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RESEARCH ARTICLE

Complete moment convergence for weighted sums of widely orthant-dependent random variables and its application in nonparametric regression models

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Abstract We establish some results on the complete moment convergence for weighted sums of widely orthant-dependent (WOD) random variables, which improve and extend the corresponding results of Y. F. Wu, M. G. Zhai, and J. Y. Peng [J. Math. Inequal., 2019, 13(1): 251–260]. As an application of the main results, we investigate the complete consistency for the estimator in a nonparametric regression model based on WOD errors and provide some simulations to verify our theoretical results.

Keywords Widely orthant-dependent random variables, complete moment convergence, nonparametric regression model

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1 Introduction

Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathscr{F}, P)$. Among many statistical problems and applications, people often assume that the random variables are independent. However, this assumption sometimes is impractical. Hence, many scholars introduced various dependence structures to obtain more accurate results in the former decades. One of the most important dependence structures is widely orthant dependence (WOD), which was introduced by Wang et al. [22]. Now, let us recall the concept of WOD random variables.

Definition 1.1 Random variables $X_1, X_2, \dots$ are said to be widely upper orthant dependent (WUOD) if for each $n \geqslant 1$, there exists some finite real

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number $g_U(n)$ such that, for all $x_i \in (-\infty, +\infty)$, $i = 1, \ldots, n$,

$$P(X_1 > x_1, \ldots, X_n > x_n) \leqslant g_U(n) \prod_{i=1}^{n} P(X_i > x_i); \tag{1.1}$$

they are said to be widely lower orthant dependent (WLOD) if for each $n \geqslant 1$, there exists some finite real number $g_L(n)$ such that, for all $x_i \in (-\infty, +\infty)$, $i = 1, \ldots, n$,

$$P(X_1 \leqslant x_1, \ldots, X_n \leqslant x_n) \leqslant g_L(n) \prod_{i=1}^{n} P(X_i \leqslant x_i); \tag{1.2}$$

they are said to be WOD if they are both WUOD and WLOD. The real numbers $g_U(n)$, $g_L(n)$, $n \geqslant 1$, are called dominating coefficients.

If $g_U(n) = g_L(n) = M$ for all $n \geqslant 1$, where $M \geqslant 1$ is a positive constant, then the sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be extended negatively dependent (END), which was introduced by Liu [15]. It is clear to see that WOD structure contains END structure.

Since the concept of WOD random variables was introduced, many authors have been devoted to the study of the probability limit properties of WOD random variables. For example, Wang and Cheng [27] presented some basic renewal theorems for a random walk with widely dependent increments; Shen [17] established the Bernstein type probability inequality of WOD random variables; Wang et al. [25] studied the complete convergence of WOD random variables, and gave its application in the nonparametric regression models; Wang and Hu [23] studied some consistency problems of the nearest neighbor kernel density estimation under WOD samples; Wu et al. [31] investigated the complete moment convergence for arrays of rowwise WOD random variables; Shen et al. [18] studied the asymptotic properties for the estimators of survival function and failure rate function based on WOD samples; Wu et al. [30] studied the L^r convergence, complete convergence, and complete moment convergence for arrays of rowwise WOD random variables under some conditions of R-h-integrability; Chen and Sung [4] established the Spizer-type law of large numbers for WOD random variables; and Lu et al. [16] studied the complete f-moment convergence for WOD random variables. The main purpose of this work is to further study the complete moment convergence for WOD random variables, which is more general than complete convergence.

The concept of complete convergence was introduced by Hsu and Robbins [7] as follows: a sequence $\{X_n, n \geqslant 1\}$ of random variables converges completely to a constant C if

$$\sum_{n=1}^{\infty} P(|X_n - C| > \varepsilon) < \infty, \quad \forall\, \varepsilon > 0.$$

By the Borel-Cantelli lemma, we immediately obtain that $X_n \to C$ almost surely (a.s.). For more details about complete convergence, one can refer to [9,11,21,24,35] among others.

Recently, Wu et al. [33] obtained the following result on complete convergence for weighted sums of END random variables.

Theorem A *Let $\{X_n,\, n \geqslant 1\}$ be a sequence of identically distributed END random variables with $EX_1 = 0$, and let $\{a_{nk},\, n \geqslant 1,\, 1 \leqslant k \leqslant n\}$ be an array of real numbers satisfying*

$$\sum_{k=1}^{n} a_{nk}^2 = O(n^{-\alpha}) \tag{1.3}$$

and

$$\max_{1\leqslant k\leqslant n} |a_{nk}| = O(n^{-\alpha}) \tag{1.4}$$

for some $p \geqslant 2$ and $\alpha \in [1/p, 1)$. If

$$E|X_1|^p < \infty, \tag{1.5}$$

then

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\biggl(\max_{1\leqslant j\leqslant n}\Bigl|\sum_{k=1}^{j} a_{nk}X_k\Bigr| > \varepsilon\biggr) < \infty, \quad \forall\, \varepsilon > 0.$$

Although conditions (1.3)–(1.5) are rather basic, the identical distribution assumption seems to be too strong and the END setting is not wide enough. On the other hand, it is also very desirable to improve the conclusion of Theorem A. The main purpose of this paper is to extend Theorem A from complete convergence to complete moment convergence for WOD random variables. In addition, the condition of identical distribution is replaced by stochastic domination. Furthermore, the meaningful case $1 < p < 2$ is also established in this paper, which was not considered by Wu et al. [33]. As an application of our main results, we present a result on complete consistency for the weighted estimator in a nonparametric regression model based on WOD errors.

The concept of complete moment convergence mentioned above was introduced by Chow [5] as follows. Let $\{X_n,\, n \geqslant 1\}$ be a sequence of random variables and $a_n, b_n, q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|X_n| - \varepsilon\}_+^q < \infty, \quad \forall\, \varepsilon > 0,$$

then q-th moment convergence is said to hold for $\{X_n,\, n \geqslant 1\}$. It is well known that complete moment convergence implies complete convergence. For more details about the complete moment convergence, we refer the readers to [3,12,14,28,32] among others.

The following concept of stochastic domination will be used in this paper.

Definition 1.2 A sequence $\{X_n,\, n \geqslant 1\}$ of random variables is said to be stochastically dominated by a random variable X, if there exists a positive constant C such that

$$P(|X_n| > x) \leqslant CP(|X| > x), \quad \forall\, x \geqslant 0,\, n \geqslant 1.$$

This paper is organized as follows. Some preliminary lemmas are stated in Section 2. In Section 3, we provide our main results and their proofs. An application to nonparametric regression models and numerical simulations are presented in Section 4.

