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RESEARCH ARTICLE

Fourier matrices and Fourier tensors

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Abstract The Fourier matrix is fundamental in discrete Fourier transforms and fast Fourier transforms. We generalize the Fourier matrix, extend the concept of Fourier matrix to higher order Fourier tensor, present the spectrum of the Fourier tensors, and use the Fourier tensor to simplify the high order Fourier analysis.

Keywords Fourier matrix, tensor, CP decomposition, Fourier analysis
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1 Introduction

Fourier analysis has been appeared in many fields such as seismology, crystallography, sonar, and many other applications [1,5,6]. Fourier transformation was first implemented on finite circles before its extension to Fourier series. Clairaut introduced the discrete Fourier transform (DFT) in 1754 and employed the DFT to determine the orbits of the astroids. Gauss initialized the fast Fourier transform (FFT) in 1805 ([3,4]) while computing the eccentricity of the orbit of the asteroid Juno [3], which was neglected and was rediscovered by Cooley and Tukey in an important paper [4], which leads to the wide adoption of DFT thereafter. Dirichlet showed by FFT the existence of infinite number of primes in any arithmetic progression (see [17]). Goertzel [7] significantly improved the efficiency of FFT by the symmetry of trigonometric functions, and Good [8] essentially developed a prime-factor FFT. The classical FFT method is implemented through Butterfly processing (also called the *divide and conquer* technique) where the summands are equally grouped iteratively. This technique can reduce the original complexity of the computation of DFT from $O(N^2)$ to $O(N\log_2 N)$.

The DFT provides an approximation to the continuous Fourier transform and can be used to solve many discrete problems in number theory [18], graph

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theory and group theory [19], as well as problems in physics, statistics, and error-correcting codes, computing the eigenvalues of the adjacency matrix of a graph. DFT can also be used to diagonalize the adjacency operators of the Cayley graphs defined on the cyclic group $\mathbb{Z}_n$.

In this paper, we want to rediscover the properties of the classical Fourier matrices and extend them to higher order tensors, i.e., the Fourier tensors. We show that the higher order Fourier transform can be handled much more easily by the introduction of the Fourier tensors. We also investigate the spectrum of the Fourier tensors based on that of the Fourier matrices.

Throughout, we use lowercase and uppercase, respectively, for vectors and matrices or groups, and slant font or the math font, e.g., $\mathbb{Z}, \mathbb{V}, \mathbb{W}$, etc. for tensors. For the detailed information on tensor theory, we refer the reader to a recent book on tensors [16]. Denote by $[n]$ the set $\{1, 2, \ldots, n\}$ and $[n]_0$ the set $[n] \cup \{0\}$ for any positive integer n. Denote by $\boldsymbol{e}_i$ the $(i+1)$th coordinate vector in $\mathscr{C}^n$ for $i \in [n-1]_0$, so

$$\boldsymbol{e}_0 = (1, 0, 0, \ldots, 0)^\top, \quad \boldsymbol{e}_1 = (0, 1, 0, 0, \ldots, 0)^\top, \ \ldots, \ \boldsymbol{e}_{n-1} = (0, 0, \ldots, 0, 1)^\top.$$

For any positive integers m, n, we denote

$$S(m, n) = \{(i_1, i_2, \ldots, i_m)\colon i_k \in [n]\}$$

and

$$S(k; m, n) = \{\sigma \in S(m, n)\colon s(\sigma) = m + k\},$$

where $s(\sigma)$ denotes the sum of all elements of σ. Let $\mathscr{G} := \{g_0, g_1, \ldots, g_{n-1}\}$ denote the additive group $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$, where $g_i = i \in \mathbb{Z}_n$ representing the class of integers $a \equiv i \pmod n$. Let $L^2(\mathscr{G})$ be the set of complex-valued functions defined on $\mathscr{G}$ and $f := (f_1, f_2, \ldots, f_n)^\top$, where $f_i = f(g_{i-1})$. The inner product on $L^2(\mathscr{G})$ is defined as

$$\langle f, g\rangle = \sum_{x \in \mathscr{G}} f(x)\overline{g}(x). \tag{1}$$

The Fourier transform (FT) of a function $f(x)$ on $\mathbb{R}$ (the field of real numbers) is defined by

$$\mathbb{F}(f)(s) := \hat{f}(s) = \int_{-\infty}^{\infty} \exp(-2\pi\imath sx) f(x)\mathrm{d}x \tag{2}$$

(the symbol $\imath$ is the imaginary unit $\sqrt{-1}\,$), while the inverse Fourier transform (IFT) of a function $g(s)$ is defined as

$$\mathbb{F}^{-1}(g) := \check{g}(t) = \int_{-\infty}^{\infty} \exp(2\pi\imath st) f(s)\mathrm{d}s. \tag{3}$$

An alternate for notation $\hat{f}$ (resp., $\check{f}$) is $(f)\hat{}$ (resp., $(f)\check{}$) or $\mathbb{F}(f)$ when f is an expression. Note that all integrals appeared here are assumed to be finite.

We note that (2) and (3) can be generalized to multivariate functions. Let f be defined on $\mathbb{R}^n$. The FT of $f(\boldsymbol{x})$ is defined as

$$\hat{f}(\boldsymbol{t}) = \int_{-\infty}^{\infty} \exp(-2\pi\imath \boldsymbol{t}^\top \boldsymbol{x}) f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \tag{4}$$

where $\boldsymbol{x}, \boldsymbol{t} \in \mathbb{R}^n$, $\mathrm{d}x := \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n$, and $\boldsymbol{t}^\top \boldsymbol{x} = \langle \boldsymbol{t}, \boldsymbol{x} \rangle$ is the inner product of $\boldsymbol{t}$ and $\boldsymbol{x}$. We list some examples to illustrate the effects of the FT on some simple functions.

Example 1 The triangle function $\Lambda(x)$ is defined by

$$\Lambda(x) = \begin{cases} 1 - |x|, & x \in [-1, 1], \\ 0, & \text{otherwise}. \end{cases} \tag{5}$$

Then

$$\hat{\Lambda} = \left(\frac{\sin \pi s}{\pi s} \right)^2 = \mathbf{sinc}^2(s).$$

As the second example, we take the exponential decay function which is defined by

$$f(x) = \begin{cases} \mathrm{e}^{-ax}, & x > 0, \\ 0, & \text{otherwise}. \end{cases} \tag{6}$$

A simple calculation tells that

$$\hat{f}(s) = \frac{1}{a + 2\pi\imath s},$$

where $a > 0$ is a positive constant.

