Front. Math.
https://doi.org/10.1007/s11464-023-0014-0

Published online: June 30, 2024

RESEARCH ARTICLE

Fixed Points of Involutions of G-Higgs Bundle Moduli Spaces over a Compact Riemann Surface with Classical Complex Structure Group

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Abstract Let X be a compact and connected Riemann surface of genus $g \geq 2$. In this paper, moduli spaces of Higgs bundles over X with structure group $\mathrm{SL}(n,\mathbb{C})$, for $n \geq 3$, and $\mathrm{Spin}(2n,\mathbb{C})$, for $n \geq 4$, are considered. In the case of structure group $\mathrm{SL}(n,\mathbb{C})$, two involutions of the Higgs bundle moduli space are defined and an alternative proof is given to show, using specific properties of the structure group, that the stable fixed points can be described as certain Higgs pairs with structure group $\mathrm{SO}(n,\mathbb{C})$ or $\mathrm{Sp}(n,\mathbb{C})$. Moreover, the notions of stability, semistability, and polystability of the obtained Higgs pairs are established. Also, two involutions of the moduli space of Higgs bundles with structure group $\mathrm{Spin}(2n,\mathbb{C})$ are defined for $n \geq 4$ analogously and it is shown, also using specific properties of the group, that their stable fixed points can be described as certain Higgs pairs whose structure group is of the form $\mathrm{Spin}(2r+1,\mathbb{C}) \times \mathrm{Spin}(2n-2r-1,\mathbb{C})$ for some $0 \leq r \leq n-1$.

Keywords Higgs bundle, Higgs pair, spin, automorphism, involution, fixed point

MSC2020 14H60, 14H10

1 Introduction

Let X be a compact and connected Riemann surface of genus $g \geq 2$. For any complex reductive Lie group G with Lie algebra $\mathfrak{g}$, a G-Higgs bundle over X is defined as a pair (E, φ) where E is a principal G-bundle over X and φ is a holomorphic global section of $E(\mathfrak{g}) \otimes K$ called Higgs field, $E(\mathfrak{g})$ being the

Received February 2, 2023; accepted February 17, 2024
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adjoint bundle of E and K being the canonical line bundle over X. Higgs bundles were first introduced by Hitchin [15] as solutions of the Yang–Mills self-duality equations, which are pairs composed by a holomorphic structure on an differentiable vector bundle $\mathcal{E}$ of rank r and degree d and the Higgs field φ, which is understood as an element of $\Omega^{1,0}(X, \operatorname{End}\mathcal{E})$. The notion of G-Higgs bundle presented here is a generalization to structure groups other than the linear group made as a result of the alternative construction by Simpson [17], for which good references are the works of Bradlow, García-Prada, and Gothen [9], Hausel's dissertation [14], or the collection of works made by Anderson, Hausel, Mazzeo, and Schaposnik [1], in which different lines of research concerning the interactions among algebraic geometry, topology, and the mathematical physics of these objects are discussed.

There are several approaches to the study of the geometry of the moduli spaces of G-Higgs bundles, such as the study of their topology and their stratifications [5, 13], or their relation to the representations of the fundamental group of the Riemann surface X [9]. Among these, a particularly fruitful line of research consists in the study of the automorphisms of the moduli space of G-Higgs bundles and the fixed point subvarieties of these automorphisms. Baraglia [8] studied the group of automorphisms of the moduli of Higgs bundles whose principal bundles are identified with rank n and degree d vector bundles over X. Other works have deepened the study of the group of automorphisms and subvarieties of fixed points of the moduli of G-Higgs bundles for particular structure groups G, such as $\operatorname{Spin}(8, \mathbb{C})$ [3] or the exceptional complex group E_6 [6]. Given any semisimple complex Lie group G and an outer automorphism σ of G, σ induces an automorphism of the moduli space of G-Higgs bundles by mapping $(E, \varphi) \mapsto (\sigma(E), ds \otimes 1_K(\varphi))$, where (E, φ) is a polystable G-Higgs bundle, $s \in \operatorname{Aut}(G)$ is a representative of σ, and $\sigma(E)$ is a principal G-bundle whose total space is the same of that of E but on which the action of G has been modified by s [3]. If σ is an outer automorphism of G of order 2, this action of σ and the new automorphism defined by the combination of the action of σ with a change of sign on the Higgs field define two nontrivial involutions of the moduli space of G-Higgs bundles. García-Prada and Ramanan [12] proved that a fixed point of each of these involutions admits a reduction of structure group to the subgroup G^s of fixed points of some order 2 representative s of σ in $\operatorname{Aut}(G)$. This result, together with the description of the subgroups of fixed points of finite order automorphisms of simple complex Lie groups given by Wolf and Gray [19], concludes a generic description of the fixed points of a certain family of involutions of the moduli space of G-Higgs bundles. In [6] a proof of an analogous result is given for the particular case of E_6-Higgs bundles, using specific techniques. In this paper the same problem is addressed for the described involutions in the particular cases of the structure groups $\operatorname{SL}(n, \mathbb{C})$, for $n \geq 3$, and $\operatorname{Spin}(2n, \mathbb{C})$, for $n \geq 4$, which are the only classical and simply connected complex Lie groups admitting outer automorphisms and, particularly, outer automorphisms of order 2. Namely, explicit constructions of the stable fixed points of the described involutions are provided (Theorems 3.1 and 4.1) and the reduced notions of stability, semistability, and polystability of the obtained fixed points are detailed (Proposition 3.2) for the $\mathrm{SL}(n,\mathbb{C})$ structure group case, which is the case in which the Higgs pairs appearing as fixed points have notions of stability and polystability not reducible to those of Higgs bundles or principal bundles. The main novelty is that the constructions provided are alternatives to the proof of García-Prada and Ramanan [12] of the general case for any semisimple complex structure group, and use is made of the specific characteristics of the Lie groups analyzed.