Throughout this paper, let C be a positive constant not depending on n, which may be different in various places. Denote

$$(\log x)_+ = \log \max\{x, \mathrm{e}\}, \quad x_+ = \max\{x, 0\}, \quad x_- = \max\{-x, 0\},$$

and let $I(A)$ be the indicator function of the set A. $A_n = O(B_n)$ stands for $|A_n| \leqslant C|B_n|$ for all $n \geqslant 1$. Let $g(n) = \max\{g_U(n), g_L(n)\}$ be the dominating coefficients of the WOD sequence.

2 Preliminary lemmas

To prove the main results of this paper, we need the following important lemmas. The first one is a basic property for WOD random variables, which was established by Wang et al. [25].

Lemma 2.1 *Let $\{X_n, n \geqslant 1\}$ be a sequence of WOD random variables with the dominating coefficients $g(n)$. If $\{f_n, n \geqslant 1\}$ is a sequence of all nondecreasing* (*or nonincreasing*) *functions, then $\{f_n(X_n), n \geqslant 1\}$ is also a sequence of WOD random variables with the same dominating coefficients $g(n)$.*

By [26, Corollary 2.3] and [20, Theorem 2.3.1], we can obtain the following Rosenthal-type maximum inequality and Marcinkiewicz-Zygmund type maximum inequality for WOD random variables.

Lemma 2.2 *Let $p \geqslant 1$ and $\{X_n, n \geqslant 1\}$ be a sequence of zero mean WOD random variables with the dominating coefficients $g(n)$ and $E|X_n|^p < \infty$ for each $n \geqslant 1$. Then there exist positive constants $C_1(p)$ and $C_2(p)$ depending only on p such that*

$$E\bigg(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k} X_i\Big|\bigg)^p \leqslant C_1(p)(\log n)_+^p \sum_{i=1}^{n} E|X_i|^p + C_2(p) g(n)(\log n)_+^p \cdot \begin{cases} \displaystyle\sum_{i=1}^{n} E|X_i|^p, & 1 \leqslant p \leqslant 2, \\ \displaystyle\bigg(\sum_{i=1}^{n} EX_i^2\bigg)^{p/2}, & p > 2. \end{cases}$$

The following lemma is essential in proving our main results, which was obtained by Wu et al. [29].

Lemma 2.3 *Let $\{Y_i, 1 \leqslant i \leqslant n\}$ and $\{Z_i, 1 \leqslant i \leqslant n\}$ be two sequences of random variables. Then, for any $q > r > 0$ and $\varepsilon, a > 0$, the following two inequalities hold*:

$$E\Big(\Big|\sum_{i=1}^{n}(Y_i+Z_i)\Big|-\varepsilon a\Big)_+^r \leqslant C_r\Big(\varepsilon^{-q}+\frac{r}{q-r}\Big)a^{r-q}E\Big|\sum_{i=1}^{n}Y_i\Big|^q+C_rE\Big|\sum_{i=1}^{n}Z_i\Big|^r,$$

$$E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k}(Y_i+Z_i)\Big|-\varepsilon a\Big)_+^r$$
$$\leqslant C_r\Big(\varepsilon^{-q}+\frac{r}{q-r}\Big)a^{r-q}E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k}Y_i\Big|^q\Big)+C_rE\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k}Z_i\Big|^r\Big),$$

where

$$C_r=\begin{cases}1, & 0<r\leqslant 1,\\ 2^{r-1}, & r>1.\end{cases}$$

By the integration by parts, we can get the following property for stochastic domination. The first inequality is due to Adler and Rosalsky [1] and the second inequality is due to Adler et al. [2].

Lemma 2.4 *Let $\{X_n,\, n\geqslant 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. Then, for any $\alpha, b>0$, the following two statements hold*:

$$E|X_n|^\alpha I(|X_n|\leqslant b)\leqslant C_1[E|X|^\alpha I(|X|\leqslant b)+b^\alpha P(|X|>b)],$$
$$E|X_n|^\alpha I(|X_n|>b)\leqslant C_2E|X|^\alpha I(|X|>b),$$

where C_1 and C_2 are positive constants. Consequently,

$$E|X_n|^\alpha\leqslant CE|X|^\alpha,$$

where C is a positive constant.

3 Main results and their proofs

Before presenting the main results, we list two assumptions as follows.

(A.1) $\{X_n,\, n\geqslant 1\}$ is a sequence of zero mean WOD random variables which is stochastically dominated by a random variable X with dominating coefficients $g(n)$, $n\geqslant 1$.

(A.2) $\{a_{nk},\, n\geqslant 1,\, 1\leqslant k\leqslant n\}$ is an array of real numbers satisfying (1.3) and (1.4) for some $p>1$ and $\alpha\in[1/p,1)$.

Theorem 3.1 *Assume that* (A.1) *and* (A.2) *are satisfied. If*

$$g(n)=O(n^\lambda)\quad\text{for some }\lambda\geqslant 0 \tag{3.1}$$

and $E|X|^p<\infty$, then

$$\sum_{n=1}^{\infty}n^{\alpha p-2}E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{k=1}^{j}a_{nk}X_k\Big|-\varepsilon\Big)_+<\infty,\quad\forall\,\varepsilon>0. \tag{3.2}$$

Proof Without loss of generality, we assume $a_{nk} \geqslant 0$ for all $k = 1, \dots, n$. Otherwise, we will use a_{nk+} and a_{nk-} instead of a_{nk}, respectively.

