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RESEARCH ARTICLE

Multipliers, covers, and stem extensions for Lie superalgebras

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Abstract Suppose that the underlying field is of characteristic different from 2 and 3. We first prove that the so-called stem deformations of a free presentation of a finite-dimensional Lie superalgebra L exhaust all the maximal stem extensions of L, up to equivalence of extensions. Then we prove that multipliers and covers always exist for a Lie superalgebra and they are unique up to superalgebra isomorphisms. Finally, we describe the multipliers, covers, and maximal stem extensions of Heisenberg superalgebras and model filiform Lie superalgebras.

Keywords Multiplier, cover, stem extension, Heisenberg superalgebra, filiform Lie supleralgebra

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1 Introduction

The notion of multipliers and covers, which is relative to stem extensions, first appeared in Schur's work in the group theory and then was generalized to Lie algebra case. It proves that the theory of multipliers, covers, and stem extensions is not only of intrinsic interest, but also of important role in characterizing algebraic structures, such as in computing second cohomology with coefficients in trivial modules for groups or Lie algebras.

The study on multipliers of Lie algebras began in 1990s (see [3,11], for example) and the theory has seen a fruitful development (see [2,4,7,8,13,14,16], for example). Among them, a typical fact analogous to the one in the group theory is that the multiplier of a finite-dimensional Lie algebra L is isomorphic to the second cohomology group of L with coefficients in the 1-dimensional trivial module (see [1], for example).

The notion of multipliers, covers, and stem extensions may also be

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naturally generalized to Lie superalgebra case. In this paper, we first introduce the notion of stem denominators and stem deformations for an extension of a Lie superalgebra and show that all the stem deformations of a free presentation of a finite-dimensional Lie superalgebra L coincide with all the maximal stem extensions of L, up to equivalence of extensions. Then we show that multipliers and covers always exist for a Lie superalgebra and they are unique up to superalgebra isomorphisms. Finally, we describe multipliers, covers, and maximal stem extensions for Heisenberg superalgebras of odd centers and model filiform Lie superalgebras.

2 Stem extensions

Unless otherwise stated, we assume that the underlying field $\mathbb{F}$ is of characteristic different from 2 and 3, and all (super)spaces and (super)algebras are defined over $\mathbb{F}$. Let $\mathbb{Z}_2 := \{\overline{0}, \overline{1}\}$ be the additive group of order 2 and $V = V_{\overline{0}} \oplus V_{\overline{1}}$ a superspace, that is, a $\mathbb{Z}_2$-graded vector space. For a homogeneous element x in V, write $|x|$ for the parity of x. The symbol $|x|$ implies that x has been assumed to be a homogeneous element.

Note that in a superspace, a subsuperspace always has a supplementary subsuperspace. Moreover, if V is a superspace and W is a subsuperspace of V, then the quotient space V/W inherits a super structure and every subsuperspace of V/W is of form X/W, where X is a subsuperspace containing W.

Let us recall the notion of extensions of Lie superalgebras and some basic properties. By definition, a Lie superalgebra homomorphism is both an even linear map and an algebra homomorphism and an ideal of a Lie superalgebra is always a $\mathbb{Z}_2$-graded ideal. An extension of a Lie superalgebra L by A is an exact sequence of Lie superalgebra homomorphisms:

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} L \to 0. \tag{2.1}$$

We usually identify A with a subalgebra of B and omit the embedding map α. Suppose that

$$0 \to C \to D \xrightarrow{\gamma} L \to 0 \tag{2.2}$$

is also an extension of L and there is a homomorphism f from extensions (2.1) to (2.2), that is, a Lie superalgebra homomorphism $f\colon B \to D$ such that $\gamma \circ f = \beta$. Then f maps A into C and $f^{-1}(C) = A$. Hereafter, we denote by f itself the restriction $f\colon A \to C$. Then the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\beta} & L & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & C & \longrightarrow & D & \xrightarrow{\gamma} & L & \longrightarrow & 0
\end{array}$$

Clearly, if $f\colon B \to D$ is surjective, then so is its restriction $f\colon A \to C$.

Recall that an extension (2.1) is said to be central if A is contained in the center $\mathrm{Z}(B)$ of B. If X is an ideal of A as well as B, then (2.1) induces an exact sequence

$$0 \to A/X \to B/X \xrightarrow{\beta} L \to 0. \tag{2.3}$$

Hereafter, we denote by β itself the induced map $B/X \to L$.

As in the Lie algebra case, a *stem extension* of a Lie superalgebra L is a central extension

$$0 \to S \to T \to L \to 0$$

such that $S \subset [T,T]$.

The following properties are analogous to the ones in Lie algebra case (see [3], for example). Hereafter, we use a partial order in $\mathbb{Z} \times \mathbb{Z}$ as follows:

$$(m,n) \leqslant (k,l) \Longleftrightarrow m \leqslant k,\ n \leqslant l. \tag{2.4}$$

For $m, n \in \mathbb{Z}$, we write $|(m,n)| = m+n$. We also view $\mathbb{Z} \times \mathbb{Z}$ as the additive group in the usual way.

Lemma 1 *Let L be a finite-dimensional Lie superalgebra. Suppose that*

$$0 \to S \to T \xrightarrow{\gamma} L \to 0 \tag{2.5}$$

is a stem extension of L. Then the following statements hold.

(1) *Suppose that* $\mathrm{sdim}L = (s,t)$. *Then both S and T are finite-dimensional and*

$$\mathrm{sdim}S \leqslant \Big(\frac{1}{2}\,s(s-1) + \frac{1}{2}\,t(t+1) + s, st\Big).$$

(2) *S is contained in every maximal subalgebra of T.*

(3) *Suppose that*

$$0 \to A \to B \xrightarrow{\beta} L \to 0 \tag{2.6}$$

is an extension of L and f is a homomorphism of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\beta} & L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & S & \longrightarrow & T & \xrightarrow{\gamma} & L & \longrightarrow & 0 \end{array} \tag{2.7}$$

Then f must be surjective.