The third one in this example is the Gaussian function furnished by $g(x) = \mathrm{e}^{-\pi x^2}$. A quick check shows that $\hat{g} = g$ since $\hat{g}(s) = \mathrm{e}^{-\pi s^2}$. The meaning of this fact is not clear so far to us.

The convolution of functions $f(x)$ and $g(x)$, denoted by $h(t) = f \circ g$, is defined as

$$h(t) = \int_{-\infty}^{\infty} f(x) g(t - x) \mathrm{d}x. \tag{7}$$

Generally speaking, the convolutions are used for smoothing and averaging. For example, the solution to the heat equation defined on a circle can be expressed as a convolution of the initial heat distribution with Green's function (see, e.g., [1,9]). For the recent work on the tensor expression of a convolution, we refer to [21]. Now, we are ready to list some basic properties which may be useful in the following argument.

Lemma 1 (i) $(\hat{f})^{\check{}} = f$, $(\check{g})^{\hat{}} = g$;

(ii) $f(0) = \int_{-\infty}^{\infty} \hat{f}(s) \mathrm{d}s$, $\hat{f}(0) = \int_{-\infty}^{\infty} f(s) \mathrm{d}s$;

(iii) $\hat{f}(-s) = \check{f}(s)$, $\check{f}(-t) = \hat{f}(t)$;

(iv) $(f(x+b))\hat{} = \mathrm{e}^{2\pi\imath sb}(f(x))\hat{}$, *where b is any real constant*;

(v) $(f(at))\hat{} = |a|^{-1}(f(s/a))\hat{}$, *where a is a nonzero real constant*;

(vi) $(f\circ g)\hat{} = \hat{f}\hat{g}$.

By Lemma 1 (iv), (v), and the fact that $(\mathrm{e}^{-\pi x^2})\hat{} = \mathrm{e}^{-\pi s^2}$ in Example 1, we can calculate the FT of the Gaussian with mean $\mu=0$ and standard deviation σ, i.e.,

$$g(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\,\mathrm{e}^{-x^2/(2\sigma^2)},$$

as

$$\hat{g}(s) = \exp(-2\pi^2\sigma^2 s^2).$$

For more information on Fourier transform, we refer the reader to [9]. In the next section, we will turn to the DFT.

2 DFT and generalized Fourier matrices

The DFT is a discrete approximation of the Fourier transform in the continuous case. It takes a vector as an input and returns another vector of the same dimension as an output. The DFT converts a finite sequence of data, usually expressed as a vector of samples of an instant function separated by sample time, into a sequence of equally-spaced samples which are a complex-valued function values of frequency. If the original sequence spans all the non-zero values of a function, then its discrete time Fourier transform (DTFT) is continuous and periodic, and the DFT provides discrete samples of one cycle. If the original sequence is one cycle of a periodic function, then the DFT provides all the non-zero values of one DTFT cycle.

To be specific, we denote $\omega := \mathrm{e}^{-2\imath\pi/n}$ and $U := \{1,\omega,\omega^2,\ldots,\omega^{n-1}\}$ is the set of n-unit roots. For simplicity, we sometimes denote $\omega_k := \omega^k$ for $k\in[n-1]_0$. Write $\boldsymbol{u}^p := (u_1^p, u_2^p, \ldots, u_n^p)^\top$ (the p-power of a vector $\boldsymbol{u}$) for a vector $\boldsymbol{u} = (u_1,u_2,\ldots,u_n)^\top \in \mathscr{C}^n$, where p is an integer. Thus, $\boldsymbol{u}^0 = \ell$ is the vector of all-ones, $\boldsymbol{u}^1 = \boldsymbol{u}$. Note that some entries of $\boldsymbol{u}^p$ can be the infinity if the corresponding coordinate of $\boldsymbol{u}$ is zero and $p<0$.

For our convenience, we index vectors and matrices from 0 to $n-1$. Thus, a column vector $\boldsymbol{x}\in\mathscr{C}^n$ is written as $\boldsymbol{x} = (x_0,x_1,\ldots,x_{n-1})^\top$ and an $n\times n$ matrix $A=(a_{ij})$ takes $i,j\in[n-1]_0$. The *Fourier matrix*

$$F_n := F_n(\omega) = (F_{jk})$$

is defined as the $n\times n$ complex matrix with

$$F_{jk} = \frac{1}{\sqrt{n}}\exp\Big(-\frac{\imath 2\pi jk}{n}\Big) = \frac{1}{\sqrt{n}}\,\omega^{jk}.$$

Note that F_n is symmetric for all positive integers n. In the following, we may use F to replace F_n for simplicity when no risk of confusion arises. Given any

$u \in U$, denote

$$\eta_u = (1, u, u^2, \ldots, u^{n-1})^\top \tag{8}$$

and

$$F(u) = n^{-1/2}[\eta_u^0, \eta_u^1, \ldots, \eta_u^{n-1}].$$

Furthermore, we denote $\eta_k := \eta_{\omega_k}$ for each $k \in [n-1]$. For any positive integers p, q, we denote (p, q) for the greatest common divisor of p and q. Then we have the following result.

Lemma 2 *For any positive integer $k \in [n-1]$, $F(\omega_k)$ is nonsingular if and only if $(k, n) = 1$, i.e., k and n are coprime.*

Proof By the formula of the Vandermonde determinant, we know that

$$\det(F(u)) = \prod_{0 \leqslant i < j \leqslant n-1} (u^j - u^i). \tag{9}$$

We now write the set of n-unit roots as $U := \{u_0, u_1, \ldots, u_{n-1}\}$ in which $u_p = \omega_p = \omega^p$ for $p \in [n-1]_0$, and let $U^k := \{u_0^k, u_1^k, \ldots, u_{n-1}^k\}$ for $k \in [n-1]$. We note by (9) that $\det(F(u_k)) \neq 0$ if and only if all the n elements in U^k are distinct, which is equivalent to condition $(k, n) = 1$ by the elementary number theory. □