For fixed $n \geqslant 1$ and for $1 \leqslant k \leqslant n$, denote

$$\begin{aligned} Y_{nk} &= -n^\alpha I(X_k < -n^\alpha) + n^\alpha I(X_k > n^\alpha) + X_k I(|X_k| \leqslant n^\alpha), \\ Z_{nk} &= n^\alpha I(X_k < -n^\alpha) - n^\alpha I(X_k > n^\alpha) + X_k I(|X_k| > n^\alpha). \end{aligned}$$

Noting that $EX_k = 0$, we have

$$X_k = Y_{nk} - EY_{nk} + Z_{nk} - EZ_{nk}. \tag{3.3}$$

Take

$$q > \begin{cases} \max\Big\{\dfrac{2(\alpha p + \lambda - 1)}{\alpha}, p\Big\}, & p \geqslant 2, \\ \max\Big\{\dfrac{2(\alpha p + \lambda - 1)}{\alpha(p-1)}, 2\Big\}, & 1 < p < 2. \end{cases}$$

Therefore, we obtain by (3.3) and Lemma 2.3 that

$$\begin{aligned} &\sum_{n=1}^\infty n^{\alpha p - 2} E\Big(\max_{1 \leqslant j \leqslant n}\Big|\sum_{k=1}^j a_{nk} X_k\Big| - \varepsilon\Big)_+ \\ &\leqslant C \sum_{n=1}^\infty n^{\alpha p - 2} E\Big(\max_{1 \leqslant j \leqslant n}\Big|\sum_{k=1}^j a_{nk}(Y_{nk} - EY_{nk})\Big|^q\Big) \\ &\quad + \sum_{n=1}^\infty n^{\alpha p - 2} E\Big(\max_{1 \leqslant j \leqslant n}\Big|\sum_{k=1}^j a_{nk}(Z_{nk} - EZ_{nk})\Big|\Big) \\ &=: I_1 + I_2. \end{aligned}$$

To prove the desired result (3.2), we only need to show $I_1, I_2 < \infty$. For I_2, it follows by (1.4), Lemma 2.4, the definition of Z_{nk}, and $E|X|^p < \infty$ that

$$\begin{aligned} I_2 &= \sum_{n=1}^\infty n^{\alpha p - 2} E\Big(\max_{1 \leqslant j \leqslant n}\Big|\sum_{k=1}^j a_{nk}(Z_{nk} - EZ_{nk})\Big|\Big) \\ &\leqslant C \sum_{n=1}^\infty n^{\alpha p - 2 - \alpha} \sum_{k=1}^n E|Z_{nk}| \\ &\leqslant C \sum_{n=1}^\infty n^{\alpha p - 1 - \alpha} E|X| I(|X| > n^\alpha) \\ &= C \sum_{n=1}^\infty n^{\alpha p - 1 - \alpha} \sum_{k=n}^\infty E|X| I(k^\alpha < |X| \leqslant (k+1)^\alpha) \\ &= C \sum_{k=1}^\infty E|X| I(k^\alpha < |X| \leqslant (k+1)^\alpha) \sum_{n=1}^k n^{\alpha p - 1 - \alpha} \end{aligned}$$

$$\begin{aligned}&\leqslant C\sum_{k=1}^{\infty}E|X|^pI(k^\alpha<|X|\leqslant(k+1)^\alpha)\\&\leqslant CE|X|^p\\&<\infty.\end{aligned}\tag{3.4}$$

Next, we estimate I_1. For each $n\geqslant 1$, it follows by Lemma 2.1 that $\{a_{nk}(Y_{nk}-EY_{nk}),\ 1\leqslant k\leqslant n\}$ is still a sequence of WOD random variables with the same dominating coefficients. Hence, by Lemma 2.2, we obtain

$$\begin{aligned}I_1&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2}(\log n)_+^q\Bigg[\sum_{k=1}^{n}E|a_{nk}(Y_{nk}-EY_{nk})|^q\\&\quad+g(n)\Big(\sum_{k=1}^{n}E(a_{nk}(Y_{nk}-EY_{nk}))^2\Big)^{q/2}\Bigg]\\&=:I_{11}+I_{12}.\end{aligned}$$

For I_{11}, by Lemma 2.4, $E|X|^p<\infty$, and Markov's inequality, we have

$$\begin{aligned}I_{11}&=C\sum_{n=1}^{\infty}n^{\alpha p-2}(\log n)_+^q\sum_{k=1}^{n}|a_{nk}|^qE|Y_{nk}-EY_{nk}|^q\\&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2}(\log n)_+^q\sum_{k=1}^{n}|a_{nk}|^qE|Y_{nk}|^q\\&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2}(\log n)_+^q\sum_{k=1}^{n}|a_{nk}|^q[n^{\alpha q}P(|X_k|>n^\alpha)+E|X_k|^qI(|X_k|\leqslant n^\alpha)]\\&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2}(\log n)_+^q\Big(\max_{1\leqslant k\leqslant n}|a_{nk}|^{q-2}\Big)\\&\quad\cdot\sum_{k=1}^{n}a_{nk}^2[n^{\alpha q}P(|X_k|>n^\alpha)+E|X_k|^qI(|X_k|\leqslant n^\alpha)]\\&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha-\alpha(q-2)+\alpha(q-p)}(\log n)_+^qE|X|^p\\&\leqslant C\sum_{n=1}^{\infty}n^{\alpha-2}(\log n)_+^q\\&<\infty.\end{aligned}\tag{3.5}$$

Now, we will show $I_{12}<\infty$.

Case 1 $p\geqslant 2$.

Noting that $q>2(\alpha p+\lambda-1)/\alpha$, by the C_r inequality, Lemma 2.4, and $EX^2<\infty$, we have

$$\begin{aligned}
I_{12} &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)_+^q g(n)\Big(\sum_{k=1}^{n} a_{nk}^2 EY_{nk}^2\Big)^{q/2}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)_+^q g(n)\\
&\quad\cdot\Big(\sum_{k=1}^{n} a_{nk}^2[n^{2\alpha}P(|X_k|>n^\alpha)+EX_k^2I(|X_k|\leqslant n^\alpha)]\Big)^{q/2}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)_+^q g(n)\\
&\quad\cdot\Big(\sum_{k=1}^{n} a_{nk}^2[EX_k^2I(|X_k|>n^\alpha)+EX_k^2I(|X_k|\leqslant n^\alpha)]\Big)^{q/2}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)_+^q g(n)\Big(\sum_{k=1}^{n} a_{nk}^2\Big)^{q/2}(EX^2)^{q/2}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\frac{\alpha q}{2}+\lambda}(\log n)_+^q\\
&<\infty. \qquad (3.6)
\end{aligned}$$

Case 2 $1<p<2$.

