Front. Math.
https://doi.org/10.1007/s11464-022-0291-z
Published online: June 30, 2024

RESEARCH ARTICLE

A Simons-type Integral Inequality for Minimal Surfaces with Constant Kähler Angle in Complex Projective Spaces

Jie FEI[1], **Xiaoxiang JIAO**[2], **Jun WANG**[3]

1 Department of Pure Mathematics, School of Mathematics and Physics, Xi'an Jiaotong–Liverpool University, Suzhou 215123, China
2 School of Mathematics Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
3 School of Mathematics Sciences and Institute of Mathematics, Nanjing Normal University, Nanjing 210023, China

Abstract In this paper, we establish a Simons-type integral inequality for minimal surfaces with constant Kähler angle in complex projective spaces, and we determine all the closed minimal surfaces with the square norm of the second fundamental form satisfying a pinching condition.

Keywords Complex projective spaces, constant Kähler angle, minimal surfaces, pinching, the second fundamental form

MSC2020 53C42, 53C55

1 Introduction

Let $\mathbb{C}P^n$ be the complex projective space endowed with the Fubini–Study metric of constant holomorphic sectional curvature 4 and let M be a Riemann surface. A conformal minimal immersion $f : M \to \mathbb{C}P^n$ satisfying some assumptions on the Gaussian curvature K and the Kähler angle θ was widely studied. It is well known that up to a rigid motion, a linearly full conformal minimal immersion of two-sphere with constant curvature in $\mathbb{C}P^n$ belongs to the Veronese sequence proved by Bando and Ohnita [1] and Bolton et al. [2]. The Kähler angle plays an important role in studying minimal surfaces in a Kähler manifold [5], as it gives a measure of the failure of f to be a holomorphic map. That is, f is holomorphic if and only if $\theta = 0$ on M, while f is anti-holomorphic if and only if $\theta = \pi$ on M. Ohnita [13] classified minimal surfaces with constant Gaussian curvature

Received October 17, 2022; accepted November 29, 2023
Corresponding author: Jun WANG, E-mail: wangjun706@njnu.edu.cn

and constant Kähler angle in $\mathbb{C}P^n$. Kenmotsu and Masuda [7] studied the local behaviour of the Kähler angle in $\mathbb{C}P^2$, which satisfies an overdetermined system of ordinary differential equations. They showed the Kähler angle must be constant if the Gaussian curvature is constant. Together with Ohnita's results [13], all such minimal surfaces in $\mathbb{C}P^2$ were classified. Bolton et al. [2] conjectured that if the Kähler angle θ of the minimal immersion $f : S^2 \to \mathbb{C}P^n$ is constant such that $\theta \neq 0, \frac{\pi}{2}, \pi$, then its Gaussian curvature K is also constant. They gave an affirmative answer to this conjecture for $n \leq 4$. Ogata [11, 12] showed that the conjecture holds if the Gaussian curvature is bounded below by a constant determined by the Kähler angle. Mo [10] verified the conjecture under the added assumption that $|\cos\theta| \geq \frac{1}{5}$. However, Li [8] gave three families of counterexamples of minimal immersion from two-sphere into $\mathbb{C}P^{10}$ with constant Kähler angle $\theta \neq 0, \frac{\pi}{2}, \pi$ and nonconstant Gaussian curvature. Therefore, the conjecture does not hold in general.

In this paper, we would like to study pinching about the square norm S of the second fundamental form for minimal surfaces with constant Kähler angle in $\mathbb{C}P^n$. This is inspired by the well-known Simons inequality, which is an integral inequality about S. More precisely, for a closed n-dimensional submanifold M^n of the unit sphere S^{n+p}, Simons [14] computed the Laplacian of the square norm of its second fundamental form S and obtain the following integral inequality

$$\int_{M^n} \left[\left(2 - \frac{1}{p}\right)S - n\right]S \ *1 \geq 0,$$

where $*1$ is the volume element of M^n with respect to the induced metric on M^n. As an application, if S satisfies the pinching condition $0 \leq S \leq \frac{n}{2-1/p}$ on M^n, then either $S = 0$ and M^n is totally geodesic, or $S = \frac{n}{2-1/p}$. All minimal submanifolds with $S = \frac{n}{2-1/p}$ were classified by Chern et al. [4]. For a totally real minimal submanifold M^n in $\mathbb{C}P^n$, Chen and Ogiue proved that if $S \leq \frac{n+1}{2-1/n}$ holds on M^n, then M^n is totally geodesic [3]. Ludden et al. [9] determined the Clifford torus that is not totally geodesic in $\mathbb{C}P^2$. Tanno [15] obtained a Simons-type inequality about a compact complex submanifold M^n immersed into $\mathbb{C}P^{n+p}$ as follows,

$$\int_{M^n} \big[3S - (n+2)\big]S \ *1 \geq 0.$$

Consequently, complex submanifolds with $S = \frac{n+2}{3}$ were completely determined such that M^n is imbedded as a complex hyperquadric Q_1 in $\mathbb{C}P^{1+p}$, where Q_1 is a complex submanifold of $\mathbb{C}P^2$.

The complex Grassmann manifold $G(k, n)$ is the space of all k-dimensional complex subspaces in $\mathbb{C}^n$. When $k = 1$, $G(1, n)$ is just the complex projective space $\mathbb{C}P^{n-1}$. The complex hyperquadric Q_n is a complex submanifold of $\mathbb{C}P^{n+1}$ and is defined by

$$Q_n = \big\{[Z = (z_1, \ldots, z_{n+2})] \in \mathbb{C}P^{n+1} \bigm| z_1^2 + \cdots + z_{n+2}^2 = 0\big\},$$

where $[Z = (z_1, z_2, \ldots, z_{n+2})]$ is the homogeneous coordinates of $\mathbb{C}P^{n+1}$. $G(k, n)$ and Q_n are two important symmetric spaces. They have natural Kählerian metric with non constant holomorphic sectional curvature when k is not equal to 1. Recently, we studied pinching results about S for holomorphic curves in the complex Grassmann manifold $G(2, n)$ [17] and minimal surfaces with constant Kähler angle in Q_n [16].

The purpose of this paper is to establish a Simons-type inequality for minimal surfaces with constant Kähler angle in $\mathbb{C}P^n$ and characterize all the associated pinching immersions. In Section 2, we study the geometry of a conformal minimal immersion $f : M \to \mathbb{C}P^n$ by moving frames, where M is a Riemann surface. In Section 3, we compute the Laplacian of S (Theorem 3.2) and obtain a Simons-type integral inequality for a closed minimal surface with constant Kähler angle (Theorem 3.4). In Section 4, if S satisfies a pinching condition, it is shown that both K and S are constant (Theorem 4.1). Moreover, we determine all the minimal surfaces as follows:

Main Theorem. *Let M be a compact Riemann surface without boundary and $f : M \to \mathbb{C}P^n$ be a conformal minimal immersion neither holomorphic nor antiholomorphic. If its Kähler angle θ is constant and the square norm S of the second fundamental form satisfies the pinching condition*

$$\frac{3}{4}S^2 - (1 + 2\cos^2\theta)S + 15\cos^2\theta\sin^2\theta - 8\kappa \leq 0$$

on M, where κ is a globally defined invariant relative to the first and second fundamental forms, then up to a rigid motion, $f(M)$ is one of following

(i) $f(T^2) \subset \mathbb{C}P^2$ *with* $\kappa = \frac{1}{8}$, $S = 2$, $\cos\theta = 0$ *and* $K = 0$, *or*
(ii) $f(S^2) \subset \mathbb{C}P^4$ *with* $\kappa = 0$, $S = \frac{4}{3}$, $\cos\theta = 0$ *and* $K = \frac{1}{3}$, *or*
(iii) $f(S^2) \subset \mathbb{C}P^2$ *with* $\kappa = 0$, $S = 0$, $\cos\theta = 0$ *and* $K = 1$, *or*
(iv) $f(S^2) \subset \mathbb{C}P^3$ *with* $\kappa = 0$, $S = \frac{48}{49}$, $\cos\theta = \frac{1}{7}$ *and* $K = \frac{4}{7}$, *or*
(v) $f(S^2) \subset \mathbb{C}P^3$ *with* $\kappa = 0$, $S = \frac{48}{49}$, $\cos\theta = -\frac{1}{7}$ *and* $K = \frac{4}{7}$.

