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RESEARCH ARTICLE

Hom-Lie Algebras with Derivations

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Abstract In this paper, we first introduce the notion of an HLieDer triple, which includes a Hom-Lie algebra and a derivation. We define a cohomology theory for HLieDer triples with coefficients in a representation. We study central extensions of an HLieDer triple. Finally, we consider homotopy derivations on HLieb_∞ algebras and 2-derivations on Hom-Lie 2-algebras, and we prove that the category of 2-term HLieb_∞ algebras with homotopy derivations and the category of Hom-Lie 2-algebras with 2-derivations are equivalent.

Keywords Hom-Lie algebra, HLieDer triple, cohomology, Hom-Lie 2-algebra, homotopy derivation

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1 Introduction

The notion of Hom-Lie algebras was introduced by Hartwig, Larsson and Silvestrov in [4] in the context of q-deformation theory of Witt and Virasoro algebras [5], which plays an important role in physics and conformal field theory. Hom-Lie algebras were widely studied in the following aspects: Hom-Yang–Baxter equation [12,13], representation and cohomology theory [7], omni-Hom-Lie algebras [10], deformation theory [1], Hom-Lie 2-algebras [9], Hom-Lie bialgebras [8] and Abelian extensions [2], restricted Hom-Lie algebras [3] and Hom-Jordan–Lie algebras [14].

Several years ago, Sheng and Chen introduced HLie_∞-algebras as well as Hom-Lie 2-algebras and showed that the category of 2-term HLie_∞-algebras and the category of Hom-Lie 2-algebras are equivalent in [9]. Recently, Tang, Frégier and Sheng [11] introduced Lie algebras with derivations, which is called a LieDer pair; they studied some properties of LieDer pairs. Due to the importance of Hom-Lie algebras and derivations, it is natural to study their relationship. More precisely, we first introduce the notion of an HLieDer triple, which includes a

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Hom-Lie algebra and a derivation. We define a cohomology theory for HLieDer triples with coefficients in a representation. We study central extensions of an HLieDer triple. Finally, we consider homotopy derivations on HLie_∞ algebras and 2-derivations on Hom-Lie 2-algebras, and we prove that the category of 2-term HLie_∞ algebras with homotopy derivations and the category of Hom-Lie 2-algebras with 2-derivations are equivalent.

The paper is organized as follows. In Section 3, we introduce the notion of an HLieDer triple, which includes a Hom-Lie algebra and a derivation. In Section 4, we define a cohomology theory for HLieDer triples with coefficients in a representation. In Section 5, we study central extensions of an HLieDer triple. In Section 6, we consider homotopy derivations on HLie_∞ algebras and 2-derivations on Hom-Lie 2-algebras. In Section 7, we prove that the category of 2-term HLie_∞ algebras with homotopy derivations is equivalent to the category of Hom-Lie 2-algebras with 2-derivations.

2 Preliminaries

In this paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic 0 and all the vector spaces are over $\mathbb{K}$ and finite-dimensional. We now recall some useful definitions in [6].

Definition 2.1. A Hom-Lie algebra is a triple $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \phi_\mathfrak{g})$ consisting of a linear space $\mathfrak{g}$, a skew-symmetric bilinear map $[\cdot,\cdot]_\mathfrak{g} : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ and an algebra homomorphism $\phi_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$[\phi_\mathfrak{g}(x), [y, z]_\mathfrak{g},]_\mathfrak{g} + [\phi_\mathfrak{g}(y), [z, x]_\mathfrak{g}]_\mathfrak{g} + [\phi_\mathfrak{g}(z), [y, x]_\mathfrak{g}]_\mathfrak{g} = 0, \quad \forall x, y, z \in \mathfrak{g}.$$

A Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \phi_\mathfrak{g})$ is said to be regular (involutive), if $\phi_\mathfrak{g}$ is nondegenerate (satisfies $\phi_\mathfrak{g}^2 = \mathrm{id}_\mathfrak{g}$).

Definition 2.2. A representation of a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \phi_\mathfrak{g})$ is a triple (V, ϕ_V, ρ), where V is a vector space, $\phi_V \in \mathfrak{gl}(V)$, $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ are linear maps such that the following equalities hold for all $x, y \in \mathfrak{g}$:

$$\begin{aligned}
&\rho(\phi_\mathfrak{g}(x)) \circ \phi_V = \phi_V \circ \rho(x);\\
&\rho([x, y]_\mathfrak{g}) \circ \phi_V = \rho(\phi_\mathfrak{g}(x)) \circ \rho(y) - \rho(\phi_\mathfrak{g}(y)) \circ \rho(x).
\end{aligned}$$

Let $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \phi_\mathfrak{g})$ be a Hom-Lie algebra with respect to a representation (V, ϕ_V, ρ). The cohomology of the Hom-Lie algebra $\mathfrak{g}$ with coefficients in V is the cohomology of the cochain complex $\{\mathrm{CH}^*(\mathfrak{g}, V), \partial\}$, where $\mathrm{CH}^n(\mathfrak{g}, V) = \mathrm{Hom}(\mathfrak{g}^{\otimes n}, V)$ for $n \geq 0$, and the coboundary operator $\partial : \mathrm{CH}^n(\mathfrak{g}, V) \to \mathrm{CH}^{n+1}$ $(\mathfrak{g}, V)$ given by

$$(\partial f)(x_1,\dots,x_{n+1}) = \sum_{i=1}^{n+1}(-1)^{i+1}\rho(\phi_{\mathfrak{g}}^{n-1}(x_i))f(x_1,\dots,\widehat{x}_i,\dots,x_{n+1})$$
$$+\sum_{1\le i<j\le n+1}(-1)^{i+j}f([x_i,x_j]_{\mathfrak{g}},\phi_{\mathfrak{g}}(x_1),\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,\phi_{\mathfrak{g}}(x_{n+1})),$$

for $x_1,\dots,x_{n+1}\in\mathfrak{g}$. The corresponding cohomology groups are denoted by $\mathrm{HH}^*(\mathfrak{g},V)$.

3 HLieDer Triples

In this section, we define the representations and cohomology of an HLieDer triple.

Let $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\phi_{\mathfrak{g}})$ be a Hom-Lie algebra. Recall from [7] that a linear map $\varphi_{\mathfrak{g}}:\mathfrak{g}\to\mathfrak{g}$ is a derivation on $\mathfrak{g}$ if it satisfies

$$\varphi_{\mathfrak{g}}\circ\phi_{\mathfrak{g}}=\phi_{\mathfrak{g}}\circ\varphi_{\mathfrak{g}},$$
$$\varphi_{\mathfrak{g}}([x,y]_{\mathfrak{g}})=[\varphi_{\mathfrak{g}}(x),y]_{\mathfrak{g}}+[x,\varphi_{\mathfrak{g}}(y)]_{\mathfrak{g}},\quad \text{for } x,y\in\mathfrak{g}.$$

We call a triple $(\mathfrak{g},\phi_{\mathfrak{g}},\varphi_{\mathfrak{g}})$ of a Hom-Lie algebra $(\mathfrak{g},\phi_{\mathfrak{g}})$ and a derivation $\phi_{\mathfrak{g}}$ on $(\mathfrak{g},\phi_{\mathfrak{g}})$, a HLieDer triple.