We call a matrix $F_n(\omega_k)$ *generalized Fourier matrix* (GF-matrix) generated by ω_k when $k \in [n-1]$ is coprime with n and denote it by $F[n, k]$. Thus, the Fourier matrix $F_n = F[n, 1]$ is a GF-matrix generated by $\omega = \omega_1$. Given any integer $n > 1$, there are exactly $\phi(n) + 1$ GF-matrices of size $n \times n$, where $\phi(n)$ is the Euler function, the number of integers coprime to n (excluding 1). For example, there are two GF-matrices of size 4×4:

$$F_4 = F[4, 1], \quad F[4, 3] = F_4(\omega^3).$$

There are 4 GF-matrices of size 5×5 since $(k, 5) = 1$ for all $k = 1, 2, 3, 4$. Actually, for any prime number n, we have $n - 1$ GF-matrices of size $n \times n$. More generally, we let

$$n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$$

be the prime decomposition of n, where $p_1, p_2, \ldots, p_s > 1$ are the prime factors of n ordered increasingly. By the Eulerian formula, we have

$$\phi(n) = n\Big(1 - \frac{1}{p_1}\Big)\Big(1 - \frac{1}{p_2}\Big) \cdots \Big(1 - \frac{1}{p_s}\Big).$$

Thus, we can calculate the number of GF-matrices for each n.

To investigate the properties of the Fourier matrices, we write $\eta := \eta_\omega$. Then

$$F_n = [\eta^0, \eta^1, \ldots, \eta^{n-1}].$$

Denote by $H_k = (h_{ij}) \in \mathbb{R}^{k \times k}$ the permutation matrix with

$$h_{ij} = 1 \Longleftrightarrow i + j = k,$$

and

$$P_n = \begin{bmatrix} 1 & & & \\ & 0 & & 1 \\ & & \iddots & \\ & 1 & & 0 \end{bmatrix}. \tag{10}$$

Thus, $P_n = \mathrm{diag}(1, H_{n-1})$ is an $n \times n$ permutation matrix. P_n is also a left-transit cyclic matrix determined by $\boldsymbol{e}_0^\top$ (the first row of P_n). We note that $P_2 = I_2$ is the identity matrix. Since $\eta_1 = \eta_{\omega_0} = n^{-1/2}\ell$, we have

$$F_n(\omega_0) = n^{-1/2} J_n,$$

where J_n is the $n \times n$ matrix of all-ones. For $k \in [n-1]$, we have the following result.

Lemma 3 *For each $k \in [n-1]$, we have*

$$[F_n(\omega_k)]^2 = P_n. \tag{11}$$

Proof It is easy to see that $F_2^2 = I_2 = P_2$. For each $k \in [n-1]$, $\omega_k = \omega^k$ satisfies

$$1 + \omega_k + \omega_k^2 + \cdots + \omega_k^{n-1} = 0, \quad \forall\, k = 1, 2, \ldots, n-1, \tag{12}$$

since

$$0 = 1 - \omega_k^n = (1 - \omega_k)(1 + \omega_k + \omega_k^2 + \cdots + \omega_k^{n-1}).$$

On the other hand,

$$1 + \omega_0 + \omega_0^2 + \cdots + \omega_0^{n-1} = n. \tag{13}$$

We show that (11) is valid for $k = 1$. For this purpose, we write $F := F_n(\omega)$ and denote $F^2 = [f_0, f_1, \ldots, f_{n-1}]$, where f_j is the $(j+1)$th column vector of F^2. Then we have

$$(f_0)_i = (F\eta^0)_i = \frac{1}{n}\sum_{k=0}^{n-1} (\eta_k)_i = \frac{1}{n}\sum_{k=0}^{n-1} \omega^{ki} = \frac{1}{n}\sum_{k=0}^{n-1} \omega_i^k.$$

Thus, $f_0 = \boldsymbol{e}_0 \in \mathbb{R}^n$ since $(f_0)_0 = 1$ by (13) and $(f_0)_i = 0$ for each $i \in [n-1]$ by (12). We note that this technique can be employed to confirm the equality $f_j = \boldsymbol{e}_{n+1-j}$ for all $j \in [n-1]$. The proof is completed. □

The DFT of an input $\boldsymbol{x} \in \mathbb{R}^n$, denoted by $\hat{\boldsymbol{x}}$ or $(\boldsymbol{x})\hat{}$, is defined by

$$\hat{\boldsymbol{x}} = (y_0, y_1, \ldots, y_{n-1})^\top = F_n \boldsymbol{x},$$

that is,

$$y_i := \sum_{k \in \mathbb{Z}_n} \omega^{ik} x_k = \langle \eta_j, \boldsymbol{x} \rangle, \quad j = 1, 2, \ldots, n,$$

where $n \equiv 0 \pmod n$ and $\eta_j = \eta_{\omega_j}$ is defined as above. We define

$$\boldsymbol{x}^- = (x_n, x_{n-1}, \ldots, x_1)^\top$$

as the reverse of $\boldsymbol{x}$, i.e., $x_i^- = x_{n-i}$ for all $i \in [n-1]_0$ ($n \equiv 0 \pmod n$). Let $A = [A_0, A_1, \ldots, A_{n-1}] \in \mathscr{C}^{m\times n}$ be any matrix and denote

$$A^- := [A_0^-, A_1^-, \ldots, A_{n-1}^-].$$

It is easy to see that

$$\boldsymbol{e}_k^- = \boldsymbol{e}_{-k}, \quad \omega_{-k} = \omega_{n-k}.$$

Here, we use the subscript $-k$ instead of $n-k$ to indicate the symmetry. Furthermore, we have

$$(\eta_j)^- = \eta_{-j}, \quad (\eta_{-j})^- = \eta_j, \quad \forall\, j = 0, 1, 2, \ldots, n-1.$$

We can now rewrite (11) as

$$F_n^2 = I_n^- \tag{14}$$

since by definition,

$$I_n^- = [e_0^-, e_1^-, \ldots, e_{n-1}^-] = [e_n, e_{n-1}, \ldots, e_{n-(n-1)}] = [e_0, e_{n-1}, \ldots, e_{n-(n-1)}],$$

which is exactly P_n. By Lemma 3, we get (14). Note that $e_n \equiv e_0$ whose unique nonzero coordinate is the first one, which is 1.

From Lemma 3, we get the following result.