To describe the stable fixed points of interest it is necessary to appeal to the concept of Higgs pair over the compact and connected Riemann surface X of genus $g \geq 2$. Given a complex reductive Lie group G and a complex representation $\rho : G \to \mathrm{GL}(V)$ of G, a (G,ρ)-Higgs pair is a pair (E,φ) where E is a principal G-bundle over X and φ is a holomorphic global section of the vector bundle over X of typical fiber V induced by E and ρ, tensored by K [11, 16]. These objects clearly extend the notion of G-Higgs bundle, which results to be a Higgs pair associated to the adjoint representation of the group G. García-Prada, Gothen, and Mundet [11] provided convenient notions of stability, semistability, and polystability appropriate to construct the moduli space of these objects, which are recalled here for semisimple Lie groups, both in their original version (Definition 2.1) and in their equivalent version in terms of filtrations (Proposition 2.1), with the aim of adapting them to the Higgs pairs that are obtained as fixed points and of finding, in this way, reduced notions for these particular pairs.

The paper is organized as follows. Section 2 establishes the notions of stability of Higgs pairs following [11] and recalls the action of $\mathrm{Out}(G)$ on the moduli space of G-Higgs bundles over X for a semisimple Lie group G, with the aim of defining the involutions to be considered. Then the stable fixed points of the defined involutions of the moduli spaces of Higgs bundles with $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{Spin}(2n,\mathbb{C})$ structure group are constructed in Sections 3 and 4, respectively, and the notions of stability and polystability of the obtained Higgs pairs are determined.

2 Higgs Bundles and Higgs Pairs over a Compact Riemann Surface

Let X be a compact and connected Riemann surface of genus $g \geq 2$, G be a complex semisimple Lie group, and $\rho : G \to \mathrm{GL}(V)$ be a complex representation of G. Given any principal G-bundle E over X, the representation ρ induces a complex holomorphic vector bundle $E(V)$ defined as a quotient of $E \times V$ where any element $(e,v) \in E \times V$ is identified with all the elements of type $(eg, \rho(g^{-1})(v))$ for $g \in G$, and whose rank is the complex dimension of V. Then a (G,ρ)-Higgs pair over X is defined as a pair (E,φ) where E is a principal

G-bundle over X and φ is a holomorphic global section of the vector bundle $E(V) \otimes K$, with K being the canonical line bundle over X. The notion of (G, ρ)-Higgs bundle generalizes the notion of G-Higgs bundle by considering any representation of G instead of the adjoint representation, which is the one that recovers the notion of G-Higgs bundle.

Although the main object of interest of this paper is G-Higgs bundles, the notions of stability and polystability for (G, ρ)-Higgs pairs will be addressed following [11], since some of the fixed points to be obtained will be described in terms of Higgs pairs. The set of conjugacy classes of parabolic subgroups of G is in bijective correspondence with the set of parts of the set Δ of roots of the Lie algebra $\mathfrak{g}$ of G. Let H be a choice of a maximal compact connected Lie subgroup of G, with Lie algebra $\mathfrak{h}$, and $\mathfrak{c}$ be the Cartan subalgebra of $\mathfrak{g}$. If P is a parabolic subgroup of G which corresponds to certain subset A of Δ, an antidominant character of P is defined as an element $\chi \in \mathfrak{c}^*$ that can be written in the form

$$\chi = \sum_{\delta \in A} \frac{2n_\delta}{(\delta, \delta)} \delta,$$

where $n_\delta \in \mathbb{R}$, $n_\delta \leq 0$ for all $\delta \in A$ and $(,)$ denotes the Killing form. An antidominant character χ of P induces an element $c_\chi \in \mathfrak{c}$ which belongs to $i\mathfrak{h}$ through the Killing form.

Given a complex representation $\rho : G \to \mathrm{GL}(V)$ of G, choices of a parabolic subgroup P of G, a Levi subgroup L of P, and an antidominant character χ of P induce the following subspaces of V, which are invariant under the actions of P and L, respectively, by ρ [11, Lemmas 2.5 and 2.6]:

$$\begin{aligned} V_\chi^- &= \{v \in V : \rho(e^{xc_\chi})v \text{ is bounded as } x \to \infty\}, \\ V_\chi^0 &= \{v \in V : \rho(e^{xc_\chi})v = v, \ \forall x\}. \end{aligned} \tag{2.1}$$

The notions of stability, semistability, and polystability of (G, ρ)-Higgs pairs over X are now introduced following [11, Definition 2.9].

Definition 2.1. Let G be a complex semisimple Lie group, $\rho : G \to \mathrm{GL}(V)$ be a complex representation of G and (E, φ) be a (G, ρ)-Higgs pair over X. The (G, ρ)-Higgs pair (E, φ) is stable (resp., semistable) if $\deg \chi_* E_P > 0$ (resp., $\deg \chi_* E_P \geq 0$) for every parabolic subgroup P of G, every antidominant character χ of P and every reduction of structure group E_P of E to P such that $\varphi \in H^0(X, E_P(V_\chi^-) \otimes L)$, where the space V_χ^- is defined in (2.1) and $\chi_* E_P$ is the push-forward of E_P by χ. It is polystable if it is semistable and for every parabolic subgroup P of G, every antidominant character χ of P, and every reduction of structure group E_P of E to P such that $\varphi \in H^0(X, E_P(V_\chi^-) \otimes K)$ and $\deg \chi_* E_P = 0$, there exists a reduction of structure group E_L of E_P to a Levi subgroup L of P such that $\varphi \in H^0(X, E_L(V_\chi^0) \otimes L)$, where the space V_χ^0 is defined in (2.1).

Equivalent notions of stability, semistability, and polystability in terms of filtrations when the representation ρ is falthful are recalled here following [11, Lemma 2.12] and [7, Proposition 2.1].