Remark. Example (i) is the Clifford torus in $\mathbb{C}P^2$ given in [9]; examples (ii) and (iii) are the middle elements of Veronese sequences in $\mathbb{C}P^4$ and $\mathbb{C}P^2$ given in [2], respectively; examples (iv) and (v) are the second and third elements of Veronese sequence in $\mathbb{C}P^3$ given in [2], respectively.

2 Preliminaries

Throughout this paper, $\mathtt{i}$ denotes the imaginary unit $\sqrt{-1}$, and we will agree on the following ranges of indices:

$$0 \leq A, B, C, \ldots \leq n, \quad 1 \leq \alpha, \beta, \gamma, \ldots \leq n.$$

We firstly study the geometry of $\mathbb{C}P^n = U(n+1)/(U(1) \times U(n))$. Let $e = (e_0, e_1, \ldots, e_n) \in U(n+1)$. Its Maurer–Cartan forms are denoted by

$(\Omega_{AB}) = e^{-1}de$. Its structure equations are given by

$$\mathrm{d}\Omega_{AB} = -\sum_{C=0}^{n} \Omega_{AC} \wedge \Omega_{CB}, \quad \Omega_{AB} + \overline{\Omega}_{BA} = 0. \tag{2.1}$$

The Fubini–Study metric on $\mathbb{C}P^n$ is

$$g = \sum_{\alpha=1}^{n} \Omega_\alpha \overline{\Omega}_\alpha, \quad \Omega_\alpha := \Omega_{\alpha 0}.$$

The structure equations of $(\mathbb{C}P^n,\, g)$ are given by

$$\mathrm{d}\Omega_\alpha = -\sum_{\beta=1}^{n} \Theta_{\alpha\beta} \wedge \Omega_\beta, \quad \Theta_{\alpha\beta} + \overline{\Theta}_{\beta\alpha} = 0,$$

where $\Theta_{\alpha\beta} = \Omega_{\alpha\beta} - \Omega_{00}\delta_{\alpha\beta}$ are the connection forms. Its curvature forms $\Psi_{\alpha\beta}$ are given by

$$\Psi_{\alpha\beta} = \mathrm{d}\Theta_{\alpha\beta} + \sum_{\gamma=1}^{n} \Theta_{\alpha\gamma} \wedge \Theta_{\gamma\beta} = \Omega_\alpha \wedge \overline{\Omega}_\beta + \delta_{\alpha\beta} \sum_{\gamma=1}^{n} \Omega_\gamma \wedge \overline{\Omega}_\gamma.$$

Let $f : M \to \mathbb{C}P^n$ be a conformal minimal immersion from a Riemann surface M, and $e : U \subset M \to U(n+1)$ be a local frame along f, i.e., $f = [e_0]$. Set $\omega = e^*\Omega$. The induced metric on M is

$$\mathrm{d}s_M^2 = f^*g = \sum_{\alpha=1}^{n} \omega_\alpha \overline{\omega}_\alpha = \varphi\overline{\varphi}, \tag{2.2}$$

where φ is a local form of $(1,0)$-type. The Levi–Civita connection form of $(M, \mathrm{d}s_M^2)$, denoted by ρ, is characterized by

$$\mathrm{d}\varphi = \mathtt{i}\rho \wedge \varphi. \tag{2.3}$$

The Gauss curvature K is given by

$$\mathrm{d}\rho = \frac{\mathtt{i}}{2} K \varphi \wedge \overline{\varphi}. \tag{2.4}$$

Define local complex-valued functions X_α and Y_α as following

$$\omega_\alpha = X_\alpha \varphi + Y_\alpha \overline{\varphi}. \tag{2.5}$$

Put $X = (X_1, \ldots, X_n)$, $Y = (Y_1, \ldots, Y_n) \in \mathbb{C}^n$. The Hermitian inner product of $\mathbb{C}^n$ is denoted by $\langle\ ,\ \rangle$. By (2.2) and (2.5), we have

$$|X|^2 + |Y|^2 = 1, \quad \langle X, Y\rangle = 0. \tag{2.6}$$

The Kähler angle $\theta : M \to [0, \pi]$ (see [5]) satisfies the equation

$$\cos\theta = |X|^2 - |Y|^2. \tag{2.7}$$

Therefore, f is holomorphic if $\theta = 0$; f is anti-holomorphic if $\theta = \pi$; and f is totally real if $\theta = \frac{\pi}{2}$. By (2.1), (2.5) and (2.7), we have

$$\mathrm{d}\omega_{00} = -\cos\theta\varphi \wedge \overline{\varphi}. \tag{2.8}$$

Taking exterior derivative on both sides of equations in (2.5) and using (2.1) and (2.3), we get

$$\mathrm{D}X_\alpha \wedge \varphi + \mathrm{D}Y_\alpha \wedge \overline{\varphi} = 0, \tag{2.9}$$

where

$$\mathrm{D}X_\alpha = \mathrm{d}X_\alpha - X_\alpha(\omega_{00} - \mathtt{i}\rho) + \sum_{\beta=1}^{n} \omega_{\alpha\beta} X_\beta,$$

$$\mathrm{D}Y_\alpha = \mathrm{d}Y_\alpha - Y_\alpha(\omega_{00} + \mathtt{i}\rho) + \sum_{\beta=1}^{n} \omega_{\alpha\beta} Y_\beta.$$

From (2.9), by using the Cartan's lemma,

$$\mathrm{D}X_\alpha = a_\alpha\varphi + b_\alpha\overline{\varphi}, \quad DY_\alpha = b_\alpha\varphi + c_\alpha\overline{\varphi},$$

where a_α, b_α and c_α are locally complex-valued smooth functions. Since f is minimal if and only if $b_\alpha = 0$ for all α, then

$$\begin{aligned} &\mathrm{d}X_\alpha - X_\alpha(\omega_{00} - \mathtt{i}\rho) + \sum_{\beta=1}^{n} \omega_{\alpha\beta} X_\beta = a_\alpha\varphi, \\ &\mathrm{d}Y_\alpha - Y_\alpha(\omega_{00} + \mathtt{i}\rho) + \sum_{\beta=1}^{n} \omega_{\alpha\beta} Y_\beta = c_\alpha\overline{\varphi}. \end{aligned} \tag{2.10}$$

Put $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{C}^n$. By the second equation in (2.6), we obtain

$$\langle X, \mathbf{c}\rangle + \overline{\langle Y, \mathbf{a}\rangle} = 0. \tag{2.11}$$

We denote by S the square norm of the second fundamental form of f, which is given by

$$S = 4(|\mathbf{a}|^2 + |\mathbf{c}|^2), \quad \text{where } |\mathbf{a}|^2 = \sum_{\alpha=1}^{n} |a_\alpha|^2,\ |\mathbf{c}|^2 = \sum_{\alpha=1}^{n} |c_\alpha|^2.$$

Let $\hat{e} : \hat{U} \to U(n+1)$ be another moving frame along f such that $\hat{e} = eU$, where $U = \mathrm{diag}\{e^{\mathtt{i}\eta}, T\}$, η is a local real-valued function and T is a local $U(n)$-valued function. If we put a hat on the corresponding quantities relative to the new frame $\hat{e}$, we have

$$\hat{X} = e^{-\mathtt{i}\eta} XT, \quad \hat{Y} = e^{-\mathtt{i}\eta} YT, \quad \hat{\mathbf{a}} = e^{-\mathtt{i}\eta}\mathbf{a}T, \quad \hat{\mathbf{c}} = e^{-\mathtt{i}\eta}\mathbf{c}T. \tag{2.12}$$

Together with (2.11) and (2.12), it is easy to check that

$$\kappa = |\langle X, \mathbf{c}\rangle|^2 = |\langle Y, \mathbf{a}\rangle|^2 \tag{2.13}$$

is a globally defined invariant on M, which will be used in the pinching condition later.

