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RESEARCH ARTICLE

Fast algorithm for viscous Cahn-Hilliard equation

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Abstract The main purpose of this paper is to solve the viscous Cahn-Hilliard equation via a fast algorithm based on the two time-mesh (TT-M) finite element (FE) method to ease the problem caused by strong nonlinearities. The TT-M FE algorithm includes the following main computing steps. First, a nonlinear FE method is applied on a coarse time-mesh τ_c. Here, the FE method is used for spatial discretization and the implicit second-order θ scheme (containing both implicit Crank-Nicolson and second-order backward difference) is used for temporal discretization. Second, based on the chosen initial iterative value, a linearized FE system on time fine mesh is solved, where some useful coarse numerical solutions are found by Lagrange's interpolation formula. The analysis for both stability and a priori error estimates is made in detail. Numerical examples are given to demonstrate the validity of the proposed algorithm. Our algorithm is compared with the traditional Galerkin FE method and it is evident that our fast algorithm can save computational time.

Keywords Fast algorithm, two time-mesh (TT-M) finite element (FE) method, viscous Cahn-Hilliard equation, stability, CPU time

MSC2020 35Q30, 74S05

1 Introduction

The Cahn-Hilliard equation describes the process of phase separation, first introduced by Cahn and Hilliard in the late 1950s [3–5]. Numerical methods for solving the Cahn-Hilliard equation provide an important tool for studying the dynamics described by the Cahn-Hilliard equation. It has been well studied and broadly used to investigate the coarsening dynamics of two immersible fluids. Recently, researchers have devoted tremendous efforts to the relaxed Cahn-Hilliard system, i.e., the viscous Cahn-Hilliard (VCH) system and its

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perturbed form with the hyperbolic relaxation (HR) effect (referred to as the perturbed viscous Cahn-Hilliard equation). Formally, the governing equation of the VCH-HR system is slightly different from the Cahn-Hilliard equation by incorporating two extra terms, including a strong damping (or viscosity) term and a hyperbolic relaxation term (or inertia). The viscous term was first proposed by Novick-Cohen [31] in order to introduce an additional regularity and some parabolic smoothing. It can be viewed as a singular limit of the phase field equations for phase transitions [9]. The hyperbolic relaxation term was proposed by Galenko et al. [10–15,23] in order to describe strong non-equilibrium decomposition generated by rapid solidification under supercooling into the spinodal region occurring in certain materials (e.g., glasses). Since the VCH system contains the viscosity, it is mathematically more intractable compared to the Cahn-Hilliard systems [21,32,44].

Before developing efficient numerical schemes to solve the VCH system, we notice that its reduced version, the Cahn-Hilliard equation has been used as a model for various problems: microphase separation of diblock copolymers (two or more different polymer chains linked together [8]); spinodal decomposition [20] (a mechanism for the phase separation of a mixture of liquids or solids from one thermodynamic phase); image inpainting (a process of reconstructing lost parts of images [2]); phase-field modeling of tumor growth simulation [38]; volume reconstruction [26]; topology optimization [45]; co-continuous binary polymer microstructures [6]; microstructures with elastic inhomogeneity [42], and multiphase fluid flows [19,24,25].

It is noticed that, despite a great deal of work done for the numerical solution of the classical Cahn-Hilliard system, almost all researches related to the VCH or VCH-HR system were focused on the theoretical analysis of partial differential equation with very few algorithm design or numerical analysis. This is due to the numerical difficulties of proper discretization for the viscous effect, besides the regular stiffness issue induced by the nonlinear double well potential. Therefore, an efficient and accurate time marching scheme is required. The invariant energy quadratization (IEQ) approach has been proposed to solve the VCH-HR equation [40].

This paper is mainly devoted to improve the speed of the numerical calculation for viscous Cahn-Hilliard system. Zhang and Qiao [43] discussed a finite difference scheme and proposed an adaptive time-stepping technique to quickly solve the 2D Cahn-Hilliard equation. He et al. [17] proposed a large time-stepping methods. Novo et al. [1], Layton [22], and Xu et al. [30,39] introduced the two-level type methods. The basic idea of the two-level method is to first solve the nonlinear equation in the coarse-level subspace and then solve the linear equation in the fine-level subspace. Therefore, the two-level method can achieve better accuracy with less CPU time. Recently, Liu et al. [29] has proposed a fast two time-mesh (TT-M) finite element (FE) algorithm for time fractional water wave model, which has been developed to deal with the time-consuming problem of nonlinear iteration used in the standard non-linear Galerkin FE method for the nonlinear term. Based on two-grid finite element discretization and a recent subgrid-scale model, Shang [33] analyzed a two-level subgrid stabilized Oseen iterative method for the convection dominated Navier-Stokes equations. Wang et al. [35] have proposed a fast time Two-Mesh Algorithm for Allen-Cahn equation.

Here, our work is to use similar idea as in [29] for solving the viscous Cahn-Hilliard equation:

$$\begin{cases} u_t = -\Delta w + g, & \text{in } \Omega \times (0,T], \\ w = \varepsilon^2 \Delta u - f(u) - \beta u_t, & \text{in } \Omega \times (0,T], \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial w}{\partial n} = 0, & \text{on } \partial\Omega \times (0,T], \\ u(\cdot,0) = u_0, & \text{in } \Omega, \end{cases} \tag{1.1}$$

where ε is a given parameter, $u_t = \dfrac{\partial u}{\partial t}$, n is the outward normal, and $\beta > 0$ is the viscosity parameter, it becomes the classical Cahn-Hilliard system when $\beta = 0$. $f(u)$ is usually of the form $f(u) = u^3 - u$ and $\Omega \in \mathbb{R}^d$, $d = 2$. Here, u is the concentration of one of the two substances in the mixture and is known as the phase variable, and g is external force. Our aim is to introduce a similar fast TT-M FE algorithm as proposed in [35] for water wave to solve viscous Cahn-Hilliard equation. The time derivative is approximated by a second-order scheme [28] derived based on the idea of the second-order-schemes in literature (see Galerkin FE method by Wang et al. [37] and finite difference schemes by Gao et al. [16]).

The organization of the rest of this paper is as follows. In Section 2, we provide some definitions of norms and lemmas. In Section 3, we give the numerical scheme of fast TT-M FE algorithm with second-order θ scheme. In Section 4, we implement the analysis of stability for the proposed scheme. In Section 5, we analyze the error estimates in detail. We present some numerical experiments in Section 6. Finally, we do some simple summaries for the numerical methods.

2 Theoretical preparation

In this section, we state the necessary abstract for the analysis of subsequent proof. We denote the inner products and the norms in space $L^2(\Omega)$ and $H^1(\Omega)$ as follows:

$$(u,v) = \int_\Omega u(x)v(x)\mathrm{d}x, \quad \|u\| = \|u\|_{L^2(\Omega)},$$

$$\|u\|_{H^1} = \left(\int_\Omega |u|^2 \mathrm{d}x + \int_\Omega |Du|^2 \mathrm{d}x \right)^{1/2}, \quad |u|_{H^1} = \left(\int_\Omega |Du|^2 \mathrm{d}x \right)^{1/2}.$$

To derive a fully discrete TT-M FE scheme, we first split the time interval $[0,T]$ into a coarse uniform partition with the nodes $t_n = nM\tau$ $(n =$

$0, 1, \ldots, N$), which satisfy $0 = t_0 < t_1 < \cdots < t_N = T$ with the fine time step size $\tau = T/(nM)$ for some positive integer $M \geqslant 2$, where $\tau_c = M\tau$ is the coarse time mesh step size. Let $u^n = u(\cdot, t_n)$. Then the time-second order θ method [28] is

$$\mathscr{D}_\tau u(t_{n-\theta}) = \frac{(3-2\theta)u^n - (4-4\theta)u^{n-1} + (1-2\theta)u^{n-2}}{2\tau}, \quad n \geqslant 2,$$

and for the first time level, we use the Crank-Nicolson discrete scheme

$$\partial_{1/2} u = \frac{u^1 - u^0}{\tau}.$$

Lemma 1 [29] *For a sufficiently smooth function $u(t) = u(\cdot, t) \in C^3[0, T]$ and any $\theta \in [0, 1/2]$, the above approximation of first-order derivative at time $t_{n-\theta}$ is of second-order convergence, i.e.,*

$$u_t(t_{n-\theta}) = \mathscr{D}_\tau u(t_{n-\theta}) + R_t^{n-\theta},\ n \geqslant 2, \quad u_t(t_{1/2}) = \partial_{1/2} u + E_1,\ n = 1,$$

where

$$\|R_t^{n-\theta}\| \leqslant C\tau^2, \quad \|E_1\| \leqslant C\tau^2,$$

with the constant C independent of τ.

