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RESEARCH ARTICLE

Exceptional sets in Waring-Goldbach problem for fifth powers

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Abstract We consider exceptional sets in the Waring-Goldbach problem for fifth powers. For example, we prove that all but $O(N^{131/132})$ integers satisfying the necessary local conditions can be represented as the sum of 11 fifth powers of primes, which improves the previous results due to A. V. Kumchev [Canad. J. Math., 2005, 57: 298–327] and Z. X. Liu [Int. J. Number Theory, 2012, 8: 1247–1256].

Keywords Exceptional sets, Waring-Goldbach problem, circle method
MSC2020 11P32, 11P55, 11P05

1 Introduction

In 1937, Vinogradov [11] found a new method for estimating sums over primes, thus he proved that every sufficiently large odd integer can be represented as the sum of three prime numbers which is known as the three prime theorem. Vinogradov's proof provided a blueprint for the subsequent applications of the circle method to additive prime number theory. Shortly after that, Vinogradov [12] and Hua [2] turned to study Waring's problem with prime variables which is known as the Waring-Goldbach problem.

In fact, the Waring-Goldbach problem is concerned with the solvability of the equation

$$n = p_1^k + p_2^k + \cdots + p_s^k, \tag{1}$$

where $p_1, p_2, \ldots, p_s$ are unknown primes. There are considerable works about this topic, and they can be classified into two categories, broadly speaking. One of them concerns the number of primes in (1). Here, the readers can refer to Vinogradov [12], Hua [2,3], Kawada and Wooley [4], Zhao [13], and so on to study the details of this direction. The other one concerns the estimates of

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exceptional sets. Let $E_{k,s}(N)$ be the number of $n \leqslant N$ satisfying some local conditions, that (1) cannot be solved in primes $p_1, p_2, \ldots, p_s$. Readers can refer to Liu [9] for the details. The recent results about $E_{5,s}(N)$ came from Kumchev [6] and Liu [9]. Combining the methods of Zhao [13] and Kawada and Wooley [5], we can establish the following result for $E_{5,s}(N)$ $(11 \leqslant s \leqslant 20)$.

Theorem 1 *For* $11 \leqslant s \leqslant 20$, *let* $E_{5,s}(N)$ *be the number of integers* $n \leqslant N$ *satisfying* $n \equiv s \pmod 2$ *for which* (1) *cannot be solved in primes* $p_1, p_2, \ldots, p_s$. *Then for arbitrary* $\varepsilon > 0$,

$$E_{5,11}(N) \ll N^{1-\theta_1+\varepsilon}, \quad E_{5,12}(N) \ll N^{1-\theta_2+\varepsilon}, \quad E_{5,13}(N) \ll N^{1-5\theta_1+\varepsilon},$$
$$E_{5,s}(N) \ll N^{1-\frac{s-12}{40}-\theta_2+\varepsilon}, \quad 14 \leqslant s \leqslant 18,$$
$$E_{5,s}(N) \ll N^{1-\frac{s-11}{40}-\theta_1+\varepsilon}, \quad s = 19, 20,$$

where
$$\theta_1 = \frac{73}{9600}, \quad \theta_2 = \frac{153}{9600}.$$

Our result can be compared with the previous results. For example, our result shows that

$$E_{5,11}(N) \ll N^{131/132}, \quad E_{5,12}(N) \ll N^{62/63},$$
$$E_{5,17}(N) \ll N^{0.8591}, \quad E_{5,18}(N) \ll N^{0.8341}.$$

This improves the results of Kumchev [6] and Liu [9]. Among these, Liu's result showed that

$$E_{5,11}(N) \ll N^{213/214}, \quad E_{5,12}(N) \ll N^{76/77},$$
$$E_{5,17}(N) \ll N^{0.9454}, \quad E_{5,18}(N) \ll N^{0.9370};$$

and Kumchev's result showed that

$$E_{5,11}(N) \ll N^{239/240}, \quad E_{5,12}(N) \ll N^{79/80},$$
$$E_{5,17}(N) \ll N^{0.9459}, \quad E_{5,18}(N) \ll N^{0.9375}.$$