Similar to the proof of (3.6), and noting that $q>2(\alpha p+\lambda-1)/(\alpha(p-1))$, we have

$$\begin{aligned}
I_{12} &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)_+^q g(n)\Big(\sum_{k=1}^{n} a_{nk}^2\Big)^{q/2}\\
&\quad\cdot[n^{2\alpha}P(|X_k|>n^\alpha)+EX_k^2I(|X_k|\leqslant n^\alpha)]^{q/2}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\frac{\alpha q}{2}+\lambda+\frac{(2-p)\alpha q}{2}}(\log n)_+^q(E|X|^p)^{q/2}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\frac{\alpha q}{2}+\lambda+\frac{(2-p)\alpha q}{2}}(\log n)_+^q\\
&<\infty.
\end{aligned}$$

The proof is completed. □

Corollary 3.1 *Assume that the conditions of Theorem* 3.1 *are satisfied. Then*

$$\sum_{n=1}^{\infty} n^{\alpha p-2}P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{k=1}^{j} a_{nk}X_k\Big|>\varepsilon\Big)<\infty,\quad \forall\,\varepsilon>0. \qquad (3.7)$$

Proof According to Theorem 3.1, we have

$$\begin{aligned}\infty &> \sum_{n=1}^{\infty} n^{\alpha p-2} E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{k=1}^{j} a_{nk}X_k\Big| - \varepsilon\Big)_+ \\ &= \sum_{n=1}^{\infty} n^{\alpha p-2}\int_0^{\infty} P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{k=1}^{j} a_{nk}X_k\Big| - \varepsilon > t\Big)\mathrm{d}t \\ &\geqslant \sum_{n=1}^{\infty} n^{\alpha p-2}\int_0^{\varepsilon} P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{k=1}^{j} a_{nk}X_k\Big| - \varepsilon > t\Big)\mathrm{d}t \\ &\geqslant \varepsilon\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{k=1}^{j} a_{nk}X_k\Big| > 2\varepsilon\Big),\end{aligned}$$

which yields (3.7). The proof is completed. □

Remark 3.1 Compared with Theorem A, we have the following extensions or improvements: (i) the END setting is extended to the WOD setting satisfying (3.1); (ii) the restriction on p is relaxed from $p \geqslant 2$ to $p > 1$; (iii) the identical distribution is weakened by stochastic domination; (iv) the complete moment convergence is stronger than complete convergence.

Theorem 3.1 deals with the complete 1-st moment convergence. The next two theorems consider the case of complete r-th moment convergence, where $r > 1$.

Theorem 3.2 *Assume that* (A.1) *and* (A.2) *are satisfied. Let* $2 \leqslant r \leqslant p$. *If*

$$g(n) = O(n^{\lambda}) \quad \text{for some } \lambda \in \Big[0, 1-\alpha + \Big(\frac{r}{2}-1\Big)(\alpha p - \alpha)\Big) \tag{3.8}$$

and $E|X|^p < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{k=1}^{j} a_{nk}X_k\Big| - \varepsilon\Big)_+^r < \infty, \quad \forall\, \varepsilon > 0. \tag{3.9}$$

Proof We use the same notations as those in the proof of Theorem 3.1. Taking $q > \max\{2(\alpha p + \lambda - 1)/\alpha, p\}$, by Lemma 2.3, we have

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{\alpha p-2} E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{k=1}^{j} a_{nk}X_k\Big| - \varepsilon\Big)_+^r \\ &\quad\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2} E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{k=1}^{j} a_{nk}(Y_{nk}-EY_{nk})\Big|^q\Big) \\ &\qquad + C\sum_{n=1}^{\infty} n^{\alpha p-2} E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{k=1}^{j} a_{nk}(Z_{nk}-EZ_{nk})\Big|^r\Big) \\ &\quad=: I_1 + I_2.\end{aligned} \tag{3.10}$$

It is clear to see that $I_1<\infty$ follows immediately from (3.5) and (3.6). Then we only need to prove $I_2<\infty$. For each $n\geqslant 1$, it follows by Lemma 2.1 that $\{a_{nk}(Z_{nk}-EZ_{nk}),\ 1\leqslant k\leqslant n\}$ is still a sequence of WOD random variables with the same dominating coefficients. Hence, by Lemma 2.2, we have

$$\begin{aligned}I_2&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2}(\log n)_+^r\bigg[\sum_{k=1}^{n}E|a_{nk}(Z_{nk}-EZ_{nk})|^r\\&\quad+g(n)\bigg(\sum_{k=1}^{n}E(a_{nk}(Z_{nk}-EZ_{nk}))^2\bigg)^{r/2}\bigg]\\&=:I_{21}+I_{22}.\end{aligned}$$

By the definition of Z_{nk} and $E|X|^p<\infty$, we have

$$\begin{aligned}I_{21}&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2}(\log n)_+^r\sum_{k=1}^{n}|a_{nk}|^rE|Z_{nk}|^r\\&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2}(\log n)_+^r\sum_{k=1}^{n}|a_{nk}|^rE|X_k|^rI(|X_k|>n^\alpha)\\&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2}(\log n)_+^r\sum_{k=1}^{n}|a_{nk}|^2|a_{nk}|^{r-2}E|X|^rI(|X|>n^\alpha)\\&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha(r-2)-\alpha+\alpha(r-p)}(\log n)_+^rE|X|^p\\&\leqslant C\sum_{n=1}^{\infty}n^{\alpha-2}(\log n)_+^r\\&<\infty.\end{aligned}\tag{3.11}$$

Similar to the proof of (3.11), and noting that $\lambda<1-\alpha+(\frac{r}{2}-1)(\alpha p-\alpha)$, we obtain

$$\begin{aligned}I_{22}&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2}(\log n)_+^rg(n)\bigg(\sum_{k=1}^{n}a_{nk}^2EZ_{nk}^2\bigg)^{r/2}\\&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2}(\log n)_+^rg(n)\bigg(\sum_{k=1}^{n}a_{nk}^2EX_k^2I(|X_k|>n^\alpha)\bigg)^{r/2}\\&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2}(\log n)_+^rg(n)\bigg(\sum_{k=1}^{n}a_{nk}^2EX^2I(|X|>n^\alpha)\bigg)^{r/2}\\&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2-\frac{\alpha r}{2}+\frac{\alpha(2-p)r}{2}}(\log n)_+^rg(n)(E|X|^p)^{r/2}\end{aligned}$$

$$\leqslant C\sum_{n=1}^{\infty} n^{(\alpha p-\alpha)(1-\frac{r}{2})+(\alpha-2)+\lambda}(\log n)_+^r$$
$$< \infty.$$

The proof is completed. □

Remark 3.2 When $p \geqslant 2$, it can been seen that condition (3.8) on the dominating coefficients is stronger than (3.1). However, Theorem 3.2 considers the complete moment convergence of higher order than that of Theorem 3.1 in the case of $p \geqslant 2$.