Lemma 2 [29] *For a sufficiently smooth function $u(t) = u(\cdot, t) \in C^2[0, T]$ and function $f(t) \in C^2[0, T]$, at time $t_{n-\theta}$, the approximate formula*

$$u(t_{n-\theta}) = (1-\theta)u(t_n) + \theta u(t_{n-1}) + E_2^{n-\theta},$$
$$f(u(t_{n-\theta})) = (1-\theta)f(u(t_n)) + \theta f(u(t_{n-1})) + E_3^{n-\theta},$$

holds for any $\theta \in [0, 1/2]$, $n \geqslant 1$, where

$$\|E_2^{n-\theta}\| \leqslant C\tau^2, \quad \|E_3^{n-\theta}\| \leqslant C\tau^2.$$

We take the following notations:

$$f^{n-\theta}(u) = (1-\theta)f(u^n) + \theta f(u^{n-1}), \quad u^{n-\theta} = (1-\theta)u^n + \theta u^{n-1}.$$

In this paper, referring [27,34,36], we assume that potential function $F(u)$ whose derivative $f(u)$ is uniformly bounded, i.e.,

$$\max_{u \in R} |f'(u)| \leqslant L.$$

3 Numerical scheme

Let

$$H_E^1(\Omega) = \Big\{ u \in H^1(\Omega) \,\Big|\, \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \Big\}.$$

Using the above θ method, the temporal semidiscrete scheme for (1.1) is as follows. Find

$$u^n\colon [0,T]\mapsto H_E^1,\quad w^n\colon [0,T]\mapsto H_E^1,$$

such that for any $v,q\in H_E^1$, when $n=1$,

$$\begin{gathered}\Big(\frac{u^1-u^0}{\tau},v\Big)-(\nabla w^{1/2}(u),\nabla v)=\Big(\frac{g^1+g^0}{2},v\Big),\\ -\varepsilon^2(\nabla u^{1/2},\nabla q)-(f^{1/2}(u),q)-\beta\Big(\frac{u^1-u^0}{\tau},q\Big)=(w^{1/2}(u),q),\end{gathered}$$

where

$$u^{1/2}=\frac{u^0+u^1}{2},\quad f^{1/2}(u)=\frac{f(u^0)+f(u^1)}{2},\quad g^{1/2}=\frac{g^0+g^1}{2};$$

when $n\geqslant 2$,

$$\begin{gathered}(\mathscr{D}_\tau u^{n-\theta},v)-(\nabla w^{n-\theta}(u),\nabla v)=(g^{n-\theta}(u),v),\\ -\varepsilon^2(\nabla u^{n-\theta},\nabla q)-(f^{n-\theta}(u),q)-\beta(\mathscr{D}_\tau u^{n-\theta},q)=(w^{n-\theta}(u),q),\end{gathered}$$

with $u^0=u_0(x,y)$.

Next, we define V_h as the subspace of $H_E^1(\Omega)$, i.e.,

$$V_h=\{v\in H_E^1\colon v|_e\in P_k(x,y)\},$$

where $P_k(x,y)$ is the space of polynomials of degree at most $k\in\mathbb{Z}^+$. The fully discrete scheme for (1.1) is as follows. Find

$$U^n\colon [0,T]\mapsto V_h,\quad W^n\colon [0,T]\mapsto V_h,$$

such that for any $v_h,q_h\in V_h$, when $n=1$,

$$\begin{gathered}\Big(\frac{U^1-U^0}{\tau},v_h\Big)-(\nabla W^{1/2}(U),\nabla v_h)=\Big(\frac{g^1+g^0}{2},v_h\Big),\\ -\varepsilon^2(\nabla U^{1/2},\nabla q_h)-(f^{1/2}(U),q_h)-\Big(\beta\frac{U^1-U^0}{\tau},q_h\Big)=(W^{1/2}(U),q_h),\end{gathered}\tag{3.1}$$

where

$$U^{1/2}=\frac{U^0+U^1}{2},\quad f^{1/2}(U)=\frac{f(U^0)+f(U^1)}{2};$$

when $n\geqslant 2$,

$$\begin{gathered}(\mathscr{D}_\tau U^{n-\theta},v_h)-(\nabla W^{n-\theta}(U),\nabla v_h)=(g^{n-\theta},v_h),\\ -\varepsilon^2(\nabla U^{n-\theta},\nabla q_h)-(f^{n-\theta}(U),q_h)-\beta(\mathscr{D}_\tau U^{n-\theta},q_h)=(W^{n-\theta}(U),q_h),\end{gathered}\tag{3.2}$$

with $U^0=u_{h_0}(x,y)$.

To improve the computation efficiency of the FE discrete system (3.1) and (3.2), we consider the following TT-M system based on FE method. Here, τ_c is the coarse time-mesh step, and τ is the fine time-mesh step. We now give the TT-M algorithm for the Cahn-Hilliard equation in three steps.

Step 1 The coarse time-mesh numerical approximations U_C^n and W_C^n are obtained by the following equations: when $n=1$,

$$\Big(\frac{U_C^1-U_C^0}{\tau_c},v_h\Big)-(\nabla W^{1/2}(U_C),\nabla v_h)=\Big(\frac{g^1+g^0}{2},v_h\Big), \tag{3.3a}$$

$$\begin{aligned}&-\varepsilon^2(\nabla U_C^{1/2},\nabla q_h)-(f^{1/2}(U_C),q_h)-\beta\Big(\frac{U_C^1-U_C^0}{\tau_c},q_h\Big)\\&\quad=(W^{1/2}(U_C),q_h),\end{aligned} \tag{3.3b}$$

where

$$U_C^{1/2}=\frac{U_C^0+U_C^1}{2},\, f^{1/2}(U_C)=\frac{f(U_C^0)+f(U_C^1)}{2};$$

when $n\geqslant 2$:

$$(\mathscr{D}_{\tau_c}U_C^{n-\theta},v_h)-(\nabla W^{n-\theta}(U_C),\nabla v_h)=(g^{n-\theta},v_h), \tag{3.4a}$$

$$\begin{aligned}&-\varepsilon^2(\nabla U_C^{n-\theta},\nabla q_h)-(f^{n-\theta}(U_C),q_h)-\beta(\mathscr{D}_{\tau_c}U_C^{n-\theta},q_h)\\&\quad=(W^{n-\theta}(U_C),q_h),\end{aligned} \tag{3.4b}$$

with $U_C^0=u_{h_0}(x,y)$, an appropriate approximation of $u_0(x,y)$.

Step 2 Obtain the values between U_C^{n-1} and U_C^n ($n=1,2,\ldots,N-1$) on the fine time-mesh by the Lagrange's interpolation, where U_I^m ($m=1,2,\ldots,M,\ldots,2M,\ldots,NM$) are the interpolated results and m is the fine time-mesh index.

Step 3 By using the interpolated results U_I^m, the following linear systems on the fine time-mesh with step τ are solved to find

$$U_F^m\colon[0,T]\mapsto V_h,\quad W_F^m\colon[0,T]\mapsto V_h,$$

for any $v_h\in V_h$, $q_h\in V_h$, when $m=1$,

$$\Big(\frac{U_F^1-U_F^0}{\tau},v_h\Big)-(\nabla W^{1/2}(U_F),\nabla v_h)=\Big(\frac{g^1+g^0}{2},v_h\Big), \tag{3.5a}$$

$$\begin{aligned}&-\varepsilon^2(\nabla U_F^{1/2},\nabla q_h)-\frac{1}{2}\,(f(U_I^1)+(U_F^1-U_I^1)f_u(U_I^1),q_h)\\&\quad-\frac{1}{2}\,(f(U_F^0),q_h)-\beta\Big(\frac{U_F^1-U_F^0}{\tau},q_h\Big)\\&\quad=(W^{1/2}(U_F),q_h);\end{aligned} \tag{3.5b}$$

when $m \geqslant 2$,

$$(\mathscr{D}_\tau U_F^{m-\theta}, v_h) - (\nabla W^{m-\theta}(U_F), \nabla v_h) = (g^{m-\theta}, v_h), \tag{3.6a}$$

$$\begin{aligned}
&-\varepsilon^2(\nabla U_F^{m-\theta}, \nabla q_h) - (1-\theta)(f(U_I^m) + (U_F^m - U_I^m) f_u(U_I^m), q_h)\\
&\quad - \theta(f(U_F^{m-1}), q_h) - \beta(\mathscr{D}_\tau U_F^{m-\theta}, q_h)\\
&= (W^{m-\theta}(U_F), q_h),
\end{aligned} \tag{3.6b}$$

where f_u is the derivative of f about u.

4 Analysis of stability

Lemma 3 [41] *For series $\{u^n\}$ and $0 \leqslant \theta \leqslant 1/2$, the following inequalities hold*:

$$(\mathscr{D}_\tau u^{n-\theta}, u^{n-\theta}) \geqslant \frac{1}{4\tau}\,(\mathbb{H}[u^n] - \mathbb{H}[u^{n-1}]), \quad n \geqslant 2,$$
$$\mathbb{H}[u^n] \geqslant \frac{1}{1-\theta}\,\|u^n\|^2, \quad n \geqslant 2,$$

where

$$\mathbb{H}[u^n] = (3-2\theta)\|u^n\|^2 - (1-2\theta)\|u^{n-1}\|^2 + (2-\theta)(1-2\theta)\|u^n - u^{n-1}\|^2, \quad n \geqslant 1.$$

Lemma 4 *For series $\{u^n\}$ and $0 \leqslant \theta \leqslant 1/2$, the following inequalities hold*:

$$(\nabla \mathscr{D}_\tau u^{n-\theta}, \nabla u^{n-\theta}) \geqslant \frac{1}{4\tau}\,(\mathbb{L}[u^n] - \mathbb{L}[u^{n-1}]), \quad n \geqslant 2, \tag{4.1}$$

$$\mathbb{L}[u^n] \geqslant \frac{1}{1-\theta}\,\|\nabla u^n\|^2, \quad n \geqslant 2,$$

where

$$\begin{aligned}
\mathbb{L}[u^n] = {}& (3-2\theta)\|\nabla u^n\|^2 - (1-2\theta)\|\nabla u^{n-1}\|^2\\
& + (2-\theta)(1-2\theta)\|\nabla u^n - \nabla u^{n-1}\|^2, \quad n \geqslant 1.
\end{aligned}$$

Proof It is obvious that the operator $\mathscr{D}_\tau u^{n-\theta}$ can be rewritten as

$$\mathscr{D}_\tau u^{n-\theta} = (2-2\theta)\,\frac{u^n - u^{n-1}}{\tau} - (1-2\theta)\,\frac{u^n - u^{n-2}}{2\tau}.$$