As usual, we abbreviate $\mathrm{e}^{2\pi \mathrm{i}\alpha}$ to $e(\alpha)$. The letter p, with or without indices, is a prime number. The letter ε denotes a sufficiently small positive real number, and the value of ε may change from statement to statement. Let N be a sufficiently large real number in terms of ε and k. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, while the implied constant may depend on ε and k.

2 Proof of Theorem 1 for $s = 11$ and $s = 12$

We will give the proof of Theorem 1 for $s = 11$ and $s = 12$ by using the

Hardy-Littlewood method in this section. Supposing that n is a sufficiently large natural number congruent to s modulo 2. We write

$$P = \frac{n^{1/5}}{2}.$$

Let

$$\upsilon_j = \left(\frac{33}{40}\right)^{j-1}, \quad j = 1, 2, \ldots, 6,$$

$$\upsilon_7 = \left(\frac{33}{40}\right)^5 \frac{136}{163}, \quad \upsilon_8 = \left(\frac{33}{40}\right)^5 \frac{576}{815}, \quad \upsilon_9 = \left(\frac{33}{40}\right)^5 \frac{512}{815}.$$

We note that

$$\sum_{i=1}^{8} \upsilon_i + 2\upsilon_9 > 4.9817431213.$$

Also, we write

$$P_j = P^{\upsilon_j}, \quad g_j(\alpha) = \sum_{P_j < p \leqslant 2P_j} (\log p) e(\alpha p^5), \quad 1 \leqslant j \leqslant 9,$$

$$\mathscr{G}(\alpha) = g_9(\alpha) \prod_{j=1}^{9} g_j(\alpha).$$

Considering the Diophantine equation and applying [4, Lemma 6.2], one has the following result.

Lemma 1 *Let $g_j(\alpha)$ be defined as above. Then*

$$\int_0^1 |\mathscr{G}(\alpha)|^2 \mathrm{d}\alpha \ll \mathscr{G}(0) P^{\varepsilon},$$

$$\int_0^1 |g_j(\alpha)\mathscr{G}(\alpha)|^2 \mathrm{d}\alpha \ll \mathscr{G}^2(0) P_j^2 P^{-5+\varepsilon}, \quad 1 \leqslant j \leqslant 9.$$

For $s \in \{11, 12\}$, let $r_s(n)$ denote the weighted number of solutions for the equation

$$n = p_1^5 + p_2^5 + \cdots + p_s^5$$

with

$$P^{\nu_i} < p_i \leqslant 2P^{\nu_i}, \quad 1 \leqslant i \leqslant 9,$$

$$P_1 < p_i \leqslant 2P_1, \quad 10 \leqslant i \leqslant s.$$

Whenever $\mathfrak{X} \subset [0, 1)$ is measurable, we put

$$r_s(n, \mathfrak{X}) = \int_{\mathfrak{X}} g_1^{s-10}(\alpha) \mathscr{G}(\alpha) e(-n\alpha) \mathrm{d}\alpha.$$

Take

$$P_0 = P_9^{\frac{2}{5} - \varepsilon}, \quad Q = N P_0^{-1}.$$

Define

$$\mathfrak{M}=\bigcup_{q\leqslant P_0}\ \bigcup_{1\leqslant a\leqslant q,\,(a,q)=1}\mathfrak{M}(q,a),\quad \mathfrak{m}=[0,1)\backslash\mathfrak{M},$$

where

$$\mathfrak{M}(q,a)=\Big\{\alpha\colon |q\alpha-a|\leqslant\frac{1}{Q}\Big\}.$$

We can get the following result by applying the standard method of enlarging major arcs (cf. [8,9]).