Theorem 3.3 *Assume that* (A.1) *is satisfied. Let* $\{a_{nk},\, n \geqslant 1,\, 1 \leqslant k \leqslant n\}$ *be an array of real numbers satisfying* (1.3). *If for some* $p \in (1, 2)$ *and* $r \in (1, p]$,

$$g(n) = O(n^{\lambda}) \quad \text{for some } \lambda \in \Big[0, \frac{(1-\alpha)r}{2}\Big) \tag{3.12}$$

and $E|X|^p < \infty$, *then* (3.9) *holds.*

Proof We use the same notations as those in the proof of Theorem 3.1. Taking $p < q < 2$, by Lemma 2.3, we have (3.10). Noting that $q < 2$, by Lemma 2.2, we have

$$I_1 \leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2}(C + Cg(n))(\log n)_+^q \sum_{k=1}^{n} E|a_{nk}(Y_{nk} - EY_{nk})|^q.$$

It follows from Hölder's inequality and (1.3) that for any $\beta \in (0, 2)$,

$$\sum_{k=1}^{n} |a_{nk}|^{\beta} \leqslant \Big(\sum_{k=1}^{n} a_{nk}^2\Big)^{\beta/2}\Big(\sum_{k=1}^{n} 1\Big)^{1-\frac{\beta}{2}} \leqslant Cn^{-(\alpha\beta-2+\beta)/2}. \tag{3.13}$$

Noting that $q > p$ and $\lambda < (1-\alpha)r/2$, we have $q > 2\lambda/(1-\alpha)$. It follows from (3.13), Lemma 2.4, and $E|X|^p < \infty$ that

$$\begin{aligned} I_1 &\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2} g(n)(\log n)_+^q \sum_{k=1}^{n} |a_{nk}|^q E|Y_{nk}|^q \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2} g(n)(\log n)_+^q \\ &\quad\cdot \sum_{k=1}^{n} |a_{nk}|^q [n^{\alpha q} P(|X_k| > n^{\alpha}) + E|X_k|^q I(|X_k| \leqslant n^{\alpha})] \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2} g(n)(\log n)_+^q \sum_{k=1}^{n} |a_{nk}|^q E|X|^p n^{\alpha(q-p)} \\ &\leqslant C\sum_{n=1}^{\infty} n^{\frac{\alpha q}{2}-1-\frac{q}{2}+\lambda}(\log n)_+^q \\ &< \infty. \end{aligned}$$

Now, we show $I_2 < \infty$. Similar to the proof of $I_1 < \infty$, it can be argued that

$$I_2 \leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2}(C + Cg(n))(\log n)_+^r \sum_{k=1}^{n} E|a_{nk}(Z_{nk} - EZ_{nk})|^r.$$

Noting that $\lambda < (1-\alpha)r/2$, by (3.13) and Lemma 2.4, we have

$$\begin{aligned} I_2 &\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2} g(n)(\log n)_+^r \sum_{k=1}^{n} |a_{nk}|^r E|Z_{nk}|^r \\ &\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2+\lambda}(\log n)_+^r \sum_{k=1}^{n} |a_{nk}|^r E|X|^r I(|X| > n^{\alpha}) \\ &\leqslant C \sum_{n=1}^{\infty} n^{\frac{\alpha r}{2}-1-\frac{r}{2}+\lambda}(\log n)_+^r E|X|^p \\ &\leqslant C \sum_{n=1}^{\infty} n^{\frac{\alpha r}{2}-1-\frac{r}{2}+\lambda}(\log n)_+^r \\ &< \infty. \end{aligned}$$

The proof is completed. □

Remark 3.3 When $1 < p < 2$, it can be seen that condition (3.12) on the dominating coefficients is stronger than that in (3.1). However, condition (1.4) is removed and Theorem 3.3 considers the complete moment convergence of higher order than that of Theorem 3.1 in the case of $1 < p < 2$.

4 An application in nonparametric regression model

4.1 Theoretical result

In what follows, we apply the result of Corollary 3.1 to a nonparametric regression model and investigate the complete consistency for the nonparametric regression estimator based on WOD errors.

Consider the nonparametric regression model

$$Y_{nk} = f(x_{nk}) + \varepsilon_{nk}, \quad k = 1, \ldots, n, \tag{4.1}$$

where x_{nk} are known fixed design points from a given compact set $A \subset \mathbb{R}^m$ for some $m \geqslant 1$, $f(\cdot)$ is an unknown regression function defined on A, and ε_{nk} are random errors. Assume that for each $n \geqslant 1$, $(\varepsilon_{n1}, \ldots, \varepsilon_{nn})$ have the same distribution as $(\varepsilon_1, \ldots, \varepsilon_n)$. As an estimator of $f(\cdot)$, the following weighted regression estimator will be considered:

$$f_{nj}(x) = \sum_{k=1}^{j} W_{nk}(x) Y_{nk}, \quad j = 1, \ldots, n, \, x \in A \subset \mathbb{R}^m, \tag{4.2}$$

where $W_{nk}(x) = W_{nk}(x; x_{n1}, \dots, x_{nn})$, $k = 1, \dots, n$, are the weight functions.

In the case of $j = n$, the above estimator with constant weight was first introduced by Stone [19] and next adapted by Georgive [6] to the fixed design case. Since then, estimator (4.2) has been researched by many scholars. One can refer to [8,13,34,36] among others.

For any function $f(x)$, we denote by $c(f)$ the set of continuity points of the function f on A. The symbol $\|X\|$ is the Euclidean norm. For any fixed point $x \in A$, the following assumptions on weight functions $W_{nk}(x)$ will be used:

(H_1) $\max_{1 \leqslant j \leqslant n} |\sum_{k=1}^{j} W_{nk}(x) - 1| \to 0$ as $n \to \infty$;

(H_2) $\sum_{k=1}^{n} |W_{nk}(x)| \leqslant C < \infty$ for all n;

(H_3) $\sum_{k=1}^{n} |W_{nk}(x)|\,|f(x_{nk}) - f(x)| I(\|x_{nk} - x\| > a) \to 0$ as $n \to \infty$ for all $a > 0$.

Based on the assumptions above, we can get the following result on complete consistency for the maximum of the nonparametric regression estimator $f_{nj}(x)$.

