \tag{4}$$

where $\Delta \ell_i$ are the compliant displacements of the i-th limb in the s_i-th direction and k_i is the i-th lumped stiffness parameter. Thus, Eq. (3) can be written as

$$\boldsymbol{W} = (k_1 \Delta \ell_1)\boldsymbol{s}_1 + (k_2 \Delta \ell_2)\boldsymbol{s}_2 + (k_3 \Delta \ell_3)\boldsymbol{s}_3. \tag{5}$$

Then, a differential of Eq. (5) can be written as

$$\delta \boldsymbol{W} = (k_1 \delta \ell_1)\boldsymbol{s}_1 + (k_2 \delta \ell_2)\boldsymbol{s}_2 + (k_3 \delta \ell_3)\boldsymbol{s}_3 + (k_1 \Delta \ell_1)\frac{\mathrm{d}\boldsymbol{s}_1}{\mathrm{d}\theta_1}\delta\theta_1 + (k_2 \Delta \ell_2)\frac{\mathrm{d}\boldsymbol{s}_2}{\mathrm{d}\theta_2}\delta\theta_2 + (k_3 \Delta \ell_3)\frac{\mathrm{d}\boldsymbol{s}_3}{\mathrm{d}\theta_3}\delta\theta_3. \tag{6}$$

If one defines $\boldsymbol{J}$, a Jacobian matrix of the system in Fig. 1(b), one can write the first three terms of Eq. (6) as reported for example in Ref. [2] in the form

$$(k_1 \delta \ell_1)\boldsymbol{s}_1 + (k_2 \delta \ell_2)\boldsymbol{s}_2 + (k_3 \delta \ell_3)\boldsymbol{s}_3 = (\boldsymbol{J}\boldsymbol{K}_\ell \boldsymbol{J}^{\mathrm{T}})\delta \boldsymbol{S}. \tag{7}$$

Therefore, Eq. (6) can be also written in the form

$$\delta \boldsymbol{W} = (\boldsymbol{J}\boldsymbol{K}_\ell \boldsymbol{J}^{\mathrm{T}})\delta \boldsymbol{S} + (k_1 \Delta \ell_1)\frac{\mathrm{d}\boldsymbol{s}_1}{\mathrm{d}\theta_1}\delta\theta_1 + (k_2 \Delta \ell_2)\frac{\mathrm{d}\boldsymbol{s}_2}{\mathrm{d}\theta_2}\delta\theta_2 + (k_3 \Delta \ell_3)\frac{\mathrm{d}\boldsymbol{s}_3}{\mathrm{d}\theta_3}\delta\theta_3, \tag{8}$$

where $\boldsymbol{K}_\ell$ is a diagonal matrix of lumped stiffness parameters, given by

$$\boldsymbol{K}_\ell = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}. \tag{9}$$

It is worth noting that the movable plate in Fig. 1(b) is assumed to be a rigid body. Thus, the distances between the points 1, 2, and 3 should not be modified by the application of any external wrench. As a consequence of this constraint, small compliant displacements on each limb do not necessarily imply negligible angular displacements of the limbs. Thus, the terms

$$(k_i \Delta \ell_i)\frac{\mathrm{d}\boldsymbol{s}_i}{\mathrm{d}\theta_i}\delta\theta_i, \tag{10}$$

with i = 1,2,3 cannot be usually considered as negligible. Moreover, they can add significant non-symmetric terms to the overall stiffness matrix $\boldsymbol{K}$. Nevertheless, for the sake of simplicity several authors consider the terms in Eq. (10) as negligible to compute the stiffness matrix as

$$\boldsymbol{K} = \boldsymbol{J}\boldsymbol{K}_\ell \boldsymbol{J}^{\mathrm{T}}. \tag{11}$$

This approach is widely used in stiffness analysis of multibody robotic systems as mentioned for example in Refs. [1–3,5–7]. In fact, if the Jacobian matrix $\boldsymbol{J}$ is available in closed form, Eq. (11) can provide a closed form equation for the stiffness matrix $\boldsymbol{K}$ that is a function only of the input parameters in terms of input angles and/or strokes.