Lemma 2 *For all positive integers n with $N<n\leqslant 2N$ satisfying $n\equiv s\ (\mathrm{mod}\ 2)$, one has*

$$r_s(n,\mathfrak{M})\gg P^{s-15}\mathscr{G}(0).$$

To estimate the integral of minor arcs, we need the following estimate proved by Ren [10].

Lemma 3 *Suppose that α is a real number, and that $1\leqslant a\leqslant q$ with $(a,q)=1$. Let $\beta=\alpha-\frac{a}{q}$. Then one has*

$$\sum_{X<p\leqslant 2X}e(\alpha p^k)$$
$$\ll d(q)^{c_k}(\log X)^c\Big(X^{1/2}\sqrt{q(1+|\beta|X^k)}+X^{4/5}+\frac{X}{\sqrt{q(1+|\beta|X^k)}}\Big),$$

where $c_k=\frac{1}{2}+\frac{\log k}{\log 2}$ and c is a constant.

First, we estimate the contribution from the minor arcs $\mathfrak{m}$. Denote

$$\mathfrak{R}=\bigcup_{q\leqslant P^{15/16}}\ \bigcup_{1\leqslant a\leqslant q,\,(a,q)=1}\mathfrak{R}(q,a),\quad \mathfrak{R}(q,a)=\{\alpha\colon |q\alpha-a|\leqslant P^{\frac{15}{16}-5}\}.$$

Applying Lemma 3, one has

$$\sup_{\alpha\in\mathfrak{m}\cap\mathfrak{R}}|g_1(\alpha)|\ll P^{1-\frac{1}{32}+\varepsilon}.$$

Thus,

$$\int_{\mathfrak{m}\cap\mathfrak{R}}|g_1^{2s-20}(\alpha)\mathscr{G}^2(\alpha)|\mathrm{d}\alpha\ll\sup_{\alpha\in\mathfrak{m}\cap\mathfrak{R}}|g_1(\alpha)|^{2s-20}\int_0^1|\mathscr{G}(\alpha)|^2\mathrm{d}\alpha$$
$$\ll P^{(1-\frac{1}{32})(2s-20)+\varepsilon}\mathscr{G}(0).$$

For the integral on $\mathfrak{m}\setminus\mathfrak{R}$, by [7, Lemma 2.2], one has

$$\sup_{\alpha\in\mathfrak{m}\backslash\mathfrak{R}}|g_j(\alpha)|\ll P_j^{1-\frac{1}{48}+\varepsilon},\quad j=1,2. \tag{2}$$