Theorem 4.1 *Let $0 < \alpha < 1$ and $\{\varepsilon_n, n \geqslant 1\}$ be a sequence of zero mean WOD random errors which is stochastically dominated by a random error X with dominating coefficients $g(n) = O(n^{\lambda})$ for some $\lambda \geqslant 0$. Suppose that conditions* (H_1)–(H_3) *hold. If*

$$\max_{1 \leqslant k \leqslant n} |W_{nk}(x)| = O(n^{-\alpha}) \tag{4.3}$$

and $E|X|^{2/\alpha} < \infty$, then for any $x \in c(f)$,

$$\max_{1 \leqslant j \leqslant n} |f_{nj}(x) - f(x)| \to 0 \quad \textit{completely as } n \to \infty. \tag{4.4}$$

Proof For any $a > 0$ and $x \in c(f)$, it follows from (4.1) and (4.2) that

$$\begin{aligned}
\max_{1 \leqslant j \leqslant n} |Ef_{nj}(x) - f(x)| &\leqslant \max_{1 \leqslant j \leqslant n} \Big| \sum_{k=1}^{j} W_{nk}(x)(f(x_{nk}) - f(x)) \Big| \\
&\quad + |f(x)| \max_{1 \leqslant j \leqslant n} \Big| \sum_{k=1}^{j} W_{nk}(x) - 1 \Big| \\
&\leqslant \sum_{k=1}^{n} |W_{nk}(x)|\,|f(x_{nk}) - f(x)| I(\|x_{nk} - x\| \leqslant a) \\
&\quad + \sum_{k=1}^{n} |W_{nk}(x)|\,|f(x_{nk}) - f(x)| I(\|x_{nk} - x\| > a) \\
&\quad + |f(x)| \max_{1 \leqslant j \leqslant n} \Big| \sum_{k=1}^{j} W_{nk}(x) - 1 \Big|.
\end{aligned} \tag{4.5}$$

Since $x \in c(f)$, for any $r > 0$, there exists a $\theta > 0$, such that $|f(x') - f(x)| < r$

when $\|x' - x\| < \theta$. Setting $0 < a < \theta$ in (4.5), we have

$$\max_{1\leqslant j\leqslant n} |Ef_{nj}(x) - f(x)| \leqslant r\sum_{k=1}^{n}|W_{nk}(x)| + |f(x)| \max_{1\leqslant j\leqslant n}\Big|\sum_{k=1}^{j} W_{nk}(x) - 1\Big| + \sum_{k=1}^{n}|W_{nk}(x)|\,|f(x_{nk}) - f(x)|I(\|x_{nk} - x\| > a),$$

which, together with (H_1)–(H_3) and the arbitrariness of $r > 0$, yields that

$$\lim_{n\to\infty}\max_{1\leqslant j\leqslant n} |Ef_{nj}(x) - f(x)| = 0. \tag{4.6}$$

In view of (4.6), to prove (4.4), it suffices to show for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{k=1}^{j} W_{nk}(x)\varepsilon_k\Big| > \varepsilon\Big) < \infty. \tag{4.7}$$

By (4.3) and (H_2), we have

$$\sum_{k=1}^{n} W_{nk}^2(x) \leqslant \max_{1\leqslant k\leqslant n}|W_{nk}(x)|\sum_{k=1}^{n}|W_{nk}(x)| \leqslant Cn^{-\alpha}. \tag{4.8}$$

We will apply Corollary 3.1 with $X_k = \varepsilon_k$, $a_{nk} = W_{nk}$, and $p = 2/\alpha$. In view of (4.8), we obtain (4.2) immediately. The proof is completed. □

Remark 4.1 Compared with [34, Theorem 2.3], we consider a more generalized WOD setting and the moment condition on random errors is weaker than that in [34]. Therefore, Theorem 4.1 extends and improves [34, Theorem 2.3] to some extent.

Remark 4.2 Compared with [36, Theorem 1.1], assumption (H_1) is stronger than that in [36]. However, we obtain a much stronger conclusion (4.4) than that in [36] under the nearly same conditions.

4.2 Numerical simulation

In this subsection, we present a simulation study based on model (4.1). Since assumption (H_1) is too strong to satisfy, we replace (H_1) by assumption (H_1') as follows:

$$\sum_{k=1}^{n} W_{nk}(x) \to 1, \quad n\to\infty. \tag{4.9}$$

Therefore, we focus on the consistency for the estimator

$$f_{nn}(x) = \sum_{k=1}^{n} W_{nk}(x)Y_{nk}. \tag{4.10}$$

It is obvious to see that (4.10) is a special case of (4.2). According to Theorem 4.1, it is easy to see that the estimator $f_{nn}(x)$ converges to $f(x)$ completely. Now, we will show the numerical performance of $f_{nn}(x)$. First, let us recall the concept of the nearest neighbor weight function as follows.

Put $A=[0,1]$ and let $x_{nk}=k/n,\ k=1,\ldots,n$. For any $x\in A$, we rewrite $|x_{n1}-x|,\ldots,|x_{nn}-x|$ as follows:

$$|x_{n,R_1(x)}-x|\leqslant\cdots\leqslant|x_{n,R_n(x)}-x|.$$

If $|x_{nk}-x|=|x_{nj}-x|$, then $|x_{nk}-x|$ is located before $|x_{nj}-x|$ when $x_{nk}<x_{nj}$.

Let $1\leqslant k_n\leqslant n$, the nearest neighbor weight function is defined as

$$W_{nk}(x)=\begin{cases}\dfrac{1}{k_n}, & |x_{nk}-x|\leqslant|x_{n,R_{k_n}(x)}-x|,\\ 0, & \text{otherwise.}\end{cases}$$

For any fixed $n\geqslant 3$, let $(\varepsilon_1,\ldots,\varepsilon_n)\sim N_n(\mathbf{0},\mathbf{\Sigma})$, where $\mathbf{0}$ represents zero vector and

$$\mathbf{\Sigma}=\begin{pmatrix}1+\theta^2 & -\theta & & 0\\ -\theta & \ddots & \ddots & \\ & \ddots & \ddots & -\theta\\ 0 & & -\theta & 1+\theta^2\end{pmatrix}_{n\times n}.$$

By [10], it can be seen that $(\varepsilon_1,\ldots,\varepsilon_n)$ is an NA vector for each $n\geqslant 3$ with finite moment of any order, and thus is a WOD vector. We choose casually that $\theta=0.7$, $k_n=\lfloor n^{1/2}\rfloor$, where $\lfloor x\rfloor$ stands for the integer part of x, and $\alpha=0.4$ in Theorem 4.1. As is stated in [25], assumptions (H_1'), (H_2), and (H_3) hold true, besides condition (4.3) is easy to be checked. Taking the sample sizes as $n=100$, 200, 300, respectively, we use Matlab software to compute the estimator $f_{nn}(x)$ of $f(x)$ for 400 times. Figs. 1–3 are the comparison of $f_{nn}(x)$ and $f(x)$ with $f(x)=\cos(2\pi x)$ and Figs. 4–6 are the comparison of $f_{nn}(x)$ and $f(x)$ with $f(x)=-x^3$.