Proof (1) See [10, Lemma 2.3].

(2) Let X be a maximal subalgebra of T. Assume conversely that $S \not\subset X$. Since $S \subset \mathrm{Z}(T)$, one sees that $S + X = T$ and then $S \subset [T,T] \subset X$, a contradiction.

(3) Assume conversely that $f(B) \neq T$. By (1), T is finite-dimensional. Then there is a maximal subalgebra X such that $f(B) \subset X$. Note that the

commutativity of the diagram implies that $T = S + f(B)$. Then, by (2), we have $T \subset X$, a contradiction. $\square$

Let us explain that an arbitrary extension (2.6) always induces naturally a central extension and then a stem extension of L. First, by factoring out the ideal $[A, B]$, we obtain a central extension

$$0 \to A/[A, B] \to B/[A, B] \xrightarrow{\beta} L \to 0.$$

Then, consider any supplementary subsuperspace of $A \cap [B, B]/[A, B]$ in $A/[A, B]$, which must be of form $X/[A, B]$, where X is a subsuperspace of A containing $[A, B]$. Clearly, X is an ideal of A as well as B. Then we have a decomposition of ideals:

$$A/[A, B] = A \cap [B, B]/[A, B] \oplus X/[A, B]. \tag{2.8}$$

Such a subsuperspace (ideal) X is called a *stem denominator* of extension (2.6). Since $X \subset A$ and $\beta(A) = 0$, we obtain an extension (2.3), which is called a *stem deformation* of extension (2.6). By the following proposition, a stem deformation of an extension is a stem extension.

Proposition 1 *Let* (2.6) *be an extension of Lie superalgebra L. Suppose that X is a subsuperspace of A and $X \supset [A, B]$.*

(1) *X is a stem denominator of extension* (2.6) *if and only if*

$$A = A \cap [B, B] + X, \quad [B, B] \cap X = [A, B]. \tag{2.9}$$

(2) *Suppose that X is a stem denominator of extension* (2.6). *Then* (2.3) *is a stem extension and*

$$A/X \cong A \cap [B, B]/[A, B]. \tag{2.10}$$

(3) *For an extension, stem denominators and stem deformations always exist.*

Proof (1) It follows from the fact that (2.9) is equivalent to (2.8).

(2) Since $[A, B] \subset X$, we have $[A/X, B/X] = 0$. By (2.9), it is clear that $A/X \subset [B/X, B/X]$. Hence, (2.3) is a stem extension. While (2.10) is a direct consequence of (2.9).

(3) By the argument before this proposition, one sees that a stem denominator always exists and so does a stem deformation by (2). $\square$

The following theorem tells us that if a stem extension (S) of a Lie superalgebra L is a homomorphic image of an extension (E) of L, then (S) must be a homomorphic image of some stem deformation of (E).

Theorem 1 *Let* (2.6) *be an extension of L and* (2.5) *a stem extension of L. Suppose that $f\colon B \to T$ is a homomorphism of extensions* (2.7). *Then there is* *a stem denominator X of extension* (2.6) *such that $f(X) = 0$ and f induces an epimorphism of extensions*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A/X & \longrightarrow & B/X & \xrightarrow{\beta} & L & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow f & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & S & \longrightarrow & T & \xrightarrow{\gamma} & L & \longrightarrow & 0 \end{array}$$

Proof By Lemma 1 (3), f is surjective. Then $f(A) = S$ and $f^{-1}(S) = A$. Since $S \subset [T, T]$, we have $S \subset f([B, B])$ and $S \subset f(A \cap [B, B])$. Hence, $S = f(A \cap [B, B])$ and then $f(A) = f(A \cap [B, B])$. Consequently,

$$A = A \cap [B, B] + \ker f. \tag{2.11}$$

Clearly, $[A, B] \subset \ker f$, since $S \subset \mathrm{Z}(T)$. By (2.11), there is a subsuperspace $X \subset \ker f$ satisfying that $[A, B] \subset X \subset A$ such that

$$A = A \cap [B, B] + X, \quad [B, B] \cap X = [A, B].$$

By Proposition 1, X is the desired stem denominator. The proof is complete. □

We shall prove an important fact, which states that a stem extension of L must be a homomorphic image of a stem deformation of any free presentation of L. Recall that a Lie superalgebra L always has a free presentation, that is, an extension

$$0 \to R \to F \xrightarrow{\pi} L \to 0 \tag{2.12}$$

with F being a free Lie superalgebra. Let (2.5) be a stem extension of L. Then there is a homomorphism of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & F & \xrightarrow{\pi} & L & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow f & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & S & \longrightarrow & T & \xrightarrow{\gamma} & L & \longrightarrow & 0 \end{array}$$

As a direct consequence of Theorem 1, we have the following result.

Theorem 2 *Suppose that* (2.12) *is a free presentation of Lie superalgebra L and* (2.5) *is a stem extension of L. Then there is a stem denominator X of* (2.12) *such that $f(X) = 0$ and f induces an epimorphism of extensions*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R/X & \longrightarrow & F/X & \xrightarrow{\pi} & L & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow f & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & S & \longrightarrow & T & \xrightarrow{\gamma} & L & \longrightarrow & 0 \end{array}$$

In view of Lemma 1 (1), the following notion makes sense. A stem extension of a finite-dimensional Lie superalgebra L,

$$0 \to S \to T \to L \to 0,$$

is called *maximal*, if among all the stem extensions of L, S is of maximal superdimension with respect to the partial order (2.4).

