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RESEARCH ARTICLE

Hoeffding's inequality for Markov processes via solution of Poisson's equation

Yuanyuan LIU, Jinpeng LIU

School of Mathematics and Statistics, New Campus, Central South University, Changsha 410083, China

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Abstract We investigate Hoeffding's inequality for both discrete-time Markov chains and continuous-time Markov processes on a general state space. Our results relax the usual aperiodicity restriction in the literature, and the explicit upper bounds in the inequalities are obtained via the solution of Poisson's equation. The results are further illustrated with applications to queueing theory and reflective diffusion processes.

Keywords Hoeffding's inequality, Markov process, Poisson's equation
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1 Introduction

Let $\{Y_i\colon i \geqslant 0\}$ be a sequence of independent and identically distributed random variables such that $\mathbb{P}\{a \leqslant Y_i \leqslant b\} = 1$ for some real numbers a and b. Since $\mathbb{E}[Y_i] < \infty$, the following strong law of large numbers holds:

$$\frac{1}{n} S_n \xrightarrow{\text{a.s.}} \mathbb{E}[Y_0], \quad n \to \infty,$$

where $S_n := \sum_{i=0}^{n-1} Y_i$. An upper bound for the tail probability of law of large numbers was first presented by Hoeffding [11] as follows:

$$\mathbb{P}\Big(\frac{1}{n} S_n - \mathbb{E}[Y_0] \geqslant \varepsilon\Big) \leqslant \exp\Big\{\frac{-2n\varepsilon^2}{(b-a)^2}\Big\}, \tag{1}$$

where $\varepsilon > 0$. Moreover, since $\mathbb{P}\{-b \leqslant -Y_i \leqslant -a\} = 1$, Hoeffding's inequality (1) immediately implies

$$\mathbb{P}\Big(\Big|\frac{1}{n} S_n - \mathbb{E}[Y_0]\Big| \geqslant \varepsilon\Big) \leqslant 2\exp\Big\{\frac{-2n\varepsilon^2}{(b-a)^2}\Big\}.$$

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Corresponding author: Yuanyuan LIU, E-mail: liuyy@csu.edu.cn

Hoeffding's inequality has an important influence on the development of probability and statistics. There are lots of applications of this inequality in various areas. For examples, Devroye et al. [8] showed that this inequality plays a key role in the construction of a performance bound, which is valid for any distribution in the error estimation methods. Inspired by the applications of Hoeffding's inequality, several researchers extended this inequality to some special settings such as Markov chains. Glynn and Ormoneit [10] and Boucher [3] established a Hoeffding's inequality for uniformly ergodic discrete-time Markov chains (DTMCs) via Doeblin's condition and the Drazin inverse, respectively. Miasojedow [20] developed a Hoeffding's inequality via the spectral gap for a geometrically ergodic Markov chain. The above literature investigate Hoeffding's inequality for DTMCs under the ergodic assumption, which implies that the chains are aperiodic. For continuous-time Markov processes (CTMPs), there are relatively few researches. Lezaud [14] investigated Hoeffding's inequality for a reversible continuous-time Markov chain (CTMC) on the finite space state. Recently, Choi and Li [7] presented a Hoeffding's inequality for uniformly ergodic diffusion processes.

In this paper, we concentrate on Hoeffding's inequality for both DTMCs and CTMPs on a general state space. In Section 2, we modify the arguments in [10] to derive a Hoeffding's inequality via the solution of Poisson's equation for DTMCs which may be periodic. We further investigate the upper bound in Hoeffding's inequality by bounding the solution of Poisson's equation through the drift condition and the ergodicity coefficient. In Section 3, we establish similar results as that for DTMCs. We first extend the arguments of Choi and Li [7] for a reflective diffusion process to a general CTMP to derive a Hoeffding's inequality via the solution of Poisson's equation. The bounds on the solution of Poisson's equation are obtained in terms of the first hitting times and the drift condition. In Section 4, these results are applied to discrete-time or continuous-time real models such as (i) single-birth processes, (ii) $GI/G/1$-type Markov chains, (iii) Markov-modulated fluid queues, and (iv) reflective diffusion processes.