By using equalities

$$(a-b)a = \frac{1}{2}\,[a^2 - b^2 + (a-b)^2], \quad (a-b)b = \frac{1}{2}\,[a^2 - b^2 - (a-b)^2],$$

the following inequality holds:

$$
\begin{aligned}
&(\nabla \mathscr{D}_\tau u^{n-\theta}, \nabla u^{n-\theta})\\
&= (\nabla \mathscr{D}_\tau u^{n-\theta}, (1-\theta)\nabla u^n + \theta \nabla u^{n-1})\\
&= (1-\theta)\Big[(2-2\theta)\Big(\frac{\nabla u^n - \nabla u^{n-1}}{\tau}, \nabla u^n\Big) - (1-2\theta)\Big(\frac{\nabla u^n - \nabla u^{n-2}}{2\tau}, \nabla u^n\Big)\Big]\\
&\quad + \theta\Big[\frac{3-2\theta}{2}\Big(\frac{\nabla u^n - \nabla u^{n-1}}{\tau}, \nabla u^{n-1}\Big) - \frac{1-2\theta}{2}\Big(\frac{\nabla u^{n-1} - \nabla u^{n-2}}{\tau}, \nabla u^{n-1}\Big)\Big]\\
&= (1-\theta)\Big[\frac{1-\theta}{\tau}(\|\nabla u^n\|^2 - \|\nabla u^{n-1}\|^2 + \|\nabla u^n - \nabla u^{n-1}\|^2)\\
&\quad - \frac{1-2\theta}{4\tau}(\|\nabla u^n\|^2 - \|\nabla u^{n-2}\|^2 + \|\nabla u^n - \nabla u^{n-2}\|^2)\Big]\\
&\quad + \theta\Big[\frac{3-2\theta}{4\tau}(\|\nabla u^n\|^2 - \|\nabla u^{n-1}\|^2 - \|\nabla u^n - \nabla u^{n-1}\|^2)\\
&\quad - \frac{1-2\theta}{4\tau}(\|\nabla u^{n-1}\|^2 - \|\nabla u^{n-2}\|^2 - \|\nabla u^{n-1} - \nabla u^{n-2}\|^2)\Big]\\
&\geqslant (1-\theta)\Big[\frac{1-\theta}{\tau}(\|\nabla u^n\|^2 - \|\nabla u^{n-1}\|^2 + \|\nabla u^n - \nabla u^{n-1}\|^2)\\
&\quad - \frac{1-2\theta}{4\tau}(\|\nabla u^n\|^2 - \|\nabla u^{n-2}\|^2 + 2\|\nabla u^n - \nabla u^{n-1}\|^2\\
&\quad + 2\|\nabla u^{n-1} - \nabla u^{n-2}\|^2)\Big]\\
&\quad + \theta\Big[\frac{3-2\theta}{4\tau}(\|\nabla u^n\|^2 - \|\nabla u^{n-1}\|^2 - \|\nabla u^n - \nabla u^{n-1}\|^2)\\
&\quad - \frac{1-2\theta}{4\tau}(\|\nabla u^{n-1}\|^2 - \|\nabla u^{n-2}\|^2 - \|\nabla u^{n-1} - \nabla u^{n-2}\|^2)\Big]\\
&= \frac{3-2\theta}{4\tau}(\|\nabla u^n\|^2 - \|\nabla u^{n-1}\|^2) - \frac{1-2\theta}{4\tau}(\|\nabla u^{n-1}\|^2 - \|\nabla u^{n-2}\|^2)\\
&\quad + \frac{2\theta^2 - 5\theta + 2}{4\tau}(\|\nabla u^n - \nabla u^{n-1}\|^2 - \|\nabla u^{n-1} - \nabla u^{n-2}\|^2)\\
&= \frac{1}{4\tau}(\mathbb{L}[u^n] - \mathbb{L}[u^{n-1}]).
\end{aligned}
$$

In addition, we have

$$
\begin{aligned}
\mathbb{L}[u^n] &= (2\theta^2 - 7\theta + 5)\|\nabla u^n\|^2 + (2\theta^2 - 3\theta + 1)\|\nabla u^{n-1}\|^2\\
&\quad - 2(2\theta^2 - 5\theta + 2)(\nabla u^n, \nabla u^{n-1})\\
&\geqslant (2\theta^2 - 7\theta + 5)\|\nabla u^n\|^2 + (2\theta^2 - 3\theta + 1)\|\nabla u^{n-1}\|^2\\
&\quad - \Big[(2\theta^2 - 3\theta + 1)\|\nabla u^{n-1}\|^2 + \frac{2\theta^2 - 5\theta + 2}{2\theta^2 - 3\theta + 1}\|\nabla u^n\|^2\Big]\\
&\geqslant \frac{1}{1-\theta}\|\nabla u^n\|^2.
\end{aligned}
$$

□

Theorem 1 *Under the condition* $\tau_c \leqslant T/N$, *for the coarse time-mesh system*

(3.3a)-(3.3b) *and* (3.4a)-(3.4b), *the following stability inequality holds*:

$$\|U_C^n\|^2 + \|\nabla U_C^n\|^2 \leqslant C\|U_C^0\|^2 + C\|\nabla U_C^0\|^2 + C\tau_c \sum_{k=0}^{n} \|g^k\|^2. \tag{4.2}$$

Proof Taking $v_h = U_C^{n-\theta}$ in (3.4a), we have

$$(\mathscr{D}_{\tau_c} U_C^{n-\theta}, U_C^{n-\theta}) - (\nabla W^{n-\theta}(U_C), \nabla U_C^{n-\theta}) = (g^{n-\theta}, U_C^{n-\theta}). \tag{4.3}$$

Taking $q_h = \Delta U_C^{n-\theta}$ in (3.4b), we have

$$\begin{aligned}&-\varepsilon^2(\nabla U_C^{n-\theta}, \nabla\Delta U_C^{n-\theta}) - (f^{n-\theta}(U_C), \Delta U_C^{n-\theta}) - (\beta\mathscr{D}_{\tau_c} U_C^{n-\theta}, \Delta U_C^{n-\theta})\\ &\quad = (W^{n-\theta}(U_C), \Delta U_C^{n-\theta}).\end{aligned} \tag{4.4}$$

Combing (4.3) and (4.4), we get

$$\begin{aligned}&(\mathscr{D}_{\tau_c} U_C^{n-\theta}, U_C^{n-\theta}) + \varepsilon^2(\Delta U_C^{n-\theta}, \Delta U_C^{n-\theta}) - (\beta\mathscr{D}_{\tau_c} U_C^{n-\theta}, \Delta U_C^{n-\theta})\\ &\quad = (f^{n-\theta}(U_C), \Delta U_C^{n-\theta}) + (g^{n-\theta}, U_C^{n-\theta}).\end{aligned}$$

Using the Cauchy-Schwarz inequality and the Young inequality, we have the following estimate:

$$\begin{aligned}|(f^{n-\theta}(U_C), \Delta U_C^{n-\theta})| &\leqslant \frac{1}{4\varepsilon^2}\|f^{n-\theta}(U_C)\|^2 + \varepsilon^2\|\Delta U_C^{n-\theta}\|^2\\ &\leqslant \frac{1}{4\varepsilon^2}\|(1-\theta)f(U_C^{n-1}) + \theta f(U_C^n)\|^2 + \varepsilon^2\|\Delta U_C^{n-\theta}\|^2\\ &\leqslant \frac{(1-\theta)^2}{2\varepsilon^2}\|f(U_C^{n-1})\|^2 + \frac{\theta^2}{2\varepsilon^2}\|f(U_C^n)\|^2 + \varepsilon^2\|\Delta U_C^{n-\theta}\|^2\\ &\leqslant C(\|U_C^{n-1}\|^2 + \|U_C^n\|^2) + \varepsilon^2\|\Delta U_C^{n-\theta}\|^2.\end{aligned} \tag{4.5}$$

By Lemmas 3, 4, and (4.5), the following inequality is obvious:

$$\begin{aligned}&\frac{1}{4\tau_c}(\mathbb{H}[U_C^n] - \mathbb{H}[U_C^{n-1}]) + \frac{\beta}{4\tau_c}(\mathbb{L}[\phi^n] - \mathbb{L}[\phi^{n-1}])\\ &\quad\leqslant |(\mathscr{D}_{\tau_c} U_C^{n-\theta}, U_C^{n-\theta})| + \beta|(\nabla\mathscr{D}_{\tau_c} U_C^{n-\theta}, \nabla U_C^{n-\theta})|\\ &\quad\leqslant C(\|U_C^n\|^2 + \|U_C^{n-1}\|^2) + C(\|g^{n-\theta}\|^2 + \|U_C^{n-\theta}\|^2)\\ &\quad\leqslant C(\|U_C^n\|^2 + \|U_C^{n-1}\|^2) + C(\|g^n\|^2 + \|g^{n-1}\|^2), \quad n \geqslant 2.\end{aligned} \tag{4.6}$$

Adding up inequality (4.6) from 2 to n, one obtains

$$\begin{aligned}&\frac{1}{4\tau_c}(\mathbb{H}[U_C^n] - \mathbb{H}[U_C^1]) + \frac{\beta}{4\tau_c}(\mathbb{L}[U_C^n] - \mathbb{L}[U_C^1])\\ &\quad\leqslant C\sum_{k=2}^{n}\|U_C^k\|^2 + C\sum_{k=2}^{n}\|U_C^{k-1}\|^2 + C\sum_{k=2}^{n}\|g^k\|^2 + C\sum_{k=2}^{n}\|g^{k-1}\|^2\\ &\quad\leqslant C\sum_{k=1}^{n}\|U_C^k\|^2 + C\sum_{k=1}^{n}\|g^k\|^2.\end{aligned}$$