Theorem 3 *Let L be a finite-dimensional Lie superalgebra and* (2.12) *a free presentation. Then, up to equivalence of extensions, the stem deformations of* (2.12) *exhaust all the maximal stem extensions of L.*

Proof Let

$$0 \to M \to C \xrightarrow{\gamma} L \to 0 \tag{2.13}$$

be a maximal stem extension of L. By Theorem 2, there is an epimorphism f of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R/X & \longrightarrow & F/X & \xrightarrow{\pi} & L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & M & \longrightarrow & C & \xrightarrow{\gamma} & L & \longrightarrow & 0 \end{array}$$

where X is a stem denominator of (2.12) such that $f(X) = 0$. By the maximality of (2.13), f is an isomorphism of extensions. By Proposition 1 (2), $M \cong R \cap [F, F]/[F, R]$ is independent of the choice of X. Hence, every stem deformation of (2.12) is a maximal stem extension of L. □

3 Multipliers and covers

Analogous to the Lie algebra case, for a finite-dimensional Lie superalgebra L, if

$$0 \to M \to C \to L \to 0$$

is a maximal stem extension of Lie superalgebra L, then M is called a *multiplier* and C a *cover* of L.

By Theorem 3, for a finite-dimensional Lie superalgebra, multipliers and covers always exist. We shall prove that both multipliers and covers are unique up to superalgebra isomorphisms. We need a technical lemma, which is a super-version of a result in [3]. Recall that a superalgebra is a superspace with a bilinear multiplication which is compatible with the $\mathbb{Z}_2$-grading structure.

Lemma 2 *Let A_1, A_2, B_1, and B_2 be superalgebras and*

$$A_1 \oplus B_1 \cong A_2 \oplus B_2 \tag{3.1}$$

as superalgebras. Suppose that A_1 and A_2 are finite-dimensional and $A_1 \cong A_2$ as superalgebras. Then $B_1 \cong B_2$ as superalgebras.

Proof By (3.1), one can view A_2 and B_2 as $\mathbb{Z}_2$-graded ideals of the superalgebra $A_1 \oplus B_1$ and then we have

$$A_1 \oplus B_1 = A_2 \oplus B_2. \tag{3.2}$$

If $A_1 \cap B_2 = 0$ or $A_2 \cap B_1 = 0$, one may easily see that $B_1 \cong B_2$ as superalgebras, since A_1 and A_2 are finite-dimensional.

Suppose

$$A_1 \cap B_2 \neq 0, \quad A_2 \cap B_1 \neq 0.$$

Note that both $A_1 \cap B_2$ and $A_2 \cap B_1$ are $\mathbb{Z}_2$-graded ideals. By (3.2), we have the following superalgebra isomorphism:

$$\frac{A_1}{A_1 \cap B_2} \oplus \frac{B_1}{A_2 \cap B_1} \cong \frac{A_2}{A_2 \cap B_1} \oplus \frac{B_2}{A_1 \cap B_2}. \tag{3.3}$$

Then we have the following superalgebra isomorphisms:

$$\begin{aligned}\frac{A_1}{A_1 \cap B_2} \oplus \frac{A_2}{A_2 \cap B_1} \oplus B_1 &\cong \frac{A_2}{A_2 \cap B_1} \oplus \frac{A_1 \oplus B_1}{A_1 \cap B_2} \\ &\cong \frac{A_2}{A_2 \cap B_1} \oplus \frac{A_2 \oplus B_2}{A_1 \cap B_2} \\ &\cong \frac{A_2}{A_2 \cap B_1} \oplus \frac{B_2}{A_1 \cap B_2} \oplus A_2.\end{aligned}$$

By the symmetry, since $A_1 \cong A_2$ as superalgebras, it follows from (3.3) that the following superalgebra isomorphism holds:

$$\frac{A_1}{A_1 \cap B_2} \oplus \frac{A_2}{A_2 \cap B_1} \oplus B_1 \cong \frac{A_1}{A_1 \cap B_2} \oplus \frac{A_2}{A_2 \cap B_1} \oplus B_2.$$

By induction on dimensions, we have $B_1 \cong B_2$ as superalgebras. □

Theorem 4 *Let L be a finite-dimensional Lie superalgebra. Up to Lie superalgebra isomorphisms, there are a unique multiplier and a unique cover of L, denoted by $\mathscr{M}(L)$ and $\mathscr{C}(L)$, respectively. Moreover, for any free presentation of L,*

$$0 \to R \to F \to L \to 0, \tag{3.4}$$

(1) $\mathscr{M}(L) \cong R \cap [F,F]/[F,R]$,

(2) $\mathscr{C}(L) \cong Y/[R,F]$, *where Y is any subsuperspace of F containing $[F,F]$ such that*

$$F/[R,F] = X/[R,F] \oplus Y/[R,F] \tag{3.5}$$

as superspaces, where X is a stem denominator of (3.4).

Proof By Theorem 3, it suffices to show that both R/X and F/X are independent of the choice of stem denominator X. Moreover, by Theorem 3 again, one may assume that F is a free Lie superalgebra generated by a finite homogeneous set, since L is finite-dimensional. Since X is a stem denominator, we have $[F,F] \cap X = [R,F]$ and then $[F,F]/[R,F] \cap X/[R,F] = 0$. Hence, there is a subsuperspace Y of F containing $[F,F]$ such that we have a direct sum decomposition of superspaces (3.5). Since $Y \supset [F,F]$, we have $Y \lhd F$.