2 Discrete-time Markov chains

Let $\{X_n : n\in\mathbb{Z}_+\}$ be a ψ-irreducible DTMC on a general state space E, with transition probability kernel P. Suppose that X_n is positive recurrent with invariant probability distribution π. For any real-valued function $g\colon E\to\mathbb{R}$, define

$$\|g\| := \sup\{|g(x)|\colon x\in E\},\quad S_n(g) := \sum_{i=0}^{n-1} g(X_i).$$

For a finite measure μ, let

$$\mu(g) := \int_E g(x)\mu(\mathrm{d}x),\quad \|\mu\| := \sup_{f\colon |f|\leqslant 1} |\mu(f)|.$$

For a finite operator B, define operator norm as $\|B\| := \sup_{\mu:\ \|\mu\|\leqslant 1} \|\mu B\|$. Note that $\|B_1 B_2\| \leqslant \|B_1\|\,\|B_2\|$ for any pair of operator B_1 and B_2. Let

$$\mathbb{E}_i[\,\cdot\,] := \mathbb{E}[\,\cdot \mid X_0 = i], \quad \mathbb{P}_i[\,\cdot\,] := \mathbb{P}[\,\cdot \mid X_0 = i],$$

be the conditional expectation and conditional probability with respect to the initial state $i \in E$, respectively.

In this section, we will derive a Hoeffding's inequality for the chain X_n in terms of the solution of Poisson's equation:

$$(I - P)\breve{g} = \bar{g}, \tag{2}$$

where g is a given function satisfying $\pi(|g|) < \infty$, and $\overline{g} = g - \pi(g)$. The function $\breve{g}$ is called the solution of Poisson's equation (2).

Theorem 1 *Let g be a bounded real-valued function. If Poisson's equation (2) admits a solution $\breve{g}$ such that $\|\breve{g}\| \leqslant c$ for a finite positive constant c, then for any $\varepsilon > 0$ and $n > 2c/\varepsilon$, we have*

$$\mathbb{P}_x\Big(\frac{1}{n}\, S_n(g) - \pi(g) \geqslant \varepsilon\Big) \leqslant \exp\Big\{\frac{-(n\varepsilon - 2c)^2}{8nc^2}\Big\}. \tag{3}$$

Proof Since g is bounded, we have $\pi(|g|) < \infty$. Following the arguments in Glynn and Ormoneit [10], we define the bounded martingale difference sequence

$$D_i := \breve{g}(X_i) - \mathbb{E}_x[\breve{g}(X_i)|X_0, \ldots, X_{i-1}]. \tag{4}$$

From (2) and (4), we have

$$S_n(g) - n\pi(g) = \sum_{i=1}^{n} (I - P)\breve{g}(X_{i-1}) = \sum_{i=1}^{n} D_i + \breve{g}(X_0) - \breve{g}(X_n). \tag{5}$$

Since $\|\breve{g}\| \leqslant c$, $|D_i| \leqslant 2c$ for any i, $1 \leqslant i \leqslant n$. Using (5), Markov's inequality, and [8, Lemma 8.1], we obtain that for any $\theta > 0$,

$$\begin{aligned}\mathbb{P}_x\Big(\frac{1}{n}\, S_n(g) - \pi(g) \geqslant \varepsilon\Big) &\leqslant \exp\{2\theta\|\breve{g}\| - \varepsilon n\theta\}\mathbb{E}_x\Big[\exp\Big\{\theta \sum_{i=1}^{n} D_i\Big\}\Big]\\ &\leqslant \exp\{-n\theta\varepsilon + 2\theta c + 2n\theta^2 c^2\}.\end{aligned} \tag{6}$$

Equation (6) is minimized at

$$\theta = \frac{n\varepsilon - 2c}{4nc^2}. \tag{7}$$

Substituting (7) into (6) yields the assertion immediately. □

Remark 1 Hoeffding's inequality had been established by Glynn and Ormoneit [10] and Boucher [3]. Their results require that the DTMCs are uniformly ergodic, which, however, fail for periodic Markov chains. Our analysis, which starts from the solution of Poisson's equation, shows that this restriction can be removed.