3 Numerical evaluation of the stiffness matrix

One of the main advantages of using models with lumped parameters instead of using finite element methods for stiffness analysis is that one can compute numerically the stiffness matrix $\boldsymbol{K}$ at any posture assumed by a multibody robotic system with a single model and algorithm.

A numerical algorithm for computing the stiffness matrix $\boldsymbol{K}$ at any posture can be composed of a first part in which the numerical values for geometrical dimensions, masses and lumped stiffness parameters are identified. In a second step one defines a kinematic model, a force transmission model and a lumped parameter model. Then, in a third step these models can be used to evaluate the stiffness matrix $\boldsymbol{K}$.

It is worth noting that the stiffness matrix $\boldsymbol{K}$ is posture dependent. Thus, one should define configuration(s) of a multibody robotic system where the stiffness matrix can be computed. The configuration(s) should be carefully chosen to have significant information on the stiffness performance of the system in its whole workspace. Then, the kinematic model can be used for computing the vector $\boldsymbol{\theta}$ that express input angles and strokes in the joint space for any posture of a multibody robotic system.

In general, a multibody robotic system can have few trajectories that are used during its operation. In these cases, a kinematic model can be used together with a proper path planning strategy for computing a vector $\boldsymbol{\theta}(t)$ that expresses input angles and strokes in the joint space as a function of time for a given trajectory. Thus, the vector $\boldsymbol{\theta}(t)$ can be used for computing the stiffness matrix as a function of time for a given end-effector trajectory. However, the whole path should be split in a given number of configurations by selecting a proper step size and the time in which the motion of the robotic system will be completed. Of course, the smaller the step size the higher the number of configurations in which stiffness matrix $\boldsymbol{K}$ is computed.

It is worth noting that the accuracy in the estimation of model data such as geometrical dimensions, and values of lumped stiffness parameters can significantly affect the accuracy of the computed stiffness matrix. Thus, experimental tests should be carried out to validate model data and overall stiffness model.

Once the stiffness matrix has been derived, it is also necessary to give synthetic evaluation of the stiffness performance both for analysis and design purposes. Thus, an index of merit can be formulated by using properties of the stiffness matrix so that it represents numerically the stiffness performance of a new multibody robotic system.

4 Indices for stiffness performance evaluation

The current standard codes for stiffness evaluation of manipulators are given as short parts of the norms ANSI/RIA15.01.11-1990 [15] and ISO9283-1995 [12], which refer explicitly to serial chain industrial robots only. In particular, Section 8.6 in Ref. [15] and Section 10 in Ref. [16] are devoted to static compliance with a very similar approach but only referring to a performance evaluation through measures of position compliant displacements. Then, a recommendation states to express the results in term of millimeters per Newton for displacements that are referred to the directions of a base coordinate system.

Thus, the standard codes do not yet consider the stiffness matrix as a performance index for the elastostatic response of multibody robotic manipulators; however, they still refer to a practical evaluation with a direct natural interpretation that is related to the compliance response of the stiffness of a manipulator structure. Of course, it is evident that compliant displacements can be considered as a measure of the manipulator stiffness because of the fundamental relationship in Eq. (1). Nevertheless, the compliance response is system posture and wrench direction dependent because one can find a 6×1 vector of compliant displacements at any posture for any wrench. Therefore, one should define a single local index of stiffness performance and then a global index expressing the stiffness performance in the overall workspace of a multibody robotic system.