Hence, (3.5) is also a direct sum decomposition of ideals. Since X is a stem denominator, we also have a direct sum decomposition of ideals:

$$R/[R,F] = R \cap [F,F]/[R,F] \oplus X/[R,F].$$

Then, up to superalgebra isomorphisms, $X/[R,F]$ is independent of the choice of X. Moreover,

$$X/[R,F] \cong R/R \cap [F,F] \cong (R+[F,F])/[F,F] \subset F/[F,F]$$

is finite-dimensional. Then by Lemma 2, it follows from (3.5) that, up to superalgebra isomorphisms, $F/X \cong Y/[R,F]$ is independent of the choice of X. □

4 Heisenberg superalgebras

In this section, we compute multipliers, covers, and maximal stem extensions for Heisenberg superalgebras of odd centers. Recall that a finite-dimensional Lie superalgebra $\mathfrak{g}$ is called a Heisenberg (Lie) superalgebra provided that $\mathfrak{g}^2 = \mathrm{Z}(\mathfrak{g})$ and $\dim \mathrm{Z}(\mathfrak{g}) = 1$. Heisenberg superalgebras consist of two types (see [15]).

(1) A Heisenberg superalgebra of even center, denoted by $H(p,q)$ with $p+q \geqslant 1$, has a homogeneous basis (called standard)

$$\{u_1, \dots, u_p, v_1, \dots, v_p, z \mid w_1, \dots, w_q\},$$

where

$$|u_i| = |v_j| = |z| = \bar{0}, \quad |w_k| = \bar{1}, \quad 1 \leqslant i,j \leqslant p,\ 1 \leqslant k \leqslant q,$$

and the multiplication is given by

$$[u_i, v_i] = -[v_i, u_i] = z, \quad [w_k, w_k] = z,$$

and the other brackets of basis elements vanishing.

(2) A Heisenberg superalgebra of odd center, denoted by $H(n)$ with $n \geqslant 1$, has a homogeneous basis (called standard)

$$\{u_1, \dots, u_n \mid z, w_1, \dots, w_n\},$$

where

$$|u_i| = \bar{0}, \quad |w_j| = |z| = \bar{1}, \quad 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n,$$

and the multiplication is given by

$$[u_i, w_i] = -[w_i, u_i] = z$$

and the other brackets of basis elements vanishing.

In [10, Proposition 4.4] (see also [12, Theorem 4.3]), the authors characterize the multipliers of Heisenberg superalgebras of even centers:

$$\mathrm{sdim}\mathscr{M}(H(p,q)) = \begin{cases} \left(2p^2 - p + \frac{1}{2}q^2 + \frac{1}{2}q - 1, 2pq\right), & p+q \geqslant 2, \\ (0,0), & p=0,\ q=1, \\ (2,0), & p=1,\ q=0. \end{cases}$$

Let us give a maximal stem extension of Heisenberg superalgebra $H(n)$. Suppose that

$$0 \to W \to K \to H(n) \to 0$$

is a stem extension. Then

$$W \subset K^2 \cap \mathrm{Z}(K), \quad K/W \cong H(n).$$

Thus, K/W has a standard basis

$$\{a_1 + W, \dots, a_n + W \mid c + W, b_1 + W, \dots, b_n + W\},$$

where $a_i, b_i, c \in K$ with

$$|a_i| = \bar{0}, \quad |b_i| = |c| = \bar{1}.$$

So one may assume that

$$[a_i, a_j] = y_{i,j}, \quad 1 \leqslant i < j \leqslant n,$$

$$[a_i, b_j] = \begin{cases} c + y_i, & 1 \leqslant i = j \leqslant n, \\ z_{i,j}, & 1 \leqslant i \neq j \leqslant n, \end{cases}$$

$$[a_i, c] = m_i, \quad [b_i, c] = n_i, \quad 1 \leqslant i \leqslant n,$$

$$[b_i, b_j] = w_{i,j}, \quad 1 \leqslant i \leqslant j \leqslant n,$$

$$[c, c] = t,$$

where $y_{i,j}$, y_i, $z_{i,j}$, m_i, $w_{i,j}$, n_i, $t \in W$ and

$$|y_{i,j}| = |n_i| = |w_{i,j}| = |t| = \bar{0}, \quad |y_i| = |z_{i,j}| = |m_i| = \bar{1}.$$

Furthermore, without loss of generality, one may assume $y_1 = 0$. Note that

$$n_i = [b_i, [a_i, b_i]] = \frac{1}{2}[a_i, [b_i, b_i]] = 0, \quad t = [c, [a_1, b_1]] = 0.$$

Then we rewrite

$$[a_i, a_j] = y_{i,j}, \quad 1 \leqslant i < j \leqslant n,$$

$$[a_1, b_1] = c, \quad [c, c] = 0,$$

$$[a_i, b_i] = c + y_i, \quad 2 \leqslant i \leqslant n,$$

$$[a_i, b_j] = z_{i,j}, \quad 1 \leqslant i \neq j \leqslant n,$$
$$[a_i, c] = m_i, \quad [b_i, c] = 0, \quad 1 \leqslant i \leqslant n,$$
$$[b_i, b_j] = w_{i,j}, \quad 1 \leqslant i \leqslant j \leqslant n.$$

Now, it is obvious that W is spanned by the following elements:

$$y_{i,j}, \quad 1 \leqslant i < j \leqslant n,$$
$$y_i, \quad 2 \leqslant i \leqslant n,$$
$$z_{i,j}, \quad 1 \leqslant i \neq j \leqslant n,$$
$$m_i, \quad 1 \leqslant i \leqslant n,$$
$$w_{i,j}, \quad 1 \leqslant i \leqslant j \leqslant n.$$

Then K is spanned by $a_1, \ldots, a_n$, $b_1, \ldots, b_n$, c, and the elements displayed above.