Example 3.1. Let $(D,\cdot,\phi_D)$ be a Hom-associative algebra. We know that (D,ϕ_D) is a Hom-Lie algebra with the bracket [6]

$$[x,y]_D := x\cdot y-y\cdot x,\quad \text{for } x,y\in D.$$

If φ_D is a derivation for the Hom-associative algebra, then φ_D is also a derivation for the induced Hom-Lie algebra structure on D.

Definition 3.1. Let $(\mathfrak{g},\phi_{\mathfrak{g}},\varphi_{\mathfrak{g}})$ and $(\mathfrak{h},\phi_{\mathfrak{h}},\varphi_{\mathfrak{h}})$ be two HLieDer triples. An HLieDer triple morphism between them is a Hom-Lie algebra morphism $f:\mathfrak{g}\to\mathfrak{h}$ satisfying $\varphi_{\mathfrak{h}}\circ f=f\circ\varphi_{\mathfrak{g}}$.

Definition 3.2. A representation of an HLieDer triple $(\mathfrak{g},\phi_{\mathfrak{g}},\varphi_{\mathfrak{g}})$ on a vector space V with respect to $\varphi_V\in\mathfrak{gl}(V)$ in which (V,ϕ_V,ρ) is a representation of $(\mathfrak{g},\phi_{\mathfrak{g}})$ such that the following equalities are satisfied:

$$\varphi_V(\rho(x)v)=\rho(\varphi_{\mathfrak{g}}(x))v+\rho(x)\varphi_V(v),$$

for all $x\in\mathfrak{g}$ and $v\in V$.

The following proposition is directly to check and we omit it.

Proposition 3.1. *Let $(\mathfrak{g},\phi_{\mathfrak{g}},\varphi_{\mathfrak{g}})$ be an HLieDer triple with respect to a representation $(V,\phi_V,\rho,\varphi_V)$. Then $(\mathfrak{g}\oplus V,\phi_{\mathfrak{g}}\oplus\phi_V,\varphi_{\mathfrak{g}}\oplus\varphi_V)$ is an HLieDer triple, where*

$$[(x,u),(y,v)]=([x,y]_{\mathfrak{g}},\rho(x)v-\rho(y)u),$$

for all $x,y\in\mathfrak{g}$ and $u,v\in V$.

4 Cohomologies of HLieDer Triples

In this section, we define a cohomology of an HLieDer triple with coefficients in a representation.

Let $(\mathfrak{g}, \phi_{\mathfrak{g}}, \varphi_{\mathfrak{g}})$ be an HLieDer triple with respect to the representation $(V, \phi_V, \rho, \varphi_V)$. We define the cochain groups by $C^0_{\text{HLieDer}}(\mathfrak{g}, V) := 0$, $C^1_{\text{HLieDer}}(\mathfrak{g}, V) := \text{Hom}(\mathfrak{g}, V)$ and

$$C^n_{\text{HLieDer}}(\mathfrak{g}, V) := \text{Hom}(\mathfrak{g}^{\otimes n}, V) \times \text{Hom}(\mathfrak{g}^{\otimes n-1}, V) \quad \text{for } n \geq 2.$$

Define the map

$$\begin{aligned}
&\delta : \text{Hom}(\mathfrak{g}^{\otimes n}, V) \to \text{Hom}(\mathfrak{g}^{\otimes n}, V),\\
&\delta(f) = \sum_{i=1}^{n} f \circ (\phi^n_{\mathfrak{g}} \otimes \cdots \otimes \varphi_{\mathfrak{g}} \otimes \cdots \otimes \phi^n_{\mathfrak{g}}) - \varphi_V \circ f.
\end{aligned}$$

Then we have the following

Lemma 4.1. *The map δ commutes with ∂, i.e., $\partial \circ \delta = \delta \circ \partial$.*

Proof. Similar to [11]. □

Finally, we define the coboundary operator $\pi : C^n_{\text{HLieDer}}(\mathfrak{g}, V) \to C^{n+1}_{\text{HLieDer}}(\mathfrak{g}, V)$ by

$$\begin{aligned}
&\pi(f) = (\partial f, -\delta f), \quad \text{for } f \in C^1_{\text{HLieDer}}(\mathfrak{g}, V),\\
&\pi(f_n, \widetilde{f_n}) = (\partial f_n, \partial \widetilde{f_n} + (-1)^n \delta f_n), \quad \text{for } (f_n, \widetilde{f_n}) \in C^n_{\text{HLieDer}}(\mathfrak{g}, V),\ n \geq 2.
\end{aligned}$$

Proposition 4.1. *The map π satisfies $\pi^2 = 0$.*

Proof. For any $f \in C^1_{\text{HLieDer}}(\mathfrak{g}, V)$, we have

$$\pi^2 f = \pi(\partial f, -\delta f) = (\partial^2 f, -\delta\partial f + \partial\delta f) = 0.$$

For any $(f_n, \widetilde{f_n}) \in C^n_{\text{HLieDer}}(\mathfrak{g}, V), n \geq 2$, we have

$$\begin{aligned}
\pi^2(f_n, \widetilde{f_n}) &= \pi(\partial f_n, \partial \widetilde{f_n} + (-1)^n \delta f_n)\\
&= (\partial^2 f_n, \partial^2 \widetilde{f_n} + (-1)^n \partial\delta f_n + (-1)^{n+1} \delta\partial f_n)\\
&= 0.
\end{aligned}$$

And the proof is finished. □

Thus, $(C^*_{\text{HLieDer}}(\mathfrak{g}, V), \pi)$ is a cochain complex. The corresponding cohomology groups are denoted by $H^*_{\text{HLieDer}}(\mathfrak{g}, V)$.

5 Central Extensions of HLieDer Triples

In this section, we study the central extensions of HLieDer triples. We show that isomorphism classes of central extensions are classified by the second cohomology group of HLieDer triples with coefficients in the trivial representation.

Let $(\mathfrak{g}, \phi_\mathfrak{g}, \varphi_\mathfrak{g})$ be an HLieDer triple and $(\mathfrak{k}, \phi_\mathfrak{k}, \varphi_\mathfrak{k})$ be an Abelian HLieDer triple, i.e., the Hom-Lie bracket $\mathfrak{k}$ is trivial.

Definition 5.1. A central extension $(\mathfrak{h}, \phi_\mathfrak{h}, \varphi_\mathfrak{h})$ of $(\mathfrak{g}, \phi_\mathfrak{g}, \varphi_\mathfrak{g})$ by the Abelian HLieDer triple $(\mathfrak{k}, \phi_\mathfrak{k}, \varphi_\mathfrak{k})$ is an exact sequence of HLieDer triples

$$0 \longrightarrow (\mathfrak{k}, \phi_\mathfrak{k}, \varphi_\mathfrak{k}) \xrightarrow{\ i\ } (\mathfrak{h}, \phi_\mathfrak{h}, \varphi_\mathfrak{h}) \xrightarrow{\ p\ } (\mathfrak{g}, \phi_\mathfrak{g}, \varphi_\mathfrak{g}) \longrightarrow 0$$

such that $[i(k), h]_\mathfrak{h} = 0 = [h, i(k)]_\mathfrak{h}$, $k \in \mathfrak{k}$, $h \in \mathfrak{h}$.

One may identify $\mathfrak{k}$ with the corresponding subalgebra of $\mathfrak{h}$ (via the map i). With this identification, we have $\varphi_\mathfrak{k} = \varphi_\mathfrak{h}|_\mathfrak{k}$.