3 Fundamental Equations

In this section, $\Delta = *\mathrm{d} * \mathrm{d}$ denotes the Laplace–Beltrami operator of $(M, \varphi\overline{\varphi})$, where $*$ is Hodge star operator of $(M, \varphi\overline{\varphi})$, and we will compute the Laplacian of $|X|^2$, $|Y|^2$, $|\mathbf{a}|^2$ and $|\mathbf{c}|^2$. Taking the exterior derivative of the first equation in (2.10), and using (2.1), (2.3), (2.8) we get the Codazzi equations of f as follows,

$$\begin{aligned} \mathrm{D}a_\alpha &= \mathrm{d}a_\alpha - a_\alpha(\omega_{00} - 2\mathtt{i}\rho) + \sum_{\beta=1}^{n} \omega_{\alpha\beta} a_\beta \\ &= a_{\alpha,1}\varphi + X_\alpha \frac{K - 3\cos\theta - 1}{2}\overline{\varphi}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \mathrm{D}c_\alpha &= \mathrm{d}c_\alpha - c_\alpha(\omega_{00} + 2\mathtt{i}\rho) + \sum_{\beta=1}^{n} \omega_{\alpha\beta} c_\beta \\ &= c_{\alpha,1}\overline{\varphi} + Y_\alpha \frac{K + 3\cos\theta - 1}{2}\varphi, \end{aligned} \tag{3.2}$$

where $a_{\alpha,1}, c_{\alpha,1}$ are local complex-valued smooth functions. From (2.10) we have

$$\mathrm{d}|X|^2 = \sum_{\alpha=1}^{n} (\overline{X}_\alpha a_\alpha \varphi + X_\alpha \overline{a}_\alpha \overline{\varphi}), \quad \mathrm{d}|Y|^2 = \sum_{\alpha=1}^{n} (\overline{Y}_\alpha c_\alpha \overline{\varphi} + Y_\alpha \overline{c}_\alpha \varphi). \tag{3.3}$$

As $*\varphi = -\mathtt{i}\varphi$, $*\overline{\varphi} = \mathtt{i}\overline{\varphi}$, from (3.3) we have

$$*\mathrm{d}|X|^2 = -\mathtt{i} \sum_{\alpha=1}^{n} (\overline{X}_\alpha a_\alpha \varphi - X_\alpha \overline{a}_\alpha \overline{\varphi}), \quad *\mathrm{d}|Y|^2 = \mathtt{i} \sum_{\alpha=1}^{n} (\overline{Y}_\alpha c_\alpha \overline{\varphi} - Y_\alpha \overline{c}_\alpha \varphi). \tag{3.4}$$

Taking exterior derivative of the first equation in (3.4), we get

$$\mathrm{d} * \mathrm{d}|X|^2 = -\mathtt{i} \sum_\alpha [\mathrm{d}\overline{X}_\alpha \wedge a_\alpha \varphi + \overline{X}_\alpha \mathrm{d}(a_\alpha \varphi) - \mathrm{d}X_\alpha \wedge \overline{a}_\alpha \overline{\varphi} - X_\alpha \mathrm{d}(\overline{a}_\alpha \overline{\varphi})].$$

Using (2.10), (3.1) and the formula $*(\varphi_1 \wedge \varphi_2) = 1$, we have

$$\Delta|X|^2 = 2|X|^2(K - 3\cos\theta - 1) + 4|\mathbf{a}|^2. \tag{3.5}$$

Similarly, we derive

$$\Delta|Y|^2 = 2|Y|^2(K + 3\cos\theta - 1) + 4|\mathbf{c}|^2. \tag{3.6}$$

From (2.6), (2.7), (3.5) and (3.6), we obtain

Theorem 3.1. *Let $f : M \to \mathbb{C}P^n$ be a conformal minimal immersion from a Riemann surface M into $\mathbb{C}P^n$. Then the Gauss equation and Laplacian of $\cos\theta$ are given by*

$$K = 1 + 3\cos^2\theta - 2(|\mathbf{a}|^2 + |\mathbf{c}|^2), \tag{3.7}$$

$$\Delta\cos\theta = 2\cos\theta(K-4) + 4(|\mathbf{a}|^2 - |\mathbf{c}|^2), \tag{3.8}$$

where K is the Gaussian curvature of the induced metric and θ is the Kähler angle of the immersion.

Remark. The two formulas (3.7) and (3.8) were firstly derived by Chern and Wolfson [5] and then by Jiao and Peng [6].

From (3.1) and (3.2) we have

$$\mathrm{d}|\mathbf{a}|^2 = P\varphi + \overline{P}\overline{\varphi}, \quad \mathrm{d}|\mathbf{c}|^2 = Q\varphi + \overline{Q}\overline{\varphi}, \tag{3.9}$$

where

$$P = \langle \mathbf{a}_{,1}, \mathbf{a}\rangle + \langle \mathbf{a}, X\rangle \frac{K - 3\cos\theta - 1}{2}, \quad Q = \langle \mathbf{c}, \mathbf{c}_{,1}\rangle + \langle Y, \mathbf{c}\rangle \frac{K + 3\cos\theta - 1}{2}, \tag{3.10}$$

with $\mathbf{a}_{,1} = (a_{1,1}, \ldots, a_{n,1})$ and $\mathbf{c}_{,1} = (c_{1,1}, \ldots, c_{n,1}) \in \mathbb{C}^n$. By Hodge star operator, we have

$$\mathrm{d} * \mathrm{d}|\mathbf{a}|^2 = \mathtt{i}\left[\mathrm{d}(\overline{P}\overline{\varphi}) - \mathrm{d}(P\varphi)\right], \quad \mathrm{d} * \mathrm{d}|\mathbf{c}|^2 = \mathtt{i}\left[\mathrm{d}(\overline{Q}\overline{\varphi}) - \mathrm{d}(Q\varphi)\right].$$

Routine computations give

$$\mathrm{d}\langle \mathbf{a}_{,1}\varphi, \mathbf{a}\rangle = \left[S_{\mathbf{a}} - \frac{\langle X, \mathbf{a}\rangle}{2\lambda}\frac{\partial(K - 3\cos\theta)}{\partial z}\right]\varphi\wedge\overline{\varphi}, \tag{3.11}$$

where

$$S_{\mathbf{a}} = |\langle X, \mathbf{a}\rangle|^2 - |\langle Y, \mathbf{a}\rangle|^2 - \frac{|\mathbf{a}|^2}{2}(3K - 5\cos\theta - 1) - |\mathbf{a}_{,1}|^2. \tag{3.12}$$