To apply Theorem 1, it is crucial for us to establish the bound on the solution of Poisson's equation. In the following, we will investigate this bound from two different aspects.

Let $\mathscr{B}(E)$ be the Borel σ-field of E. A set C is called a υ_m-small set, if there is a nontrivial measure $\upsilon_m(B)$ on $\mathscr{B}(E)$ such that

$$P^m(i,B) \geqslant \upsilon_m(B), \quad \forall\, i \in C, \forall\, B \in \mathscr{B}(E).$$

Moreover, a small set α is called an atom, if there exists a measure υ on $\mathscr{B}(E)$ such that $P(i,B) = \upsilon(B)$ for any $i \in \alpha$ and $B \in \mathscr{B}(E)$. Define $\tau_C := \inf\{i \geqslant 1\colon X_i \in C\}$ to be the first return time on C.

1. Drift condition First, we present the following drift condition.

D1(*V*, *b*, *C*) Suppose that there exists a set C, a positive constant $b < \infty$, and a non-negative bounded function V, such that

$$PV(x) \leqslant V(x) - 1 + b\mathbb{I}_C(x), \quad x \in E,$$

where $\mathbb{I}_C$ is the indicator function with respect to the set C.

Assumption 1 Let $C \in \mathscr{B}(E)$, and let $\alpha \subseteq C$ be an atom. Assume that C and α satisfy one of the following relations:

(i) the set $C = \alpha$ is an atom;

(ii) the set C is a υ_m-small set and $\upsilon_m(\alpha) > 0$;

(iii) there exist some $N < \infty$ and $\gamma_C > 0$ such that $\sum_{j=1}^N P^j(x,\alpha) \geqslant \gamma_C$ for any $x \in C$.

Proposition 1 *Suppose that Assumption* 1 *and* D1(V, b, C) *hold. Then, for any bounded real-valued function* g, *Poisson's equation* (2) *admits a solution* $\breve{g}$ *such that*

$$\|\breve{g}\| \leqslant (\|g\| + |\pi(g)|)(\|V\| + d) \leqslant 2\|g\|(\|V\| + d),$$

where the constant d *is specified as follows*:

$$d = \begin{cases} 0, & \textit{if (i) of Assumption 1 holds,} \\ \dfrac{bm}{\upsilon_m(\alpha)}, & \textit{if (ii) of Assumption 1 holds,} \\ \dfrac{bN(N+1)}{2\delta_C}, & \textit{if (iii) of Assumption 1 holds.} \end{cases}$$

Proof By Glynn and Meyn [9], a solution of Poisson's equation (2) is given by

$$\breve{g}(x) = E_x\Big[\sum_{k=0}^{\tau_\alpha - 1} \bar{g}(X_k)\Big], \quad x \in E.$$

Applying [16, Proposition 1] yields the result directly. □

2. Ergodicity coefficient Now, we adopt the norm ergodicity coefficient to derive the bounds of the solution of Poisson's equation instead of the drift condition. The classical ergodicity coefficient (see, e.g., Seneta [22]) is defined by

$$\tau(P) = \sup_{x,y\in E} \frac{\|P(x,\cdot) - P(y,\cdot)\|}{2}.$$

Two basic properties about the ergodicity coefficient will be used later:

(i) for stochastic kernels P_1 and P_2,

$$\tau(P_1P_2) \leqslant \tau(P_1)\tau(P_2); \tag{8}$$

(ii) for stochastic kernel P and (possible negative) kernel R such that $R(x,E) = 0$,

$$\|RP\| \leqslant \|R\|\tau(P). \tag{9}$$

Proposition 2 *Let g be a bounded real-valued function.*

(i) *If there exists an integer $m \geqslant 1$ such that $\tau(P^m) \leqslant \rho_m < 1$ for some positive constant ρ_m, then Poisson's equation* (2) *admits a solution $\breve{g}$ such that*

$$\|\breve{g}\| \leqslant \frac{2m\|g\|}{1-\rho_m}.$$

(ii) *If the state space E is υ_m-small for some positive integer m and some nontrivial measure υ_m, then Poisson's equation* (2) *admits a solution $\breve{g}$ such that*