Definition 5.2. Two central extensions $(\mathfrak{h}, \phi_\mathfrak{h}, \varphi_\mathfrak{h})$ and $(\mathfrak{h}', \phi_{\mathfrak{h}'}, \varphi_{\mathfrak{h}'})$ are said to be isomorphic if there exists an HLieDer triple isomorphism $\eta : (\mathfrak{h}, \phi_\mathfrak{h}, \varphi_\mathfrak{h}) \to (\mathfrak{h}', \phi_{\mathfrak{h}'}, \varphi_{\mathfrak{h}'})$ such that the following commutative diagram holds:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathfrak{k}, \phi_\mathfrak{k}, \varphi_\mathfrak{k}) & \xrightarrow{\ i\ } & (\mathfrak{h}, \phi_\mathfrak{h}, \varphi_\mathfrak{h}) & \xrightarrow{\ p\ } & (\mathfrak{g}, \phi_\mathfrak{g}, \varphi_\mathfrak{g}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \eta} & & & & \\
0 & \longrightarrow & (\mathfrak{k}, \phi_\mathfrak{k}, \varphi_\mathfrak{k}) & \xrightarrow{\ i'\ } & (\mathfrak{h}', \phi_{\mathfrak{h}'}, \varphi_{\mathfrak{h}'}) & \xrightarrow{\ q\ } & (\mathfrak{g}', \phi_{\mathfrak{g}'}, \varphi_{\mathfrak{g}'}) & \longrightarrow & 0.
\end{array}$$

A section of a central extension $(\mathfrak{h}, \phi_\mathfrak{h}, \varphi_\mathfrak{h})$ of $(\mathfrak{g}, \phi_\mathfrak{g}, \varphi_\mathfrak{g})$ by $(\mathfrak{k}, \phi_\mathfrak{k}, \varphi_\mathfrak{k})$ is a linear map $s : \mathfrak{g} \to \mathfrak{h}$ such that $p \circ s = \mathrm{id}_\mathfrak{g}$, $s \circ \phi_\mathfrak{g} = \phi_\mathfrak{h} \circ s$.

Let $s : \mathfrak{g} \to \mathfrak{h}$ be any section of p. Define linear maps $\psi : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{k}$ and $\chi : \mathfrak{g} \to \mathfrak{k}$ by

$$\psi(x, y) := [s(x), s(y)]_\mathfrak{h} - s[x, y]_\mathfrak{g}, \quad \chi(x) := \varphi_\mathfrak{h}(s(x)) - s(\varphi_\mathfrak{g}(x)), \quad \forall x, y \in \mathfrak{g}.$$

One can obtain an HLieDer triple $(\mathfrak{g} \oplus \mathfrak{k}, \phi_\mathfrak{g} + \phi_\mathfrak{k}, \varphi_\chi)$, where $[\cdot, \cdot]_\psi$ and φ_χ are given by

$$\begin{aligned}
&[x + k, y + l]_\psi = [x, y]_\mathfrak{g} + \psi(x, y), \\
&\varphi_\chi(x + k) = \varphi_\mathfrak{g}(x) + \varphi_\mathfrak{k}(k) + \chi(x), \quad \forall x, y \in \mathfrak{g},\ k, l \in \mathfrak{k}.
\end{aligned}$$

Proposition 5.1. *With notions as above, the vector space* $(\mathfrak{g} \oplus \mathfrak{k}, \phi_\mathfrak{g} + \phi_\mathfrak{k})$ *is a Hom-Lie algebra if and only if* ψ *is a 2-cocycle in the Hom-Lie algebra cohomology of* $(\mathfrak{g}, \phi_\mathfrak{g})$ *with coefficients in the trivial representation* $(\mathfrak{k}, \phi_\mathfrak{k})$*.* φ_χ *is a derivation for the above Hom-Lie algebra if and only if* χ *satisfies* $\partial(\chi) + \delta\psi = 0$.

Proof. The bracket $[\cdot,\cdot]_\psi$ is a Hom-Lie bracket if it satisfies

$$\begin{aligned}&[[x+j,y+k]_\psi,\phi_{\mathfrak{g}}(z)+\phi_{\mathfrak{k}}(l)]_\psi\\&\quad=[[x+j,z+l]_\psi,\phi_{\mathfrak{g}}(y)+\phi_{\mathfrak{k}}(k)]_\psi+[\phi_{\mathfrak{g}}(x)+\phi_{\mathfrak{k}}(j),[y+k,z+l]_\psi]_\psi.\end{aligned}$$

This is equivalent to

$$\psi([x,y]_{\mathfrak{g}},\phi_{\mathfrak{g}}(z))=\psi([x,z]_{\mathfrak{g}},\phi_{\mathfrak{g}}(y))+\psi(\phi_{\mathfrak{g}}(x),[y,z]_{\mathfrak{g}}),$$

which is same as $\partial(\psi)=0$, where ∂ is the Hom-Lie algebra coboundary operator of $(\mathfrak{g},\phi_{\mathfrak{g}})$ with coefficients in the trivial representation $(\mathfrak{k},\phi_{\mathfrak{k}})$.

The map φ_χ is a derivation for the bracket $[\cdot,\cdot]_\psi$ if

$$\varphi_\chi([x+j,y+k]_\psi)=[\varphi_\chi(x+j),y+k]_\psi+[x+j,\varphi_\chi(y+k)]_\psi.$$

This is equivalent to

$$\varphi_{\mathfrak{k}}(\psi(x,y))+\chi([x,y]_{\mathfrak{g}})=\psi(\varphi_{\mathfrak{g}}(x),y)+\psi(x,\varphi_{\mathfrak{g}}(y)),$$

which is same to $\partial(\chi)+\delta\psi=0$. And the proof is finished. □

Proposition 5.2. *The cohomology class of the* 2*-cocycle* (ψ,χ) *does not depend on the choice of sections of* p.

Proof. Let s_1 and s_2 be two sections of p. Consider the map $u:\mathfrak{g}\to\mathfrak{k}$ by $u(x):=s_1(x)-s_2(x)$. Then we have

$$\begin{aligned}\psi_1(x,y)&=[s_1(x),s_2(y)]_{\mathfrak{g}}-s_1[x,y]_{\mathfrak{g}}\\&=[s_2(x)+u(x),s_2(y)+u(y)]_{\mathfrak{g}}-s_2[x,y]_{\mathfrak{g}}-u[x,y]_{\mathfrak{g}}\\&=\psi_2(x,y)-u[x,y]_{\mathfrak{g}}\end{aligned}$$

and

$$\begin{aligned}\chi_1(x)&=\varphi_{\mathfrak{h}}(s_1(x))-s_1(\varphi_{\mathfrak{g}}(x))\\&=\varphi_{\mathfrak{h}}(s_2(x)+u(x))-s_2(\varphi_{\mathfrak{g}}(x))-u(\varphi_{\mathfrak{g}}(x))\\&=\chi_2(x)+\varphi_{\mathfrak{k}}(u(x))-u(\varphi_{\mathfrak{g}}(x)).\end{aligned}$$

This shows that $(\psi_1,\chi_1)-(\psi_2,\chi_2)=\pi u$. Therefore, the cohomology class of the 2-cocycle (ψ,χ) does not depend on the choice of sections of p. □

Theorem 5.1. *Let* $(\mathfrak{g},\phi_{\mathfrak{g}},\varphi_{\mathfrak{g}})$ *be an HLieDer triple and* $(\mathfrak{k},\phi_{\mathfrak{k}},\varphi_{\mathfrak{k}})$ *be an abelian HLieDer triple. Then the isomorphism classes of central extensions of* $(\mathfrak{g},\phi_{\mathfrak{g}},\varphi_{\mathfrak{g}})$ *by* $(\mathfrak{k},\phi_{\mathfrak{k}},\varphi_{\mathfrak{k}})$ *are classified by the second cohomology group* $H^2_{\mathrm{LieDer}}(\mathfrak{g},\mathfrak{k})$.