Corollary 1 *For each $j \in [n]$, define $\eta_j = \eta_{\omega_j}$ as*

$$\eta_j := \frac{1}{\sqrt{n}}(1, \omega_j, \omega_j^2, \ldots, \omega_j^{n-1})^\top \in \mathbb{C}^n \tag{15}$$

and denote $\omega_j = \omega^j$. Then we have

(i) $F_n(\omega_k)^4 = I_n$ *for* $k \in [n-1]$;
(ii) $F_n(\omega_{-k})^{-1} = F_n(\omega_k)^- = F_n(\omega_k)P_n$;
(iii) $(\eta_k)\hat{} = \boldsymbol{e}_{-k}$ *for all* $k \in [n-1]_0$;
(iv) $(\boldsymbol{x}^-)\hat{} = (\hat{\boldsymbol{x}})^-$ *for all* $\boldsymbol{x} \in \mathbb{R}^n$.

From Corollary 1 (ii), we have

$$F_n^- = F_n^{-1} = [\eta_0, \eta_{n-1}, \eta_{n-2}, \ldots, \eta_1].$$

Furthermore, we have $\hat{\eta_k} = \boldsymbol{e}_{n-k}$ for all $k \in [n-1]_0$.

Given a vector $\boldsymbol{x} \in \mathscr{C}^n$, a *Vandermonde matrix* (V-matrix) associated with $\boldsymbol{x}$ is defined as an $n\times n$ matrix $V(\boldsymbol{x}) := (V_{ij})$, where $V_{ij} = x_i^j$ for all $i, j \in [n-1]_0$. The GF-matrix $F_n(\omega_k)$ is exactly the Vandermonde matrix $V(\eta_k)$. An m-order *Vandermonde tensor* or *V-tensor* of *type* I generated by $\boldsymbol{x}$ is defined as a tensor $\mathbb{V} = (V_{i_1 i_2 \cdots i_m})$, where

$$V_{i_1 i_2 \cdots i_m} = x_{i_1}^{(i_2-1)(i_3-1)\cdots(i_m-1)}. \tag{16}$$

$\mathbb{W}$ is called a *V-tensor of type* II, if there exists an $(m-1)$-order n-dimensional tensor $\mathscr{B} = (B_{i_1 i_2 \cdots i_{m-1}})$ such that

$$W_{i_1 i_2 \cdots i_m} = B_{i_1 i_2 \cdots i_{m-1}}^{i_m - 1}.$$

Recall that an mth order n-dimensional symmetric tensor $\mathscr{A} = (A_\tau)$ is called a *Hankel tensor* (associated with $\boldsymbol{v}$), if there exists $\boldsymbol{v} = (v_0, v_1, \ldots, v_N)^\top$ such that $\forall\, k \in [N]_0$, $A_\tau = v_k$ for all $\tau \in S(k; m, n)$. A Hankel tensor $\mathscr{A}$ is an *elementary Hankel tensor*, if it is associated with a coordinate vector $\boldsymbol{e}_r \in \mathbb{R}^{N+1}$. Denote $\mathscr{H}_r$ for the elementary Hankel tensor associated with $\boldsymbol{e}_r$ ([20,21]). The elementary Hankel tensor $\mathscr{H}_r$ is closely related to the convolution operators [21].

The Fourier matrix is a kind of V-matrix. To see this, we denote

$$\omega_k := \omega^k = \exp\Big\{ -\frac{2\pi\imath k}{n} \Big\},$$

where

$$\omega := \exp\Big\{ -\frac{2\pi\imath}{n} \Big\}, \quad W := \{\omega_k \colon k \in [n]\},$$

noting that $\omega_n = 1$. Then W is a cyclic group generated by ω_1 and each ω_k has a unique inverse ω_{n-k} for $k \in [n]$. Recall η_j in (15) and note that $\eta_n = (1, 1, \ldots, 1)^\top$ is the vector of all-ones. The following lemma will be used later.

Lemma 4

$$\langle \eta_i, \eta_j \rangle = \delta_{ij}, \quad \forall\, i, j \in [n], \tag{17}$$

where the Kronecker delta δ_{ij} takes value in $\{0,1\}$ and is equal to 1 if and only if $i = j$. Here, the inner product is defined as

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{y}^* \boldsymbol{x} = \sum_{k=1}^{n} x_k \overline{y}_k,$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^\top$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^\top \in \mathbb{C}^n$.

Proof We note that $\overline{\omega}_i = \omega_{n-i}$. By (15), we have

$$\langle \eta_i, \eta_j \rangle = \frac{1}{n} \sum_{k=1}^{n} (\omega_i \overline{\omega}_j)^{k-1} = \sum_{k=1}^{n} (\omega_{i-j})^{k-1} \tag{18}$$

with $j - i \pmod n \in [n]$. If $i \neq j$, then $i - j \pmod n \in [n-1]$, thus, $\omega_{i-j} \in \{\omega_1, \omega_2, \ldots, \omega_{n-1}\}$, and so

$$\sum_{k=1}^{n} (\omega_{i-j})^{k-1} = 0.$$

For $i = j$, we have $\langle \eta_i, \eta_i \rangle = 1$. Thus, (17) is proved. $\square$

The Fourier matrix is useful in the DFT of one-dimensional signals. For the multi-order DFT (or the multi-dimensional DFT), say, an m-order DFT, with length n along each mode, we need to calculate mn numbers of one-dimensional (1-D) DFT if the DFT is based upon the 1-D formula.

In the following, we extend Fourier matrices to Fourier tensors in order to define a higher order DFT. The DFT on a square image comprising of $n \times n$ pixels requires a 4-order Fourier tensor to implement the calculation, which can be done compactly in one formula.

In the next section, we introduce the high order Fourier tensor and its spectrum, and then use the high order Fourier tensors to implement the DFT.

3 Fourier tensors

In signal processing, we sometimes encounter the case when the input signal is multi-dimensional. A multi-dimensional signal is a function of more than two variables. For example, a video signal is a function of three independent variables which are time and two spatial coordinates (X, Y). The high order (not higher dimensional) Fourier analysis arises when we consider the Fourier transform of a multi-dimensional input signal. In 2012, Tao used the high order FT to deal with the higher order linear patterns. As put in [18], he thought that 'The full theory of the high order patterns is still rather complicated $\cdots$.' Tao mainly investigated the behavior of polynomial patterns on arithmetic progressions as $f(n)$, $f(n+r)$, $f(n+2r), \ldots$ through the high order Fourier analysis. The reader is referred to [18,19] for more detail.