Let
$$\mathscr{I}(h)=\int_{\mathfrak{m}\backslash\mathfrak{R}}|g_1^h(\alpha)\mathscr{G}^2(\alpha)|\mathrm{d}\alpha,\quad h\geqslant 2.$$
By the definition of $g_1(\alpha)$, we have
$$\begin{aligned}\mathscr{I}(h)&=\sum_{P<p_1,p_2\leqslant 2P}\int_{\mathfrak{m}\backslash\mathfrak{R}}e((p_1^5-p_2^5)\alpha)|g_1^{h-2}(\alpha)\mathscr{G}^2(\alpha)|\mathrm{d}\alpha\\&\leqslant\sum_{P<x_1,x_2\leqslant 2P}\Big|\int_{\mathfrak{m}\backslash\mathfrak{R}}e((x_1^5-x_2^5)\alpha)|g_1^{h-2}(\alpha)\mathscr{G}^2(\alpha)|\mathrm{d}\alpha\Big|\\&\leqslant P\Upsilon^{1/2}(h),\end{aligned}$$
where
$$\begin{aligned}\Upsilon(h)&=\sum_{P<x_1,x_2\leqslant 2P}\Big|\int_{\mathfrak{m}\backslash\mathfrak{R}}e((x_1^5-x_2^5)\alpha)|g_1^{h-2}(\alpha)\mathscr{G}^2(\alpha)|\mathrm{d}\alpha\Big|^2\\&=\int_{\mathfrak{m}\backslash\mathfrak{R}}\int_{\mathfrak{m}\backslash\mathfrak{R}}|f_5^2(\alpha-\beta;P)g_1^{h-2}(\alpha)g_1^{h-2}(\beta)\mathscr{G}^2(\alpha)\mathscr{G}^2(\beta)|\mathrm{d}\alpha\mathrm{d}\beta.\end{aligned}$$
Let
$$\mathfrak{N}=\bigcup_{q\leqslant P^{5/16}}\ \bigcup_{1\leqslant a\leqslant q,\,(a,q)=1}\mathfrak{N}(q,a),\quad \mathfrak{n}=[0,1)\backslash\mathfrak{N},$$
where
$$\mathfrak{N}(q,a)=\{\alpha\colon |q\alpha-a|\leqslant P^{\frac{5}{16}-5}\}.$$
We denote by $\mathfrak{B}$ the set of ordered pairs $(\alpha,\beta)\in(\mathfrak{m}\setminus\mathfrak{R})^2$ for which $\alpha-\beta\in\mathfrak{N}$ (mod 1), and put $\mathfrak{b}=\mathfrak{m}^2\backslash\mathfrak{B}$. Next, we define the function $\Psi\colon[0,1)\to[0,\infty)$ as
$$\Psi(\alpha)=\omega_5(q)P\Big(1+P^5\Big|\alpha-\frac{a}{q}\Big|\Big)^{-1},$$
when $\alpha\in\mathfrak{N}(q,a)\subseteq\mathfrak{N}$, otherwise by taking $\Psi(\alpha)=0$. Then
$$\Upsilon(h)\leqslant\Big(\iint_{\mathfrak{b}}+\iint_{\mathfrak{B}}\Big)|f_5^2(\alpha-\beta;P)g_1^{h-2}(\alpha)g_1^{h-2}(\beta)\mathscr{G}^2(\alpha)\mathscr{G}^2(\beta)|\mathrm{d}\alpha\mathrm{d}\beta$$
Applying [7, Lemma 2.1] for $f_5(\alpha-\beta;P)$, one has
$$\Upsilon(h)\ll\Upsilon_1(h)+\Upsilon_2(h),\tag{3}$$
where
$$\Upsilon_1(h)=P^{2-\frac{1}{8}+\varepsilon}\int_{\mathfrak{m}\backslash\mathfrak{R}}\int_{\mathfrak{m}\backslash\mathfrak{R}}|g_1^{h-2}(\alpha)g_1^{h-2}(\beta)\mathscr{G}^2(\alpha)\mathscr{G}^2(\beta)|\mathrm{d}\alpha\mathrm{d}\beta,$$
$$\Upsilon_2(h)=\iint_{\mathfrak{B}}\Psi^2(\alpha-\beta)|g_1^{h-2}(\alpha)g_1^{h-2}(\beta)\mathscr{G}^2(\alpha)\mathscr{G}^2(\beta)|\mathrm{d}\alpha\mathrm{d}\beta.$$

Next, we first deal with $\Upsilon_2(h)$ for $h=2$ and $h=4$. It is easily to see that