Besides, one can compute that

$$\begin{aligned}&\frac{K - 3\cos\theta - 1}{2}\mathrm{d}\langle \mathbf{a}\varphi, X\rangle \\ &\quad = -\frac{K - 3\cos\theta - 1}{2}\left[|\mathbf{a}|^2 + \frac{|X|^2}{2}(K - 3\cos\theta - 1)\right]\varphi\wedge\overline{\varphi},\end{aligned} \tag{3.13}$$

and

$$\mathrm{d}\left(\frac{K - 3\cos\theta - 1}{2}\right)\wedge\langle \mathbf{a}\varphi, X\rangle = -\frac{\overline{\langle X, \mathbf{a}\rangle}}{2\lambda}\frac{\partial(K - 3\cos\theta)}{\partial\overline{z}}\varphi\wedge\overline{\varphi}. \tag{3.14}$$

Define a new operator

$$\mathrm{D}_{\mathbf{a}} = \frac{\langle X, \mathbf{a}\rangle}{2\lambda}\frac{\partial}{\partial z} + \frac{\overline{\langle X, \mathbf{a}\rangle}}{2\lambda}\frac{\partial}{\partial \overline{z}}. \tag{3.15}$$

From (3.11)–(3.15) it follows that

$$\begin{aligned}\Delta|\mathbf{a}|^2 &= 4|\mathbf{a}|^2(2K - 4\cos\theta - 1) + |X|^2(K - 3\cos\theta - 1)^2 \\ &\quad - 4|\langle X, \mathbf{a}\rangle|^2 + 4|\langle Y, \mathbf{a}\rangle|^2 + 4\mathrm{D}_{\mathbf{a}}(K - 3\cos\theta) + 4|\mathbf{a}_{,1}|^2.\end{aligned} \tag{3.16}$$

Similarly, we can compute that

$$\mathrm{d}\langle \mathbf{c}\varphi, \mathbf{c}_{,1}\rangle = \left[S_{\mathbf{c}} - \frac{\overline{\langle Y, \mathbf{c}\rangle}}{2\lambda}\frac{\partial(K + 3\cos\theta)}{\partial z}\right]\varphi\wedge\overline{\varphi}, \tag{3.17}$$

where

$$S_{\mathbf{c}} = |\langle Y, \mathbf{c}\rangle|^2 - |\langle X, \mathbf{c}\rangle|^2 - \frac{|\mathbf{c}|^2}{2}(3K + 5\cos\theta - 1) - |\mathbf{c}_{,1}|^2. \tag{3.18}$$

We also have

$$\begin{aligned}&\frac{K + 3\cos\theta - 1}{2}\mathrm{d}\langle Y\varphi, \mathbf{c}\rangle \\ &= -\frac{K + 3\cos\theta - 1}{2}\left[|\mathbf{c}|^2 + |Y|^2\left(\frac{K}{2} + \cos\theta - |Y|^2\right)\right]\varphi\wedge\overline{\varphi},\end{aligned} \tag{3.19}$$

and

$$\mathrm{d}\left(\frac{K + 3\cos\theta - 1}{2}\right)\wedge\langle Y\varphi, \mathbf{c}\rangle = -\frac{\langle Y, \mathbf{c}\rangle}{2\lambda}\frac{\partial(K + 3\cos\theta)}{\partial\overline{z}}\varphi\wedge\overline{\varphi}. \tag{3.20}$$

Define

$$\mathrm{D}_{\mathbf{c}} = \frac{\overline{\langle Y, \mathbf{c}\rangle}}{2\lambda}\frac{\partial}{\partial z} + \frac{\langle Y, \mathbf{c}\rangle}{2\lambda}\frac{\partial}{\partial\overline{z}}. \tag{3.21}$$

From (3.9), (3.10), (3.17)–(3.21), we obtain

$$\begin{aligned}\Delta|\mathbf{c}|^2 &= 4|\mathbf{c}|^2(2K + 4\cos\theta - 1) + |Y|^2(K + 3\cos\theta - 1)^2 - 4|\langle Y, \mathbf{c}\rangle|^2 \\ &\quad + 4|\langle X, \mathbf{c}\rangle|^2 + 4\mathrm{D}_{\mathbf{c}}(K + 3\cos\theta) + 4|\mathbf{c}_{,1}|^2.\end{aligned} \tag{3.22}$$

From (3.16) and (3.22), we get

Theorem 3.2. *Let $f: M \to \mathbb{C}P^n$ be a conformal minimal immersion. Then*

$$\begin{aligned}\frac{1}{4}\Delta S &= (2K - 1)S - 16\cos\theta(|\mathbf{a}|^2 - |\mathbf{c}|^2) + (K - 1)^2 - 6K\cos^2\theta \\ &\quad + 15\cos^2\theta - 4|\langle X, \mathbf{a}\rangle|^2 - 4|\langle Y, \mathbf{c}\rangle|^2 + 4|\langle Y, \mathbf{a}\rangle|^2 + 4|\langle X, \mathbf{c}\rangle|^2 \\ &\quad + 4|\mathbf{a}_{,1}|^2 + 4|\mathbf{c}_{,1}|^2 + 2\mathrm{D}_{\mathbf{a}}(K - 2\cos\theta) + 2\mathrm{D}_{\mathbf{c}}(K + 2\cos\theta).\end{aligned} \tag{3.23}$$

In the following, we always assume that the Kähler angle θ is constant. Form (2.6) and (2.7), both $|X|$ and $|Y|$ are constant. Then (3.3) implies that

$$\langle X, \mathbf{a}\rangle = \langle Y, \mathbf{c}\rangle = 0. \tag{3.24}$$

From (3.15) and (3.21), $\mathrm{D}_{\mathbf{a}} = \mathrm{D}_{\mathbf{c}} = 0$. Besides, from (3.8) we have

$$|\mathbf{a}|^2 - |\mathbf{c}|^2 = \frac{1}{2}\cos\theta(4-K). \tag{3.25}$$

By (3.24) and (3.25), equation (3.23) becomes

$$\begin{aligned}\frac{1}{4}\Delta S &= (2K-1)S + (K-1)^2 + (2K-17)\cos^2\theta \\ &\quad + 4|\langle Y, \mathbf{a}\rangle|^2 + 4|\langle X, \mathbf{c}\rangle|^2 + 4|\mathbf{a}_{,1}|^2 + 4|\mathbf{c}_{,1}|^2. \end{aligned}\tag{3.26}$$

Substituting Gauss equation (3.7) into (3.26) and using (2.13), we obtain

Theorem 3.3. *Let $f : M \to \mathbb{C}P^n$ be a conformal minimal immersion with constant Kähler angle. Then*

$$\frac{1}{4}\Delta S = -\frac{3}{4}S^2 + (1+2\cos^2\theta)S + 15\cos^2\theta(\cos^2\theta - 1) + 8\kappa + 4|\mathbf{a}_{,1}|^2 + 4|\mathbf{c}_{,1}|^2, \tag{3.27}$$

where κ is a globally defined invariant on M given by (2.13).

Integrating (3.27) on M and using Stokes' theorem, we get

Theorem 3.4. *Let M be a compact Riemann surface without boundary and $f : M \to \mathbb{C}P^n$ be a conformal minimal immersion. If its Kähler angle θ is constant, then*

$$\int_M \left[\frac{3}{4}S^2 - (1+2\cos^2\theta)S + 15\cos^2\theta\sin^2\theta - 8\kappa\right] * 1 \geq 0.$$

For $\theta = 0$ or π, by Theorem 3.4, it is easy to get

Theorem 3.5. *Let M be a compact Riemann surface without boundary and $f : M \to \mathbb{C}P^n$ be a holomorphic or anti-holomorphic immersion. Then*

$$\int_M S(S-4) \; * 1 \geq 0.$$

Remark. Theorem 3.5 is a special case for $m = 1$ of Theorem 1 in [15].