Proposition 2.1. *Let G be a complex semisimple Lie group, $\rho : G \to \mathrm{GL}(V)$ be a faithful complex representation of G and (E, φ) be a (G, ρ)-Higgs pair over X. Suppose that there exists a representation $\rho_G : G \to \mathrm{GL}(W)$, with $W \cong \mathbb{C}^n$ for some $n \in \mathbb{N}$, such that for any $a, b \in (\mathrm{Ker}\, d\rho_G)^\perp$ one has that $\langle a, b\rangle = \mathrm{Tr}\, d\rho_G(a) d\rho_G(b)$, where the product is the Euclidean product of W. Denote $E' = E(W)$. Then*

1. *The (G, ρ)-Higgs pair (E, φ) is stable (resp., semistable) if for every parabolic subgroup P of G, every antidominant character χ of P, and every filtration $E_0 = 0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_k = E'$ induced by a reduction of structure group of E to P and such that φ takes values, in each fiber over X, in the space V_χ^- defined in (2.1), it is satisfied that the degree of the filtration, defined by*

$$\lambda_k \deg E + \sum_{j=1}^{k-1} (\lambda_j - \lambda_{j+1}) \deg E_j, \tag{2.2}$$

is greater than 0 (resp., greater than or equal to 0), where $\lambda_1 < \cdots < \lambda_k$ are the eigenvalues of $d\rho(c_\chi)$.

2. *The (G, ρ)-Higgs pair (E, φ) is polystable if it is semistable and there exists a parabolic subgroup P of G and an antidominant character χ of P such that E' admits a decomposition of the form $E' = \bigoplus_{j=1}^k E_j/E_{j-1}$ into vector subbundles where $E_0 = 0$ and E_j/E_{j-1} is the eigenspace of $d\rho(c_\chi)$ with eigenvalue λ_j for all $j = 1, \ldots, k$, the degree defined in (2.2) equals 0 and φ takes values, in each fiber over X, in the space V_χ^0 defined in (2.1).*

Remark. What is proved in [11, Lemma 2.12] is that, under the assumptions of Proposition 2.1, the degree considered in Definition 2.1 coincides with the degree defined in (2.2).

Given a complex semisimple Lie group G, the group $\mathrm{Out}(G)$ of outer automorphisms of G acts on the set of isomorphism classes of principal G-bundles over X according to the following action [2–4, 12]: if E is a principal G-bundle over X, σ is an outer automorphism of G, and s is an automorphism of G that represents σ, then $\sigma(E)$ is the principal G-bundle over X whose total space coincides with that of E and such that the action of G in it is defined by

$$e \cdot g = es^{-1}(g) \tag{2.3}$$

for $g \in G$ and $e \in \sigma(E)$. From this action, a new action of σ on the moduli space of G-Higgs bundles is defined: if (E, φ) is a G-Higgs bundle over X,

$\sigma(E,\varphi) = (\sigma(E), ds \otimes 1_K(\varphi))$. In this way, any outer automorphism of G induces an automorphism of the moduli space of G-Higgs bundles over X, whose order coincides with the order of the outer automorphism. Thus an outer involution of G induces an involution of the moduli space of G-Higgs bundles. The combination of these automorphisms with the change of sign on the Higgs field defines a new involution of the moduli space.

In [12, Proposition 3.9] it is proved that the fixed points of an involution such as those described above, defined from an outer involution σ of G, admit a reduction of structure group to the subgroup G^s of fixed points of one of the order 2 representatives of σ, $s \in \mathrm{Aut}(G)$. In [2, 3], the following equivalence relation (which will be called inner equivalence relation) in the group $\mathrm{Aut}(G)$ is defined: two automorphisms, s and t, of G are inner equivalent if there exists an inner automorphism i of G such that $t = i \circ s \circ i^{-1}$. Observe that, if s and t are inner equivelent, then they represent the same element in $\mathrm{Out}(G)$ [2] and, if they are inner equivalent by way of an inner automorphism i, then the subgroups of fixed points G^s and G^t of s and t, respectively, are conjugated, hence isomorphic, through the restriction to them of the inner automorphism i. This means that only representatives of an outer automorphism σ which are not inner equivalent are essentially different. In this sense, the description provided by [19, Theorem 5.10] of the equivalence classes of the inner equivalence relation and its corresponding subgroups of fixed points of finite order automorphisms of simple complex Lie groups, together with the description of fixed points given in [12, Proposition 3.9] completes a description of fixed points of the involutions of the moduli space of G-Higgs bundles induced by outer involutions of G for a simple complex Lie group G. In what follows, an alternative construction of these fixed points for classical simple groups admitting outer involutions is provided.

3 Involutions of $\mathrm{SL}(n,\mathbb{C})$-Higgs Bundles

For $n \geq 3$, the special linear group $\mathrm{SL}(n,\mathbb{C})$ is given by determinant 1 $n \times n$ complex matrices and is the simply connected complex Lie group whose Lie algebra is the algebra $\mathfrak{a}_{n-1} = \mathfrak{sl}(n,\mathbb{C})$ of trace 0 $n \times n$ matrices over the complex numbers and whose center is the finite group of n-th roots of unity. An $\mathrm{SL}(n,\mathbb{C})$-Higgs bundle over X can be understood as a pair (E,φ), where E is a rank n and trivial determinant complex vector bundle over X and $\varphi : E \to E \otimes K$ is a holomorphic homomorphism. With this notation, a subbundle F of E is said to be φ-invariant if $\varphi(F) \subset F \otimes K$. The notions of stable and polystable Higgs bundles were established by Simpson [17] as a generalization of that of vector bundles:

Proposition 3.1. *Let (E,φ) be an $\mathrm{SL}(n,\mathbb{C})$-Higgs bundle over X. Then (E,φ) is stable (resp., semistable) if $\deg F < 0$ (resp., $\deg F \leq 0$) for every proper φ-*

invariant subbundle F of E. It is polystable if E can be written as a direct sum of degree 0 proper φ-invariant subbundles of E.