By the triangle inequality and the Young inequality, we have

$$\begin{aligned}\mathbb{H}[U_C^1] &= (3-2\theta)\|U_C^1\|^2 - (1-2\theta)\|U_C^0\|^2 + (2-\theta)(1-2\theta)\|U_C^1 - U_C^0\|^2 \\ &\leqslant C\|U_C^1\|^2 + C\|U_C^0\|^2, \\ \mathbb{L}[U_C^1] &= (3-2\theta)\|\nabla U_C^1\|^2 - (1-2\theta)\|\nabla U_C^0\|^2 + (2-\theta)(1-2\theta)\|\nabla U_C^1 - \nabla U_C^0\|^2 \\ &\leqslant C\|\nabla U_C^1\|^2 + C\|\nabla U_C^0\|^2.\end{aligned}$$

By Lemmas 3, 4, and the above inequality, we have

$$\begin{aligned}&\frac{1}{1-\theta}\|U_C^n\|^2 + \frac{\beta}{1-\theta}\|\nabla U_C^n\|^2 \leqslant \mathbb{H}[U_C^n] + \beta\mathbb{L}[U_C^n] \\ &\leqslant C\tau_c\sum_{k=1}^n \|U_C^k\|^2 + C\tau_c\sum_{k=1}^n \|g^k\|^2 + \mathbb{H}[U_C^1] + \beta\mathbb{L}[U_C^1] \\ &\leqslant C\tau_c\sum_{k=1}^n \|U_C^k\|^2 + C\tau_c\sum_{k=1}^n \|g^k\|^2 + C\|U_C^1\|^2 + C\|U_C^0\|^2 \\ &\quad + C\|\nabla U_C^1\|^2 + C\|\nabla U_C^0\|^2.\end{aligned}$$

Next, to estimate $\|U_C^1\|^2 + \|\nabla U_C^1\|^2$, taking

$$v_h = \frac{U_C^1 + U_C^0}{2}, \quad q_h = \Delta\frac{U_C^1 + U_C^0}{2},$$

in (3.3a) and (3.3b), respectively, and adding the results yields

$$\begin{aligned}&\Big(\frac{U_C^1 - U_C^0}{\tau_c}, \frac{U_C^1 + U_C^0}{2}\Big) + \varepsilon^2\Big(\Delta\frac{U_C^1 + U_C^0}{2}, \Delta\frac{U_C^0 + U_C^1}{2}\Big) \\ &\quad - \Big(\frac{f(U_C^0) + f(U_C^1)}{2}, \Delta\frac{U_C^0 + U_C^1}{2}\Big) - \beta\Big(\frac{U_C^1 - U_C^0}{\tau_c}, \Delta\frac{U_C^0 + U_C^1}{2}\Big) \\ &= \Big(\frac{g^1 + g^0}{2}, \frac{U_C^1 + U_C^0}{2}\Big).\end{aligned} \tag{4.7}$$

And using the above similar analysis, one has

$$\|U_C^1\|^2 + \|\nabla U_C^1\|^2 \leqslant C\|U_C^0\|^2 + C\|\nabla U_C^0\|^2 + C\tau_c(\|g^0\|^2 + \|g^1\|^2). \tag{4.8}$$

By combining (4.1) and (4.7) with (4.8), the following inequality holds, i.e., for sufficiently small τ_c and β, one has

$$\|U_C^n\|^2 + \|\nabla U_C^n\|^2 \leqslant C\|U_C^0\|^2 + C\|\nabla U_C^0\|^2 + C\tau_c\sum_{k=1}^n \|U_C^k\|^2 + C\tau_c\sum_{k=0}^n \|g^k\|^2.$$

Finally, the use of the discrete Gronwall inequality proves (4.2). □

Theorem 2 *Under the condition* $\tau \leqslant T/(MN)$, *for the fine time-mesh system* (3.5a)-(3.5b) *and* (3.6a)-(3.6b), *the following stability inequality holds*:

$$\|U_F^m\|^2 + \|\nabla U_F^m\|^2 \leqslant C\bigg(\|U_C^0\|^2 + \|\nabla U_C^0\|^2 + \|U_F^0\|^2 + \|\nabla U_F^0\|^2 + \tau \sum_{k=0}^{m+M} \|g^k\|^2\bigg). \tag{4.9}$$

Proof Taking $v_h = U_F^{m-\theta}$ in (3.6a), we have

$$(\mathscr{D}_\tau U_F^{m-\theta}, U_F^{m-\theta}) - (\nabla W^{m-\theta}(U_F), \nabla U_F^{m-\theta}) = (g^{m-\theta}, U_F^{m-\theta}). \tag{4.10}$$

Taking $q_h = \Delta U_F^{m-\theta}$ in (3.6b), we have

$$\begin{aligned} &-\varepsilon^2(\nabla U_F^{m-\theta}, \nabla\Delta U_F^{m-\theta}) - (1-\theta)(f(U_I^m) + (U_F^m - U_I^m)f_u(U_I^m), \Delta U_F^{m-\theta}) \\ &\quad - \theta(f(U_F^{m-1}), \Delta U_F^{m-\theta}) - \beta(\mathscr{D}_\tau U_F^{m-\theta}, \Delta U_F^{m-\theta}) \\ &= (W^{m-\theta}(U_F), \Delta U_F^{m-\theta}). \end{aligned} \tag{4.11}$$

Combining (4.10) and (4.11), we get

$$\begin{aligned} &(\mathscr{D}_\tau U_F^{m-\theta}, U_F^{m-\theta}) + \varepsilon^2(\Delta U_F^{m-\theta}, \Delta U_F^{m-\theta}) \\ &- (1-\theta)(f(U_I^m) + (U_F^m - U_I^m)f_u(U_I^m), \Delta U_F^{m-\theta}) \\ &- \theta(f(U_F^{m-1}), \Delta U_F^{m-\theta}) - \beta(\mathscr{D}_\tau U_F^{m-\theta}, \Delta U_F^{m-\theta}) = (g^{m-\theta}, U_F^{m-\theta}). \end{aligned}$$

By the similar method applied to (4.2), we can obtain the following inequality easily:

$$\begin{aligned} &\mathbb{H}[U_F^m] - \mathbb{H}[U_F^1] + \beta\mathbb{L}[U_F^m] - \beta\mathbb{L}[U_F^1] \\ &\leqslant C\tau\sum_{k=2}^{m}(\|U_F^k\|^2 + \|U_F^{k-1}\|^2) + C\tau\sum_{k=2}^{m}(\|g^k\|^2 + \|g^{k-1}\|^2) + C\tau\sum_{k=1}^{m}\|U_I^k\|^2 \\ &\leqslant C\tau\sum_{k=1}^{m}\|U_F^k\|^2 + C\tau\sum_{k=1}^{m}\|g^k\|^2 + C\tau\sum_{k=1}^{m}\|U_I^k\|^2. \end{aligned} \tag{4.12}$$

Let $n = \lceil k/M \rceil$, the smallest integer that is equal to or greater than k/M. Using the interpolation, we can estimate the Lagrange interpolation term $\|U_I^k\|$, i.e.,

$$U_I^k = \lambda_k U_C^{n-1} + (1-\lambda_k)U_C^n, \quad \lambda_k = n - \frac{k}{M} \in [0, 1), \tag{4.13}$$

$$
\begin{aligned}
\tau\sum_{k=1}^{m}\|U_I^k\|^2 &\leqslant \tau\sum_{k=1}^{m}\|\lambda_k U_C^{n-1}+(1-\lambda_k)U_C^n\|^2\\
&\leqslant C\tau\sum_{k=1}^{m}(\|U_C^{n-1}\|^2+\|U_C^n\|^2)\\
&\leqslant C\tau\sum_{k=1}^{M\lceil m/M\rceil}(\|U_C^{n-1}\|^2+\|U_C^n\|^2)\\
&= C\tau\sum_{l=0}^{\lceil m/M\rceil-1}\ \sum_{k=1+lM}^{(l+1)M}(\|U_C^{n-1}\|^2+\|U_C^n\|^2)\\
&= C\tau\sum_{l=0}^{\lceil m/M\rceil-1}\ \sum_{k=1+lM}^{(l+1)M}(\|U_C^{l}\|^2+\|U_C^{l+1}\|^2)\\
&= CM\tau\sum_{l=0}^{\lceil m/M\rceil-1}(\|U_C^{l}\|^2+\|U_C^{l+1}\|^2)\\
&\leqslant C\tau\sum_{l=0}^{n}\|U_C^l\|^2\\
&\leqslant C\tau\sum_{l=0}^{n}\left(\|U_C^0\|^2+\|\nabla U_C^0\|^2+C\tau_c\sum_{k=0}^{l}\|g^k\|^2\right)\\
&\leqslant Ct_n\|U_C^0\|^2+Ct_n\|\nabla U_C^0\|^2+C\tau_c\tau\sum_{k=0}^{n}\sum_{l=k}^{n}\|g^k\|^2\\
&\leqslant C\|U_C^0\|^2+C\|\nabla U_C^0\|^2+C\tau\sum_{k=0}^{n}\|g^k\|^2\\
&\leqslant C\|U_C^0\|^2+C\|\nabla U_C^0\|^2+C\tau\sum_{k=0}^{m+M}\|g^k\|^2. \qquad (4.14)
\end{aligned}
$$

By (4.12) and (4.14), the following inequality holds:

$$
\begin{aligned}
\frac{1}{1-\theta}\|U_F^m\|^2+\frac{\beta}{1-\theta}\|\nabla U_F^m\|^2 &\leqslant \mathbb{H}[U_F^m]+\beta\mathbb{L}[U_F^m]\\
&\leqslant \mathbb{H}[U_F^1]+\beta\mathbb{L}[U_F^1]+C\|U_C^0\|^2+C\|\nabla U_C^0\|^2\\
&\quad + C\tau\sum_{k=1}^{m}\|U_F^k\|^2+C\tau\sum_{k=0}^{m+M}\|g^k\|^2.
\end{aligned}
$$