4 Proof of Main Theorem

In this section, we will characterize minimal surfaces with constant Kähler angle $\theta \in (0, \pi)$ in $\mathbb{C}P^n$ satisfying a pinching condition as follows.

Theorem 4.1. *Let M be a compact Riemann surface without boundary and $f : M \to \mathbb{C}P^n$ be a conformal minimal immersion with constant Kähler angle $\theta \in (0, \pi)$. If*

$$\frac{3}{4}S^2 - (1 + 2\cos^2\theta)S + 15\cos^2\theta\sin^2\theta - 8\kappa \le 0 \tag{4.1}$$

holds on M, then K, $|\mathbf{a}|$ and $|\mathbf{c}|$ are all constant.

Proof. By (3.27), Theorem 3.4 and (4.1), we have

$$\frac{3}{4}S^2 - (1 + 2\cos^2\theta)S + 15\cos^2\theta\sin^2\theta - 8\kappa = 0,$$

and

$$\mathbf{a}_{,1} = \mathbf{c}_{,1} = 0. \tag{4.2}$$

Using (4.2), equations in (3.1) and (3.2) become

$$\mathrm{D}a_\alpha = \mathrm{d}a_\alpha - a_\alpha(\omega_{00} - 2\mathtt{i}\rho) + \sum_{\beta=1}^{n} \omega_{\alpha\beta}a_\beta = X_\alpha \frac{K - 3\cos\theta - 1}{2}\overline{\varphi}, \tag{4.3}$$

and

$$\mathrm{D}c_\alpha = \mathrm{d}c_\alpha - c_\alpha(\omega_{00} + 2\mathtt{i}\rho) + \sum_{\beta=1}^{n} \omega_{\alpha\beta}c_\beta = Y_\alpha \frac{K + 3\cos\theta - 1}{2}\varphi. \tag{4.4}$$

From (3.24) and (4.3),

$$\sum_{\alpha=1}^{n} \overline{a}_\alpha \mathrm{d}a_\alpha = |\mathbf{a}|^2(\omega_{00} - 2\mathtt{i}\rho) - \sum_{\alpha,\beta=1}^{n} \overline{a}_\alpha\omega_{\alpha\beta}a_\beta.$$

Combining with (2.1), we have

$$\mathrm{d}|\mathbf{a}|^2 = \sum_{\alpha=1}^{n}(\overline{a}_\alpha\mathrm{d}a_\alpha + a_\alpha\mathrm{d}\overline{a}_\alpha) = 0,$$

which implies $|\mathbf{a}|^2$ is constant. By the same way, from (4.4), one can prove that $|\mathbf{c}|^2$ is constant, too. Therefore, S is constant. By (3.7), K is also constant. □

By (2.6) and (2.12), there exists a moving frame [5] such that

$$\omega_{10} = \cos\frac{\theta}{2}\varphi, \quad \omega_{20} = \sin\frac{\theta}{2}\overline{\varphi}, \quad \omega_{\alpha 0} = 0, \quad 3 \le \alpha \le n. \tag{4.5}$$

As θ is constant, substituting (4.5) in (2.10), we have

$$\begin{cases} \mathrm{D}X_1 = \cos\frac{\theta}{2}(\mathtt{i}\rho - \omega_{00} + \omega_{11}) = a_1\varphi, \\ \mathrm{D}X_2 = \cos\frac{\theta}{2}\omega_{21} = a_2\varphi, \\ \mathrm{D}X_\alpha = \cos\frac{\theta}{2}\omega_{\alpha 1} = a_\alpha\varphi, \quad 3 \le \alpha \le n, \end{cases} \tag{4.6}$$

and

$$\begin{cases} \mathrm{D}Y_1 = \sin\frac{\theta}{2}\omega_{12} = c_1\overline{\varphi}, \\ \mathrm{D}Y_2 = -\sin\frac{\theta}{2}(\mathtt{i}\rho + \omega_{00} - \omega_{22}) = c_2\overline{\varphi}, \\ \mathrm{D}Y_\alpha = \sin\frac{\theta}{2}\omega_{\alpha 2} = c_\alpha\overline{\varphi}, \quad 3 \le \alpha \le n. \end{cases} \tag{4.7}$$

Since $\sin\theta \neq 0$, from (4.6) and (4.7) it follows that

$$\mathtt{i}\rho - \omega_{00} + \omega_{11} = 0, \quad \mathtt{i}\rho + \omega_{00} - \omega_{22} = 0, \tag{4.8}$$

and

$$a_1 = c_2 = 0. \tag{4.9}$$

Furthermore, there exists a moving frame such that

$$\begin{cases} \omega_{31} = \sec\frac{\theta}{2}a_3\varphi, \quad \omega_{32} = \csc\frac{\theta}{2}c_3\overline{\varphi}, \\ \omega_{41} = 0, \quad \omega_{42} = \csc\frac{\theta}{2}c_4\overline{\varphi}, \quad a_3 \ge c_4 \ge 0, \\ \omega_{\alpha 1} = 0, \quad \omega_{\alpha 2} = 0, \quad 5 \le \alpha \le n. \end{cases} \tag{4.10}$$

Taking exterior derivative of equations in (4.8), we have

$$K - 3\cos\theta - 1 + 2\sec^2\frac{\theta}{2}|\mathbf{a}|^2 = 0, \quad |\mathbf{a}|^2 = |a_2|^2 + |a_3|^2,$$

and

$$K + 3\cos\theta - 1 + 2\csc^2\frac{\theta}{2}|\mathbf{c}|^2 = 0, \quad |\mathbf{c}|^2 = |c_1|^2 + |c_3|^2 + |c_4|^2.$$

It follows that

$$K - 1 + \sec^2\frac{\theta}{2}|\mathbf{a}|^2 + \csc^2\frac{\theta}{2}|\mathbf{c}|^2 = 0.$$

Using (4.3), (4.5), (4.9) and (4.10), equations in (4.3) become

$$\begin{cases} \mathrm{D}a_1 = \omega_{12}a_2 + \omega_{13}a_3 = \cos\frac{\theta}{2}\frac{K - 3\cos\theta - 1}{2}\overline{\varphi}, \\ \mathrm{D}a_2 = \mathrm{d}a_2 + a_2(2\mathtt{i}\rho - \omega_{00} + \omega_{22}) + \omega_{23}a_3 = 0, \\ \mathrm{D}a_3 = \mathrm{d}a_3 + a_3(2\mathtt{i}\rho - \omega_{00} + \omega_{33}) + \omega_{32}a_2 = 0, \\ \mathrm{D}a_4 = \omega_{42}a_2 + \omega_{43}a_3 = 0, \\ \mathrm{D}a_\alpha = \omega_{\alpha 3}a_3 = 0, \quad 5 \le \alpha \le n, \end{cases} \tag{4.11}$$

and

$$\begin{cases} \mathrm{D}c_1 = \mathrm{d}c_1 - c_1(\omega_{00} + 2\mathtt{i}\rho - \omega_{11}) + \omega_{13}c_3 = 0, \\ \mathrm{D}c_2 = \omega_{21}c_1 + \omega_{23}c_3 + \omega_{24}c_4 = \sin\frac{\theta}{2}\frac{K+3\cos\theta-1}{2}\varphi, \\ \mathrm{D}c_3 = \mathrm{d}c_3 - c_3(\omega_{00} + 2\mathtt{i}\rho) + \omega_{31}c_1 + \omega_{33}c_3 + \omega_{34}c_4 = 0, \\ \mathrm{D}c_4 = \mathrm{d}c_4 - c_4(\omega_{00} + 2\mathtt{i}\rho) + \omega_{43}c_3 + \omega_{44}c_4 = 0, \\ \mathrm{D}c_\alpha = \omega_{\alpha 3}c_3 + \omega_{\alpha 4}c_4 = 0, \quad 5 \le \alpha \le n. \end{cases} \tag{4.12}$$