The group of outer automorphisms of $\mathrm{SL}(n,\mathbb{C})$ is isomorphic to $\mathbb{Z}/(2)$. Then $\mathrm{SL}(n,\mathbb{C})$ admits only one nontrivial outer automorphism σ which is an involution. The action of σ on the moduli space of $\mathrm{SL}(n,\mathbb{C})$-Higgs bundles over the curve X and the composition of this action with the action which consists of a change of sign on the Higgs field define the following order 2 automorphisms of the moduli space of $\mathrm{SL}(n,\mathbb{C})$-Higgs bundles over X [12]:

$$F_+(E,\varphi) = (E^*,\varphi^t), \tag{3.1}$$
$$F_-(E,\varphi) = (E^*,-\varphi^t). \tag{3.2}$$

Theorem 3.1. *Let (E,φ) be a stable $\mathrm{SL}(n,\mathbb{C})$-Higgs bundle over X for $n \geq 3$. If (E,φ) is fixed by the automorphism F_+ defined in (3.1) (resp., by the automorphism F_- defined in (3.2)), then E admits a reduction of structure group E_0 to $\mathrm{SO}(n,\mathbb{C})$ or to $\mathrm{Sp}(n,\mathbb{C})$ such that φ takes values in $\mathrm{Sym}^2 E_0 \otimes K$ (resp., in $\wedge^2 E_0 \otimes K$).*

Proof. If (E,φ) is stable and fixed by F_+ (resp., F_-, then there exists an isomorphism $f : E \to E^*$ of vector bundles such that $(f \otimes 1_K) \circ \varphi = \pm\varphi^t \circ f$, where 1_K denotes the identity on the canonical line bundle K, $+$ corresponds to F_+ and $-$ corresponds to F_-. Consider the automorphism of Higgs bundles $(f^{-1})^t \circ f : E \to E$. Since (E,φ) is stable, it is simple, so there exists an n-th root of unity λ such than $(f^{-1})^t \circ f = \lambda 1_E$. By transposing and inversing the last expression, one has that $(f^t)^{-1} \circ f = \lambda^{-1} 1_E$. Since $(f^{-1})^t \circ f = (f^t)^{-1} \circ f$, it follows that $\lambda = \lambda^{-1}$, so $\lambda = \pm 1$. Observe that $\lambda = -1$ can only occur if n is even, since $\lambda^n = 1$.

Suppose first that $\lambda = 1$. By choosing a point $x \in X$, it is clear that the restriction f_x of f to the fiber E_x defines an isomorphism $E_x \cong E_x^*$ such that $f_x = f_x^t$, so it defines a nondegenerate symmetric bilinear form by $\langle v, w\rangle = f_x(v)(w)$, for $v, w \in E_x$. This then gives a globally defined nondegenerate symmetric bilinear form on E, so a reduction of structure group E_0 of E to $\mathrm{SO}(n,\mathbb{C})$.

For $\lambda = -1$ (in this case it must be assumed that n is even), it is analogously proved that f defines a globally defined symplectic form on E, since in this case $f^t = -f$, so a reduction of structure group E_0 to $\mathrm{Sp}(n,\mathbb{C})$ is defined.

Now, since $(f \otimes 1_K) \circ \varphi = \pm\varphi^t \circ f$, if $\langle , \rangle$ denotes the orthogonal or symplextic form with which E is equipped and $v, w \in E$, then

$$\langle \varphi(v), \varphi(w)\rangle = f \otimes 1_K(\varphi(v))(\varphi(w)) = \pm\varphi^t(f(v))(\varphi(w)) = \pm f(v)(w) = \pm \langle v, w\rangle,$$

so φ takes values in $\mathrm{Sym}^2 E_0$ if the considered automorphism is F_+ and in $\wedge^2 E_0$ if the automorphism is F_-. □

The fixed points obtained in Theorem 3.1 can be described in terms of special orthogonal or symplectic Higgs pairs over X, which are precisely the subgroups of fixed points of all possible not inner-equivalent involutions of $\mathrm{SL}(n,\mathbb{C})$ representing its nontrivial outer automorphism [19, Theorem 5.10]. Let G be the group $\mathrm{SO}(n,\mathbb{C})$ or $\mathrm{Sp}(n,\mathbb{C})$ for any $n \geq 3$ (the symplectic group is only considered if n is even). The following faithful complex representations of G are defined:

$$\rho_{G,S} = \mathrm{Sym}^2 \iota_{n,G}, \tag{3.3}$$

$$\rho_{G,\wedge} = \wedge^2 \iota_{n,G}, \tag{3.4}$$

where $\iota_{n,G} : G \to \mathrm{GL}(n,\mathbb{C})$ denotes the obvious inclusion of groups. Observe that any given $(\mathrm{SO}(n,\mathbb{C}), \rho_{\mathrm{SO},S})$-Higgs pair (resp., $(\mathrm{SO}(n,\mathbb{C}), \rho_{\mathrm{SO},\wedge})$-Higgs pair) over X can be understood as a holomorphic rank n orthogonal vector bundle E over X together with a holomorphic global section of the vector bundle $\mathrm{Sym}^2 E \otimes K$ (resp., $\wedge^2 E^r \otimes K$). The same is true for $\mathrm{Sp}(n,\mathbb{C})$, with n even, substituting that E is orthogonal for symplectic. In the next result, the reduced stability, semistability, and polystability conditions for these types of orthogonal and symplectic Higgs pairs over X are obtained.

Proposition 3.2. *Let G be the complex Lie group $\mathrm{SO}(n,\mathbb{C})$ or $\mathrm{Sp}(n,\mathbb{C})$ (the latter only when n is even) and $\rho_{G,S}$ (resp., $\rho_{G,\wedge}$) be the representation of G defined in (3.3) (resp., in (3.4)). Let (E,φ) be a $(G,\rho_{G,S})$-Higgs pair (resp., $(G,\rho_{G,\wedge})$-Higgs pair) over X. Then (E,φ) is semistable if $\deg F \leq 0$ for every isotropic subbundle F of E satisfying that φ takes values in $(F \otimes_S E) \otimes K$ (resp., $(F \wedge E) \otimes K$) (the stability condition requires $\deg F < 0$ in the same circumstances). The pair (E,φ) is polystable if E can be expressed as a direct sum of degree 0 vector subbundles of the form*

$$E = E_1 \oplus E_2/E_1 \oplus \cdots \oplus E_m^{\perp}/E_m \oplus \cdots \oplus E/E_1^{\perp},$$

where $E_1, \ldots, E_m$ are isotropic and φ takes values in the subbundle

$$\left(\bigoplus_{j=1}^{m} E_j/E_{j-1} \otimes_S E_{j-1}^{\perp}/E_j^{\perp} \right) \otimes K$$

of $\mathrm{Sym}^2 E \otimes K$ (resp., in the subbundle $(\bigoplus_{j=1}^{m} E_j/E_{j-1} \wedge E_{j-1}^{\perp}/E_j^{\perp}) \otimes K$ of $\wedge^2 E \otimes K$), where $E_0 = 0$ and $E_0^{\perp} = E$.