Case 1 $n = 1$.

Suppressing all the subscripts, we have

$$[a, b] = c, \quad [a, c] = m, \quad [b, b] = w, \quad [b, c] = [c, c] = 0.$$

Then W is generated by w and m. Hence, $\mathrm{sdim} W \leqslant (1 \mid 1)$.

Now, let $\widehat{H}(1)$ be a superspace with a basis $(\widehat{a}, \widehat{w} \mid \widehat{b}, \widehat{c}, \widehat{m})$. Then $\widehat{H}(1)$ becomes a Lie superalgebra by letting

$$[\widehat{a}, \widehat{b}] = -[\widehat{b}, \widehat{a}] = \widehat{c}, \quad [\widehat{a}, \widehat{c}] = -[\widehat{c}, \widehat{a}] = \widehat{m}, \quad [\widehat{b}, \widehat{b}] = \widehat{w},$$

and the other brackets of basis elements vanish. Let $\widehat{MH}(1)$ be the subsuperspace spanned by $\widehat{w}$ and $\widehat{m}$. Then

$$\widehat{MH}(1) \subset \widehat{H}(1)^2 \cap \mathrm{Z}(\widehat{H}(1)), \quad \widehat{H}(1)/\widehat{MH}(1) \cong H(1).$$

Since $\mathrm{sdim}\widehat{MH}(1) = (1 \mid 1)$, one sees that

$$0 \to \widehat{MH}(1) \to \widehat{H}(1) \to H(1) \to 0$$

is a maximal stem extension of $H(1)$. In particular, $\widehat{MH}(1)$ is a multiplier and $\widehat{H}(1)$ a cover of $H(1)$.

Case 2 $n \geqslant 2$.

Fix any i and take $j \neq i$. One may check that

$$m_i = [a_i, [a_j, b_j]] = 0.$$

Then W is spanned by the following elements:

$$y_{i,j}, \quad 1 \leqslant i < j \leqslant n,$$

$$y_i, \quad 2 \leqslant i \leqslant n,$$

$$z_{i,j}, \quad 1 \leqslant i \neq j \leqslant n,$$

$$w_{i,j}, \quad 1 \leqslant i \leqslant j \leqslant n.$$

Hence, $\mathrm{sdim} W \leqslant (n^2 \mid n^2 - 1)$.

Now, let $\widehat{H}(n)$ with $n \geqslant 2$ be a superspace with a basis consisting of even elements

$$\widehat{a}_i, \quad 1 \leqslant i \leqslant n,$$

$$\widehat{y}_{i,j}, \quad 1 \leqslant i < j \leqslant n,$$

$$\widehat{w}_{i,j}, \quad 1 \leqslant i \leqslant j \leqslant n,$$

and odd elements $\widehat{c}$ as well as

$$\widehat{b}_i, \quad 1 \leqslant i \leqslant n,$$

$$\widehat{y}_i, \quad 2 \leqslant i \leqslant n,$$

$$\widehat{z}_{i,j}, \quad 1 \leqslant i \neq j \leqslant n.$$

Then one may check that $\widehat{H}(n)$ becomes a Lie superalgebra by letting

$$[\widehat{a}_i, \widehat{a}_j] = -[\widehat{a}_j, \widehat{a}_i] = \widehat{y}_{i,j}, \quad [\widehat{b}_i, \widehat{b}_j] = -[\widehat{b}_j, \widehat{b}_i] = \widehat{w}_{i,j}, \quad 1 \leqslant i < j \leqslant n,$$

$$[\widehat{a}_1, \widehat{b}_1] = -[\widehat{b}_1, \widehat{a}_1] = \widehat{c},$$

$$[\widehat{a}_i, \widehat{b}_i] = -[\widehat{b}_i, \widehat{a}_i] = \widehat{c} + \widehat{y}_i, \quad 2 \leqslant i \leqslant n,$$

$$[\widehat{a}_i, \widehat{b}_j] = -[\widehat{b}_j, \widehat{a}_i] = \widehat{z}_{i,j}, \quad 1 \leqslant i \neq j \leqslant n,$$

and the other brackets of basis elements vanish. Let $\widehat{MH}(n)$ be the subsuperspace spanned by $\widehat{y}_{i,j}, \widehat{w}_{i,j}, \widehat{y}_i, \widehat{z}_{i,j}$. Then

$$\widehat{MH}(n) \subset \widehat{H}(n)^2 \cap \mathrm{Z}(\widehat{H}(n)), \quad \widehat{H}(n)/\widehat{MH}(n) \cong H(n).$$

Since $\mathrm{sdim}\widehat{MH}(n) = (n^2 \mid n^2 - 1)$, one sees that

$$0 \to \widehat{MH}(n) \to \widehat{H}(n) \to H(n) \to 0 \tag{4.1}$$

is a maximal stem extension and then $\widehat{MH}(n)$ is a multiplier and $\widehat{H}(n)$ a cover of $H(n)$. Summarizing, we have the following result.