Proof. Let $(\mathfrak{h}, \phi_{\mathfrak{h}}, \varphi_{\mathfrak{h}})$ and $(\mathfrak{h}', \phi_{\mathfrak{h}'}, \varphi_{\mathfrak{h}'})$ be two isomorphic central extensions. Suppose the isomorphism is given by a map $\eta : (\mathfrak{h}, \phi_{\mathfrak{h}}, \varphi_{\mathfrak{h}}) \to (\mathfrak{h}', \phi_{\mathfrak{h}'}, \varphi_{\mathfrak{h}'})$. For any section s of the map p, we have

$$p' \circ (\eta \circ s) = (p' \circ \eta) \circ s = p \circ s = \mathrm{id}_{\mathfrak{g}}.$$

This shows that $s' := \eta \circ s$ is a section of the map p'. Since η is a morphism of HLieDer triples, we have $\eta|_{\mathfrak{k}} = \mathrm{id}_{\mathfrak{k}}$. Hence, we have

$$\begin{aligned}
&\psi'(x, y) = [s'(x), s'(y)]_{\mathfrak{h}'} - s'[x, y]_{\mathfrak{g}} = \psi(x, y),\\
&\chi'(x) = \varphi_{\mathfrak{h}'}(s'(x)) - s'(\varphi_{\mathfrak{g}}(x)) = \chi(x).
\end{aligned}$$

Therefore, the isomorphic central extensions give rise to the same element in $H^2_{\mathrm{LieDer}}(\mathfrak{g}, \mathfrak{k})$.

Conversely, consider two cohomologous 2-cocycles (ψ, χ) and (ψ', χ'). There exists a linear map $v : \mathfrak{g} \to \mathfrak{k}$ such that $(\psi, \chi) - (\psi', \chi') = dv$. Then we have the corresponding HLieDer triples $(\mathfrak{g} \oplus \mathfrak{k}, \phi_{\mathfrak{g}} + \phi_{\mathfrak{k}}, \varphi_{\chi})$ and $(\mathfrak{g} \oplus \mathfrak{k}, \phi_{\mathfrak{g}} + \phi_{\mathfrak{k}}, \varphi_{\chi'})$. These two HLieDer triples are isomorphic via the map $\eta : \mathfrak{g} \oplus \mathfrak{k} \to \mathfrak{g} \oplus \mathfrak{k}$ given by $\eta(x, k) = x + k + v(x)$. The map η is an isomorphism of central extensions. □

6 Homotopy Derivations on 2-term HLie$_\infty$-algebras

The notion of HLie$_\infty$-algebras was introduced by [9]. In this section, we pay our attention to those HLie$_\infty$-algebras whose underlying graded vector space L is concentrated in degrees 0 and 1. We call them 2-term HLie$_\infty$-algebras. They are related to categorification of Hom-Lie algebras.

Definition 6.1. A 2-term HLie$_\infty$-algebra consists of the following data:

- a complex of vector spaces $L_1 \xrightarrow{d} L_0$,
- bilinear maps $l_2 : L_i \otimes L_j \to L_{i+j}$, where $i + j \leq 1$,
- a trilinear map $l_3 : L_0 \otimes L_0 \otimes L_0 \to L_1$,
- two linear maps $\phi_0 : L_0 \to L_0$ and $\phi_1 : L_1 \to L_1$ satisfying $\phi_0 \circ d = d \circ \phi_1$ and

$$\phi_1 \circ l_3 = l_3 \circ (\phi_0 \otimes \phi_0 \otimes \phi_0)$$

such that for any $x, y, z, w \in L_0$ and $m, n \in L_1$, the following equalities are satisfied:

(a) $l_2(x, y) = -l_2(y, x)$,

(b) $l_2(x, m) = -l_2(m, x)$,

(c) $l_2(m, n) = 0$,

(d) $dl_2(x,m) = l_2(x,dm)$,

(e) $l_2(dm,n) = l_2(m,dn)$,

(f) $\phi_0(l_2(x,y)) = l_2(\phi_0(x),\phi_0(y))$,

(g) $\phi_1(l_2(x,m)) = l_2(\phi_0(x),\phi_1(m))$,

(h) $dl_3(x,y,z) = l_2(l_2(x,y),\phi_0(z)) - l_2(l_2(x,z),\phi_0(y)) - l_2(\phi_0(x),l_2(y,z))$,

(i) $l_3(x,y,dm) = l_2(l_2(x,y),\phi_1(m)) - l_2(\phi_0(x),l_2(y,m)) - l_2(l_2(x,m),\phi_0(y))$,

(j)
$$\begin{aligned} &l_2(\phi_0^2(x),l_3(y,z,w)) + l_2(l_3(x,z,w),\phi_0^2(y)) \\ &\quad - l_2(l_3(x,y,w),\phi_0^2(z)) + l_2(l_3(x,y,z),\phi_0^2(w)) \\ &= l_3(l_2(x,y),\phi_0(z),\phi_0(w)) - l_3(l_2(x,z),\phi_0(y),\phi_0(w)) \\ &\quad + l_3(l_2(x,w),\phi_0(y),\phi_0(z)) - l_3(\phi_0(x),l_2(y,z),\phi_0(w)) \\ &\quad + l_3(\phi_0(x),l_2(y,w),\phi_0(z)) + l_3(\phi_0(x),\phi_0(y),l_2(z,w)). \end{aligned}$$

We denote a 2-term HLie$_\infty$-algebra as above by $(L_1 \xrightarrow{d} L_0, l_2, l_3, \phi_0, \phi_1)$.

Definition 6.2. Let $L = (L_1 \xrightarrow{d} L_0, l_2, l_3, \phi_0, \phi_1)$ and $L' = (L'_1 \xrightarrow{d'} L'_0, l'_2, l'_3, \phi'_0, \phi'_1)$ be two 2-term HLie$_\infty$-algebras. A morphism $f : L \to L'$ consists of

- a chain map $f : L \to L'$ (which consists of linear maps $f : L_0 \to L'_0$ and $f_1 : L_1 \to L'_1$ with $f_0 \circ d = d' \circ f_1$) satisfying
$$\phi_0 \circ f_0 = f_0 \circ \phi_0, \quad \phi_1 \circ f_1 = f_1 \circ \phi_1,$$
- a bilinear map $f_2 : L_0 \otimes L_0 \to L'_1$ satisfying $\phi'_1 \circ f_2 = f_2 \circ (\phi_0 \otimes \phi_0)$, such that for any $x,y,z \in L_0$ and $m \in L_1$, the following conditions hold:

(a) $d(f_2(x,y)) = f_0(l_2(x,y)) - l'_2(f_0(x),f_0(y))$,

(b) $f_2(x,dm) = f_1(l_2(x,m)) - l'_2(f_0(x),f_1(m))$,

(c)
$$\begin{aligned} &f_1(l_3(x,y,z)) + l'_2(f_0(x,y),\phi'_0 f_0(z)) - l'_2(f_2(x,z),\phi'_0 f_0(y)) \\ &\quad - l'_2(\phi'_0 f_0(x),f_2(y,z)) + f_2(l_2(x,y),\phi_0(z)) - f_2(l_2(x,z),\phi_0(y)) \\ &\quad - f_2(\phi_0(x),l_2(y,z)) - l'_3(f_0(x),f_0(y),f_0(z)) = 0. \end{aligned}$$

If $f = (f_0,f_1,f_2) : L \to L'$ and $g = (g_0,g_1,g_2) : L' \to L''$ are two morphisms of 2-term HLieb$_\infty$-algebras, their composition $g \circ f : L \to L''$ is defined by $(g \circ f)_0 = g_0 \circ f_0$, $(g \circ f)_1 = g_1 \circ f_1$ and
$$(g \circ f)_2(x,y) = g_2(f_0(x),f_0(y)) + g_1(f_2(x,y)), \quad \forall x,y \in L_0.$$
For any 2-term HLie$_\infty$-algebra L, the identity morphism $\mathrm{id}_L : L \to L$ is given by the identity chain map $L \to L$ together with $(1_L)_2 = 0$.

The collection of 2-term HLie$_\infty$-algebras and morphisms between them form a category. We denote this category by *2HLie*$_\infty$.

Definition 6.3. Let $L = (L_1 \xrightarrow{d} L_0, l_2, l_3, \phi_0, \phi_1)$ be a 2-term HLie_∞-algebra. A homotopy derivation on it consists of a chain map of the underlying chain complex (i.e., linear maps $\theta_0 : L_0 \to L_0$ and $\theta_1 : L_1 \to L_1'$ with $\theta_0 \circ d = d' \circ \theta_1$) satisfying

$$\phi_0 \circ \theta_0 = \theta_0 \circ \phi_0, \quad \phi_1 \circ \theta_1 = \theta_1 \circ \phi_1,$$

and a bilinear map $\theta_2 : L_0 \otimes L_0 \to L_1'$ satisfying $\phi_1' \circ \theta_2 = \theta_2 \circ (\phi_0 \otimes \phi_0)$, such that for any $x, y, z \in L_0$ and $m \in L_1$, the following conditions hold:

(a) $d(\theta_2(x, y)) = \theta_0(l_2(x, y)) - l_2(\theta_0(x), y) - l_2(x, \theta_0(y))$,

(b) $\theta_2(x, dm) = \theta_1(l_2(x, m)) - l_2(\theta_0(x), m) - l_2(x, \theta_1(m))$,

(c) $$\begin{aligned} &l_3(\theta_0(x), y, z) + l_3(x, \theta_0(y), z) + l_3(x, y, \theta_0(z)) - \theta_1(l_3(x, y, z)) \\ &= l_2(\theta_2(x, y), \phi_0(z)) - l_2(\theta_2(x, z), \phi_0(y)) - l_2(\phi_0(x), \theta_2(y, z)) \\ &+\theta_2(l_2(x, y), \phi_0(z)) - \theta_2(l_2(x, z), \phi_0(y)) - \theta_2(\phi_0(x), l_2(y, z)). \end{aligned}$$

A 2-term HLie_∞-algebra with a homotopy derivation as above is denoted by the triple $((L_1 \xrightarrow{d} L_0, l_2, l_3), (\phi_0, \phi_1), (\theta_0, \theta_1, \theta_2))$. Such a triple is called a $\mathrm{2HLieDer}_\infty$ triple.

Definition 6.4. Let $((L_1 \xrightarrow{d} L_0, l_2, l_3), (\phi_0, \phi_1), (\theta_0, \theta_1, \theta_2))$ and $((L_1' \xrightarrow{d'} L_0', l_2', l_3'), (\phi_0', \phi_1'), (\theta_0', \theta_1', \theta_2'))$ be $\mathrm{2HLieDer}_\infty$ triples. A morphism between them consists of a morphism (f_0, f_1, f_2) between the underlying 2-term HLie_∞-algebras and a linear map $\Psi : L_0 \to L_1'$ satisfying

(1) $\Psi \circ \phi_0 = \phi_1' \circ \Psi$,

(2) $f_0(\theta_0(x)) - \theta_0'(f_0(x)) = d'(\Psi(x))$,

(3) $f_1(\theta_1(m)) - \theta_1'(f_1(m)) = \Psi(dx)$,

(4) $$\begin{aligned} f_1(\theta_2(x, y)) - \theta_2'(f_0(x), f_0(y)) &= \theta_1'(f_2(x, y)) - f_2(\theta_0(x), y) - f_2(x, \theta_0(y)) \\ &+ \Psi(l_2(x, y)) - l_2'(\Psi(x), f_0(y)) - l_2'(f_0(x), \Psi(y)). \end{aligned}$$

We denote the category of $\mathrm{2HLieDer}_\infty$ triples and morphisms between them by *2HLieDer*$_\infty$. An HLie_∞ triple is said to be skeletal if the underlying 2-term HLie_∞-algebra is skeletal, i.e., $d = 0$.

Theorem 6.1. *There is a one-to-one correspondence between skeletal 2-term* HLie_∞*-algebras with homotopy derivations and triple* $((\mathfrak{g}, \phi_\mathfrak{g}, \varphi_\mathfrak{g}), (V, \phi_V, \varphi_V), (\theta, \overline{\theta}))$*, where* $(\mathfrak{g}, \phi_\mathfrak{g}, \varphi_\mathfrak{g})$ *is an HLieDer triple,* (V, ϕ_V, φ_V) *is a representation and* $(\theta, \overline{\theta})$ *is a 3-cocycle of the HLieDer triple with coefficients in the representation.*

Proof. Let $(L_1 \xrightarrow{0} L_0, l_2, l_3, (\phi_0, \phi_1), (\theta_0, \theta_1, \theta_2))$ be a skeletal 2-term HLie$_\infty$-algebra with a homotopy derivation. Then θ_0 is a derivation for the Hom-Lie algebra (L_0, l_2, ϕ_0). We have that (L_1, ϕ_1, θ_1) is a representation of the HLieDer triple (L_0, ϕ_0, θ_0) from Definition 5.3. It remains to find out a 3-cocycle of the HLieDer triple (L_0, ϕ_0, θ_0) with coefficients in the representation (L_1, ϕ_1, θ_1). Note that the condition (c) in Definition 5.3 is same as $\partial(\theta_2) + \delta(l_3) = 0$. Therefore $(l_3, -\theta_2)$ is the required 3-cocycle.