$$|g_3(\alpha)g_3(\beta)|^2 \ll |g_3(\alpha)|^4 + |g_3(\beta)|^4. \tag{4}$$

Combining (4) with trivial estimates and symmetry, we can establish

$$\begin{aligned}\Upsilon_2(h) &\ll \int_{\mathfrak{m}\setminus\mathfrak{R}}\int_{\mathfrak{m}\setminus\mathfrak{R}} \Psi^2(\alpha-\beta)|g_1^{h-2}(\alpha)g_1^{h-2}(\beta)g_3^2(\alpha)\mathscr{G}^2(\alpha)\\ &\quad \times g_1^2(\beta)g_2^2(\beta)g_4^2(\beta)g_5^2(\beta)\cdots g_8^2(\beta)g_9^4(\beta)|\mathrm{d}\alpha\mathrm{d}\beta\\ &\ll \sup_{\beta\in\mathfrak{m}\setminus\mathfrak{R}} |g_1^{2h-2}(\beta)g_2^2(\beta)g_5^2(\beta)g_6^2(\beta)g_7^2(\beta)g_8^2(\beta)g_9^4(\beta)|\\ &\quad \times \int_0^1 \Psi^2(\alpha-\beta)|g_4^2(\beta)|\mathrm{d}\beta \int_0^1 |g_3^2(\alpha)\mathscr{G}^2(\alpha)|\mathrm{d}\alpha.\end{aligned}$$

By [13, Lemma 2.2], one has

$$\int_0^1 \Psi^2(\alpha-\beta)|g_4^2(\beta)|\mathrm{d}\beta \ll P_4^2P^{-3+\varepsilon}.$$

Thus, by Lemma 1 and (2), we have

$$\begin{aligned}\Upsilon_2(2) &\ll P^{-8-\frac{73}{960}+\varepsilon}\mathscr{G}^4(0), \qquad (5)\\ \Upsilon_2(4) &\ll P^{-4-\frac{51}{320}+\varepsilon}\mathscr{G}^4(0).\end{aligned}$$

Now, we turn to apply Lemma 1 for $\Upsilon_1(h)$ when $h=2$. One has

$$\Upsilon_1(2) \ll P^{2-\frac{1}{8}+\varepsilon}\left(\int_0^1 \mathscr{G}^2(\alpha)\mathrm{d}\alpha\right)^2 \ll P^{2-\frac{1}{8}+\varepsilon}\mathscr{G}^2(0). \tag{6}$$

By (3), (6), and (5), one has

$$\mathscr{I}(2) \ll P^{-3-\frac{73}{1920}+\varepsilon}\mathscr{G}^2(0). \tag{7}$$

However, when $h=4$, it follows that

$$\Upsilon_1(4) \ll P^{2-\frac{1}{8}+\varepsilon}\left(\mathscr{I}(2)\right)^2 \ll P^{-4-\frac{193}{960}+\varepsilon}\mathscr{G}^4(0),$$

and $\Upsilon_2(4)$ dominates $\Upsilon_1(4)$. Thus,

$$\mathscr{I}(4) \ll P^{-1-\frac{51}{640}+\varepsilon}\mathscr{G}^2(0).$$

Proof of Theorem 1 *for* $s=11$ *and* $s=12$

Case $s=11$ Applying Bessel's inequality, one has

$$\begin{aligned}\sum_{N<n\leqslant 2N}\left|\int_{\mathfrak{m}} g_1(\alpha)\mathscr{G}(\alpha)e(-n\alpha)\mathrm{d}\alpha\right|^2 &\leqslant \int_{\mathfrak{m}} |g_1(\alpha)\mathscr{G}(\alpha)|^2\mathrm{d}\alpha\\ &\ll P^{-3-\frac{73}{1920}+\varepsilon}\mathscr{G}^2(0). \qquad (8)\end{aligned}$$

By a standard argument, we have

$$E_{5,11}(N) \ll N^{1-\theta_1+\varepsilon}, \quad \theta_1 = \frac{73}{9600}.$$

Case $s = 12$ Also applying Bessel's inequality, one has

$$\sum_{N<n\leqslant 2N} \left| \int_{\mathfrak{m}} g_1^2(\alpha)\mathscr{G}(\alpha)e(-n\alpha)\mathrm{d}\alpha \right|^2 \leqslant \int_{\mathfrak{m}} |g_1^2(\alpha)\mathscr{G}(\alpha)|^2\mathrm{d}\alpha \ll P^{-1-\frac{51}{640}+\varepsilon}\mathscr{G}^2(0).$$

And one has

$$E_{5,12}(N) \ll N^{1-\theta_2+\varepsilon}, \quad \theta_2 = \frac{153}{9600}. \qquad \square$$

Now, we have finished the proof of Theorem 1 for $s = 11$ and $s = 12$. Next, we will use the relations between exceptional sets to prove Theorem 1 for $13 \leqslant s \leqslant 20$.