Proof. If F is a proper isotropic subbundle of E, then, given an antidominant character χ of the parabolic subgroup of G corresponding to the choice of this subbundle F, the induced element c_χ diagonalizes in the form

$$\begin{pmatrix} \lambda I_F & & \\ & 0 I_{F^{\perp}/F} & \\ & & -\lambda I_{E/F^{\perp}} \end{pmatrix}$$

for some $\lambda < 0$ and, if $\mathbf{v} = v_1 \otimes_S v_2$ is a generic element of E expressed with eigenvectors of c_χ, then

$$\rho_{G,S}(e^{xc_\chi})(\mathbf{v}) = e^{x(l_1+l_2-l_1'-l_2')\lambda}\mathbf{v},$$

where l_k is the number of v_k's which belong to F and l_k' is the number of v_k's which belong to $E/F^\perp$ (analogous expressions are obtained for $\rho_{G,\wedge}$). The degree of the induced filtration $0 \subset F \subset F^\perp \subset E$ defined in (2.2) is $\lambda \deg F$. The condition of semistability then requires $\lambda \deg F \geq 0$ (that is, $\deg F \leq 0$ whenever $l_1 + l_2 \geq l_1' + l_2'$, that is, whenever φ takes values in a subbundle of $\operatorname{Sym}^2 E \otimes K$ (resp., $\wedge^2 E \otimes K$) such that the number of factors which are copies of F is greater than or equal to the number of factors which are copies of $E/F^\perp$ (for stability, $\deg F < 0$ is required). The proof has been done for $\rho_{G,S}$, but of course the same proof works for $\rho_{G,\wedge}$. Observe that the conditions obtained have considered only filtrations coming from reductions of E to maximal parabolic subgroups of G, which are choices of proper isotropic subbundles. In the following it will be shown that the conditions obtained from reductions to maximal parabolic subgroups are sufficient to check stability and semistability, because the conditions established in the proposition for filtrations induced by reductions to any parabolic are covered by the conditions arising from the consideration of filtrations induced by reductions to maximal parabolics.

Observe that a reduction of structure group of E to a parabolic subgroup P of G corresponds to a choice of a filtration of E into subbundles of the form

$$0 \subset E_1 \subset \cdots \subset E_m \subseteq E_m^\perp \subset \cdots \subset E_1^\perp \subset E.$$

An antidominant character χ of P then diagonalizes in the form

$$\begin{pmatrix} \lambda_1 I_{E_1} & & & & & & & \\ & \lambda_2 I_{E_2/E_1} & & & & & & \\ & & \ddots & & & & & \\ & & & \lambda_m I_{E_m/E_{m-1}} & & & & \\ & & & & 0 \cdot I_{E_m^\perp/E_m} & & & \\ & & & & & -\lambda_m I_{E_{m-1}^\perp/E_m^\perp} & & \\ & & & & & & \ddots & \\ & & & & & & & -\lambda_1 I_{E/E_1^\perp} \end{pmatrix}$$

in terms of the differentiable decomposition $E_1 \oplus E_2/E_1 \oplus \cdots \oplus E_m/E_{m-1} \oplus E_m^\perp/E_m \oplus \cdots \oplus E/E_1^\perp$ of E, where $\lambda_i < 0$ for $i = 1, \ldots, m$. Then the degree expressed in (2.2) turns to demand

$$\sum_{i=1}^{m} (\lambda_i - \lambda_{i+1})(\deg E_i + \deg E_i^\perp) \geq 0,$$

that is, $\deg E_i \leq 0$ for all i for (E, φ) to be semistable. Now, if $e_1, \dots, e_{2n}$ is a basis of E given in terms of the differentiable decomposition of E induced by the considered filtration, then

$$\rho_{G,S}(e^{xc_\chi})(e_i \otimes_S e_j) = e^{2x\sum_{i=1}^m (l_i - l_i')\lambda}(e_i \otimes_S e_j),$$

where l_i and l_i' are the number of vectors of the elements e_i and e_j, respectively, which belong to E_i. Of course, the expression $e^{2\sum_{i=1}^m (l_i - l_i')\lambda x}$ is bounded as $x \to \infty$ if and only if $l_i \geq l_i'$ for all i, since $\lambda_i < 0$. Therefore, (E, φ) is semistable if $\deg E_i \leq 0$ for every isotropic subbundle E_i of E of every filtration into isotropic subbundles such that φ takes values in the subbundle $E_i \otimes E$ of $\operatorname{Sym}^2 E$, by Proposition 2.1. The stability condition requires $\deg E_i < 0$ whenever φ takes values in $E_i \otimes E$. Then, of course, the satisfaction of this condition in filtrations of the form $0 \subset F \subseteq F^\perp \subset \overline{E}$, which correspond to reductions to a maximal parabolic subgroup, leads to the satisfaction of it in all other filtration corresponding to reductions to any parabolic subgroup of G. Again, $\rho_{G,S}$ has been considered, but the same works for $\rho_{G,\wedge}$.