A local stiffness index can be directly related with the Cartesian stiffness matrix by means of different mathematical operators that can be applied to a matrix. Feasible choices can be the determinant, trace, and norm at a given posture, as proposed for example in Refs. [1–3,5–7,22,23]. In particular, the determinant of a stiffness matrix $\boldsymbol{K}$ is invariant in similarity transformations. Thus, it does not rely on the choice of reference frame. Moreover, it can be computed as

$$\det(\boldsymbol{K}) = (-1)^6 + P_1(-1)^5 + P_2(-1)^4 + P_3(-1)^3 + P_4(-1)^2 + P_5(-1) + P_6, \quad (12)$$

where P_i (with i = 1,2,…,6) is the sum of the principal minors of order i of the matrix $\boldsymbol{K}$. However, the determinant can be expressed also as the product of matrix eigenvalues as given in Matrix Algebra [23]. Each entry K_{ij}^{-1} of the inverse matrix of $\boldsymbol{K}$ can be computed as

$$K_{ij}^{-1} = \frac{(K)_{ji}}{\det(\boldsymbol{K})}, \quad (13)$$

where $(K)_{ji}$ is the algebraic complement of the entry K_{ij} of the matrix $\boldsymbol{K}$ with i, j = 1,2,…,6. Thus, if the determinant det($\boldsymbol{K}$) is zero, the Eq. (13) gives singular values and Eq. (12) cannot be computed. Therefore, the determinant of $\boldsymbol{K}$ can be used as a performance index to investigate synthetically the effect of the design parameters on the stiffness behavior, since it is easy to compute and it is particularly significant for determining stiffness singularity properties.

The trace tr($\boldsymbol{K}$) of $\boldsymbol{K}$ can be expressed as

$$\mathrm{tr}(\boldsymbol{K}) \equiv \sum_{i,j=1}^{n} K_{ij}. \quad (14)$$

The trace can be seen as the sum of the components of compliance displacements along the principal directions. It gives a measure of the compliant displacements per unit of external wrench. Nevertheless, it is worth noting that its components do not have the same dimensions and thus the sum does not have a full physical interpretation. A possible solution to this problem is the definition of a dimensionless or dimensionally consistent stiffness matrix as proposed for example in Refs. [24–26]. However, this would require the definition of a characteristic length L, whose choice is usually questionable but significantly affecting the results.

A norm of a matrix is similar to what an absolute value is for a real number or what a modulus is for a complex number. The norm of $\boldsymbol{K}$ can be also very useful as stiffness index because it provides a measure on how much the stiffness matrix differs from zero. In particular, the norm can be defined in various forms as outlined in the following.

The Euclidean norm, which is also called as the 2-norm, is the square root of the largest (nonnegative) eigenvalue of the positive-semidefinite product of the matrix by its transpose, regardless of the ordering of the factors, as

$$\|\boldsymbol{K}\|_E \equiv \max_i \{\sqrt{\lambda_i}\}, \quad (15)$$

where symbol $\|\ \|$ is the norm operator; $\{\sqrt{\lambda_i}\}$ is the set of nonnegative eigenvalues of $\boldsymbol{KK}^{\mathrm{T}}$. This norm is also called the spectral norm and it is frame invariant. Notice that λ_i is identical to the square of the module of the i-th eigenvalue of $\boldsymbol{K}$ itself. In this case, the norm expresses the spectral radius whose length is related with the maximum eigenvalue which is the value of the stiffness in the stiffest direction. A similar norm can be defined as related with the minimum eigenvalue being the value of the stiffness in the most compliant direction.

The Frobenius norm is the square root of the sum of the squares of the entries of the matrix $\boldsymbol{K}$. It can be expressed in the form

$$\|\boldsymbol{K}\|_F \equiv \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} K_{ij}^2} \equiv \sqrt{\mathrm{tr}(\boldsymbol{KK}^{\mathrm{T}})}. \quad (16)$$

The Chebyshev norm or infinity norm is the maximum absolute value of the entries of the matrix $\boldsymbol{K}$ as

$$\|\boldsymbol{K}\|_{\infty} \equiv \max_{i,j} \{|K_{ij}|\}. \tag{17}$$

The main advantage of this norm is that it does not require floating point operations that can increase the computational costs and roundoff errors.

The p-norm is a general definition of a norm. It can be written as

$$\|\boldsymbol{K}\|_p \equiv \left(\sum_{j=1}^{n}\sum_{i=1}^{n}|K_{ij}|p\right)^{1/p}, \tag{18}$$

which becomes the Frobenius norm for p = 2 or the Chebyshev norm for p going to infinity, respectively.