For fixed points $x=0.25,0.5,0.75$, we take the sample sizes as $n=800,1200$, 1600 and use the Matlab software to compute the Mean Square Error (MSE) of the $f_{nn}(x)$ with 400 times experiments, which is shown in Table 1.

Table 1 MSE of estimator $f_{nn}(x)$

$f(x)$	x	n		
		800	1200	1600
$\cos(2\pi x)$	0.25	0.004981199	0.003750069	0.002819992
	0.5	0.005251874	0.004430218	0.003085777
	0.75	0.004597972	0.003594320	0.003366705
$-x^3$	0.25	0.005267617	0.003654640	0.003501542
	0.5	0.004772634	0.003786893	0.002803920
	0.75	0.005263979	0.004598840	0.003023554

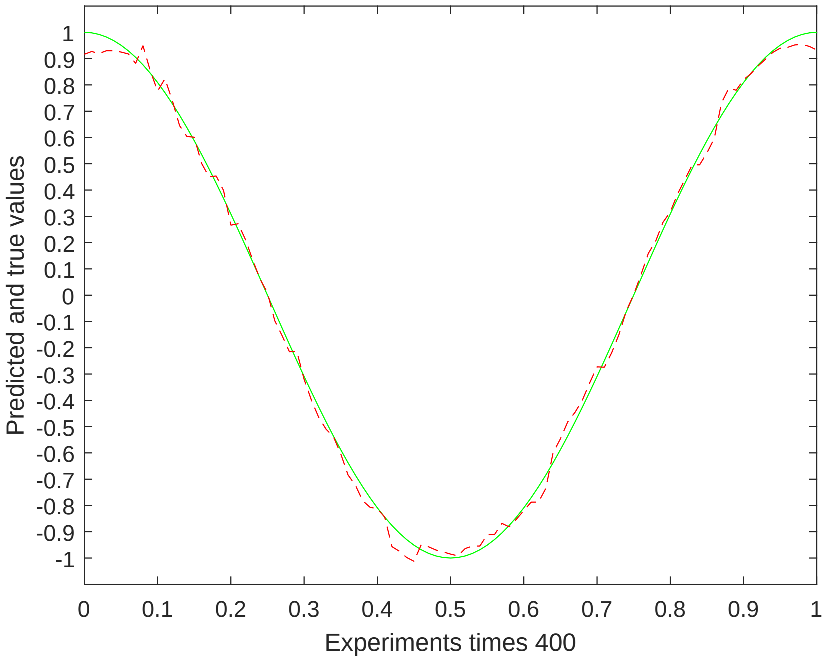

Fig. 1 Comparison of $f_{nn}(x)$ and $f(x) = \cos(2\pi x)$ with $n = 100$

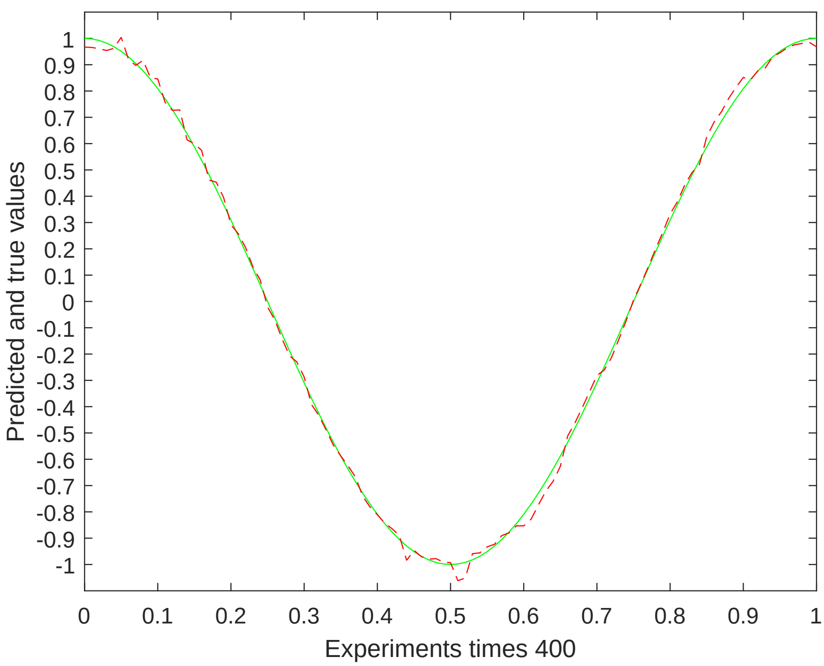

Fig. 2 Comparison of $f_{nn}(x)$ and $f(x) = \cos(2\pi x)$ with $n = 200$

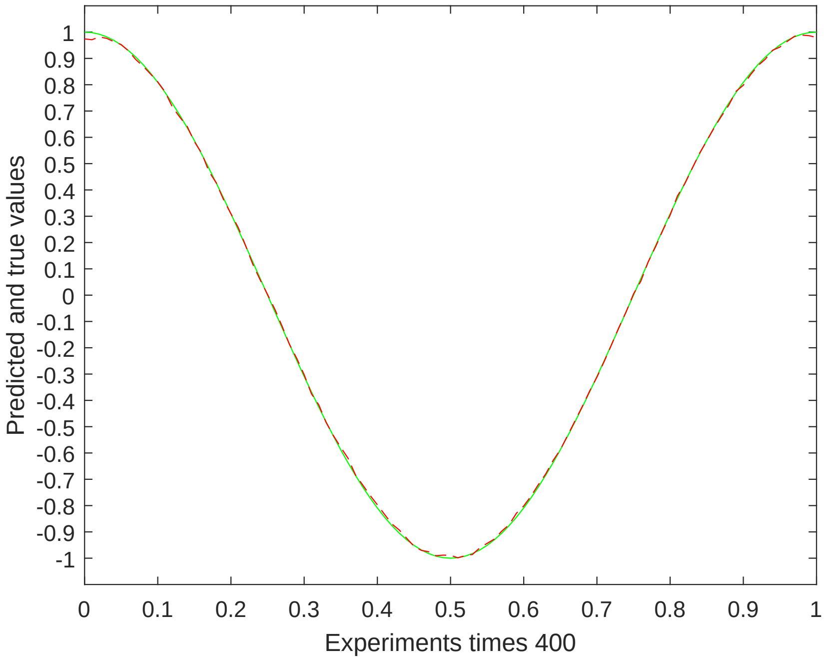