Conversely, given a triple $((\mathfrak{g}, \phi_{\mathfrak{g}}, \varphi_{\mathfrak{g}}), (V, \phi_V, \varphi_V), (\theta, \overline{\theta}))$ as in the statement, define $L_0 = L$, $L_1 = V$, $\phi_0 = \phi_{\mathfrak{g}}$, $\phi_1 = \phi_V$ and $\theta_1 = \varphi_{\mathfrak{g}}$, $\theta_2 = \varphi_V$, $\theta_2 = -\overline{\theta}$. We define multiplications $l_2 : L_i \otimes L_j \to L_{i+j}$ and $l_3 : L_0 \otimes L_0 \otimes L_0 \to L_1$ by

$$l_2(x, y) = [x, y], \quad l_2(x, m) = [x, m], \quad l_2(m, x) = [m, x], \quad l_3 = 0,$$

for $x, y, z \in L_0 = L$ and $m \in L_1 = V$. Then it is easy to verify that $((L_1 \xrightarrow{0} L_0, l_2, l_3), (\phi_0, \phi_1), (\theta_0, \theta_1, \theta_2))$ is a skeletal 2-term HLie$_\infty$-algebra. The above two correspondences are inverses to each other. □

An HLie$_\infty$ pair is said to be strict if the underlying 2-term HLie$_\infty$-algebra is strict, i.e., $l_3 = 0$, $\theta_2 = 0$. Next we introduce crossed modules of HLieDer triples and show that strict 2-term HLie$_\infty$-algebras are in one-to-one correspondence with crossed module of HLieDer triples.

Definition 6.5. A crossed module of HLieDer triples consists of a tuple $((\mathfrak{g}, \phi_{\mathfrak{g}}, \varphi_{\mathfrak{g}}), (\mathfrak{h}, \phi_{\mathfrak{h}}, \varphi_{\mathfrak{h}}), dt, \Lambda)$ where $(\mathfrak{g}, \phi_{\mathfrak{g}}, \varphi_{\mathfrak{g}})$ and $(\mathfrak{h}, \phi_{\mathfrak{h}}, \varphi_{\mathfrak{h}})$ are HLieDer triples, $dt : \mathfrak{g} \to \mathfrak{h}$ is an HLieDer triple morphism and

$$\Lambda : \mathfrak{h} \to gl(\mathfrak{g}), \quad x \mapsto \Lambda_x,$$

satisfying $\Lambda_{\phi_{\mathfrak{h}}(x)} \circ \phi_{\mathfrak{g}} = \phi_{\mathfrak{g}} \circ \Lambda_x$ such that for $m, n \in \mathfrak{g}$, $x, y \in \mathfrak{h}$,

(a) $dt(\Lambda_x(m)) = [x, dt(m)]_{\mathfrak{h}}$,

(b) $\Lambda_{dt(m)}(n) = [m, n]_{\mathfrak{g}}$,

(c) $\Lambda_{[x,y]_{\mathfrak{h}}} \circ \phi_{\mathfrak{g}} = \Lambda_{\phi_{\mathfrak{h}}(x)} \Lambda_y - \Lambda_{\phi_{\mathfrak{h}}(y)} \Lambda_x$,

(d) $\varphi_{\mathfrak{g}}(\Lambda_x(m)) = \Lambda_{\varphi_{\mathfrak{h}}(x)}(m) + \Lambda_x(\varphi_{\mathfrak{g}}(m))$.

When $\varphi_{\mathfrak{g}} = \varphi_{\mathfrak{h}} = 0$, we can obtain the crossed module of Hom-Lie algebras [2].

Theorem 6.2. *There is a one-to-one correspondence between strict 2-term HLieDer$_\infty$-algebras and crossed module of HLieDer triples.*

Proof. Let $(L_1 \xrightarrow{d} L_0,\ l_2,\ l_3 = 0,\ (\phi_0, \phi_1), (\theta_0, \theta_1, \theta_2))$ be a strict 2-term HLieDer$_\infty$-algebra. Then (L, ϕ_0) is a Hom-Lie algebra. θ_0 is a derivation for the Hom-Lie algebra (L_0, l_2, ϕ_0) and θ_1 is a derivation for the Hom-Lie algebra (L_1, l_2, ϕ_1) from Definition 5.3. Thus (L_0, ϕ_0, θ_0) and (L_1, ϕ_1, θ_1) are both HLieDer triples. Since $\theta_0 \circ d = d \circ \theta_1$, the map $dt = d : L_1 \to L_0$ is a morphism of HLieDer triples. Finally, the condition (b) of Definition 5.3 is equivalent to the condition (d) of Definition 5.6. And the proof is finished. □

7 Categorification of HLieDer Triples

In [9] the authors introduced HLie_∞-algebras as well as Hom-Lie 2-algebras and showed that the category of 2-term HLie_∞-algebras and the category of Hom-Lie 2-algebras are equivalent. In this section, we introduce categorified derivations (also called 2-derivations) on Hom-Lie 2-algebras.

Definition 7.1. A Hom-Lie 2-algebra is a 2-vector space L equipped with
- a bilinear functor $[\cdot,\cdot] : L \otimes L \to L$,
- a linear functor $\Phi = (\Phi_0, \Phi_1) : L \to L$ satisfying

$$\Phi([x, y]) = [\Phi(x, y)], \quad \forall x, y \in L,$$

- a trilinear natural isomorphism, called the Hom-Jacobiator

$$\mathscr{J}_{x,y,z} : [[x, y], \Phi_0(z)] \to [[x, z], \Phi_0(y)] + [\Phi_0(x), [y, z]],$$

satisfying

$$\mathscr{J}_{\Phi_0(x),\Phi_0(y),\Phi_0(z)} = \Phi_1 \mathscr{J}_{x,y,z},$$

and such that the following commutative diagram holds

$$\begin{array}{ccc}
[[[x,y],\Phi_0(z)],\Phi_0^2(w)] & \xrightarrow{\mathscr{J}_{[x,y],\Phi_0(z),\Phi_0(w)}} & [[[x,y],\Phi_0(w)],\Phi_0^2(z)] + [\Phi_0[x,y],[\Phi_0(z),\Phi_0(w)]] \\
\Big\downarrow {\scriptstyle [\mathscr{J}_{x,y,z},\Phi_0^2(w)]} & & \Big\downarrow {\scriptstyle [\mathscr{J}_{x,y,w},\Phi_0^2(z)]+1} \\
[[[x,z],\Phi_0(y)] + [\Phi_0(x),[y,z]],\Phi_0^2(w)] & & R \\
\Big\downarrow {\scriptstyle \mathscr{J}_{[x,z],\Phi_0(y),\Phi_0(w)}+\mathscr{J}_{\Phi_0(x),[y,z],\Phi_0(w)}} & & \Big\downarrow {\scriptstyle \Theta} \\
P & \xrightarrow[{[\mathscr{J}_{x,z,w},\Phi_0^2(y)]+1+1+[\Phi_0^2(x),\mathscr{J}_{y,z,w}]}]{} & Q,
\end{array}$$

where Θ, R, P and Q are given by

$$\begin{aligned}
\Theta &= \mathscr{J}_{[x,w],\Phi_0(y),\Phi_0(z)} + \mathscr{J}_{\Phi_0(x),[y,w],\Phi_0(z)} + \mathscr{J}_{\Phi_0(x),\Phi_0(y),[z,w]}, \\
R &= [[[x, w], \Phi_0(y)], \Phi_0^2(z)] + [[\Phi_0(x), [y, w]], \Phi_0^2(z)] + [\Phi_0[x, y], [\Phi_0(z), \Phi_0(w)]], \\
P &= [[[x, z], \Phi_0(w)], \Phi_0^2(y)] + [\Phi_0[x, z], [\Phi_0(y), \Phi_0(w)]] \\
&\quad + [[\Phi_0(x), \Phi_0(w)], \Phi_0[y, z]] + [\Phi_0^2(x), [[y, z], \Phi_0(w)]], \\
Q &= [[[x, w], \Phi_0(z)], \Phi_0^2(y)] + [[\Phi_0(x), [z, w]], \Phi_0^2(y)] + [\Phi_0[x, z], [\Phi_0(y), \Phi_0(w)]] \\
&\quad + [[\Phi_0(x), \Phi_0(w)], \Phi_0[y, z]] + [\Phi_0^2(x), [[y, w], \Phi_0(z)]] + [\Phi_0^2(x), [\Phi_0(y), [z, w]]].
\end{aligned}$$