The following inequality is obvious by the triangle inequality and the Young inequality:

$$
\begin{aligned}
\mathbb{H}[U_F^1] &= (3-2\theta)\|U_F^1\|^2-(1-2\theta)\|U_F^0\|^2+(2-\theta)(1-2\theta)\|U_F^1-U_F^0\|^2\\
&\leqslant C\|U_F^1\|^2+C\|U_F^0\|^2,
\end{aligned}
$$

$$\begin{aligned}\mathbb{L}[U_F^1] &= (3-2\theta)\|\nabla U_F^1\|^2 - (1-2\theta)\|\nabla U_F^0\|^2 + (2-\theta)(1-2\theta)\|\nabla U_F^1 - \nabla U_F^0\|^2\\ &\leqslant C\|\nabla U_F^1\|^2 + C\|\nabla U_F^0\|^2.\end{aligned}$$

By the above inequality, we have

$$\begin{aligned}&\frac{1}{1-\theta}\|U_F^m\|^2 + \frac{\beta}{1-\theta}\|\nabla U_F^m\|^2\\ &\leqslant \mathbb{H}[U_F^m] + \beta\mathbb{L}[U_F^m]\\ &\leqslant \mathbb{H}[U_F^1] + \beta\mathbb{L}[U_F^1] + C\tau\sum_{k=1}^{m}\|U_F^k\|^2 + C\tau\sum_{k=0}^{m+M}\|g^k\|^2 + C\|U_C^0\|^2 + C\|\nabla U_C^0\|^2\\ &\leqslant C\|U_F^1\|^2 + C\|U_F^0\|^2 + C\|\nabla U_F^1\|^2 + C\|\nabla U_F^0\|^2\\ &\quad + C\tau\sum_{k=1}^{n}\|U_F^k\|^2 + C\tau\sum_{k=0}^{m+M}\|g^k\|^2 + C\|U_C^0\|^2 + C\|\nabla U_C^0\|^2. \end{aligned}\tag{4.15}$$

By taking

$$v_h = \frac{U_F^1 + U_F^0}{2}, \quad q_h = \Delta\frac{U_F^1 + U_F^0}{2},$$

in (3.5a) and (3.5b), respectively, and adding the results, $\|U_F^1\|^2$ and $\|\nabla U_F^1\|^2$ can be estimated using the above similar analysis:

$$\|U_F^1\|^2 + \|\nabla U_F^1\|^2 \leqslant C\|U_F^0\|^2 + C\|\nabla U_F^0\|^2 + C\tau(\|g^0\|^2 + \|g^1\|^2). \tag{4.16}$$

By (4.15) and (4.16), and combing (4.10) and (4.11), we have

$$\begin{aligned}\|U_F^m\|^2 + \|\nabla U_F^m\|^2 &\leqslant C\|U_F^0\|^2 + C\|U_C^0\|^2 + C\|\nabla U_C^0\|^2 + C\|\nabla U_F^0\|^2\\ &\quad + C\tau\sum_{k=1}^{m}\|U_F^k\|^2 + C\tau\sum_{k=1}^{m}\|\nabla U_F^k\|^2 + C\tau\sum_{k=0}^{m+M}\|g^k\|^2.\end{aligned}$$

By using the discrete Gronwall inequality, the proof of (4.9) is done. □

5 Error analysis

Definition 1 The orthogonal projection operator $P_h\colon H_E^1(\Omega) \to V_h$ is defined as

$$B(u - P_h(u), v) = 0, \quad u \in H_E^1(\Omega), \forall\, v \in V_h.$$

Lemma 5 [7] *For any $u, u_t \in L^2(0,T;H^2)$, there exists a unique $P_h u \in V_h$ such that for $0 < t \leqslant T$,*

$$\|u - P_h u\| + h\|u - P_h u\|_1 + \|(u - P_h u)_t\| + h\|(u - P_h u)_t\|_1 \leqslant Ch^2.$$

Theorem 3 *Suppose that the solution of initial problem (1.1) and the fine time-mesh problem (3.4a)-(3.4b) are u and U_C^m, respectively. Then, with the* *assumption* $u \in C^3(0,T,H^2(\Omega))$, *there exists a positive constant* C *independent of* τ_c, τ, *and* h *such that*

$$\|u(t_m) - U_C^m\| \leqslant C(\tau_c^2 + h^2). \tag{5.1}$$

Proof First, we rewrite the weak form of the initial system (1.1) is as follows. For any $v, q \in H_E^1(\Omega)$, when $n = 1$,

$$(u_t(t_{1/2}), v) - (\nabla w(t_{1/2}), \nabla v) = (g(t_{1/2}), v),$$
$$-\varepsilon^2(\nabla u(t_{1/2}), \nabla q) - (f(u(t_{1/2})), q) - \beta(u_t(t_{1/2}), q) = (w(t_{1/2}), q),$$

or

$$(\partial_{1/2}u, v) - (\nabla w(t_{1/2}), \nabla v) + (E_1, v) = (g^{1/2}, v) + (E_2^{1/2}, v), \tag{5.2a}$$
$$\begin{aligned} &-\varepsilon^2(\nabla u(t_{1/2}), \nabla q) - (f^{1/2}(u), q) - \beta(\partial_{1/2}u, q) \\ &\quad = (E_3^{1/2}, q) + \beta(E_1, q) + (w(t_{1/2}), q); \end{aligned} \tag{5.2b}$$

when $n \geqslant 2$,

$$(u_t(t_{n-\theta}), v) - (\nabla w(t_{n-\theta}), \nabla v) = (g(t_{n-\theta}), v),$$
$$-\varepsilon^2(\nabla u(t_{n-\theta}), \nabla q) - (f(u(t_{n-\theta}), q) - \beta(u_t(t_{n-\theta}), q) = (w(t_{n-\theta}), q),$$

or

$$(D_{\tau_c}u^{n-\theta}, v) - (\nabla w(t_{n-\theta}), \nabla v) + (R_t^{n-\theta}, v) = (g^{n-\theta}, v) + (E_2^{n-\theta}, v), \tag{5.3a}$$
$$\begin{aligned} &-\varepsilon^2(\nabla u(t_{n-\theta}), \nabla q) - (f^{n-\theta}(u), q) - \beta(D_{\tau_c}u^{n-\theta}, q) \\ &\quad = (E_3^{n-\theta}, q) + \beta(R_t^{n-\theta}, q) + (w(t_{n-\theta}), q), \end{aligned} \tag{5.3b}$$

where

$$u_t(t_{1/2}) = \partial_{1/2}u + E_1, \quad g(t_{n-\theta}) = g^{n-\theta} + E_2^{n-\theta}, \quad f(u(t_{n-\theta})) = f^{n-\theta}(u) + E_3^{n-\theta}.$$

Subtracting (5.3a)-(5.3b) from (3.4a)-(3.4b), and letting

$$U_C^n - u(t_n) = (U_C^n - P_h u(t_n)) + (P_h u(t_n) - u(t_n)) =: \xi_c^n + \rho_c^n,$$

we have

$$\begin{aligned} &(D_{\tau_c}\xi_c^{n-\theta}, v) - (\nabla W^{n-\theta}(U_C) - \nabla w(t_{n-\theta})(u), \nabla v) + (D_{\tau_c}\rho_c^{n-\theta}, v) \\ &\quad = (R_t^{n-\theta}, v) - (E_2^{n-\theta}, v), \end{aligned} \tag{5.4a}$$
$$\begin{aligned} &-\varepsilon^2(\nabla\xi_c^{n-\theta}, \nabla q) - (f^{n-\theta}(U_C) - f^{n-\theta}(u), q) - \beta(D_{\tau_c}\xi_c^{n-\theta}, q) - \beta(D_{\tau_c}\rho_c^{n-\theta}, q) \\ &\quad = (W^{n-\theta}(U_C) - w(t_{n-\theta})(u), q) - (E_3^{n-\theta}, q) - \beta(R_t^{n-\theta}, q). \end{aligned} \tag{5.4b}$$

By using $v = \xi_c^{n-\theta}$ in (5.4a), $q = \Delta\xi_c^{n-\theta}$ in (5.4b), and adding them, we have

$$
\begin{aligned}
&(\mathscr{D}_{\tau_c}\xi_c^{n-\theta}, \xi_c^{n-\theta}) + \varepsilon^2(\Delta\xi_c^{n-\theta}, \Delta\xi_c^{n-\theta}) \\
&\quad = (f^{n-\theta}(U_C) - f^{n-\theta}(u), \Delta\xi_c^{n-\theta}) + \beta(\mathscr{D}_{\tau_c}\rho_c^{n-\theta} + \mathscr{D}_{\tau_c}\xi_c^{n-\theta}, \Delta\xi_c^{n-\theta}) \\
&\qquad + (R_t^{n-\theta} - E_2^{n-\theta} - \mathscr{D}_{\tau_c}\rho_c^{n-\theta}, \xi_c^{n-\theta}) - (E_3^{n-\theta}, \Delta\xi_c^{n-\theta}) - \beta(R_t^{n-\theta}, \Delta\xi_c^{n-\theta}) \\
&\quad =: J_1 + J_2 + \cdots + J_5.
\end{aligned}
$$