From (2.11), (4.5)–(4.7), (4.9) and (4.10) we get

$$\cos\frac{\theta}{2}\bar{c}_1 = \langle X, \mathbf{c}\rangle = -\overline{\langle Y, \mathbf{a}\rangle} = -\sin\frac{\theta}{2}a_2. \tag{4.13}$$

From (4.10), $\omega_{23} = -\csc\frac{\theta}{2}\bar{c}_3\varphi$. By the second equation in (4.11),

$$\mathrm{d}a_2 + a_2(2\mathtt{i}\rho + \omega_{11} - \omega_{22}) \equiv 0 \pmod{\varphi}. \tag{4.14}$$

On the other hand, taking exterior derivative of the second equation in (4.6), and using (4.5) and (4.10) give

$$\mathrm{d}a_2 + a_2(\mathtt{i}\rho + \omega_{11} - \omega_{22}) \equiv 0 \pmod{\varphi}. \tag{4.15}$$

Therefore, a_2 is a function of analytic type, i.e., either a_2 is identically zero, or it only vanishes at finitely many points.

4.1 Case $a_2 \neq 0$

Case I. Suppose a_2 vanishes at finitely many points. By (4.13), c_1 vanishes at finitely many points, too. By (4.14) and (4.15), we have

$$\rho = 0, \quad K = 0. \tag{4.16}$$

By (4.8),

$$\omega_{00} = \omega_{11} = \omega_{22}. \tag{4.17}$$

From (3.7) and (3.25),

$$|\mathbf{a}|^2 + |\mathbf{c}|^2 = \frac{1}{2}(1 + 3\cos^2\theta), \quad |\mathbf{a}|^2 - |\mathbf{c}|^2 = 2\cos\theta.$$

By (4.14), (4.16) and (4.17), we get $\mathrm{d}a_2 \equiv 0 \pmod{\varphi}$. Thus a_2, $a_3 = \sqrt{|\mathbf{a}|^2 - |a_2|^2}$ and c_1 are constant. By (4.10), $\omega_{23} = -\csc\frac{\theta}{2}\bar{c}_3\varphi$. From (4.11),

$$\begin{cases} a_3\bar{c}_3\csc\frac{\theta}{2}\varphi = 0, \\ a_3(\omega_{33} - \omega_{00}) = -a_2\bar{c}_3\csc\frac{\theta}{2}\varphi, \\ \omega_{42}a_2 + \omega_{43}a_3 = 0, \\ \omega_{\alpha 3}a_3 = 0, \quad 5 \le \alpha \le n. \end{cases} \tag{4.18}$$

As $\omega_{33}-\omega_{00}$ is a pure imaginary 1-form, from the second equation in (4.18), we have $c_3=0$ because $a_2\neq 0$. So that $c_4=\sqrt{|\mathbf{c}|^2-|c_1|^2}$ is constant. From (4.12),

$$\begin{cases} \omega_{34}c_4 = -c_1\sec\frac{\theta}{2}a_3\varphi, \\ c_4(\omega_{44}-\omega_{00}) = 0, \\ \omega_{\alpha 4}c_4 = 0, \quad 5\leq\alpha\leq n. \end{cases} \tag{4.19}$$

We claim that a_3 must be identically zero. In fact, if $a_3\neq 0$, by the first equation in (4.19) we have $c_4\neq 0$. From (4.18) and (4.19),

$$\omega_{44}=\omega_{33}=\omega_{00}, \quad \omega_{\alpha 3}=\omega_{\alpha 4}=0, \quad 5\leq\alpha\leq n. \tag{4.20}$$

In summary, by (4.5), (4.10), (4.17) and (4.20), we have

$$\omega_{00}=\omega_{11}=\omega_{22}=\omega_{33}=\omega_{44},\ \omega_{\alpha A}=0, \quad 0\leq A\leq 4, \quad 5\leq\alpha\leq n. \tag{4.21}$$

and its Maure–Cartan forms are given by

$$\begin{pmatrix} \omega_{00} & -\cos\frac{\theta}{2}\overline{\varphi} & -\sin\frac{\theta}{2}\varphi & 0 & 0 \\ \cos\frac{\theta}{2}\varphi & \omega_{11} & -\sec\frac{\theta}{2}\overline{a}_2\overline{\varphi} & -\sec\frac{\theta}{2}a_3\overline{\varphi} & 0 \\ \sin\frac{\theta}{2}\overline{\varphi} & \sec\frac{\theta}{2}a_2\varphi & \omega_{22} & 0 & -\csc\frac{\theta}{2}c_4\varphi \\ 0 & \sec\frac{\theta}{2}a_3\varphi & 0 & \omega_{33} & -u\varphi \\ 0 & 0 & \csc\frac{\theta}{2}c_4\overline{\varphi} & \overline{u\varphi} & \omega_{44} \end{pmatrix}, \tag{4.22}$$

where $u=\sec\frac{\theta}{2}\frac{a_3}{c_4}c_1$. By (4.22), one can compute that

$$\begin{cases} \mathrm{d}\omega_{00} = -\cos\theta\varphi\wedge\overline{\varphi}, \\ \mathrm{d}\omega_{11} = (\cos^2\frac{\theta}{2}-\sec^2\frac{\theta}{2}|\mathbf{a}|^2)\varphi\wedge\overline{\varphi}, \\ \mathrm{d}\omega_{22} = (-\sin^2\frac{\theta}{2}+\csc^2\frac{\theta}{2}|\mathbf{c}|^2)\varphi\wedge\overline{\varphi}, \\ \mathrm{d}\omega_{33} = (\sec^2\frac{\theta}{2}|a_3|^2+|u|^2)\varphi\wedge\overline{\varphi}, \\ \mathrm{d}\omega_{44} = -(\csc^2\frac{\theta}{2}|c_4|^2+|u|^2)\varphi\wedge\overline{\varphi}. \end{cases} \tag{4.23}$$

From (4.21) and (4.23),

$$\sec^2\frac{\theta}{2}|a_3|^2+|u|^2=-\csc^2\frac{\theta}{2}|c_4|^2-|u|^2. \tag{4.24}$$

From (4.24), we get a contradiction $a_3=c_4=0$.