3 Proof of Theorem 1 for $13 \leqslant s \leqslant 20$

Here, we introduce some notations basing on [5]. When $\mathscr{C} \subseteq \mathbb{N}$, we define $\overline{\mathscr{C}}$ as the complement $\mathbb{N}\backslash\mathscr{C}$ of $\mathscr{C}$ within $\mathbb{N}$, and $(\mathscr{C})_a^b$ is the set $\mathscr{C} \cap (a, b]$ when a and b are non-negative integers, and $|\mathscr{C}|_a^b$ denotes the cardinality of $\mathscr{C} \cap (a, b]$. For $\mathscr{C}, \mathscr{D} \subseteq \mathbb{N}$, we define their sum and difference as the following form:

$$\mathscr{C} \pm \mathscr{D} = \{c \pm d\colon c \in \mathscr{C},\, d \in \mathscr{D}\}.$$

$h\mathscr{D}$ denotes the h-fold sum $\mathscr{D} + \cdots + \mathscr{D}$.

If $q \in \mathbb{N}$ and $\mathfrak{a} \in \{0, 1, \ldots, q-1\}$, we define $\mathscr{P}_{\mathfrak{a},q}$ by

$$\mathscr{P}_{\mathfrak{a},q} = \{\mathfrak{a} + mq\colon m \in \mathbb{Z}\}.$$

Also, we describe a set $\mathscr{L}$ as being a union of arithmetic progressions modulo q when

$$\mathscr{L} = \bigcup_{\mathfrak{l}\in\mathfrak{L}} \mathscr{P}_{\mathfrak{l},q}$$

for some subset $\mathfrak{L}$ of $\{0, 1, \ldots, q-1\}$. Furthermore, it is convenient to write

$$\langle \mathscr{C} \wedge \mathscr{L} \rangle_a^b = \min_{\mathfrak{l}\in\mathfrak{L}} |\mathscr{C} \cap \mathscr{P}_{\mathfrak{l},q}|_a^b,$$

where a and b are integers.

For $k \in \mathbb{N}$, we describe a subset $\mathscr{Q} \subset \mathbb{N}$ as being a high-density subset of the kth powers relative to $\mathscr{L}$ when

(i) $\mathscr{Q} \subset \{n^k\colon n \in \mathbb{N}\}$,

(ii) for each $\varepsilon > 0$,

$$\langle \mathscr{Q} \wedge \mathscr{L} \rangle_0^N \gg_q N^{\frac{1}{k}-\varepsilon}$$

when N is a natural number sufficiently large in terms of ε.

Moreover, if there exists $\delta > 0$ such that

$$|\overline{\mathscr{R}} \cap \mathscr{L}|_0^N < N^{\theta-\delta}$$

holds for all sufficiently large natural numbers N, then we say that a set $\mathscr{R} \subset \mathbb{N}$ has $\mathscr{L}$-complementary density growth exponent smaller than θ.