To check the reduced polystability condition observe that φ takes values in the space V_χ^0 defined in (2.1) if and only if it takes values in the subbundle $(\bigoplus_{j=1}^m E_j/E_{j-1} \otimes_S E_{j-1}^\perp/E_j^\perp) \otimes K$ of $\operatorname{Sym}^2 E \otimes K$ or in the subbundle $(\bigoplus_{j=1}^m E_j/E_{j-1} \wedge E_{j-1}^\perp/E_j^\perp) \otimes K$ of $\wedge^2 E \otimes K$, where it is denoted by $E_0 = 0$ and $E_0^\perp = E$. This together with the fact that a reduction of structure group of E to a Levi subgroup of P corresponds to a holomorphic decomposition

$$E = E_1 \oplus E_2/E_1 \oplus \cdots \oplus E_m/E_{m-1} \oplus E_m^\perp/E_m \oplus \cdots \oplus \overline{E}/E_1^\perp$$

concludes the result. □

4 Involutions of Spin$(2n, \mathbb{C})$-Higgs Bundles

Let $n \geq 2$ be any integer number. The group $\operatorname{Spin}(2n, \mathbb{C})$ is the simply connected complex Lie group of type D_n, that is, whose Lie algebra is $\mathfrak{so}(2n, \mathbb{C})$. Spin groups can also be considered with odd rank, $\operatorname{Spin}(2m+1, \mathbb{C})$, which are also simple and simply connected Lie groups, but these groups are not of type D_n but of type B_m for $m \geq 2$, that is, their Lie algebra is $\mathfrak{so}(2m+1, \mathbb{C})$. In any case, $\operatorname{Spin}(r, \mathbb{C})$ is the universal cover of the special orthogonal group $\operatorname{SO}(r, \mathbb{C})$ through a 2-to-1 covering map

$$\pi : \operatorname{Spin}(r, \mathbb{C}) \to \operatorname{SO}(r, \mathbb{C}), \tag{4.1}$$

whose kernel is $\mathbb{Z}/(2)$, for any $r \geq 4$. The center of $\operatorname{Spin}(r, \mathbb{C})$ is isomorphic to $\mathbb{Z}/(2)$ if r is odd, to $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$ if r is multiple of 4, and to $\mathbb{Z}/(4)$ if r is even but not multiple of 4 [18].

A simple construction of Spin groups can be provided from Clifford algebras [2, 10]. If V is a rank r complex vector space equipped with a quadratic form

q, the Clifford algebra $\mathrm{Cl}(V)$ associated to V and q is the quotient of the tensor algebra of V by the subalgebra $J(V)$, generated by elements of the form $v \otimes v - q(v)$ for $v \in V$. It decomposes as the direct sum of the subalgebras of elements with an even number of factors, $\mathrm{Cl}(V)_0$, and those with an odd number of factors, $\mathrm{Cl}(V)_1$. The Clifford algebra admits an automorphism $*$ which acts by $(v_1 \otimes \cdots \otimes v_r)^* = (-1)^r v_r \otimes \cdots \otimes v_1$. For every $x \in \mathrm{Cl}(V)_0$, the norm of x is defined as $N(x) = xx^*$ and is a character on $\mathrm{Cl}(V)_0$. Then the group $\mathrm{Pin}(V)$, called also Clifford group, is defined as the subset of $\mathrm{Cl}(V)$ of those invertible elements x such that $xx^* = 1$ and $xVx^{-1} \subset V$. The group $\mathrm{Spin}(V)$ is the intersection of $\mathrm{Pin}(V)$ with $\mathrm{Cl}(V)$. The homomorphism of the Pin group onto $\mathrm{SO}(r,\mathbb{C})$ has $\mathbb{C}^*$ as kernel, from which it is easily checked that the obstruction to an $\mathrm{SO}(r,\mathbb{C})$-bundle to be liftable to a $\mathrm{Pin}(V)$-bundle is an element of $H^2(X, \mathcal{O}^*) = 0$, so any special orthogonal bundle can be lifted to a Clifford bundle. The Picard group of line bundles acts on the set of isomorphism classes of $\mathrm{Pin}(V)$-bundles via the action of $\mathbb{C}^*$ on $\mathrm{Pin}(V)$. The lift M of an $\mathrm{SO}(r,\mathbb{C})$-bundle is determined upto the action of a line bundle L on it (call this action $L \circ M$). Then $N(L \circ M) = N(M) \otimes L^2$, so the degree of the lifted bundle is defined modulo 2. Therefore, a principal $\mathrm{SO}(r,\mathbb{C})$ bundle lifts to a principal $\mathrm{Spin}(r,\mathbb{C})$-bundle if and only if the lifted $\mathrm{Pin}(V)$-bundle has norm of even degree. In this case, all Spin structures on it are obtained from one of them by the action of order 2 elements of the Picard group. For this reason, from the short exact sequence of groups $1 \to \mathbb{Z}/(2) \to \mathrm{Spin}(r,\mathbb{C}) \to \mathrm{SO}(r,\mathbb{C}) \to 1$ it follows that, although every principal $\mathrm{Spin}(r,\mathbb{C})$-bundle over X gives obviously a principal $\mathrm{SO}(r,\mathbb{C})$ through the covering map π defined in (4.1), the obstruction for an $\mathrm{SO}(r,\mathbb{C})$-bundle to be lifted to a $\mathrm{Spin}(r,\mathbb{C})$-bundle is an element of $H^1(X,\mathbb{Z}/(2)) \cong \mathbb{Z}/(2)$.

A $\mathrm{Spin}(r,\mathbb{C})$ over the curve X is then a pair (E,φ), where E is a principal $\mathrm{Spin}(r,\mathbb{C})$-bundle over X and φ is a holomorphic global section of the vector bundle $E(\mathfrak{so}(r,\mathbb{C})) \otimes K$ defined by the adjoint representation of $\mathrm{Spin}(r,\mathbb{C})$. Since this adjoint representation is the same as the composition of π defined in (4.1) with the adjoint representation of $\mathrm{SO}(r,\mathbb{C})$, it follows that φ is also a holomorphic global section of the vector bundle $E_0(\mathfrak{so}(r,\mathbb{C})) \otimes K$, where E_0 is the $\mathrm{SO}(r,\mathbb{C})$-bundle induced by E. So φ can be understood as a holomorphic homomorphism $\varphi : E_0 \to E_0 \otimes K$ such that $\varphi \in \mathfrak{so}(r,\mathbb{C})$ in each fiber. Likewise, an $\mathrm{SO}(r,\mathbb{C})$-Higgs bundle is a pair (E_0,φ) where E_0 is a special orthogonal rank r bundle (in other words, a rank r and trivial determinant vector bundle equipped with a globally defined holomorphic symmetric bilinear form) and $\varphi : E_0 \to E_0 \otimes K$ is, in each fiber, an element of $\mathfrak{so}(r,\mathbb{C})$.