Theorem 5 *Let n be a positive integer. Then (4.1) is a maximal stem extension of $H(n)$. In particular, $\widehat{H}(n)$ is the cover of $H(n)$ and*

$$\mathrm{sdim}\, \mathscr{M}(H(n)) = \begin{cases} (n^2 \mid n^2 - 1), & n \geqslant 2, \\ (1 \mid 1), & n = 1. \end{cases}$$

5 Model filiform Lie superalgebras

Suppose that n is a positive integer and m a nonnegative integer. Let $F(n,m)$ be a Lie superalgebra with basis

$$\{x_0,\dots,x_n \mid y_1,\dots,y_m\},$$

where

$$|x_i|=\bar{0},\quad |y_j|=\bar{1},\quad 0\leqslant i\leqslant n,\ 1\leqslant j\leqslant m,$$

and multiplication given by

$$[x_0,x_i]=x_{i+1},\quad [x_0,y_j]=y_{j+1},$$

and the other brackets of basis elements vanishing. It is easy to see that $F(n,m)$ is a nilpotent Lie superalgebra of super-nilindex (n,m) (see [9], for example). Note that $F(1,0)$ is an abelian Lie algebra and $F(1,1)$ is an abelian Lie superalgebra. Note that $F(n,0)$ with $n>1$ is just the model filiform Lie algebra of nilindex n (see [5], for example). We call $F(n,m)$ with $(n,m)\neq(1,0)$, $(1,1)$ the *model filiform Lie superalgebra* of super-nilindex (n,m) (see [5,6], for example).

Let us give a maximal stem extension of $F(n,m)$. Suppose that

$$0\to W\to K\to F(n,m)\to 0$$

is a stem extension. Then

$$W\subset K^2\cap \mathrm{Z}(K),\quad K/W\cong F(n,m).$$

Thus, K/W has a standard basis

$$\{a_0+W,\dots,a_n+W \mid b_1+W,\dots,b_m+W\},$$

where $a_i,b_j\in K$ with $|a_i|=\bar{0}$, $|b_j|=\bar{1}$. Then we have

$$[a_0,a_i]=a_{i+1}+x_i,\quad 1\leqslant i\leqslant n-1,$$
$$[a_i,a_j]=y_{i,j},\quad 1\leqslant i<j\leqslant n,$$
$$[a_0,b_j]=b_{j+1}+y_j,\quad 1\leqslant j\leqslant m-1,$$
$$[b_i,b_j]=z_{i,j},\quad 1\leqslant i\leqslant j\leqslant m,$$
$$[a_i,b_j]=t_{i,j},\quad 1\leqslant i\leqslant n, 1\leqslant j\leqslant m,$$

where x_i, $y_{i,j}$, y_j, $z_{i,j}$, $t_{i,j}\in W$ and

$$|x_i|=|y_{i,j}|=|z_{i,j}|=\bar{0},\quad |y_j|=|t_{i,j}|=\bar{1}.$$

Since $W\subset \mathrm{Z}(K)$, without loss of generality, one may assume that $x_i=y_j=0$. Moreover, using the super Jacobi identity, one may check the following identities:

(1) $y_{i,j+1} = -y_{i+1,j}$ for $1 \leqslant i, j \leqslant n-1$;
(2) $t_{i,j+1} = -t_{i+1,j}$ for $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant m-1$;
(3) $t_{1,j+1} = 0$ for $1 \leqslant j \leqslant m-1$;
(4) $t_{i+1,1} = 0$ for $1 \leqslant i \leqslant n-1$;
(5) $z_{i,j+1} = -z_{i+1,j}$ for $1 \leqslant i \neq j \leqslant m-1$;
(6) $2z_{j,j+1} = 0$ for $1 \leqslant j \leqslant m-1$.

Case 1 $n \geqslant 2$, $m = 0$.

In this case, $F(n,0)$ is a model filiform Lie algebra. By (1), we rewrite

$$[a_0, a_i] = a_{i+1}, \quad 1 \leqslant i \leqslant n-1,$$
$$[a_1, a_2] = y_{1,2}, \quad [a_{n-1}, a_n] = y_{n-1,n},$$
$$[a_i, a_j] = y_{i,j} = -y_{i+1,j-1} = -[a_{i+1}, a_{j-1}], \quad 2 < j-i \text{ being odd.}$$

Now, it is obvious that W is spanned by the elements $y_{i,i+1}$, $1 \leqslant i \leqslant n-1$. Hence, $\mathrm{sdim} W \leqslant (n-1 \mid 0)$. Then K is spanned by $a_0, \ldots, a_n$ and the elements displayed above.

Now, let $\widehat{F}(n,0)$ with $n \geqslant 2$ be a superspace with a basis consisting of even elements $\widehat{a}_i$, $0 \leqslant i \leqslant n$; $\widehat{y}_j$, $2 \leqslant j \leqslant n$. Then one may check that $\widehat{F}(n,0)$ becomes a Lie superalgebra by letting

$$[\widehat{a}_0, \widehat{a}_i] = -[\widehat{a}_i, \widehat{a}_0] = \widehat{a}_{i+1}, \quad 1 \leqslant i \leqslant n-1,$$
$$[\widehat{a}_1, \widehat{a}_2] = -[\widehat{a}_2, \widehat{a}_1] = \widehat{y}_2, \quad [\widehat{a}_{n-1}, \widehat{a}_n] = -[\widehat{a}_n, \widehat{a}_{n-1}] = \widehat{y}_n,$$
$$[\widehat{a}_i, \widehat{a}_j] = -[\widehat{a}_j, \widehat{a}_i] = -[\widehat{a}_{i+1}, \widehat{a}_{j-1}] = [\widehat{a}_{j-1}, \widehat{a}_{i+1}] = \widehat{y}_{j-1}, \quad 2 < j-i \text{ being odd,}$$

and the other brackets of basis elements vanish. Let $\widehat{MF}(n,0)$ be the subsuperspace spanned by all $\widehat{y}_j$. Then

$$\widehat{MF}(n,0) \subset \widehat{F}(n,0)^2 \cap \mathrm{Z}(\widehat{F}(n,0)), \quad \widehat{F}(n,0)/\widehat{MF}(n,0) \cong F(n,0).$$

Since $\mathrm{sdim}\widehat{MF}(n,0) = (n-1 \mid 0)$, one sees that

$$0 \to \widehat{MF}(n,0) \to \widehat{F}(n,0) \to F(n,0) \to 0$$

is a maximal stem extension of $F(n,0)$. In particular, $\widehat{MF}(n,0)$ is a multiplier and $\widehat{F}(n,0)$ a cover of $F(n,0)$.