Next, we estimate J_1, J_2, $\ldots$, J_5:

$$
\begin{aligned}
J_1 &= (f^{n-\theta}(U_C) - f^{n-\theta}(u), \Delta\xi_c^{n-\theta}) \\
&= \frac{1}{\varepsilon^2}\,\|(1-\theta)(f(U_C^n) - f(u^n)) + \theta(f(U_C^{n-1}) - f(u^{n-1}))\|^2 + \frac{\varepsilon^2}{4}\,\|\Delta\xi_c^{n-\theta}\|^2 \\
&\leqslant \frac{C}{\varepsilon^2}\,\|U_C^n - u^n\|^2 + \frac{C}{\varepsilon^2}\,\|U_C^{n-1} - u^{n-1}\|^2 + \frac{\varepsilon^2}{4}\,\|\Delta\xi_c^{n-\theta}\|^2,
\end{aligned}
$$

$$
\begin{aligned}
J_2 &= \beta(\mathscr{D}_{\tau_c}\rho_c^{n-\theta} + \mathscr{D}_{\tau_c}\xi_c^{n-\theta}, \Delta\xi_c^{n-\theta}) \\
&\leqslant \frac{2\beta^2}{\varepsilon^2}\,(\|\mathscr{D}_{\tau_c}\rho_c^{n-\theta}\|^2 + \|\mathscr{D}_{\tau_c}\xi_c^{n-\theta}\|^2) + \frac{\varepsilon^2}{4}\,\|\Delta\xi_c^{n-\theta}\|^2, \\
J_3 &= (R_t^{n-\theta} - E_2^{n-\theta} - \mathscr{D}_{\tau_c}\rho_c^{n-\theta}, \xi_c^{n-\theta}) \\
&\leqslant \|R_t^{n-\theta}\|^2 + \|E_2^{n-\theta}\|^2 + \|\mathscr{D}_{\tau_c}\rho_c^{n-\theta}\|^2 + \frac{1}{2}\,\|\xi_c^{n-\theta}\|^2, \\
J_4 &= -(E_3^{n-\theta}, \Delta\xi_c^{n-\theta}) \leqslant \frac{1}{\varepsilon^2}\,\|E_3^{n-\theta}\|^2 + \frac{\varepsilon^2}{4}\,\|\Delta\xi_c^{n-\theta}\|^2, \\
J_5 &= -\beta(R_t^{n-\theta}, \Delta\xi_c^{n-\theta}) \leqslant \frac{\beta^2}{\varepsilon^2}\,\|R_t^{n-\theta}\|^2 + \frac{\varepsilon^2}{4}\,\|\Delta\xi_c^{n-\theta}\|^2.
\end{aligned}
$$

Putting everything together, we have

$$
\begin{aligned}
|(\mathscr{D}_{\tau_c}\xi_c^{n-\theta}, \xi_c^{n-\theta})| \leqslant{}& \frac{C}{\varepsilon^2}\,\|U_C^n - u^n\|^2 + \frac{C}{\varepsilon^2}\,\|U_C^{n-1} - u^{n-1}\|^2 \\
&+ \frac{2\beta^2}{\varepsilon^2}\,(\|\mathscr{D}_{\tau_c}\rho_c^{n-\theta}\|^2 + \|\mathscr{D}_{\tau_c}\xi_c^{n-\theta}\|^2) + \|R_t^{n-\theta}\|^2 + \|E_2^{n-\theta}\|^2 \\
&+ \|\mathscr{D}_{\tau_c}\rho_c^{n-\theta}\|^2 + \frac{1}{2}\,\|\xi_c^{n-\theta}\|^2 + \frac{1}{\varepsilon^2}\,\|E_3^{n-\theta}\|^2 + \frac{\beta^2}{\varepsilon^2}\,\|R_t^{n-\theta}\|^2.
\end{aligned}
$$

By using Lemma 3, the following estimate is obtained:

$$
\begin{aligned}
\frac{1}{4\tau_c}\,(\mathbb{H}[\xi_c^n] - \mathbb{H}[\xi_c^{n-1}]) &\leqslant |(\mathscr{D}_{\tau_c}\xi_c^{n-\theta}, \xi_c^{n-\theta})| \\
&\leqslant C\|U_C^n - u(t_n)\|^2 + C\|U_C^{n-1} - u(t_{n-1})\|^2 \\
&\quad + C(\|D_{\tau_c}\rho_c^{n-\theta}\|^2 + \|D_{\tau_c}\xi_c^{n-\theta}\|^2 + \|\xi_c^{n-\theta}\|^2) \\
&\quad + C(\|E_2^{n-\theta}\|^2 + \|E_3^{n-\theta}\|^2 + \|R_t^{n-\theta}\|^2). \qquad (5.5)
\end{aligned}
$$

Adding up inequality (5.5) from 2 to n, we get

$$\begin{aligned}
&\mathbb{H}[\xi_c^n]-\mathbb{H}[\xi_c^1]\\
&\leqslant C\tau_c\sum_{j=1}^n\|U_C^j-u(t_j)\|^2+C\tau_c\sum_{j=2}^n(\|D_{\tau_c}\rho_c^{j-\theta}\|^2+\|D_{\tau_c}\xi_c^{j-\theta}\|^2+\|\xi_c^{j-\theta}\|^2)\\
&\quad+C\tau_c\sum_{j=2}^n(\|E_2^{j-\theta}\|^2+\|E_3^{j-\theta}\|^2+\|R_t^{j-\theta}\|^2)\\
&\leqslant C\tau_c\sum_{j=1}^n(\|\xi_c^j\|^2+\|\rho_c^j\|^2)+C\tau_c\sum_{j=2}^n\|D_{\tau_c}\rho_c^{j-\theta}\|^2+C\tau_c\sum_{j=2}^n\|D_{\tau_c}\xi_c^{j-\theta}\|^2\\
&\quad+C\tau_c\sum_{j=2}^n\|\xi_c^{j-\theta}\|^2+C\tau_c\sum_{j=2}^n(\|E_2^{j-\theta}\|^2+\|E_3^{j-\theta}\|^2+\|R_t^{j-\theta}\|^2).
\end{aligned}\tag{5.6}$$

By Lemma 3 and using the discrete Gronwall inequality, (5.6) is formulated as ($n\geqslant 2$)

$$\begin{aligned}
\|\xi_c^n\|^2&\leqslant C\tau_c\sum_{j=1}^n\|\rho_c^j\|^2+C\tau_c\sum_{j=2}^n(\|E_2^{j-\theta}\|^2+\|E_3^{j-\theta}\|^2+\|R_t^{j-\theta}\|^2)\\
&\quad+C\tau_c\sum_{j=2}^n\|D_{\tau_c}\rho_c^{j-\theta}\|^2+C\|\xi_c^1\|^2\\
&\leqslant C\tau_c\sum_{j=1}^n\|\rho_c^j\|^2+C\tau_c\sum_{j=2}^n(\|E_2^{j-\theta}\|^2+\|E_3^{j-\theta}\|^2+\|R_t^{j-\theta}\|^2)\\
&\quad+C\tau_c\sum_{j=2}^n\|D_{\tau_c}\rho_c^{j-\theta}-\partial_t\rho_c^{j-\theta}\|^2+C\tau_c\sum_{j=2}^n\|\partial_t\rho_c^{j-\theta}\|^2+C\|\xi_c^1\|^2.
\end{aligned}$$

By Lemmas 2 and 5, one obtains

$$\|\xi_c^n\|^2\leqslant Ct_nh^4+Ct_n\tau_c^4+Ct_n\tau_c^4+C\|\xi_c^1\|^2\leqslant Ch^4+C\tau_c^4+C\|\xi_c^1\|^2.\tag{5.7}$$

Next, as in the previous analysis to estimate $\|\xi_c^1\|^2$, subtracting (5.2a)-(5.2b) from (3.3a)-(3.3b), we get

$$\begin{aligned}
&(\partial_{1/2}\xi_c,v)-(\nabla W^{1/2}(U_C)-\nabla w(t_{1/2})(u),\nabla v)+(\partial_{1/2}\rho_c,v)\\
&\quad=(E_1,v)-(E_2^{1/2},v),
\end{aligned}\tag{5.8a}$$

$$\begin{aligned}
&-\varepsilon^2(\nabla\xi_c^{1/2},\nabla q)-(f^{1/2}(U_C)-f^{1/2}(u),q)-\beta(\partial_{1/2}\xi_c,q)-\beta(\partial_{1/2}\rho_c,q)\\
&\quad=(W^{1/2}(U_C)-w(t_{1/2})(u),q)-(E_3^{1/2},q)-\beta(E_1,q).
\end{aligned}\tag{5.8b}$$

By using $v = \xi_c^{1/2}$ in (5.8a), $q = \Delta\xi_c^{1/2}$ in (5.8b), and adding them, we have

$$\begin{aligned}
&(\partial_{1/2}\xi_c, \xi_c^{1/2}) + \varepsilon^2(\Delta\xi_c^{1/2}, \Delta\xi_c^{1/2})\\
&\quad - (f^{1/2}(U_C) - f^{1/2}(u), \Delta\xi_c^{1/2}) - \beta(\partial_{1/2}\xi_c, \Delta\xi_c^{1/2})\\
&= \beta(\partial_{1/2}\rho_c, \Delta\xi_c^{1/2}) + (E_1 - E_2^{1/2}, \xi_c^{1/2}) - (E_3^{1/2}, \Delta\xi_c^{1/2})\\
&\quad - (\partial_{1/2}\rho_c, \xi_c^{1/2}) - \beta(E_1, \Delta\xi_c^{1/2}).
\end{aligned}$$

By the similar analysis, one can derive

$$\|\xi_c^1\|^2 \leqslant C(\tau_c^4 + h^4) + C\|U_C^0 - u_0\|^2. \tag{5.9}$$

By (5.7) and (5.9), the following inequality holds:

$$\|\xi_c^n\|^2 \leqslant C(\tau_c^4 + h^4). \tag{5.10}$$

Combining (5.10) with the property of the orthogonal projector P_h, we complete the proof of (5.1). □