For our purpose, we introduce some fundamental knowledge and the multiplications defined on the tensors. Let $\alpha_j := (a_{1j}, a_{2j}, \ldots, a_{nj})^\top \in \mathbb{C}^n$ for $j \in [m]$. The tensor product

$$\mathscr{X} := \alpha_1 \times \alpha_2 \times \cdots \times \alpha_m$$

is defined as an m-order n-dimensional tensor

$$\mathscr{X} := (X_{i_1 i_2 \cdots i_m}),$$

where

$$X_{i_1 i_2 \cdots i_m} = a_{i_1 1} a_{i_2 2} \cdots a_{i_m m}.$$

$\mathscr{X}$ is a rank-1 tensor when all α_k's are nonzero. Similarly, the tensor product

$$\mathscr{A} := A_1 \times A_2 \times \cdots \times A_m$$

of a sequence of matrices $A_k = (a_{ij}^{(k)})$ is defined as a $2m$-order tensor

$$\mathscr{A} := (A_{i_1 i_2 \cdots i_m, j_1 j_2 \cdots j_m}),$$

where

$$A_{i_1 i_2 \cdots i_m, j_1 j_2 \cdots j_m} = a_{i_1 j_1}^{(1)} a_{i_2 j_2}^{(2)} \cdots a_{i_m j_m}^{(m)}, \tag{19}$$

where A_k's are not required to be the same size. A $2m$-order n-dimensional *Fourier tensor*, denoted by $\mathbb{W}_{m,n} \in \mathscr{T}_{2m;n}$, is defined by

$$W_{i_1 i_2 \cdots i_m, j_1 j_2 \cdots j_m} := n^{-m/2} \omega^{\sum_{k=1}^{m} (i_k - 1)(j_k - 1)}, \tag{20}$$

where $i_k, j_k \in [n]$, $\omega = \exp(2\pi\imath/n)$. It is easy to see that (20) is equivalent to

$$\mathbb{W}_{m,n} = F_n^{[m]} := \overbrace{F_n \times F_n \times \cdots \times F_n}^{m}, \tag{21}$$

which is also called the mth *tensor power* of Fourier matrix F_n.

Recall that a 2-dimensional discretized signal can be expressed as a matrix, say,

$$f = f(x, y) = (f_{ij}), \quad i \in [m],\, j \in [n].$$

Similarly, an m-dimensional discretized signal can be expressed as an m-order tensor

$$f = (f_{j_1 j_2 \cdots j_m}) \in \mathscr{T}_{m;n},$$

which may be generated by the gridding of an m-variate function $f = f(x_1, x_2, \ldots, x_m)$. Here, we only concern with the hyper-cubic case though the dimensions of each mode (direction) of f can be different. We call f an *m-order signal*, or briefly, an *m-signal*. The Fourier transform on an m-signal f can be described as the tensor multiplication

$$F = \hat{f} = \mathbb{W}_{m,n} \times f$$

with

$$\hat{f}_{i_1 i_2 \cdots i_m} = \sum_{j_1, j_2, \ldots, j_m} W_{i_1 i_2 \ldots i_m, j_1 j_2 \ldots j_m} f_{j_1 j_2 \ldots j_m}. \tag{22}$$

This is called the *Fourier tensor transform* of f, where the output $\hat{f}$ is also an m-order n-dimensional tensor.

Example 2 Let $f = \delta = (\delta_{i_1 i_2 \cdots i_m})$ be an m-order n-dimensional Kronecker (δ-) tensor defined by

$$\delta_{i_1 i_2 \cdots i_m} = 1 \Longleftrightarrow i_1 = i_2 = \cdots = i_m \pmod{n}.$$

Then $\hat{\delta}_\sigma = n$ if $s(\sigma) \equiv m \pmod{n}$ and $\hat{\delta}_\sigma = 0$ otherwise. In fact, for any $\sigma := (i_1, i_2, \ldots, i_m) \in S(k, m, n)$, we have

$$\begin{aligned}
\hat{\delta}_\sigma &= \sum_{j_1, j_2, \ldots, j_m} W_{i_1 i_2 \cdots i_m, j_1 j_2 \cdots j_m} \delta_{j_1 j_2 \cdots j_m} \\
&= \sum_{j_1, j_2, \ldots, j_m} \omega^{-\sum_{s=1}^{m} (i_s - 1)(j_s - 1)} \delta_{j_1 j_2 \cdots j_m} \\
&= \sum_{j=1}^{n} \omega^{-k(j-1)} \\
&= \sum_{j=1}^{n} \omega(k)^{(j-1)},
\end{aligned}$$

where $\omega(k) := \omega^{-k} = \exp(-2k\imath\pi/n)$. It follows that

$$\hat{\delta}_\sigma \neq 0 \Longleftrightarrow \omega(k) = 1 \Longleftrightarrow k = nq \Longleftrightarrow i_1 + i_2 + \cdots + i_m - m \equiv 0 \pmod{n}.$$

Example 2 shows that the Fourier transformation on a δ-tensor produces an elementary Hankel tensor (see [21]). Conversely, an even-order elementary Hankel tensor can be transformed into the δ-tensor by a Fourier transform.

We now study the spectrum of a Fourier tensor $\mathbb{W}_{m,n}$. By Lemma 4, F_n is a unitary matrix, so each eigenvalue of F_n lies in the set $S := \{1, -1, \imath, -\imath\}$. This conclusion also applies to the higher order case.

For convenience, we denote $\mu_k := \imath^k$. Then $S = \{\mu_1, \mu_2, \mu_3, \mu_4\}$ can be regarded as a cyclic group of order 4 generated by μ_1. Furthermore, if we denote $S^{(m)} = \{\mu_1^m, \mu_2^m, \mu_3^m, \mu_4^m\}$ and let $\texttt{rem}(m,n)$ denote the remainder of m divided by an integer n, then we have

(i) $S^{(m)} = S^{(0)} := \{\mu_1\}$ for all m with $\texttt{rem}(m,4) = 0$;

(ii) $S^{(m)} = S^{(1)} := S$ for all $m = 4k$ with $\texttt{rem}(m,4) = \pm 1$;

(iii) $S^{(m)} = S^{(2)} := \{\mu_1, \mu_2\} = \{1, -1\}$ for m with $\texttt{rem}(m,4) = 2$.