The condition number has been also proposed as a potential candidate for a local index of stiffness performance. In fact, the condition number of a stiffness matrix $\boldsymbol{K}$ can be computed as

$$\kappa(\boldsymbol{K}) = \|\boldsymbol{K}\|\|\boldsymbol{K}^{-1}\|. \tag{19}$$

If one refers to Euclidian norm, Eq. (18) can be written as

$$\kappa_E(\boldsymbol{K}) \equiv \sqrt{\frac{\lambda_l}{\lambda_s}}, \tag{20}$$

where λ_l and λ_s are the larger and smaller eigenvalue of $\boldsymbol{K}\boldsymbol{K}^T$. It is worth noting that due to Eq. (20) the minimum value of the condition number is one and it tends to ∞ when the matrix $\boldsymbol{K}$ is singular.

It is worth noting that if one considers the simplified expression Eq. (11) one can define performance indices also by using the Jacobian matrix to characterize the stiffness performance. A significant example is the manipulability index that has been defined as the square root of the determinant of the product of the manipulator Jacobian by its transpose [25]. Other useful indices are reported for example in Refs. [26–28] such as alternative expressions for kinematic performance that can be related to stiffness through Jacobian.

Compliant displacements can also provide an insight on local stiffness performance due to their simple physical interpretation, as indeed suggested by ISO and ANSI codes [15,16]. In fact, one can compute the compliant displacements for a given configuration by multiplying the computed stiffness matrix for a given external wrench $\boldsymbol{W}_G$. Reasonable choices for $\boldsymbol{W}_G$ can be a unit vector or a vector equal to the expected payload for a multibody robotic system as proposed for example in Ref. [7]. The first choice gives a measure of the compliant displacements per unit of external wrench. The second choice provides a measure of the maximum compliant displacements for the system in specific applications. Nevertheless, compliant displacements have usually six components. Thus, they cannot be treated as a single merit index.

Eigenvalues and eigenvectors of a stiffness matrix are also very useful for their physical interpretation with respect to local stiffness performance. In fact, the eigenvectors are related with the maximum and minimum eigenvalue and they provide the directions of maximum and minimum stiffness performance, respectively. Moreover, a smaller difference among the eigenvalues stands for a smaller anisotropic stiffness behavior at a given posture. Nevertheless, eigenvalues and eigenvectors cannot be treated as a single merit index. However, their values can be used for drawing graphical local representations of the stiffness performance such as compliance/stiffness ellipses and ellipsoids, as reported for example in Refs. [1–3,5]. These graphical local representations also provide a graphical tool for the comparison of stiffness performance along and about different directions. These graphical representations can be very useful when specific design requirements arise. In particular, they are useful if there is a requirement of the best stiffness performance only in a given direction or if equal stiffness is preferred in all directions.

Other graphical tools for a comparison of stiffness performance can be obtained through the definition of the so-called center-of-stiffness or the center-of compliance and by means of stiffness or compliant axes that can be used for defining directions and orientations in which a robotic system acts as a simple spring, as mentioned for example in Ref. [24].

A local index of stiffness performance is neither suitable for an accurate design analysis nor useful for a comparison of different designs. In fact, even if a multibody robotic system has suitable stiffness for a given system posture it can have inadequate stiffness at other postures. Therefore, one should look at stiffness performance at all points of the workspace or define a single global stiffness index over the whole workspace.

A global index of stiffness performance for a multibody robotic system can be defined by means of graphical methods that are based on plotting curves connecting postures having the same value of the local stiffness index (ISO-stiffness curves or surfaces), as proposed for example in Refs. [3,5]. Nevertheless, the number of ISO-stiffness curves or surfaces that one can plot is graphically limited. Moreover, few curves or surfaces usually do not provide sufficient insight of the overall stiffness behavior of a multibody robotic system. These aspects significantly reduce the effectiveness of ISO-stiffness curves or surfaces.