Theorem 4 *Suppose that the solutions of initial problem* (1.1) *and the fine time-mesh problem* (3.5a)-(3.5b) *and* (3.6a)-(3.6b) *are u and U_F^m, respectively. Then, with the assumption $u \in C^3(0, T, H^2(\Omega))$, there exists a positive constant C independent of τ_c, τ, and h such that*

$$\|u(t_m) - U_F^m\| \leqslant C(\tau_c^4 + \tau^2 + h^2). \tag{5.11}$$

Proof We first estimate the error $\|u(t_m) - U_I^m\|$ on the fine time-mesh. By the notations introduced in (4.13), we have

$$\begin{aligned}
U_I^m &= \lambda_m U_C^{n-1} + (1-\lambda_m)U_C^n,\\
u(t_m) &= \lambda_m u^{n-1} + (1-\lambda_m)u^n + C\tau_c^2 u_{tt}(\vartheta_m), \quad \vartheta_m \in (t_{n-1}, t_n).
\end{aligned} \tag{5.12}$$

With (5.12) and (5.1), the following result holds by the triangle inequality:

$$\|u(t_m) - U_I^m\| \leqslant C(\tau_c^2 + h^2).$$

Next, we replace n with m and τ_c with τ in (5.3a)-(5.3b), and subtract it from (3.6a)-(3.6b). We have

$$\begin{aligned}
&(\mathscr{D}_\tau\xi_f^{m-\theta}, v) + (\mathscr{D}_\tau\rho_f^{m-\theta}, v) - (\nabla W^{m-\theta}(U_F) - \nabla w(t_{m-\theta})(u)), \nabla v)\\
&\quad = (R_t^{m-\theta}, v) - (E_2^{m-\theta}, v),\\
&-\varepsilon^2(\nabla\xi_f^{m-\theta}, \nabla q) - \beta(\mathscr{D}_\tau\xi_f^{m-\theta}, q) - \beta(\mathscr{D}_\tau\rho_f^{m-\theta}, \triangle\xi_f^{m-\theta})\\
&\qquad - (1-\theta)(f(U_I^m) + (U_F^m - U_I^m)f_u(U_I^m) - f(u^m), q)\\
&\qquad - \theta((f(U_F^{m-1}) - f(u^{m-1})), q)\\
&\quad = (W^{m-\theta}(U_C) - w(t_{m-\theta})(u), q) - (E_3^{m-\theta}, \nabla q) - \beta(R_t^{m-\theta}, q).
\end{aligned}$$

By choosing $v=\xi_f^{m-\theta}$, $q=\Delta\xi_f^{m-\theta}$, and adding them, we have

$$\begin{aligned}
&(\mathscr{D}_\tau\xi_f^{m-\theta},\xi_f^{m-\theta})+(\mathscr{D}_\tau\rho_f^{m-\theta},\xi_f^{m-\theta})+\varepsilon^2(\Delta\xi_f^{m-\theta},\Delta\xi_f^{m-\theta})\\
&\quad=(1-\theta)(f(U_I^m)+(U_F^m-U_I^m)f_u(U_I^m)-f(u^m),\Delta\xi_f^{m-\theta})\\
&\qquad+\theta(f(U_F^{m-1})-f(u^{m-1}),\Delta\xi_f^{m-\theta})+\beta(\mathscr{D}_\tau\xi_f^{m-\theta},\Delta\xi_f^{m-\theta})\\
&\qquad+\beta(\mathscr{D}_\tau\rho_f^{m-\theta},\Delta\xi_f^{m-\theta})-(E_3^{m-\theta},\Delta\xi_f^{m-\theta})\\
&\qquad-\beta(R_t^{m-\theta},\Delta\xi_f^{m-\theta})+(R_t^{m-\theta},\xi_f^{m-\theta})-(E_2^{m-\theta},\xi_f^{m-\theta}),
\end{aligned}\tag{5.13}$$

where

$$U_F^m-u(t_m)=(U_F^m-P_hu(t_m))+(P_hu(t_m)-u(t_m))=\xi_f^m+\rho_f^m.$$

Using Taylor expansion, we estimate the first term on the right-hand-side of (5.13) as follows

$$\begin{aligned}
&\|f(u^m)-f(U_I^m)-(U_F^m-U_I^m)f_u(U_I^m)\|\\
&\quad=\|f_u(U_I^m)[(u^m-U_I^m)-(U_F^m-U_I^m)]+Cf_{uu}(\eta_m)(u^m-U_I^m)^2\|\\
&\quad=\|f_u(U_I^m)(u^m-U_F^m)+Cf_{uu}(\eta_m)(u^m-U_I^m)^2\|\\
&\quad=\|f_u(U_I^m)(\xi_f^m+\rho_f^m)+Cf_{uu}(\eta_m)(u^m-U_I^m)^2\|\\
&\quad\leqslant C(\|\xi_f^m\|+\|\rho_f^m\|)+C\|u^m-U_I^m\|_{L^4(\Omega)}^2.
\end{aligned}\tag{5.14}$$

Combining (5.13) and (5.14) with the similar analysis applied to (5.1), for $n\geqslant 2$, we have

$$\|\xi_f^m\|^2\leqslant C(\tau^4+\tau_c^8+h^4)+C\|U_F^0-u_0\|^2+C\|\xi_f^1\|^2.\tag{5.15}$$

Next, as in the previous analysis to estimate $\|\xi_f^1\|^2$, we replace n with m, τ_c with τ in (3.3a)-(3.3b), then we subtract (5.2a)-(5.2b) from (3.3a)-(3.3b):

$$\begin{aligned}
&(\partial_{1/2}\xi_f,v)-(\nabla W^{1/2}(U_F)-\nabla w(t_{1/2})(u),\nabla v)+(\partial_{1/2}\rho_f,v)\\
&\quad=(E_1,v)-(E_2^{1/2},v),
\end{aligned}\tag{5.16a}$$

$$\begin{aligned}
&-\varepsilon^2(\nabla\xi_f^{1/2},\nabla q)-(f^{1/2}(U_F)-f^{1/2}(u),q)-\beta(\partial_{1/2}\xi_f,q)-\beta(\partial_{1/2}\rho_f,q)\\
&\quad=(W^{1/2}(U_F)-w(t_{1/2})(u),q)-(E_3^{1/2},q)-\beta(E_1,q).
\end{aligned}\tag{5.16b}$$

By using $v=\xi_f^{1/2}$ in (5.16a), $q=\Delta\xi_f^{1/2}$ in (5.16b), and adding them, we have

$$\begin{aligned}
&(\partial_{1/2}\xi_f,\xi_f^{1/2})+\varepsilon^2(\Delta\xi_f^{1/2},\Delta\xi_f^{1/2})\\
&\qquad-(f^{1/2}(U_f)-f^{1/2}(u),\Delta\xi_f^{1/2})-\beta(\partial_{1/2}\xi_f,\Delta\xi_f^{1/2})\\
&\quad=(E_1,\xi_f^{1/2})-(E_2^{1/2},\xi_f^{1/2})-(E_3^{1/2},\Delta\xi_f^{1/2})\\
&\qquad-\beta(E_1,\Delta\xi_f^{1/2})-(\partial_{1/2}\xi_f,\rho_f^{1/2})+\beta(\partial_{1/2}\rho_f,\xi_f^{1/2}).
\end{aligned}$$

Using the above technique again, we can get

$$\|\xi_f^1\|^2 \leqslant C(\tau^4 + \tau_c^8 + h^4) + C\|U_F^0 - u_0\|^2. \tag{5.17}$$

Combining (5.15) and (5.17) with the property of the orthogonal projector P_h, we complete the proof of (5.11). □

6 Numerical experiments

In this section, we present some numerical examples to illustrate the theoretical results obtained in the previous section. We study the effects of θ and β values on the spatial convergence orders and compare the CPU time of the Galerkin method and the TT-M FE method. The behavior of the exact solution and the TT-M numerical solution are demonstrated via visualization. All tests are obtained by the package Freefem++ [18].

6.1 Convergence results

In this part, we verify the theoretical error estimates by numerical examples, i.e., verification of the order of spatial convergence of the viscous Cahn-Hilliard equation using the TT-M method by examples. We take the $P1$ finite element space, and choose the initial condition

$$u_0 = \mathrm{e}\cos(\pi x)\cos(\pi y),$$

the exact solution

$$u(x, y, t) = \mathrm{e}^{\cos t}\cos(\pi x)\cos(\pi y).$$

We can compute the external force term in the equation.

6.1.1 Order of convergence in space

Tables 1–3 show that the spatial convergence order is consistent with the theoretical value for different β value. The parameters used in our simulation are

$$\varepsilon = 0.3, \quad \theta = 0.2, \quad \tau_c = 10\tau = \frac{1}{20}, \quad T = 1,$$
$$\beta = 0,\ 0.001,\ 0.01, \quad h = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}.$$

Table 1 Spatial convergence rate of numerical results of TT-M: $\beta = 0$

h	$\|u - U_F\|/\|u\|$ relative error	rate	$\|u - U_F\|_{H^1}/\|u\|_{H^1}$ relative error	rate
1/16	0.104671		0.201608	
1/32	0.0243144	2.105	0.0966592	1.065
1/64	0.00603877	2.009	0.0479912	1.010

Table 2 Spatial convergence rate of numerical results of TT-M: $\beta = 0.001$

h	$\|u - U_F\|/\|u\|$		$\|u - U_F\|_{H^1}/\|u\|_{H^1}$	
	relative error	rate	relative error	rate
1/16	0.104686		0.201582	
1/32	0.0243551	2.103	0.0966518	1.060
1/64	0.00608312	2.001	0.047988	1.010

Table 3 Spatial convergence rate of numerical results of TT-M: $\beta = 0.01$

h	$\|u - U_F\|/\|u\|$		$\|u - U_F\|_{H^1}/\|u\|_{H^1}$	
	relative error	rate	relative error	rate
1/16	0.104837		0.201357	
1/32	0.0247716	2.081	0.0966	1.059
1/64	0.00666528	1.893	0.0479904	1.009

Table 1 lists the results based on TT-M FE method for $\beta = 0$ and $h = 1/16, 1/32, 1/64$. The spatial convergence orders computed from relative errors $\|u - U_F\|/\|u\|$ and $\|u - U_F\|_{H^1}/\|u\|_{H^1}$ are close to 2 and 1, respectively. Results for $\beta = 0.001$ and 0.01 are listed in Tables 2 and 3, respectively. We get the same conclusion according to the similar analysis as that discussed for Table 1. We also observe that the spacial convergence rate is almost same, when β takes different values. These numerical results imply that our numerical algorithm is correct.