Given a tensor $\mathscr{A} = (A_{i_1 i_2 \cdots i_m}) \in \mathscr{T}_{m;n}$ and a vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^\top \in \mathscr{C}^n$, we define $\boldsymbol{y} = \mathscr{A}\boldsymbol{x}^{m-1}$ as a vector $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^\top$ with

$$y_i = \sum_{i_2, i_3, \ldots, i_m} A_{i i_2 i_3 \cdots i_m} x_{i_2} x_{i_3} \cdots x_{i_m}$$

for each $i \in [n]$. A scalar $\lambda \in \mathbb{C}$ is called an *eigenvalue* of $\mathscr{A}$, if there exists a nonzero vector $\boldsymbol{x} \in \mathbb{C}^n$, which is called an eigenvector of $\mathscr{A}$ corresponding to λ, such that

$$\mathscr{A}\boldsymbol{x}^{m-1} = \lambda \boldsymbol{x}^{m-1}, \tag{23}$$

where $\boldsymbol{x}^k$ is defined as a vector $(x_1^k, x_2^k, \ldots, x_n^k)^\top$. The pair $(\lambda, \boldsymbol{x})$ is called an eigenpair of $\mathscr{A}$. The pair $(\lambda, \boldsymbol{x})$ is called an E-eigenpair (λ is called an E-eigenvalue) of $\mathscr{A}$, if $\mathscr{A}\boldsymbol{x}^{m-1} = \lambda\boldsymbol{x}$, where $\boldsymbol{x} \in \mathbb{C}^n$ is a unit vector. λ is called a Z-eigenvalue, if there is a real vector $\boldsymbol{x}$ satisfying $\mathscr{A}\boldsymbol{x}^{m-1} = \lambda\boldsymbol{x}$.

Denote the spectrum of a tensor (matrix) $\mathscr{A}$ by $\pi(\mathscr{A})$. It is easy to see that

$$\pi(F_1) = S^{(0)}, \quad \pi(F_2) = S^{(2)}, \quad \pi(F_4) = S^{(1)}.$$

We use $\pi_z(\mathscr{A})$ to denote the Z-spectrum of a tensor. In the following, we only consider the case when $m > 1$.

Theorem 1 *Let $m, n > 1$ be two positive integers. Then*

(i) $\pi(\mathbb{W}_{m,n}) = S^{(0)}$ *if $m = 4k$ for any positive integer k;*

(ii) $\pi(\mathbb{W}_{m,n}) = S^{(2)}$ *if $m = 4k + 2$ for any nonnegative integer k;*

(iii) $S^{(m)} = S^{(1)}$ *if $m > 1$ is an odd number.*

Proof Let $0 \neq \boldsymbol{x} \in \mathbb{C}^n$ be a unit eigenvector of F_n corresponding to an eigenvalue $\lambda \in S$, i.e., $F_n\boldsymbol{x} = \lambda\boldsymbol{x}$. Then

$$\mathbb{W}_{m,n}\boldsymbol{x}^{2m-1} = F_n^{[m]}\boldsymbol{x}^{2m-1} = (\boldsymbol{x}^* F_n \boldsymbol{x})^{m-1} F_n \boldsymbol{x} = \lambda^m \boldsymbol{x}. \tag{24}$$

The second equality is due to the fact that

$$\begin{aligned} F_n^{[m]}\boldsymbol{x}^{2m-1} &= \overbrace{F_n \times F_n \times \cdots \times F_n}^{m} \times \boldsymbol{x}^{2m-1} \\ &= (F_n \times_1 \boldsymbol{x} \times_2 \boldsymbol{x})^{m-1} F_n \times \boldsymbol{x} \\ &= (\boldsymbol{x}^* F_n \boldsymbol{x})^{m-1} F_n \boldsymbol{x}. \end{aligned}$$

Thus, for any given $k \in [4]$, $\lambda_k \in \pi(F_n)$ implies $\lambda_k^m \in \pi(\mathbb{W}_{m,n})$. The result follows by observing the set consisting of λ_k^m. □

Now, we let

$$A^{(k)} = (a_{ij}^{(k)}) \in \mathscr{C}^{n\times n}, \quad k \in [m],$$

and define the tensor product $A^{(1)} \times A^{(2)} \times \cdots \times A^{(m)}$ of $A^{(k)}$'s as the $2m$-order n-dimensional tensor

$$\mathscr{G} = (G_{i_1 i_2 \cdots i_m; j_1 j_2 \cdots j_m})$$

such that

$$G_{i_1 i_2 \cdots i_m; j_1 j_2 \cdots j_m} = a_{i_1 j_1}^{(1)} a_{i_2 j_2}^{(2)} \cdots a_{i_m j_m}^{(m)}.$$

When

$$A := A^{(1)} = \cdots = A^{(m)} \in \mathbb{R}^{m\times n},$$

we call $\mathscr{G}$ the *m-tensor power of A* and denote it by $\mathscr{G} = A^{[m]}$.

A *paired symmetric tensor* was defined by Huang and Qi [11] as an even-order tensor $\mathscr{A}$ whose entries, indexed as $A_{i_1 j_1 i_2 j_2 \cdots i_m j_m}$, are invariant under the swapping of indices in any block(s) $(i_k j_k)$. $\mathscr{A}$ is called *strong paired symmetric*, if, additionally, it also satisfies

$$A_{i_1 j_1 i_2 j_2 \cdots i_m j_m} = A_{i_2 j_2 i_3 j_3 \cdots i_m j_m i_1 j_1}$$

for all possible indices. Denote $\mathbb{W}_{m,n}$ for the $2m$-order n-dimensional Fourier tensor, i.e., $\mathbb{W}_{m,n} = F_n^{[m]}$. By [11], $\mathbb{W}_{m,n}$ is a strong paired symmetric tensor. For any $2m$-order n-dimensional complex tensor $\mathscr{A} = (A_{i_1 j_1 i_2 j_2 \cdots i_m j_m})$, we define the $2m$-degree homogeneous polynomial associated with $\mathscr{A}$ by

$$f_{\mathscr{A}}(\boldsymbol{x}, \boldsymbol{y}) := \sum_{j=1} \left[A_{i_1 j_1 i_2 j_2 \cdots i_m j_m} \prod_{k=1}^{m} \{\overline{x}_{i_k} y_{j_k}\} \right], \tag{25}$$

where $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{C}^n$ and $\overline{\boldsymbol{x}}$ is the conjugate of $\boldsymbol{x}$. $\mathscr{A}$ is called *positive* (*semi-*) *definite* or a *pd* (*psd*) tensor if $f_{\mathscr{A}}(\boldsymbol{x}, \boldsymbol{x}) \geqslant 0$ (> 0) for all nonzero complex vectors $\boldsymbol{x} \in \mathscr{C}^n$. This is in fact the extension of positive semidefinite tensor, a symmetric tensor whose corresponding polynomial is nonnegative (see, e.g., [11,15]) from the real field to the complex case. For the properties of the symmetric tensors, we refer to [14,15].