Global stiffness indices can be defined also in a mathematical form by using minimum, maximum, average or statistic evaluations of a local stiffness index. For example, one can compute a global index in the form

$$GI_d = \min|\det(\boldsymbol{K})|. \tag{21}$$

It is worth noting that a GI_d index equal to zero means that at least one singular configuration is within the workspace of a multibody robotic system. This is a critical situation that should be avoided at the design stage.

A global performance index similar to Eq. (21) can be obtained by referring to local indices in Eqs. (14)–(20). In addition, mean and standard dispersion of the local indices in Eqs. (14)–(20) or of the compliant displacements can be considered as merit indices since they provide a useful insight of the global stiffness performance of a multibody robotic system with a direct physical interpretation.

Another possible approach for defining a global stiffness index is based on something similar to the magnitude of a vector. In particular, one can compute the integral of a certain power of a local stiffness index in Eqs. (12)–(20). For example, if a design requirement is to have an isotropic behavior in terms of stiffness performance a new useful global stiffness index can be obtained as the integral of the stiffness condition number over the whole workspace in the form

$$GI_{\mathrm{C}} = \frac{\int \kappa^*(\boldsymbol{K})\mathrm{d}V}{L^3}, \tag{22}$$

where V is the workspace volume; L is a characteristic length used to obtain information that is independent from the workspace volume. Alternatively to L^3, the denominator can be expressed as the volume V of the workspace. Moreover, the dimensional inconsistency can be solved by using a proper dimensionless value of the merit index (which is indicated with a superscript *) that can be obtained by dividing the length entries by a characteristic length L.

Another interesting new global stiffness index can be obtained from Eq. (15) in the form

$$GI_{\mathrm{MN}} = \frac{\int \max_{i=1,\dots,6}\{\sqrt{\lambda_i^*}\}\mathrm{d}V}{L^3}. \tag{23}$$

This global index can be useful when the design goal is to maximize the stiffness performance along or about one or more specific direction(s). A similar new global stiffness index can be defined by referring to the minimum eigenvalue as

$$GI_{\mathrm{mN}} = \frac{\int \min_{i=1,\dots,6}\{\sqrt{\lambda_i^*}\}\mathrm{d}V}{L^3}. \tag{24}$$

This global stiffness index can be useful to detect and avoid a design with weak stiffness performance along or about a specific direction. A global index can be defined also as the difference between GI_{MN} and GI_{mN} as

$$GI_{\mathrm{RN}} = \frac{\int \max_{i=1,\dots,6}\{\sqrt{\lambda_i^*}\}\mathrm{d}V - \int \min_{i=1,\dots,6}\{\sqrt{\lambda_i^*}\}\mathrm{d}V}{L^3}, \tag{25}$$

which can be considered as a global index providing information on the range of stiffness performance within the workspace. Equations (22)–(25) can be considered new synthetic evaluation measures of stiffness performance in multibody robotic systems.

5 Numerical example

A numerical example is reported as referring to CaPaMan 2bis in order to compute and compare some of the indices of stiffness performance that are proposed in Eqs. (12)–(24) for a real multibody robotic system.

CaPaMan 2bis is a parallel manipulator that has been designed and built at LARM in Cassino [29], Fig. 2. A kinematic scheme of CaPaMan 2bis is shown in Fig. 2(a), where the fixed platform is *FP* and the moving platform is *MP*. *MP* is connected to *FP* through three identical leg mechanisms and is driven by the corresponding articulation points. An articulated parallelogram *AP*, a revolute joint *RJ* and a connecting bar *CB* compose each leg mechanism. *AP*'s coupler carries the *RJ* and *CB* transmits the motion from *AP* to *MP* through *RJ*; *CB* is connected to