We will use the following lemma, which is [1, Lemma 4.3]. Here, we denote

$$\sigma_k^{-1} = \min\{2^{k-1}, k(k-1)\}.$$

Lemma 4 *Let $\mathscr{L}$, $\mathscr{M}$, and $\mathscr{N}$ be unions of arithmetic progressions modulo q for some natural number q. Suppose that $\mathscr{C}$ is a high-density subset of k-th powers relative to $\mathscr{L}$, and that $\mathscr{A} \subset \mathbb{N}$ has $\mathscr{L}$-complementary density growth exponent smaller than θ. Then, whenever $\varepsilon > 0$ and N is a natural number sufficiently large in terms of ε, one has the following estimates.*

(a) *If $\mathscr{N} \subset \mathscr{L} + \mathscr{M}$ and $k \geqslant 4$, then, without any condition on θ, one has*

$$|\overline{\mathscr{A}+\mathscr{C}} \cap \mathscr{N}|_{2N}^{3N} \ll_q N^{\varepsilon - \frac{2\sigma_k}{k}} |\overline{\mathscr{A}} \cap \mathscr{M}|_N^{3N} + N^{\varepsilon - \frac{2}{k-2}} (|\overline{\mathscr{A}} \cap \mathscr{M}|_N^{3N})^{k/(k-2)}.$$

(b) *If $\mathscr{N} \subset 2\mathscr{L} + \mathscr{M}$ and $k \geqslant 5$, then, without any condition on θ, one has*

$$|\overline{\mathscr{A}+2\mathscr{C}} \cap \mathscr{N}|_{2N}^{3N} \ll_q N^{\varepsilon - \frac{4\sigma_k}{k}} |\overline{\mathscr{A}} \cap \mathscr{M}|_N^{3N} + N^{\varepsilon - \frac{4}{k-4}} (|\overline{\mathscr{A}} \cap \mathscr{M}|_N^{3N})^{k/(k-4)}.$$

Proof of Theorem 1 *for* $13 \leqslant s \leqslant 20$ Denote

$$\mathscr{C} = \{p^5 \colon p > 6 \text{ is a prime}\},$$

and

$$\mathscr{N}_s = \{n \in \mathbb{N} \colon n \equiv s \pmod 2\},$$

which is a union of arithmetic progressions modulo 240. Then one has

$$p^5 \equiv 1 \pmod 2, \quad p \in \mathscr{C},$$

and

$$s\mathscr{C} \subseteq \mathscr{N}_s, \quad s \geqslant 11.$$

To use Lemma 4, we need to define another union of arithmetic progressions $\mathscr{L}$. Let

$$\mathscr{L} = \{l \in \mathbb{N} \colon l \equiv 1 \pmod 2\}.$$

One can deduce

$$\langle \mathscr{C} \wedge \mathscr{L} \rangle_0^N \gg N^{1/5} (\log N)^{-1}$$

by applying the Prime Number Theorem in arithmetic progressions. Thus, $\mathscr{C}$ is a high-density subset of the fifth powers relative to $\mathscr{L}$. Also, basing on the definition of $\mathscr{N}_s$ and $\mathscr{L}$, it follows that

$$\mathscr{N}_{s+1} = \mathscr{L} + \mathscr{N}_s, \quad \mathscr{N}_{s+2} = 2\mathscr{L} + \mathscr{N}_s, \quad s \geqslant 9.$$

It follows from part (b) of Lemma 4 that

$$\begin{aligned}|\overline{11\mathscr{C}+2\mathscr{C}}\cap\mathscr{N}_{13}|_{2N}^{3N} &\ll N^{\varepsilon-\frac{1}{20}}|\overline{11\mathscr{C}}\cap\mathscr{N}_{11}|_{N}^{3N}+N^{\varepsilon-4}(|\overline{11\mathscr{C}}\cap\mathscr{N}_{11}|_{N}^{3N})^5\\ &\ll N^{\varepsilon-\frac{1}{20}}E_{5,11}(3N)+N^{\varepsilon-4}(E_{5,11}(3N))^5\\ &\ll N^{1-5\theta_1+\varepsilon},\\ |\overline{12\mathscr{C}+2\mathscr{C}}\cap\mathscr{N}_{14}|_{2N}^{3N} &\ll N^{\varepsilon-\frac{1}{20}}|\overline{12\mathscr{C}}\cap\mathscr{N}_{12}|_{N}^{3N}+N^{\varepsilon-4}(|\overline{12\mathscr{C}}\cap\mathscr{N}_{12}|_{N}^{3N})^5\\ &\ll N^{\varepsilon-\frac{1}{20}}E_{5,12}(3N)+N^{\varepsilon-4}(E_{5,12}(3N))^5\\ &\ll N^{1-\frac{1}{20}-\theta_2+\varepsilon},\end{aligned}$$

where θ_1 and θ_2 are defined in Theorem 1.