Let $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathscr{C}^n$. The tensor product of α_j's, denoted by $\mathscr{A} := \alpha_1 \times \alpha_2 \times \cdots \times \alpha_m$, is an mth order n-dimensional tensor, and it is a rank-1 tensor if $\alpha_i \neq 0$ for all i. $\mathscr{A}$ is written as α^m if $\alpha := \alpha_1 = \alpha_2 = \cdots = \alpha_m$. Note that a tensor $\mathscr{A} \in \mathscr{T}_{m;n}$ is a symmetric rank-1 tensor if and only if $\mathscr{A} = \lambda \boldsymbol{x}^m$ for some $0 \neq \boldsymbol{x} \in \mathbb{R}^n$ and scaler $\lambda \neq 0$. In this case, $A_\sigma = x_{i_1} x_{i_2} \cdots x_{i_m}$ for any $\sigma = (i_1, i_2, \ldots, i_m) \in S(m,n)$. It is shown that [10,13] an mth order n-dimensional real tensor can always be decomposed into a rank-1 tensor as stated in the following lemma (see, e.g., [2]).

Lemma 5 *Let $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathscr{T}_{m;n}$ be an mth order n-dimensional real tensor. Then $\mathscr{A}$ can be decomposed as*

$$\mathscr{A} = \sum_{j=1}^{r} \alpha_j^{(1)} \times \alpha_j^{(2)} \times \cdots \times \alpha_j^{(m)}, \tag{26}$$

where $\alpha_j^{(i)} \in \mathbb{R}^n$ for $i \in [m]$, $j \in [r]$. The smallest positive integer r is called the rank of $\mathscr{A}$. If $\mathscr{A}$ is symmetric, then (26) *can be reduced to*

$$\mathscr{A} = \sum_{j=1}^{r} (\alpha_j)^m, \tag{27}$$

where $\alpha_j \in \mathbb{R}^n$ is a nonzero vector, see, e.g., [2,12].

Now, we assume that $\mathscr{A} = (A_{i_1 i_2 \cdots i_m}) \in \mathscr{T}_{m;n}$ is an input signal. By Lemma 5, $\mathscr{A}$ can be written as (26), where $\alpha_j^{(i)} \in \mathbb{R}^n$ are all nonzero input vectors.

Theorem 2 *Let the input signal $\mathscr{A} = (a_{i_1 \cdots i_m}) \in \mathscr{T}_{m;n}$ be an mth order n-dimensional real tensor with decomposition* (26)*. Then*

$$\hat{\mathscr{A}} = \sum_{j=1}^{r} \hat{\alpha}_j^{(1)} \times \hat{\alpha}_j^{(2)} \times \cdots \times \hat{\alpha}_j^{(m)}. \tag{28}$$

If $\mathscr{A}$ is a symmetric tensor with decomposition (27)*, then*

$$\hat{\mathscr{A}} = \sum_{j=1}^{r} (\hat{\alpha}_j)^m, \tag{29}$$

where $\hat{\alpha}_j \in \mathbb{R}^n$ is a Fourier transform.

Proof By (21) and Lemma 5, we have

$$\begin{aligned}
\hat{\mathscr{A}} &= \mathbb{W}_{m,n} \times \mathscr{A} \\
&= F_n^{[m]} \times \mathscr{A} \\
&= F_n^{[m]} \times \left(\sum_{j=1}^{r} \alpha_j^{(1)} \times \alpha_j^{(2)} \times \cdots \times \alpha_j^{(m)} \right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{r}[F_n^{[m]} \times (\alpha_j^{(1)} \times \alpha_j^{(2)} \times \cdots \times \alpha_j^{(m)})] \\
&= \sum_{j=1}^{r}[(F_n\alpha_j^{(1)}) \times (F_n\alpha_j^{(2)}) \times \cdots \times (F_n\alpha_j^{(m)})] \\
&= \sum_{j=1}^{r}[(\hat{\alpha}_j^{(1)}) \times (\hat{\alpha}_j^{(2)}) \times \cdots \times (\hat{\alpha}_j^{(m)})].
\end{aligned}
$$

For the special case when the input signal $\mathscr{A}$ is a symmetric tensor, we can show expression (29) by employing the same technique. □

Theorem 2 allows us to transfer the implementation of a complex high order Fourier transform into a one-dimensional Fourier transform. We shall mention that the introduction of a Fourier tensor can make easy even the 2-order Fourier transforms. We now consider the second order Fourier transform on matrices, i.e., the 2-order tensor input signal. Note that a 2-order Fourier transform corresponds to a 4-order Fourier tensor $\mathbb{W}_{4,n}$ which is defined by $F_n^{[2]} = F_n \times F_n$.

Theorem 3 *Let $A \in \mathbb{R}^{n\times n}$ be any $n \times n$ real matrix. Then $\hat{A} = F^\top AF$.*

Proof Since $\hat{A} = \mathbb{W}_{4,n} \times A$, we have $\hat{A} \in \mathscr{C}^{n\times n}$. By the definition,

$$
\begin{aligned}
\hat{A}_{ij} &= (\mathbb{W}_{4,n} \times A)_{ij} \\
&= \sum_{i',j'}(W_{ii',jj'}a_{i'j'}) \\
&= \sum_{i',j'}(F_{ii'}F_{jj'}a_{i'j'}) \\
&= \sum_{i',j'} a_{i'j'}\omega_i^{i'}\omega_j^{j'} \\
&= \eta_i^\top A\eta_j.
\end{aligned}
$$

Consequently, we get

$$
\hat{A} = F^\top AF = FAF
$$

due to the symmetry of F. The proof is completed. □

From Theorem 3, we can deduce the Fourier transforms of some special input signals. For example, an identity input is exactly P_n defined by (10), as stated in the following corollary. Here, we supply an alternative proof to it.