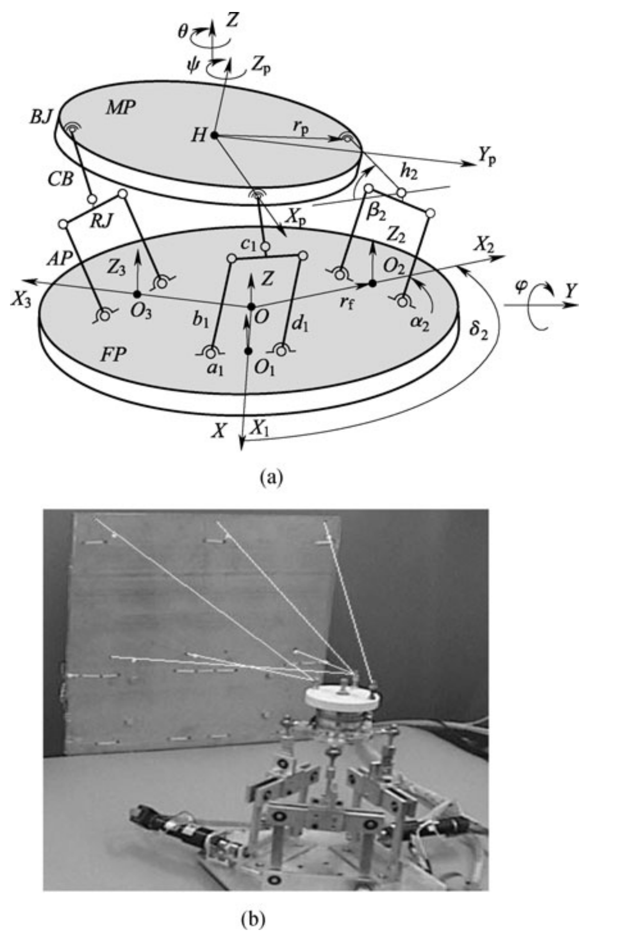

Fig. 2 CaPaMan 2bis. (a) Kinematic scheme; (b) prototype with Milli-CaTraSys set up at LARM

the *MP* by a spherical joint *BJ*, which is installed on *MP*. Each plane, which contains *AP*, is rotated of π/3 with respect to the neighbor one. Design parameters of a k-th leg are identified through: a_k, which is the length of the frame link; b_k, which is the length of the input crank; c_k, which is the length of the coupler link; d_k, which is the length of the follower crank; h_k, which is the length of the connecting bar. Kinematic input variables are the crank angles α_k $(k = 1,2,3)$. Sizes of *MP* and *FP* are given by r_p and r_f, respectively. Other sizes of design parameters are reported in Table 1.

The stiffness matrix of CaPaMan 2bis has been numerically computed by means of models with lumped parameters as reported in Ref. [7]. In particular, a lumped parameter has been used for modeling the axial compliance of the links of the articulated parallelograms *AP* and a torsional spring has modeled the compliance of the actuators and corresponding bearings.

Table 1 Sizes of design parameters for CaPaMan 2bis.

$a_k = c_k$/mm	$b_k = d_k$/mm	h_k/mm	$r_P = r_f$/mm	$\alpha_k/(°)$
100	100	50	65	45 : 135

Then, the stiffness performance has been experimentally measured by means of Milli-CaTraSys as reported in Refs. [7,14]. The experimental results confirm the numerical computations that are reported in the following. For example, when the three legs of CaPaMan 2bis are in the vertical configuration the stiffness matrix is given by

$$\boldsymbol{K} = 10^8 \begin{bmatrix} 0.018 & 0.008 & -0.027 & -1.17 & 0.031 & 0.339 \\ 0.000 & 0.005 & 0.015 & -0.930 & 0.001 & -0.272 \\ 0.000 & 0.006 & 0.019 & -1.141 & -0.001 & 0.333 \\ 0.000 & -0.000 & 0.000 & 0.022 & 0.000 & -0.006 \\ 0.000 & -0.000 & 0.001 & 0.042 & 0.000 & -0.001 \\ -0.000 & -0.000 & 0.000 & -0.029 & 0.000 & 0.008 \end{bmatrix}. \tag{26}$$