Since Borel subgroups and parabolic and Levi subgroups of $\mathrm{Spin}(r,\mathbb{C})$ and $\mathrm{SO}(r,\mathbb{C})$ are in bijective correspondence through π defined in (4.1) in a way that the spaces defined in (2.1) and the degrees considered in Definition 2.1, it follows that a $\mathrm{Spin}(r,\mathbb{C})$-Higgs bundle over X is stable (resp., semistable, polystable) if and only if the corresponding $\mathrm{SO}(r,\mathbb{C})$-bundle is stable (resp., semistable, polystable) [3]. The following reduced notion of stability and polystability for special orthogonal Higgs bundles is well known [3].

Proposition 4.1. *Let* (E, φ) *be an* $\mathrm{SO}(r, \mathbb{C})$*-Higgs bundle over* X *for some* $r \geq 4$*. Then* (E, φ) *is stable (resp., semistable) if* $\deg F < 0$ *(resp.,* $\deg F \leq 0$*) for every proper isotropic subbundle* F *of* E *such that* $\varphi(F) \subset F \otimes K$*. It is polystable if it can be written as a direc sum of degree* 0 *isotropic* φ*-invariant vector subbundles of* E*.*

For $n \geq 4$, the group of outer automorphisms of $\mathrm{Spin}(2n, \mathbb{C})$ is isomorphic to $\mathbb{Z}/(2)$, except for $n = 4$, for which it is the group S_3 of permutations of three elements. Then, for $n > 4$ $\mathrm{Spin}(2n, \mathbb{C})$ admits only one nontrivial outer involution, σ, and $\mathrm{Spin}(8, \mathbb{C})$ admits three different nontrivial outer involutions, which can be constructed from one of them by composing with the powers of one of the order 3 outer automorphisms that the group admits. If this is the case, call σ a choice of a nontrivial outer involution of $\mathrm{Spin}(8, \mathbb{C})$. In any case, the action of σ on the $\mathrm{Spin}(2n, \mathbb{C})$-Higgs bundles and the combination of this action with a change of sign on the Higgs field give rise to the definition of two automorphisms of order 2 of the moduli space of $\mathrm{Spin}(2n, \mathbb{C})$-Higgs bundles:

$$F_+(E, \varphi) = (\sigma(E), ds \otimes 1_K(\varphi)), \tag{4.2}$$

$$F_-(E, \varphi) = (\sigma(E), -ds \otimes 1_K(\varphi)), \tag{4.3}$$

for an order 2 representative s of σ in $\mathrm{Aut}(\mathrm{Spin}(2n, \mathbb{C}))$.

Lemma 4.1. *Let* (E, φ) *be a stable* $\mathrm{Spin}(2n, \mathbb{C})$*-Higgs bundle over* X*,* (E_0, φ) *be the corresponding* $\mathrm{SO}(2n, \mathbb{C})$*-Higgs bundle,* f_0 *be an automorphism of* (E_0, φ) *and* V *be the* (-1)*-eigenspace of* f_0*. Then* f_0 *lifts to an automorphism of* (E, φ) *if and only if* V *has even rank and trivial determinant. In this case, there are two elements in* $\mathrm{Aut}(E, \varphi)$ *over* f*.*

Proof. Observe that any automorphism of (E, φ) is a section of the group scheme whose fiber at any $x \in X$ is the group $\mathrm{Spin}(2n, \mathbb{C})$ corresponding to the quadratic space V_x. Now, the restriction $f_{0,x}$ of f_0 to the fiber at x is a special orthogonal transformation of $E_{0,x}$. The two elements of the Spin group lifting this automorphism are given by the elements of norm 1 in $\det V$, imbedded in the Clifford algebra of the quadratic space in question. These are in the even graded component of the Clifford algebra if and only if the rank of V is even. It has a nonzero section if and only if $\det V$ is trivial. If $e_1, \ldots, e_{2r}$ is an orthonormal bases of V_x then the lifted elements are $\pm e_1 \cdot \cdots \cdot e_{2r}$ in the Clifford algebra, and its square is $(-1)^r$. Of course, since the kernel of the projection map $\mathrm{Spin}(2n, \mathbb{C}) \to \mathrm{SO}(2n, \mathbb{C})$ is $\mathbb{Z}/(2)$, also the kernel of the obvious homomorphism $\mathrm{Aut}(E, \varphi) \to \mathrm{Aut}(E_0, \varphi)$ is $\mathbb{Z}/(2)$. □

Theorem 4.1. *Let* (E, φ) *be a stable* $\mathrm{Spin}(2n, \mathbb{C})$*-Higgs bundle over* X *for* $n \geq 4$. *If* (E, φ) *is fixed by the automorphism* F_+ *defined in* (4.2) (*resp., the automorphism* F_- *defined in* (4.3)), *then* E *admits a reduction of structure group* $E^1 \oplus E^2$ *to* $\mathrm{Spin}(2r + 1, \mathbb{C}) \times \mathrm{Spin}(2n - 2r - 1, \mathbb{C})$ *for some integer* r *with* $0 \leq r \leq n - 1$ *such that* φ *maps* E_0^1 *to* $E_0^1 \otimes K$ *and* E_0^2 *to* $E_0^2 \otimes K$ (*resp.,* E_0^1 *to* $E_0^2 \otimes K$ *and* E_0^2 *to* $E_0^1 \otimes K$), *where* E_0^1 *and* E_0^2 *are the principal special orthogonal bundles corresponding to* E^1 *and* E^2, *respectively.*