Corollary 2 $\hat{I}_n = P_n$.

Proof Let $A = I_n$, the $n \times n$ identity matrix. Then A can be written as

$$
A = \sum_{k=0}^{n-1} \boldsymbol{e}_k \times \boldsymbol{e}_k,
$$

where $\boldsymbol{e}_k$ is the kth coordinate vector in $\mathscr{C}^n$ for $i \in [n-1]_0$ as defined in Section 1. By Theorem 2, we have

$$\hat{A} = \sum_{k=0}^{n-1} \hat{\boldsymbol{e}}_k \times \hat{\boldsymbol{e}}_k. \tag{30}$$

By the definition of F_n and (15), for each $k \in [n-1]_0$, we have

$$\hat{\boldsymbol{e}}_k = F_n \boldsymbol{e}_k = \eta_k.$$

Therefore,

$$\hat{A} = \sum_{k=0}^{n-1} \eta_k \times \eta_k = F_n F_n^\top.$$

Since F_n is symmetric, by Lemma 3, we get $\hat{A} = P_n$. □

Given an mth order n-dimensional tensor $\mathscr{A} = (A_{i_1 i_2 \cdots i_m}) \in \mathscr{T}_{m;n}$ and a matrix $B = (B_{ij}) \in \mathscr{C}^{n \times n}$, for any index $k \in [n-1]_0$, the product $\mathscr{A} \times_k B$, called $\mathscr{A}$ multiplied by matrix B on the right side along mode-k (or direction k), is also an mth order n-dimensional tensor defined by

$$(\mathscr{A} \times_k B)_{i_1 i_2 \cdots i_m} = \sum_{i'_k=0}^{n-1} (A_{i_1 \cdots i'_k \cdots i_m} B_{i'_k i_k}).$$

This can be naturally extended to the product $\mathscr{A} \times_k \mathscr{B}$ for the tensors $\mathscr{A}$ and $\mathscr{B}$, where the dimensionalities of the k-mode of $\mathscr{A}$'s and $\mathscr{B}$'s are consistent. Similarly, we define $B \times_k \mathscr{A}$ for any $k \in [n-1]_0$. We have the following properties (see, e.g., [13]).

Lemma 6 *Let $\mathscr{A} \in \mathscr{T}_{m;n}$, $B_1, B_2 \in \mathscr{C}^{n \times n}$, and let $p, q \in [n-1]_0$ be any two distinct positive integers. Then we have*

$$(\mathscr{A} \times_p B_1) \times_q B_2 = (\mathscr{A} \times_q B_2) \times_p B_1 \tag{31}$$

and

$$\mathscr{A} \times_p B_1 \times_p B_2 = \mathscr{A} \times_p (B_2 B_1). \tag{32}$$

Here, we focus on the situation $\mathscr{A} \in \mathscr{T}_{m;n}$ and $B \in \mathscr{C}^{n \times n}$. Denote by $\mathscr{A}[B]$ for the product

$$\mathscr{A}[B] = \mathscr{A} \times_0 B \times_1 B \times_2 \cdots \times_{n-1} B.$$

By (31), we assert that $\mathscr{A}[B] \in \mathscr{T}_{m;n}$ is well defined. Analogously, we can define $[B]\mathscr{A}$, i.e., $\mathscr{A}$ multiplied by B from the left side along all directions. When B is a symmetric matrix, we have

$$\mathscr{A} \times_k B = B \times_k \mathscr{A}, \quad \forall\, k,$$

and thus,

$$\mathscr{A}[B] = [B]\mathscr{A}.$$

Now, we are ready to state the following theorem, which is the generalization of Theorem 3 for the general m-order Fourier transform.

Theorem 4 *Let $\mathscr{X} \in \mathscr{T}_{m;n}$ be any m-order input signal tensor. Then the Fourier transform of $\mathscr{X}$ is*

$$\hat{\mathscr{X}} = [F_n]\mathscr{X} = \mathscr{X}[F_n].$$

Proof By the definition, we have

$$\hat{\mathscr{X}} = \mathbb{W}_{m,n} \times \mathscr{X} \in \mathscr{T}_{m;n}.$$

Denote $\hat{\mathscr{X}} = (\hat{X}_{i_1 i_2 \cdots i_m})$. For any $\sigma := (i_1, i_2, \ldots, i_m) \in S(m,n)$, by the definition, we have

$$\begin{aligned}
\hat{X}_{i_1 i_2 \cdots i_m} &= \sum_{j_1, j_2, \ldots, j_m} (W_{i_1 i_2 \cdots i_m; j_1 j_2 \cdots j_m} X_{j_1 j_2 \cdots j_m}) \\
&= \sum_{j_1, j_2, \ldots, j_m} (F_{i_1 j_1} F_{i_2 j_2} \cdots F_{i_m j_m} X_{j_1 j_2 \cdots j_m}) \\
&= \sum_{j_1, j_2, \ldots, j_m} [F_n]\mathscr{X} \\
&= \mathscr{X}[F_n].
\end{aligned}$$

The proof is completed. □

We end the paper by remarking that since the vector space $\mathbb{R}^{n\times n}$ is isometric to vector space $\mathbb{R}^{n^2}$ under the vectorization ($\mathtt{Vec}$), the 4-order Fourier tensor as a Fourier transform defined on the vector space $\mathbb{R}^{n\times n}$ is actually equivalent to a 2-order Fourier tensor. This can be shown as follows:

$$\mathtt{Vec}(\hat{A}) = \mathtt{Vec}(\mathbb{W}_{4,n} \times A) = \mathtt{Vec}(FAF) = (F^\top \otimes F)\mathtt{Vec}(A) = (F \otimes F)\mathtt{Vec}(A),$$

where $X \otimes Y$ denotes the Kronecker product of two matrices and $\mathtt{Vec}(A)$ is the vectorization of a matrix A. Note that the last equality is due to the symmetry of F.

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