Thus, we have $a_3=0$. By the second equation in (4.18), $\omega_{32}=0$ and $c_3=0$. By the third equation in (4.18), $\omega_{42}=0$. Thus $c_4=0$. In summary, if f is linearly full, then $n=3$ and there exists a moving frame such that

$$\begin{pmatrix} \omega_{00} & -\cos\frac{\theta}{2}\overline{\varphi} & -\sin\frac{\theta}{2}\varphi \\ \cos\frac{\theta}{2}\varphi & \omega_{11} & -\sec\frac{\theta}{2}\overline{a}_2\overline{\varphi} \\ \sin\frac{\theta}{2}\overline{\varphi} & \sec\frac{\theta}{2}a_2\varphi & \omega_{22} \end{pmatrix},$$

and

$$\mathrm{d}\omega_{00} = \mathrm{d}\omega_{11} = \mathrm{d}\omega_{22}.$$

From (4.23),

$$-\cos\theta = \cos^2\frac{\theta}{2} - \sec^2\frac{\theta}{2}|\mathbf{a}|^2 = -\sin^2\frac{\theta}{2} + \sec^2\frac{\theta}{2}|\mathbf{a}|^2.$$

One can get

$$\cos\theta = 0, \quad |\mathbf{a}|^2 = |\mathbf{c}|^2 = \frac{1}{4}, \quad \kappa = \frac{1}{8}.$$

Up to a rigid motion, $f(M)$ is the Clifford torus in $\mathbb{C}P^2$ given in [9], which is explicitly given by

$$f : T^2 = S^1 \times S^1 \longrightarrow \mathbb{C}P^2, \quad (e^{\mathtt{i}u}, e^{\mathtt{i}v}) \longmapsto [1, e^{\mathtt{i}u}, e^{\mathtt{i}v}].$$

4.2 Case $a_2 \equiv 0$

If $a_2 \equiv 0$, then $\omega_{12} = 0$, $c_1 = 0$ and $a_3 = |\mathbf{a}|$ is constant. From (4.11) and (4.12) we have

$$\begin{cases} \mathrm{D}a_2 = \omega_{23}a_3 = 0, \\ \mathrm{D}a_3 = a_3(2\mathtt{i}\rho - \omega_{00} + \omega_{33}) = 0, \\ \mathrm{D}a_\alpha = \omega_{\alpha 3}a_3 = 0, \quad 4 \le \alpha \le n, \end{cases} \tag{4.25}$$

and

$$\begin{cases} \mathrm{D}c_1 = \omega_{13}c_3 = 0, \\ \mathrm{D}c_3 = \mathrm{d}c_3 - c_3(\omega_{00} + 2\mathtt{i}\rho - \omega_{33}) + \omega_{34}c_4 = 0, \\ \mathrm{D}c_4 = \mathrm{d}c_4 - c_4(\omega_{00} + 2\mathtt{i}\rho - \omega_{44}) + \omega_{43}c_3 = 0, \\ \mathrm{D}c_\alpha = \omega_{\alpha 3}c_3 + \omega_{\alpha 4}c_4 = 0, \quad 5 \le \alpha \le n. \end{cases} \tag{4.26}$$

Case II. If $a_3 \neq 0$, by (4.10) and (4.25), we get

$$2\mathtt{i}\rho - \omega_{00} + \omega_{33} = 0, \quad \omega_{\alpha 3} = 0, \quad 4 \le \alpha \le n,$$

and

$$\omega_{32} = 0, \quad c_3 = 0.$$

Thus $c_4 = |\mathbf{c}|$ is constant. From (4.26) we have

$$\omega_{34}c_4 = 0, \quad \omega_{\alpha 4}c_4 = 0, \quad 5 \le \alpha \le n,$$

and

$$c_4(2\mathtt{i}\rho + \omega_{00} - \omega_{44}) = 0.$$

Case IIa. If $c_4 \neq 0$, then $\omega_{43} = 0$ and $\omega_{\alpha 4} = 0$ for all $\alpha \geq 5$. So, there exists a moving frame such that

$$\begin{pmatrix} \omega_{00} & -\cos\frac{\theta}{2}\overline{\varphi} & -\sin\frac{\theta}{2}\varphi & 0 & 0 \\ \cos\frac{\theta}{2}\varphi & \omega_{11} & 0 & -\sec\frac{\theta}{2}|\mathbf{a}|\overline{\varphi} & 0 \\ \sin\frac{\theta}{2}\overline{\varphi} & 0 & \omega_{22} & 0 & -\csc\frac{\theta}{2}|\mathbf{c}|\varphi \\ 0 & \sec\frac{\theta}{2}|\mathbf{a}|\varphi & 0 & \omega_{33} & 0 \\ 0 & 0 & \csc\frac{\theta}{2}|\mathbf{c}|\overline{\varphi} & 0 & \omega_{44} \end{pmatrix}, \tag{4.27}$$

where

$$\begin{cases} \mathtt{i}\rho - \omega_{00} + \omega_{11} = 0, \\ \mathtt{i}\rho + \omega_{00} - \omega_{22} = 0, \\ 2\mathtt{i}\rho - \omega_{00} + \omega_{33} = 0, \\ 2\mathtt{i}\rho + \omega_{00} - \omega_{44} = 0. \end{cases} \tag{4.28}$$

From (4.27), it is easy to get

$$\begin{cases} \mathrm{d}\omega_{00} = -\cos\theta\varphi \wedge \overline{\varphi}, \\ \mathrm{d}\omega_{11} = (\cos^2\frac{\theta}{2} - \sec^2\frac{\theta}{2}|\mathbf{a}|^2)\varphi \wedge \overline{\varphi}, \\ \mathrm{d}\omega_{22} = (-\sin^2\frac{\theta}{2} + \csc^2\frac{\theta}{2}|\mathbf{c}|^2)\varphi \wedge \overline{\varphi}, \\ \mathrm{d}\omega_{33} = \sec^2\frac{\theta}{2}|\mathbf{a}|^2\varphi \wedge \overline{\varphi}, \\ \mathrm{d}\omega_{44} = -\csc^2\frac{\theta}{2}|\mathbf{c}|^2\varphi \wedge \overline{\varphi}. \end{cases} \tag{4.29}$$

From (2.4), (4.28) and (4.29),

$$\begin{cases} -\frac{1}{2}K + \cos\theta + \cos^2\frac{\theta}{2} - \sec^2\frac{\theta}{2}|\mathbf{a}|^2 = 0, \\ -\frac{1}{2}K - \cos\theta + \sin^2\frac{\theta}{2} - \csc^2\frac{\theta}{2}|\mathbf{c}|^2 = 0, \\ -K + \cos\theta + \sec^2\frac{\theta}{2}|\mathbf{a}|^2 = 0, \\ -K - \cos\theta + \csc^2\frac{\theta}{2}|\mathbf{c}|^2 = 0. \end{cases} \tag{4.30}$$

One can solve (4.30) to get

$$K = \frac{1}{3}, \quad \cos\theta = 0, \quad |\mathbf{a}|^2 = |\mathbf{c}|^2 = \frac{1}{6}, \quad \kappa = 0.$$

By the first equation in (4.29), ω_{00} is a closed. Rotating a suitable angle for e_0 gives $\omega_{00} = 0$. So there exists a moving frame such that

$$\begin{pmatrix} 0 & -\frac{1}{\sqrt{2}}\overline{\varphi} & -\frac{1}{\sqrt{2}}\varphi & 0 & 0 \\ \frac{1}{\sqrt{2}}\varphi & -\mathtt{i}\rho & 0 & -\frac{1}{\sqrt{3}}\overline{\varphi} & 0 \\ \frac{1}{\sqrt{2}}\overline{\varphi} & 0 & \mathtt{i}\rho & 0 & -\frac{1}{\sqrt{3}}\varphi \\ 0 & \frac{1}{\sqrt{3}}\varphi & 0 & -2\mathtt{i}\rho & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}}\overline{\varphi} & 0 & 2\mathtt{i}\rho \end{pmatrix}.$$

Up to a rigid motion, $f(M)$ is the middle element of Veronese sequence in $\mathbb{C}P^4$ explicitly given by

$$
\begin{aligned}
f : S^2 &\longrightarrow \mathbb{C}P^4,\\
z &\longmapsto \big[\sqrt{6}\overline{z}^2, \sqrt{6}(|z|^2-1)\overline{z}, (1-|z|^2)^2 - 2|z|^2, \sqrt{6}(|z|^2-1)z, \sqrt{6}z^2\big],
\end{aligned}
$$

where z is the local coordinate of S^2.