Fig. 3 Comparison of $f_{nn}(x)$ and $f(x) = \cos(2\pi x)$ with $n = 300$

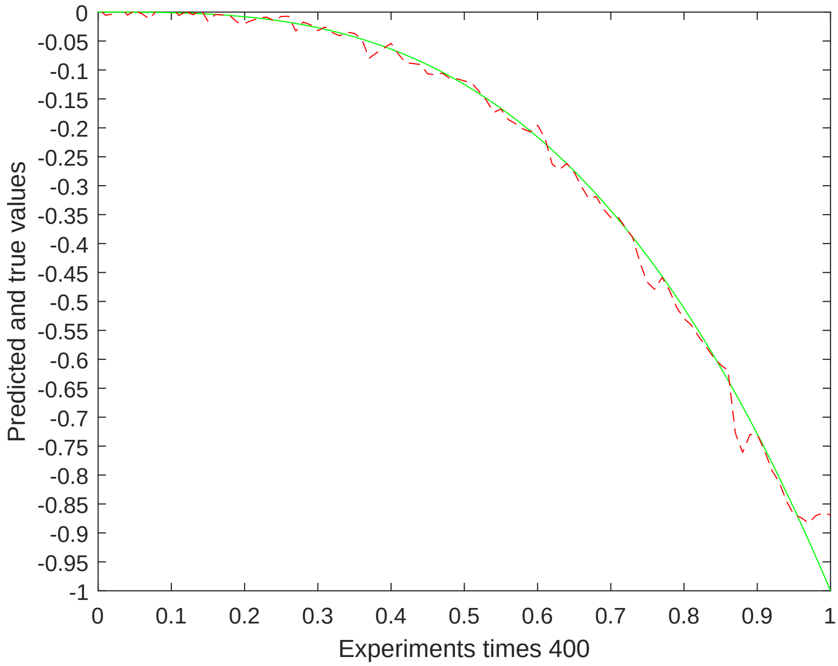

Fig. 4 Comparison of $f_{nn}(x)$ and $f(x) = -x^3$ with $n = 100$

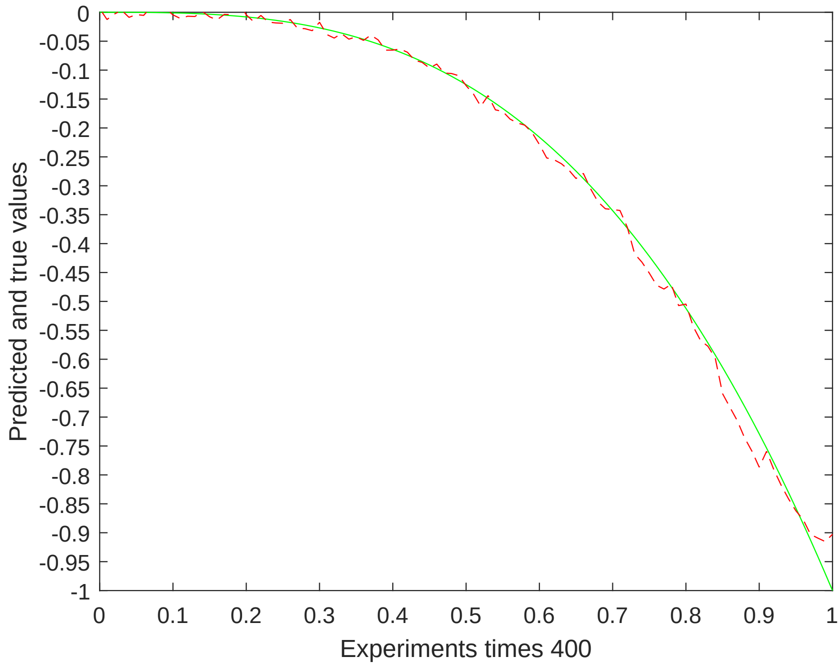

Fig. 5 Comparison of $f_{nn}(x)$ and $f(x) = -x^3$ with $n = 200$

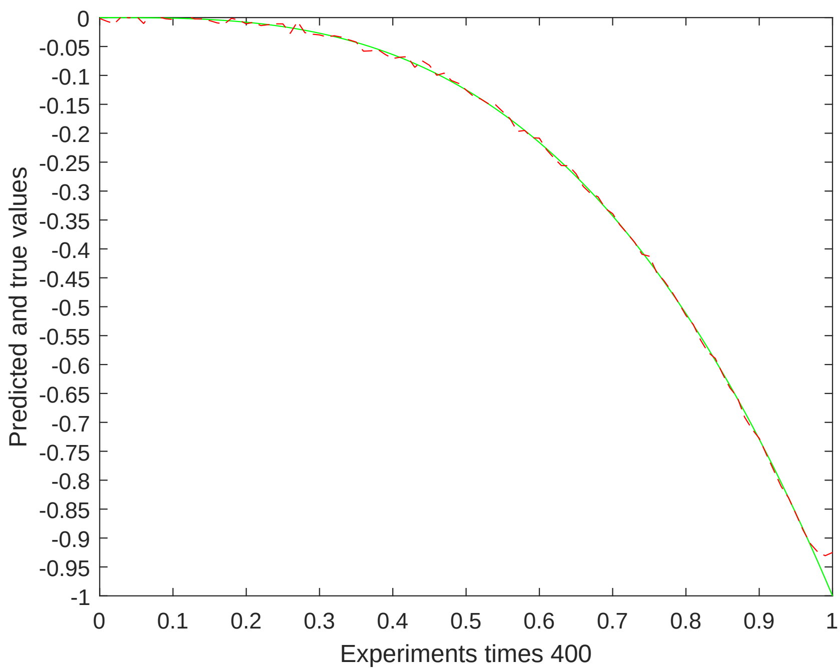

Fig. 6 Comparison of $f_{nn}(x)$ and $f(x) = -x^3$ with $n = 300$

It can be seen that no matter whether $f(x) = \cos(2\pi x)$ or $f(x) = -x^3$, the goodness of fit increases with the increasing of sample size. On the other hand, for fixed points $x = 0.25, 0.5, 0.75$, the MSE of the estimator $f_{nn}(x)$ decreases as the sample size increases. These results basically consistent with our theoretical result obtained in this paper.

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