Proof. Since (E, φ) is fixed by F_+ (or F_-), there exists an isomorphism $f : E \to \sigma(E)$ such that $(f \otimes 1_K) \circ \varphi = \pm ds \otimes 1_K(\varphi) \circ f$ for an order 2 representative s of σ in $\mathrm{Aut}(\mathrm{Spin}(2n, \mathbb{C}))$. If E_0 and $\sigma(E)_0$ denote the principal $\mathrm{SO}(2n, \mathbb{C})$-bundles induced by E and $\sigma(E)$, respectively, then f defines an isomorphism $f_0 : E_0 \to \sigma(E)_0$, so there is an automorphism $\sigma(f)_0 \circ f_0 : E_0 \to E_0$, which comes from the iterated map $\sigma(f) : \sigma(f) \to \sigma^2(E) \cong E$. Since (E, φ) is stable, (E_0, φ) is also stable, so there exist subbundles V and W of E_0 such that $E_0 = V \oplus W$ and

$$\sigma(f)_0 \circ f_0 = \begin{pmatrix} I_V & 0 \\ 0 & -I_W \end{pmatrix}.$$

In fact, any eigenvalue of $\sigma(f)_0 \circ f_0$ should by ± 1 by orthogonality, and the quadratic form of E_0 is nondegenerate when restricted to any eigenspace, by stability of (E_0, φ). Moreover, the decomposition $E_0 = V \oplus W$ is orthogonal, in the sense that V and W are mutually orthogonal.

Take any $x \in X$. Suppose that $v \in V_x$ is an eigenvector of f_0 whose eigenvalue is λ. Then,

$$v = \sigma(f)_0 \circ f_0(v) = \sigma(f)_0(\lambda v) = \lambda \sigma(f)_0,$$

so v is a λ^{-1}-eigenvector of $\sigma(f)_0$. By orthogonality of f_0 and $\sigma(f)_0$, it must be $\lambda = \lambda^{-1}$. This proves that $\sigma(f)_0 \circ f_0 = 1_{E_0}$. From this, there exists an element $\alpha \in Z(\mathrm{Spin}(2n, \mathbb{C}))$ such that $\sigma(f) \circ f = \alpha 1_E$.

Now, again by orthogonality and stability, there exist subbundles U_1 and U_2 of E_0 such that the restriction $f_{0,x}$ of f_0 to $E_{0,x}$ admits the form

$$\begin{pmatrix} I_{U_1} & 0 \\ 0 & -I_{U_2} \end{pmatrix}.$$

The ranks of U_1 and U_2 must be odd. If those ranks were even, since by tensoring by a convenient line bundle it may be assumed that U_2 has trivial determinant, by Lemma 4.1 there exists an automorphism F of E whose descent to $E_{0,x}$ coincides with $f_{0,x}$. Then F and f differ on a central element of $\mathrm{Spin}(2n, \mathbb{C})$, which is not possible. Then $\operatorname{rk} U_1 = 2r + 1$ and $\operatorname{rk} U_2 = 2n - 2r - 1$ for some $0 \leq r \leq n - 1$, so this defines the reduction claimed. The requirement on φ follows immediately from the condition $(f \otimes 1_K) \circ \varphi = \pm ds \otimes 1_K(\varphi) \circ f$, by observing that U_1 is the 1-eigenspace of f_0 and U_2 is the (-1)-eigenspace of f_0. □

Remark. Observe that the groups $\mathrm{Spin}(2r+1,\mathbb{C}) \times \mathrm{Spin}(2n-2r-1,\mathbb{C})$, for $0 \le r \le n-1$ are precisely the subgroups of fixed points of all possible not inner-equivalent involutions of $\mathrm{Spin}(2n,\mathbb{C})$ which represent an outer automorphism of order 2 [19, Theorem 5.10]. On the other hand, the stability condition of the fixed points of the automorpjhism F_+ defined in (4.2) is exactly the stability condition of $\mathrm{Spin}(k,\mathbb{C})$-Higgs bundles over X with k odd. In the case of the automorphism F_- defined in (4.3), the stability condition of its fixed points reduces to the stability condition of principal $\mathrm{Spin}(k,\mathbb{C})$-bundles with k odd.

5 Conclusion

It has been proved, by an alternative way to the proof given by García-Prada and Ramanan, that the stable fixed points of the involution $(E,\varphi) \mapsto (E^*,\varphi^t)$ or the involution $(E,\varphi) \mapsto (E^*,-\varphi^t)$ defined in (3.1) and (3.2) of the moduli space of $\mathrm{SL}(n,\mathbb{C})$-Higgs bundles over the compact Riemann surface X, for $n \ge 3$, can be described as Higgs pairs (E,φ) over X with structure group $\mathrm{SO}(n,\mathbb{C})$ or $\mathrm{Sp}(n,\mathbb{C})$ whose Higgs field takes values, respectively, in $\mathrm{Sym}^2 E$ (for the first involution) or $\wedge^2 E$ (for the second involution). On the other hand, if σ denotes a choice of a nontrivial outer automorphism of $\mathrm{Spin}(2n,\mathbb{C})$, for $n \ge 4$, and $s \in \mathrm{Aut}(\mathrm{Spin}(2n,\mathbb{C}))$ is a representative of σ, then any stable fixed point (E,φ) of the involution $(E,\varphi) \mapsto (\sigma(E), ds \otimes 1_K(\varphi))$ or the involution $(E,\varphi) \mapsto (\sigma(E), -ds \otimes 1_K(\varphi))$ admits a reduction of structure group to a subgroup of the form $\mathrm{Spin}(2r+1,\mathbb{C}) \times \mathrm{Spin}(2n-2r-1,\mathbb{C})$ of $\mathrm{Spin}(2n,\mathbb{C})$ for some integer r with $0 \le r \le n-1$ such that φ maps E_0^1 to $E_0^1 \otimes K$ and E_0^2 to $E_0^2 \otimes K$ (for the first involution) or maps E_0^1 to $E_0^2 \otimes K$ and E_0^2 to $E_0^1 \otimes K$ (for the second involution), where E_0^1 and E_0^2 are the principal special orthogonal bundles corresponding to the principal $\mathrm{Spin}(2r+1,\mathbb{C})$-bundle E^1 and the principal $\mathrm{Spin}(2n-2r-1,\mathbb{C})$-bundle E^2, respectively, defined by the reduction of structure group claimed.

Conflict of Interest The author declares no conflict of interest.

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