Case IIb. If $c_4 = 0$, then $|\mathbf{c}|^2 = 0$. There exists a moving frame such that

$$
\begin{pmatrix}
\omega_{00} & -\cos\frac{\theta}{2}\overline{\varphi} & -\sin\frac{\theta}{2}\varphi & 0\\
\cos\frac{\theta}{2}\varphi & \omega_{11} & 0 & -\sec\frac{\theta}{2}|\mathbf{a}|\overline{\varphi}\\
\sin\frac{\theta}{2}\overline{\varphi} & 0 & \omega_{22} & 0\\
0 & \sec\frac{\theta}{2}|\mathbf{a}|\varphi & 0 & \omega_{33}
\end{pmatrix}.
$$

We have

$$
\begin{cases}
-\frac{K}{2} + \cos\theta + \cos^2\frac{\theta}{2} - \sec^2\frac{\theta}{2}|\mathbf{a}|^2 = 0,\\
-\frac{K}{2} - \cos\theta + \sin^2\frac{\theta}{2} = 0,\\
-K + \cos\theta + \sec^2\frac{\theta}{2}|\mathbf{a}|^2 = 0.
\end{cases}
$$

One can compute that

$$
K = \frac{4}{7}, \quad \cos\theta = \frac{1}{7}, \quad |\mathbf{a}|^2 = \frac{12}{49}, \quad \kappa = 0.
$$

Up to a rigid motion, $f(M)$ is the second element of Veronese sequence in $\mathbb{C}P^3$ explicitly given by

$$
\begin{aligned}
f : S^2 &\longrightarrow \mathbb{C}P^3,\\
z &\longmapsto \big[-\sqrt{3}\overline{z}, 1-2|z|^2, (2-|z|^2)z, \sqrt{3}z^2\big],
\end{aligned}
$$

where z is the local coordinate of S^2.

Case III. If $a_3 = 0$, then $|\mathbf{a}| = 0$. By choosing a new moving frame such that $c_3 \geq 0$, and $c_\alpha = 0$ for $\alpha \geq 4$. Thus $c_3 = |\mathbf{c}|$ is constant. Equations (4.26) become

$$
c_3(2\mathtt{i}\rho + \omega_{00} - \omega_{33}) = 0, \quad \omega_{\alpha 3}c_3 = 0, \quad 5 \leq \alpha \leq n.
$$

Case IIIa. If $c_3 \neq 0$, by the same arguments as **Case IIb**, One can compute that

$$
K = \frac{4}{7}, \quad \cos\theta = -\frac{1}{7}, \quad |\mathbf{c}|^2 = \frac{12}{49}, \quad \kappa = 0.
$$

Up to a rigid motion, $f(M)$ is the third element of Veronese sequence in $\mathbb{C}P^3$ explicitly given by

$$
\begin{aligned}
f : S^2 &\longrightarrow \mathbb{C}P^3,\\
z &\longmapsto \big[\sqrt{3}\overline{z}^2, (|z|^2-2)\overline{z}, 1-2|z|^2, \sqrt{3}z\big],
\end{aligned}
$$

where z is the local coordinate of S^2.

Case IIIb. If $c_3 = 0$, then $S = 0$. We then have

$$\begin{cases} -\frac{K}{2} + \cos\theta + \cos^2\frac{\theta}{2} = 0, \\ -\frac{K}{2} - \cos\theta + \sin^2\frac{\theta}{2} = 0. \end{cases}$$

It follows that

$$K = 1, \quad \cos\theta = 0, \quad \kappa = 0.$$

As ω_{00} is closed, similarly, there exists a moving frame such that

$$\begin{pmatrix} 0 & -\frac{1}{\sqrt{2}}\overline{\varphi} & -\frac{1}{\sqrt{2}}\varphi \\ \frac{1}{\sqrt{2}}\varphi & -\mathtt{i}\rho & 0 \\ \frac{1}{\sqrt{2}}\overline{\varphi} & 0 & \mathtt{i}\rho \end{pmatrix}.$$

Up to a rigid motion, $f(M)$ is the middle element of Veronese sequence in $\mathbb{C}P^2$ explicitly given by

$$\begin{aligned} f : S^2 &\longrightarrow \mathbb{C}P^2, \\ z &\longmapsto \big[-\sqrt{2}\overline{z}, 1-|z|^2, \sqrt{2}z\big]. \end{aligned}$$

Therefore, the proof of Main Theorem is finished.

Acknowledgements This work was supported by NSFC (Nos. 11401481, 12371055, 11301273, 11971237). The first named author was also supported by the Research Enhancement Fund of Xi'an Jiaotong–Liverpool University (No. REF-18-01-03). The third named author was also supported by the Natural Science Foundation of Jiangsu Province (No. BK20221320).

Conflict of Interest The authors declare no conflict of interest.

References

1. Bando S., Ohnita Y., Minimal 2-spheres with constant curvature in $\mathbb{C}P^n$. J. Math. Soc. Japan, 1987, 39(3): 477–487
2. Bolton J., Jensen G.R., Rigoli M., Woodward L.M., On conformal minimal immersions of S^2 into $\mathbb{C}P^n$. Math. Ann., 1988, 279(4): 599–620
3. Chen B.Y., Ogiue K., On totally real submanifolds. Trans. Amer. Math. Soc., 1974, 193: 257–266
4. Chern S., do Carmo M., Kobayashi S., Minimal submanifolds of a sphere with second fundamental form of constant length. New York–Berlin: Springer, 1970: 59–75
5. Chern S., Wolfson J., Minimal surfaces by moving frames. Amer. J. Math., 1983, 105(1): 59–83
6. Jiao X.X., Peng J.G., Minimal 2-spheres in a complex projective space. Differential Geom. Appl., 2007, 25(5): 506–517
7. Kenmotsu K., Masuda K., On minimal surfaces of constant curvature in two-dimensional complex space form. J. Reine Angew. Math., 2000, 523: 69–101

8. Li Z.Q., Counterexample to the conjecture on minimal S^2 in $\mathbb{C}P^n$ with constant Kähler angle. Manuscripta Math., 1995, 88(4): 417–431
9. Ludden G., Okumura M., Yano K., A totally real surface in $\mathbb{C}P^2$ that is not totally geodesic. Proc. Amer. Math. Soc., 1975, 53(1): 186–190
10. Mo X.H., Minimal surfaces with constant Kähler angle in complex projective spaces. Proc. Amer. Math. Soc., 1994, 121(2): 569–571
11. Ogata T., Curvature pinching theorem for minimal surfaces with constant Kähler angle in complex projective spaces. Tohoku Math. J. (2), 1991, 43(3): 361–374
12. Ogata T., Curvature pinching theorem for minimal surfaces with constant Kähler angle in complex projective spaces, II. Tohoku Math. J. (2), 1993, 45(2): 271–283
13. Ohnita Y., Minimal surfaces with constant curvature and Kähler angle in complex space forms. Tsukuba J. Math., 1989, 13(1): 191–207
14. Simons J., Minimal varieties in Riemannian manifolds. Ann. of Math. (2), 1968, 88: 62–105
15. Tanno S., Compact complex submanifolds immersed in complex projective spaces. J. Differential Geometry, 1973, 8: 629–641
16. Wang J., Fei J., Jiao X.X., Simons-type inequalities for minimal surfaces with constant Kähler angle in a complex hyperquadric. Differential Geom. Appl., 2023, 88: Paper No. 102001, 21 pp.
17. Wang J., Fei J., Xu X.W., Pinching for holomorphic curves in a complex Grassmann manifold $G(2,n;\mathbb{C})$. Differential Geom. Appl., 2022, 80: Paper No. 101840, 15 pp.