Case 2 $n \geqslant 2$, $m = 1$.

By (1) and (4), we rewrite

$$[a_0, a_i] = a_{i+1}, \quad 1 \leqslant i \leqslant n-1,$$
$$[a_1, a_2] = y_{1,2}, \quad [a_{n-1}, a_n] = y_{n-1,n},$$
$$[a_i, a_j] = y_{i,j} = -y_{i+1,j-1} = -[a_{i+1}, a_{j-1}], \quad 2 < j-i \text{ being odd,}$$
$$[a_1, b_1] = t_{1,1}, \quad [b_1, b_1] = z_{1,1}.$$

Now, it is obvious that W is spanned by the elements $t_{1,1}$, $z_{1,1}$, $y_{i,i+1}$, $1 \leqslant i \leqslant n-1$. Hence, $\mathrm{sdim} W \leqslant (n \mid 1)$. Then K is spanned by $a_0, \ldots, a_n$ and the elements displayed above.

Now, let $\widehat{F}(n,1)$ with $n \geqslant 2$ be a superspace with a basis consisting of even elements $\widehat{z}$,

$$\widehat{a}_i,\ 0 \leqslant i \leqslant n; \quad \widehat{y}_j,\ 2 \leqslant j \leqslant n,$$

and odd element $\widehat{t}$. Then one may check that $\widehat{F}(n,1)$ becomes a Lie superalgebra by letting

$$[\widehat{a}_0, \widehat{a}_i] = -[\widehat{a}_i, \widehat{a}_0] = \widehat{a}_{i+1}, \quad 1 \leqslant i \leqslant n-1,$$
$$[\widehat{a}_1, \widehat{a}_2] = -[\widehat{a}_2, \widehat{a}_1] = \widehat{y}_2, \quad [\widehat{a}_{n-1}, \widehat{a}_n] = -[\widehat{a}_n, \widehat{a}_{n-1}] = \widehat{y}_n,$$
$$[\widehat{a}_i, \widehat{a}_j] = -[\widehat{a}_j, \widehat{a}_i] = -[\widehat{a}_{i+1}, \widehat{a}_{j-1}] = [\widehat{a}_{j-1}, \widehat{a}_{i+1}] = \widehat{y}_{j-1}, \quad 2 < j-i \text{ being odd},$$
$$[\widehat{a}_1, \widehat{b}_1] = -[\widehat{b}_1, \widehat{a}_1] = \widehat{t}, \quad [\widehat{b}_1, \widehat{b}_1] = -[\widehat{b}_1, \widehat{b}_1] = \widehat{z},$$

and the other brackets of basis elements vanish. Let $\widehat{MF}(n,1)$ be the subsuperspace spanned by $\widehat{t}$, $\widehat{z}$, and all $\widehat{y}_j$. Then

$$\widehat{MF}(n,1) \subset \widehat{F}(n,1)^2 \cap \mathrm{Z}(\widehat{F}(n,1)), \quad \widehat{F}(n,1)/\widehat{MF}(n,1) \cong F(n,1).$$

Since $\mathrm{sdim}\widehat{MF}(n,1) = (n \mid 1)$, one sees that

$$0 \to \widehat{MF}(n,1) \to \widehat{F}(n,1) \to F(n,1) \to 0$$

is a maximal stem extension of $F(n,1)$. Then $\widehat{MF}(n,1)$ is a multiplier and $\widehat{F}(n,1)$ a cover of $F(n,1)$.

Case 3 $n, m \geqslant 2$.

By (1), (2), (5), and (6), we rewrite

$$[a_0, a_i] = a_{i+1}, \quad 1 \leqslant i \leqslant n-1,$$
$$[a_1, a_2] = y_{1,2}, \quad [a_{n-1}, a_n] = y_{n-1,n},$$
$$[a_i, a_j] = y_{i,j} = -y_{i+1,j-1} = -[a_{i+1}, a_{j-1}], \quad 2 < j-i \text{ being odd},$$
$$[a_0, b_j] = b_{j+1}, \quad 1 \leqslant j \leqslant m-1,$$
$$[a_1, b_1] = t_{1,1}, \quad [a_n, b_m] = t_{n,m},$$
$$[a_i, b_j] = t_{i,j} = -t_{i+1,j-1} = -[a_{i+1}, b_{j-1}], \quad 1 \leqslant i \leqslant n-1,\ 2 \leqslant j \leqslant m.$$

Now, it is obvious that W is spanned by the following elements:

$$y_{i,i+1}, \quad 1 \leqslant i \leqslant n-1,$$
$$t_{i,1}, \quad 1 \leqslant i \leqslant n,$$
$$t_{n,j}, \quad 2 \leqslant j \leqslant m.$$

Hence, $\mathrm{sdim}W \leqslant (n-1 \mid n+m-1)$. Then K is spanned by $a_0, \dots, a_n$, $b_1, \dots, b_m$, and the elements displayed above.