Definition 7.2. Let $(L, [\cdot,\cdot], \Phi, \mathscr{J})$ and $(L', [\cdot,\cdot]', \Phi', \mathscr{J}')$ be two Hom-Lie 2-algebras. A Hom-Lie 2-algebra morphism consists of

- a linear functor (F_0, F_1) from the underlying 2-vector space L to L' such that

$$\Phi' \circ (F_0, F_1) = (F_0, F_1) \circ \Phi,$$

- a bilinear natural transformation

$$F_2(x,y) : [F_0(x), F_0(y)]' \to F_0([x,y]),$$

satisfying $F_2(\Phi_0(x), \Phi_0(y)) = \Phi'_1(F_2(x,y))$ and such that the following diagram commutes

$$\begin{array}{ccc}
[[F_0(x),F_0(y)]',\Phi'F_0(z)]' & \xrightarrow{\mathscr{J}_{F_0(x),F_0(y),F_0(z)}} & [[F_0(x),F_0(z)]',\Phi'F_0(y)]' + [\Phi'F_0(x),[F_0(y),F_0(z)]']' \\
\Big\downarrow {\scriptstyle [F_2(x,y),1]'} & & \Big\downarrow {\scriptstyle [F_2(x,z),1]'+[1,F_2(y,z)]'} \\
[F_0[x,y], F_0\Phi(z)]' & & [F_0[x,z],\Phi'F_0(y)]' + [\Phi'F_0(x),F_0[y,z]]' \\
\Big\downarrow {\scriptstyle F_2([x,y],\Phi(z))} & & \Big\downarrow {\scriptstyle F_2([x,z],\Phi(y))+F_2(\Phi(x),[y,z])} \\
F_0[[x,y],\Phi(z)] & \xrightarrow[F_0(\mathscr{J}_{x,y,z})]{} & F_0([[x,z],\Phi_0(y)] + [\Phi_0(x),[y,z]]).
\end{array}$$

The composition of two Hom-Lie 2-algebra morphisms is again a Hom-Lie 2-algebra morphism. More precisely, let L, L' and L'' be three Hom-Lie 2-algebras and $F : L \to L'$, $G : L' \to L''$ be Hom-Lie 2-algebra morphisms. Their composition $G \circ F : L \to L''$ is a Hom-Lie 2-algebra morphism whose components are given by $(G\circ F)_0 = G_0 \circ F_0$, $(G \circ F)_1 = G_1 \circ F_1$ and $(G \circ F)_2$ is given by

$$\begin{array}{ccc}
[G_0\circ F_0(\xi), G_0\circ F_0(\eta)]'' & \xrightarrow{(G\circ F)_2(\xi,\eta)} & (G_0\circ F_0)([\xi,\eta]). \\
{\scriptstyle G_2(F_0(\xi),F_0(\eta))} \searrow & & \nearrow {\scriptstyle G_0(F_2(\xi,\eta))} \\
& G_0([F_0(\xi),F_0(\eta)]') &
\end{array}$$

For any Hom-Lie 2-algebra L, the identity morphism $\mathrm{id}_L : L \to L$ is given by the identity functor as its linear functor together with the identity natural transformation as $(\mathrm{id}_L)_2$.

Hom-Lie 2-algebras and Hom-Lie 2-algebra morphisms form a category. We denote this category by *HLie2*.

In the next, we define 2-derivations on Hom-Lie 2-algebras. They are a categorification of derivations on Hom-Lie algebras.

Definition 7.3. Let $(L, [\cdot,\cdot], \Phi, \mathscr{J})$ be a Hom-Lie 2-algebra. A 2-derivation on it consists of a linear map functor $D : L \to L$ satisfying $D \circ \Phi_0 = \Phi_0 \circ D$ and a natural isomorphism

$$\mathscr{D}_{x,y} : D[x,y] \to [Dx,y] + [x,Dy], \quad \forall x, y \in L$$

such that the following commutative diagram holds

$$\begin{array}{ccc}
D[[x,y],\Phi_0(z)] & \xrightarrow{\mathscr{J}} & D([[x,z],\Phi_0(y)] + [\Phi_0(x),[y,z]]) \\
\Big\downarrow {\scriptstyle \mathscr{D}_{[x,y],\Phi_0(z)}} & & \Big\downarrow \\
[D[x,y],\Phi_0(z)] + [[x,y],D\Phi_0(z)] & & [D[x,z],\Phi_0(y)] + [[x,z],D\Phi_0(y)] + [D\Phi_0(x),[y,z]] + [\Phi_0(x),D[y,z]] \\
\Big\downarrow {\scriptstyle [\mathscr{D},1]+1} & & \Big\downarrow {\scriptstyle [\mathscr{D},1]+1+1+[1,\mathscr{D}]} \\
[[Dx,y]+[x,Dy],\Phi_0(z)] + [[x,y],D\Phi_0(z)] & \xrightarrow[\mathscr{J}+\mathscr{J}+\mathscr{J}]{} & Q,
\end{array}$$

where

$$\begin{aligned}
Q = {} & [[Dx,z],\Phi_0(y)] + [[x,Dz],\Phi_0(y)] + [[x,z],D\Phi_0(y)] + [D\Phi_0(x),[y,z]] \\
& + [\Phi_0(x),[Dy,z]] + [\Phi_0(x),[y,Dz]].
\end{aligned}$$

We call a Hom-Lie 2-algebra with a 2-derivation, an HLieDer2 triple.

Definition 7.4. Let $(L, [\cdot,\cdot], \Phi, \mathscr{J}, D, \mathscr{D})$ and $(L', [\cdot,\cdot]', \Phi, \mathscr{J}', D', \mathscr{D}')$ be two HLieDer2 triples. A morphism between them consists of a Hom-Lie 2-algebra morphism $(F = (F_0, F_1), F_2)$ and a natural isomorphism

$$\Theta_x : D'(F_0(x)) \to F_0(D(x)), \quad \forall x \in L_0$$

such that the following diagram commutes:

$$\begin{array}{ccc}
D'([F_0(x),F_0(y)]') & \xrightarrow{F_2} & D'(F_0[x,y]) \\
\Big\downarrow {\scriptstyle \mathscr{D}'} & & \Big\downarrow {\scriptstyle \Theta_{[x,y]}} \\
[D'(F_0(x)),F_0(y)]' + [F_0(x),D'(F_0(y))]' & & F_0(D[x,y]) \\
\Big\downarrow {\scriptstyle [\Theta_x,1]'+[1,\Theta_y]'} & & \Big\downarrow {\scriptstyle \mathscr{D}} \\
[F_0(D(x)),F_0(y)]' + [F_0(x),F_0(D(y))]' & \xrightarrow[F_2+F_2]{} & F_0([Dx,y]+[x,Dy]).
\end{array}$$

We denote the category of HLieDer2 triples and morphisms between them by *HLieDer2*.

Now we are ready to prove our main result of this section.