When a wrench is obtained by using additional masses of 50 g on each wire once in tension, similarly, when the three legs of CaPaMan 2bis are inclined 45°, the stiffness matrix is given by

$$\boldsymbol{K} = 10^7 \begin{bmatrix} 0.013 & 0.002 & 0.040 & -0.013 & 0.031 & 3.058 \\ 0.003 & 0.001 & 0.010 & -2.595 & 0.012 & 0.782 \\ 0.000 & 0.000 & 0.001 & -0.093 & -0.013 & 0.028 \\ 0.000 & 0.000 & 0.000 & -0.027 & 0.000 & 0.008 \\ -0.000 & -0.000 & -0.001 & 0.195 & -0.001 & -0.059 \\ 0.000 & 0.000 & 0.000 & -0.051 & 0.000 & 0.015 \end{bmatrix}. \tag{27}$$

The stiffness matrix of CaPaMan 2bis can be computed also when two legs are inclined 60°, and one leg is in vertical configuration in the form

$$\boldsymbol{K} = 10^5 \begin{bmatrix} 0.014 & -0.094 & 0.268 & 7.416 & 4.410 & -2.051 \\ 0.009 & -0.174 & 0.606 & 1.740 & 13.99 & -5.299 \\ 0.008 & -0.138 & 0.500 & 20.09 & 7.603 & -5.923 \\ -0.000 & 0.000 & -0.000 & 0.003 & 0.007 & -0.000 \\ -0.000 & 0.000 & -0.001 & 0.169 & -0.026 & -0.050 \\ -0.000 & 0.000 & -0.000 & -0.307 & -0.126 & 0.092 \end{bmatrix}. \tag{28}$$

The stiffness matrices in Eqs. (26)–(28) have been used for computing the local stiffness indices in Eqs. (12)–(20). The obtained results are listed in Table 2.

It is worth noting that for all the local indices in Table 2 (with the exception of the condition number) a higher value means a stiffer behavior. In fact, the closer is the robot configuration to the vertical the better is the stiffness performance as physically expected. The only exception is

Table 2 Evaluation of local stiffness indices for CaPaMan 2bis

local stiffness index	posture		
	0/(°)	45/(°)	60/(°)
determinant	2.9920e + 034	˜6.8292e + 028	˜8.8743e + 021
trace	5790000	580000	80900
Frobenius norm	1.9622e + 008	3.3120e + 007	2.5677e + 006
Chebyshev norm	333980000	39500000	2972510
maximum eigenvalue	2.4831e + 006	2.8513e + 005	4.1461e + 004
minimum eigenvalue	1.1278e + 004	5.1926e + 003	117.7264
condition number	2.6095e + 005	4.2335e + 005	3.5142e + 006

the condition number. In fact, a lower condition number means a better conditioned stiffness matrix. The global indices in Eqs. (21)–(25) have been also computed as listed in Table 3. It is worth noting that GI_d or GI_c indices should be selected as global stiffness indices when a specific application requires higher attention to global properties. Instead, GI_{MN} and GI_{mN} can be selected as global indices when a specific application requires higher attention to local properties.

For example, a system that is very stiff in most part of its workspace but very weak in one configuration cannot be used as a machine tool. In this case, it can be useful to compute GI_{mN} and GI_{MN}. Moreover, one should note that for GI_d, GI_{MN}, and GI_{mN} the higher the value the better the stiffness performance. Instead, for GI_C, and GI_{RN} the lower the value the better the stiffness performance.

Table 3 Evaluation of global stiffness indices for CaPaMan 2bis

global stiffness index	value
GI_d	8.8743e + 021
GI_c	1.3995e + 006
GI_{MN}	771.1297
GI_{mN}	63.0357
GI_{RN}	708.0940

6 Conclusions

This paper addresses the problem of a numerical evaluation of the stiffness performance of multibody robotic systems in general terms. In particular, we have overviewed the possibilities for local merit indices of stiffness performance. Then, synthetic measures of global stiffness performance are proposed for characterizing numerically the overall stiffness performance of a multibody robotic system even by proposing the new expressions in Eqs. (22)–(25). The reported case of study compares the stiffness indices showing feasibility of their numerical computation and effectiveness of their practical use.

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