Now, let $\widehat{F}(n,m)$ with $n, m \geqslant 2$ be a superspace with a basis consisting of even elements

$$\widehat{a}_i,\ 0 \leqslant i \leqslant n; \quad \widehat{y}_j,\ 2 \leqslant j \leqslant n,$$

and odd elements

$$\widehat{b}_p,\ 1 \leqslant p \leqslant m; \quad \widehat{t}_q,\ 2 \leqslant q \leqslant m+n.$$

Then one may check that $\widehat{F}(n,m)$ becomes a Lie superalgebra by letting

$$[\widehat{a}_0, \widehat{a}_i] = -[\widehat{a}_i, \widehat{a}_0] = \widehat{a}_{i+1}, \quad 1 \leqslant i \leqslant n-1,$$

$$[\widehat{a}_1, \widehat{a}_2] = -[\widehat{a}_2, \widehat{a}_1] = \widehat{y}_2, \quad [\widehat{a}_{n-1}, \widehat{a}_n] = -[\widehat{a}_n, \widehat{a}_{n-1}] = \widehat{y}_n,$$

$$[\widehat{a}_i, \widehat{a}_j] = -[\widehat{a}_j, \widehat{a}_i] = -[\widehat{a}_{i+1}, \widehat{a}_{j-1}] = [\widehat{a}_{j-1}, \widehat{a}_{i+1}] = \widehat{y}_{j-1}, \quad 2 < j-i \text{ being odd},$$

$$[\widehat{a}_0, \widehat{b}_p] = -[\widehat{b}_p, \widehat{a}_0] = \widehat{b}_{p+1}, \quad 1 \leqslant p \leqslant m-1,$$

$$[\widehat{a}_1, \widehat{b}_1] = -[\widehat{b}_1, \widehat{a}_1] = \widehat{t}_2, \quad [\widehat{a}_n, \widehat{b}_m] = -[\widehat{b}_m, \widehat{a}_n] = \widehat{t}_{m+n},$$

$$[\widehat{a}_i, \widehat{b}_p] = -[\widehat{b}_p, \widehat{a}_i] = -[\widehat{a}_{i+1}, \widehat{b}_{p-1}] = [\widehat{b}_{p-1}, \widehat{a}_{i+1}] = \widehat{t}_{i+p},$$
$$1 \leqslant i \leqslant n-1,\ 2 \leqslant p \leqslant m.$$

and the other brackets of basis elements vanish. Let $\widehat{MF}(n,m)$ be the subsuperspace spanned by all $\widehat{y}_j$ and $\widehat{t}_q$. Then

$$\widehat{MF}(n,m) \subset \widehat{F}(n,m)^2 \cap \mathrm{Z}(\widehat{F}(n,m)), \quad \widehat{F}(n,m)/\widehat{MF}(n,m) \cong F(n,m).$$

Since $\mathrm{sdim}\widehat{MF}(n,m) = (n-1 \mid m+n-1)$, one sees that

$$0 \to \widehat{MF}(n,m) \to \widehat{F}(n,m) \to F(n,m) \to 0$$

is a maximal stem extension of $F(n,m)$. In particular, $\widehat{MF}(n,m)$ is a multiplier and $\widehat{F}(n,m)$ a cover of $F(n,m)$.

Case 4 $n = 1$, $m \geqslant 2$.

By (3), (5), and (6), we rewrite

$$[a_0, b_j] = b_{j+1}, \quad 1 \leqslant j \leqslant m-1,$$

$$[a_1, b_1] = t_{1,1}.$$

Now, it is obvious that W is spanned by the element $t_{1,1}$. Hence, $\mathrm{sdim}\, W \leqslant (0 \mid 1)$. Then K is spanned by $a_0, a_1, b_1, \dots, b_m$ and the elements displayed above.

Now, let $\widehat{F}(1,m)$ with $m \geqslant 2$ be a superspace with a basis consisting of even elements $\widehat{a}_0$, $\widehat{a}_1$ and odd elements $\widehat{t}$, $\widehat{b}_p$, $1 \leqslant p \leqslant m$.

Then one may check that $\widehat{F}(1,m)$ becomes a Lie superalgebra by letting

$$[\widehat{a}_0, \widehat{b}_p] = -[\widehat{b}_p, \widehat{a}_0] = \widehat{b}_{p+1}, \quad 1 \leqslant p \leqslant m-1,$$

$$[\widehat{a}_1, \widehat{b_1}] = -[\widehat{b_1}, \widehat{a}_1] = \widehat{t},$$

and the other brackets of basis elements vanish. Let $\widehat{MF}(1,m)$ be the sub-superspace spanned by $\widehat{t}$. Then

$$\widehat{MF}(1,m) \subset \widehat{F}(1,m)^2 \cap \mathrm{Z}(\widehat{F}(1,m)), \quad \widehat{F}(1,m)/\widehat{MF}(1,m) \cong F(1,m).$$

Since $\mathrm{sdim}\widehat{MF}(1,m) = (0 \mid 1)$, one sees that

$$0 \to \widehat{MF}(1,m) \to \widehat{F}(1,m) \to F(1,m) \to 0$$

is a maximal stem extension of $F(1,m)$. In particular, $\widehat{MF}(1,m)$ is a multiplier and $\widehat{F}(1,m)$ a cover of $F(1,m)$.

Summarizing, we have the following result.

Theorem 6 *Let n and m be positive integers. Then*

$$0 \to \widehat{MF}(n,m) \to \widehat{F}(n,m) \to F(n,m) \to 0$$

is a maximal stem extension of the model filiform Lie superalgebra $F(n,m)$. In particular, $\widehat{F}(n,m)$ is the cover of $F(n,m)$ and

$$\mathrm{sdim}\,\mathscr{M}(F(n,m)) = \begin{cases} (n-1 \mid 0), & n \geqslant 2,\ m = 0, \\ (n \mid 1), & n \geqslant 2,\ m = 1, \\ (n-1 \mid n+m-1), & n \geqslant 2,\ m \geqslant 2, \\ (0 \mid 1), & n = 1,\ m \geqslant 2. \end{cases}$$

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