For an arbitrary natural number s, we define

$$E_s(N)=|\overline{s\mathscr{C}}\cap\mathscr{N}_s|_0^N.$$

It follows that the exceptional sets in the Waring-Goldbach problem for fifth powers, given in the preamble to Theorem 1, can be covered by $E_s(N)$.

Let $\lceil x\rceil$ denote the least integer not smaller than x, and define the integers N_j for $j\geqslant 0$ by means of the iterative formula

$$N_0=\left\lceil\frac{1}{2}N\right\rceil,\quad N_{j+1}=\left\lceil\frac{2}{3}N_j\right\rceil,\ j\geqslant 0.$$

Moreover, let J be the least positive integer j with the property that $N_j=2$, and note that $J=O(\log N)$. Hence, it follows that

$$E_{5,13}(N)\leqslant 3+\sum_{j=1}^{J}|\overline{11\mathscr{C}+2\mathscr{C}}\cap\mathscr{N}_{13}|_{2N_j}^{3N_j}\ll N^{1-5\theta_1+\varepsilon},$$

$$E_{5,14}(N)\leqslant 3+\sum_{j=1}^{J}|\overline{12\mathscr{C}+2\mathscr{C}}\cap\mathscr{N}_{14}|_{2N_j}^{3N_j}\ll N^{1-\frac{1}{20}-\theta_2+\varepsilon}.$$

Likewise, for $15\leqslant s\leqslant 18$, from part (a) in Lemma 4, one finds that

$$\begin{aligned}&|\overline{(s-1)\mathscr{C}+\mathscr{C}}\cap\mathscr{N}_s|_{2N}^{3N}\\ &\quad\ll N^{\varepsilon-\frac{1}{40}}|\overline{(s-1)\mathscr{C}}\cap\mathscr{N}_{s-1}|_{N}^{3N}+N^{\varepsilon-\frac{2}{3}}(|\overline{(s-1)\mathscr{C}}\cap\mathscr{N}_{s-1}|_{N}^{3N})^{5/3}\\ &\quad\ll N^{\varepsilon-\frac{1}{40}}E_{5,s-1}(3N)+N^{\varepsilon-\frac{2}{3}}(E_{5,s-1}(3N))^{5/3}\\ &\quad\ll N^{1-\frac{s-14}{40}-\theta_2+\varepsilon}.\end{aligned}$$

Consequently, one has

$$E_{5,s}(N)\leqslant 3+\sum_{j=1}^{J}|\overline{(s-1)\mathscr{C}+\mathscr{C}}\cap\mathscr{N}_{s-1}|_{2N_j}^{3N_j}\ll N^{1-\frac{s-14}{40}-\theta_2+\varepsilon},\quad 15\leqslant s\leqslant 18.$$

For $s=19$, applying results on [9, pp. 1255, 1256], one has

$$\mathscr{G}(0)P^4E_{5,19}(N) \ll (P^{15}E_{5,19}(N) + P^{11+\varepsilon}E_{5,19}^2(N))^{1/2}\mathscr{I}^{1/2}(2).$$

Then, by (7), one can obtain

$$E_{5,19}(N) \ll N^{1-\frac{1}{5}-\theta_1+\varepsilon}.$$

Likewise, again from part (a) in Lemma 4, one eventually gets

$$E_{5,20}(N) \ll N^{1-\frac{9}{40}-\theta_1+\varepsilon}. \qquad \square$$

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