Theorem 7.1. *The categories 2HLieDer$_\infty$ and HLieDer2 are equivalent.*

Proof. First we construct a functor $T : \mathit{2HLieDer}_\infty \to \mathit{HLieDer2}$ as follows. Given a 2-term HLieb$_\infty$-algebra with a homotopy derivation $((L_1 \xrightarrow{d} L_0, l_2, l_3), (\phi_0, \phi_1), (\theta_0, \theta_1, \theta_2))$, we have the 2-vector space $C = (L_0 \oplus L_1 \rightrightarrows L_0)$. Define a bilinear functor $[\cdot,\cdot] : C \otimes C \to C$ by

$$[(x,m),(y,n)] = (l_2(x,y), l_2(x,n) + l_2(m,y) + l_2(dm,n)),$$

for $(x,m),(y,n) \in C_1 = L_0 \oplus L_1$. The linear functor $\Phi : C \to C$ is given by $\Phi = (\Phi_0, \Phi_1) := (\phi_0, \phi_0 + \phi_1)$. Moreover, we have

$$\begin{aligned}
&\Phi([(x,m),(y,n)]) = (\phi_0 l_2(x,y), \phi_1(l_2(x,n) + l_2(m,y) + l_2(dm,n)))\\
&\quad = (l_2(\phi_0(x),\phi_0(y)), l_2(\phi_0(x),\phi_1(n)) + l_2(\phi_1(m),\phi_0(y)) + l_2(d\phi_1(m),\phi_1(n)))\\
&\quad = [(\phi_0(x),\phi_1(m)),(\phi_0(y),\phi_1(n))]\\
&\quad = [\Phi(x,m),\Phi(y,n)].
\end{aligned}$$

Define the Hom-Jacobiator by

$$\mathscr{J}_{x,y,z} = ([[x,y],\phi_0(z)], l_3(x,y,z)).$$

Note that

$$\begin{aligned}
\mathscr{J}_{\Phi_0(x),\Phi_0(y),\Phi_0(z)} &= ([[\phi_0(x),\phi_0(y)],\phi_0^2(z)], l_3(\phi_0(x),\phi_0(y),\phi_0(z)))\\
&= (\phi_0[[x,y],\phi_0(z)], \phi_1 l_3(x,y,z))\\
&= \Phi_1 \mathscr{J}_{x,y,z}.
\end{aligned}$$

By using identities (a)–(j), one can also verify that the diagram in Definition 5.1 commutes. Therefore, $(C, [\cdot,\cdot], \mathscr{J}, \Phi)$ is a Hom-Lie 2-algebra. Moreover, we define a 2-derivation $(D, \mathscr{D}')$ by

$$D(x,m) := (\theta_0(x), \theta_1(m)), \quad \mathscr{D}_{x,y} := ([x,y], \theta_2(x,y)).$$

For any HLieDer$_\infty$-morphism (f_0, f_1, f_2, Ψ) from L to L', we define a morphism F from $C = T(L)$ to $C' = T(L')$ as follows. Take $F_0 = f_0$, $F_1 = f_1$ and

$$F_2(x,y) = ([f_0(x), f_0(y)]', f_2(x,y)), \quad \Theta = \Psi.$$

It is easy to verify that F is a morphism from C to C'. Moreover, one can verify that T preserves the identity morphisms and composition of morphisms. Therefore, T is a functor from $\mathit{2HLieDer}_\infty$ to $\mathit{HLieDer2}$.

In the next, we construct a functor $S : \mathit{HLieDer2} \to \mathit{2HLieDer}_\infty$ as follows. Given an HLieDer2 triple $C = (C_1 \oplus C_0, \Phi, \mathscr{J}, D, \mathscr{D})$, we have the 2-term chain complex

$$L_1 = \operatorname{Ker} s \xrightarrow{d = t|_{\operatorname{Ker} s}} C_0 = L_0.$$

Define $l_2 : L_i \otimes L_j \to L_{i+j}$ by

$$l_2(x,y) = [x,y], \quad l_2(x,m) = [x,m], \quad l_2(m,x) = [m,x].$$

The map $l_3 : L_0 \otimes L_0 \otimes L_0 \to L_1$ is defined by

$$l_3(x,y,z) = pr(\mathscr{J}_{x,y,z}), \quad \forall x,y,z \in L_0,$$

where pr denotes the projection on $\operatorname{Ker} s$. Moreover, we define a homotopy derivation by

$$\theta_0(x) := D(i(m)), \quad \theta_1(m) := D|_{\operatorname{Ker} s}(m), \quad \theta_2(x,y) := pr(\mathscr{D}_{x,y}).$$

For any HLieDer2 triple morphism $(F_0, F_1, F_2, \Theta) : C \to C'$, then $f_0 = F_0$, $f_1 = F_1|_{L_1} = \operatorname{Ker} s$ and define f_2 by

$$f_2(x,y) = prF_2(x,y), \quad \Psi = \Theta.$$

Moreover, S preserves the identity morphisms and composition of morphisms. Therefore, S is a functor from *HLieDer2* to *2HLieDer*$_\infty$.

It is not hard to check that the composite functor $T \circ S$ is naturally isomorphic to the identity functor $1_{HLieDer2}$, and the composite $S \circ T$ is naturally isomorphic to $1_{2HLieDer_\infty}$. We omit them. □

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References

1. Ammar F., Ejbehi Z., Makhlouf A., Cohomology and deformations of Hom-algebras. J. Lie Theory, 2011, 21(4): 813–836
2. Casas J.M., García-Martínez X., Abelian extensions and crossed modules of Hom-Lie algebras. J. Pure Appl. Algebra, 2020, 224(3): 987–1008
3. Guan B.L., Chen L.Y., Restricted hom-Lie algebras. Hacet. J. Math. Stat., 2015, 44(4): 823–837
4. Hartwig J.T., Larsson D., Silvestrov S.D., Deformations of Lie algebras using σ-derivations. J. Algebra, 2006, 295(2): 314–361
5. Hu N.H., q-Witt algebras, q-Lie algebras, q-holomorph structure and representations. Algebra Colloq., 1999, 6(1): 51–70
6. Makhlouf A., Silvestrov S.D., Hom-algebra structures. J. Gen. Lie Theory Appl., 2008, 2(2): 51–64
7. Sheng Y.H., Representations of hom-Lie algebras. Algebr. Represent. Theory, 2012, 15(6): 1081–1098
8. Sheng Y.H., Bai C.M., A new approach to hom-Lie bialgebras. J. Algebra, 2014, 399: 232–250

9. Sheng Y.H., Chen D.H., Hom-Lie 2-algebras. J. Algebra, 2013, 376: 174–195

10. Sheng Y.H., Xiong Z., On Hom-Lie algebras. Linear Multilinear Algebra, 2015, 63(12): 2379–2395

11. Tang R., Frégier Y., Sheng Y.H., Cohomologies of a Lie algebra with a derivation and applications. J. Algebra, 2019, 534: 65–99

12. Yau D., The Hom-Yang–Baxter equation, Hom-Lie algebras, and quasi-triangular bialgebras. J. Phys. A, 2009, 42(16): 165202, 12 pp.

13. Yau D., The classical Hom-Yang–Baxter equation and Hom-Lie bialgebras. Int. Electron. J. Algebra, 2015, 17: 11–45

14. Zhao J., Chen L.Y., Ma L.L., Representations and T^*-extensions of hom-Jordan–Lie algebras. Comm. Algebra, 2016, 44(7): 2786–2812