Tables 4–7 show that the spatial convergence order is consistent with the theoretical value for different θ value. The parameters used in our simulation are

$$\varepsilon = 0.3, \quad \beta = 0.01, \quad \tau_c = 10\tau = \frac{1}{20}, \quad T = 1,$$
$$\theta = 0,\ 0.2,\ 0.4,\ 0.5, \quad h = \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32}.$$

Table 4 lists the results based on TT-M FE method for $\theta = 0$ and $h = 1/8, 1/16, 1/32$. The spatial convergence orders computed from relative errors $\|u - U_F\|/\|u\|$ and $\|u - U_F\|_{H^1}/\|u\|_{H^1}$ are close to 2 and 1, respectively. Results for $\theta = 0.2$, 0.4, and 0.5 are listed on Tables 5, 6, and 7, respectively. The same conclusion can be obtained. These numerical results imply that our numerical algorithm is correct.

Table 4 Spatial convergence rate of numerical results of TT-M: $\theta = 0$

h	$\|u - U_F\|/\|u\|$		$\|u - U_F\|_{H^1}/\|u\|_{H^1}$	
	relative error	rate	relative error	rate
1/8	0.858158		0.616371	
1/16	0.236553	1.859	0.299096	1.043
1/32	0.0324659	2.865	0.0991513	1.59

Table 5 Spatial convergence rate of numerical results of TT-M: $\theta = 0.2$

h	$\|u-U_F\|/\|u\|$		$\|u-U_F\|_{H^1}/\|u\|_{H^1}$	
	relative error	rate	relative error	rate
1/8	0.388401		0.448404	
1/16	0.104837	1.889	0.201357	1.155
1/32	0.0247716	2.081	0.0966	1.059

Table 6 Spatial convergence rate of numerical results of TT-M: $\theta = 0.4$

h	$\|u-U_F\|/\|u\|$		$\|u-U_F\|_{H^1}/\|u\|_{H^1}$	
	relative error	rate	relative error	rate
1/8	0.335496		0.40981	
1/16	0.0950239	1.819	0.197356	1.054
1/32	0.0245207	1.954	0.0965279	1.031

Table 7 Spatial convergence rate of numerical results of TT-M: $\theta = 0.5$

h	$\|u-U_F\|/\|u\|$		$\|u-U_F\|_{H^1}/\|u\|_{H^1}$	
	relative error	rate	relative error	rate
1/8	0.333468		0.408707	
1/16	0.0948542	1.813	0.197361	1.050
1/32	0.0245138	1.952	0.0965818	1.031

6.1.2 Order of convergence in time

Table 8 shows that the time convergence order is consistent with the theoretical value for different θ value. We choose parameters as follows:

$$\varepsilon = 0.3, \quad \theta = 0,\, 0.2,\, 0.4,\, 0.5, \quad \beta = 0.1, \quad \tau = h = \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \quad T = 1.$$

The time convergence orders computed from errors $\|u-U_F\|$ is close to 2. These numerical results imply that our numerical algorithm is correct.

Table 9 shows that the time convergence order is consistent with the theoretical value for different β value. We choose parameters as follows:

$$\varepsilon = 0.3, \quad \tau_c^2 = \tau = h = \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \quad \beta = 0,\, 0.01, \quad \theta = 0.5, \quad T = 1.$$

The time convergence orders computed from errors $\|u - U_F\|$ is close to 2.

6.1.3 Comparison between TT-M FE method and Galerkin FE method on CPU time

In this part, the CPU time of the traditional Galerkin FE method and the TT-M FE method are compared. We take parameters

$$\theta = 0.5, \quad \beta = 0.01, \quad \varepsilon = 0.3, \quad \tau = 10\tau_c = \frac{1}{200}, \quad h = \frac{1}{8}, \frac{1}{16}, \frac{1}{32}.$$

From Table 10, one can see that our numerical algorithm can substantially reduce the CPU time while producing equally accurate numerical results.

Table 8 Temporal convergence rate of numerical results of TT-M: $\beta = 0.1$

θ	$\tau = h$	$\|u - U_F\|$	rate
0	1/9	0.271629	
	1/16	0.0950813	1.824
	1/25	0.0407967	1.895
	1/36	0.0209129	1.832
0.2	1/9	0.27205	
	1/16	0.0936853	1.852
	1/25	0.0387008	1.980
	1/36	0.0186608	2.000
0.4	1/9	0.272465	
	1/16	0.0940023	1.849
	1/25	0.039034	1.969
	1/36	0.0188599	1.994
0.5	1/9	0.27231	
	1/16	0.0940378	1.847
	1/25	0.0390698	1.968
	1/36	0.0189	1.991

Table 9 Temporal convergence rate of numerical results of TT-M: $\theta = 0.5$

β	$\tau = h$	$\|u - U_F\|$	rate
0	1/9	0.278024	
	1/16	0.0943529	1.878
	1/25	0.0396985	1.939
	1/36	0.0189656	2.025
0.01	1/9	0.278574	
	1/16	0.0948477	1.872
	1/25	0.0401712	1.925
	1/36	0.01947771	1.985

Table 10 CPU time of TT-M FE method and Galerkin FE method: $\theta = 0.5$

h	FE(s)			TT-M(s)		
	$\|u - U_F\|/\|u\|$	$\|u - U_F\|_{H^1}/\|u\|_{H^1}$	CPU	$\|u - U_F\|/\|u\|$	$\|u - U_F\|_{H^1}/\|u\|_{H^1}$	CPU
1/8	0.333532	0.408677	36.74	0.333468	0.408707	11.86
1/16	0.0948615	0.197356	92.518	0.0948542	0.197361	36.858
1/32	0.0245146	0.0965814	395.162	0.0245138	0.0965818	162.789

6.2 Influence of M on CPU time and error

To check the computational efficiency of the fast TT-M FE method, we consider the impact of parameter M on CPU time. Fig. 1 plots the CPU time versus the value of M for parameters

$$\varepsilon = 1, \quad \beta = 0.01, \quad \theta = 0.5, \quad \tau = \frac{1}{400}, \quad h = \frac{1}{20}, \quad T = 1.$$

One can see that the computing time of the fast TT-M FE method gradually decreases as M increases from 2 to 20, which indicates higher computational efficiency for greater value of M, and one can know that the most efficient calculation is produced at $M = 20$, and when M tends to value $M = 20$, the CPU changes very slowly.

Next, we check the possible influence of M on the computational accuracy. Using the same set of parameters as in Fig. 1, we compute the error versus value of M in Fig. 2. One clearly sees that the parameter M has little influence on the computational accuracy. Therefore, when using the TT-M FE method, we can select the appropriate value of M to improve the computational efficiency without affecting the accuracy.

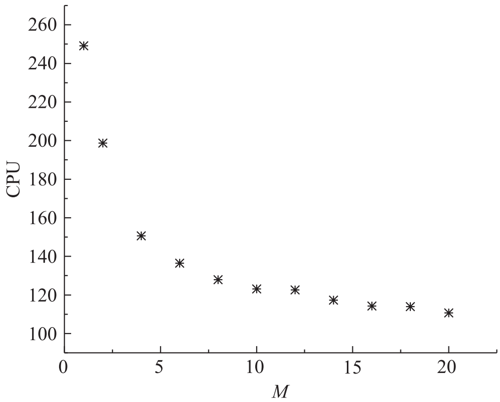

Fig. 1 CPU(M) based on $\theta = 0.5$

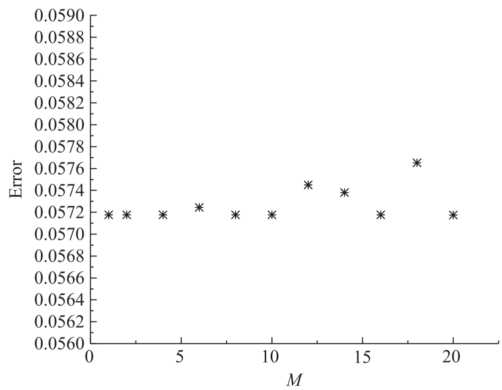

Fig. 2 Error(M) based on $\theta = 0.5$

7 Summary

In this paper, we apply the fast TT-M FE method to the nonlinear viscous Cahn-Hilliard equation. Theoretical stability analysis and error estimates of the method are provided in detail. Numerical examples are given to verify these theoretical results. The comparisons of CPU time is made between the TT-M FE method and the Galerkin FE method, and we study the effect of the parameter M. It is worth noticing that for the second-order θ scheme with $\beta = 0$, the spatial convergence orders computed from $\|u-U_F\|$ and $\|u-U_F\|_{H^1}$ are close to 2 and 1, respectively. When $\beta = 0.001$, 0.01 have similar results. All of the numerical results show that our fast TT-M FE method is effective and efficient.

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