$$\|\breve{g}\| \leqslant \frac{2m\|g\|}{\upsilon_m(E)}.$$

Proof (i) Since $\tau(P^m) \leqslant \rho_m < 1$, we know that the chain P^m is aperiodic. Now, we consider the deviation operator for the transition kernel P^m, defined by

$$\widetilde{D} := \sum_{t=0}^{\infty} (P^{mt} - \Pi),$$

where Π is the stationary operator satisfying $\Pi(x,B) = \pi(B)$ for any $x \in E$ and $B \in \mathscr{B}(E)$. It follows from the basic property of deviation matrix that

$$(I - P^m)\widetilde{D} = I - \Pi,$$

from which, we know that $\breve{g} = (\sum_{n=0}^{m-1} P^n)\widetilde{D}g$ is a solution of Poisson's equation (2) since

$$I - P^m = (I-P)\sum_{n=0}^{m-1} P^n.$$

Furthermore, by (8) and (9), we obtain

$$\begin{aligned}\|\check{g}\| &\leqslant \|g\|\sum_{n=0}^{m-1}\|P^n\|\sum_{t=0}^{\infty}\|P^{mt}-\Pi\| \\ &= m\|g\|\sum_{t=0}^{\infty}\|(I-\Pi)P^{mt}\| \\ &\leqslant m\|g\|\,\|I-\Pi\|\sum_{t=0}^{\infty}\tau(P^{mt}) \\ &\leqslant 2m\|g\|\sum_{t=0}^{\infty}(\tau(P^m))^t \\ &\leqslant \frac{2m\|g\|}{1-\rho_m}.\end{aligned}$$

(ii) Since the state space E is υ_m-small, for any $x\in E$ and $A\in\mathscr{B}(E)$, we have

$$P^m(x,A)\geqslant \upsilon_m(A).$$

Hence, for any fixed x and A, we can find a nonnegative real number $d(x,A)$ such that

$$P^m(x,A)=\upsilon_m(A)+d(x,A),$$

from which we obtain

$$\begin{aligned}\tau(P^m) &= \frac{1}{2}\sup_{x,y\in E}\|P^m(x,\cdot)-P^m(y,\cdot)\| \\ &= \frac{1}{2}\sup_{x,y\in E}\sup_{f\colon |f|\leqslant 1}\int_E [P^m(x,\mathrm{d}z)-P^m(y,\mathrm{d}z)]f(z) \\ &\leqslant \frac{1}{2}\sup_{f\colon |f|\leqslant 1}\sup_{x,y\in E}\int_E [d(x,\mathrm{d}z)+d(y,\mathrm{d}z)]f(z) \\ &= \sup_{f\colon |f|\leqslant 1}\sup_{x\in E}\int_E d(x,\mathrm{d}z)f(z) \\ &= 1-\upsilon_m(E).\end{aligned}$$

Thus, we finish the proof. □

3 Continuous-time Markov processes

Let $\{X_t\colon t\in\mathbb{R}_+\}$ be a ψ-irreducible CTMP on a general state space E with transition kernel P^t. Suppose that X_t is positive recurrent with invariant probability distribution π. Furthermore, we assume that X_t is a Borel right process (see, e.g., Sharpe [23]) and is non-explosive, i.e., the escape time is infinite.

Denote by $D(\widetilde{\mathscr{A}}\,)$ the set of all functions $V\colon E \times \mathbb{R}_+ \to \mathbb{R}$ for which there exists a measurable function $U\colon E \to \mathbb{R}$ such that, for each $i \in E$, $t > 0$,

$$P^t V(i) = V(i) + \int_0^t P^s U(i)\mathrm{d}s, \quad \int_0^t P^s |U|(i)\mathrm{d}s < \infty.$$

We write $\widetilde{\mathscr{A}}\,V := U$ and call $\widetilde{\mathscr{A}}$ the extended generator of the process X_t .

We now show how the discrete-time results can be extended to analogue results for CTMPs. For a real-valued function g, let

$$S_t(g) := \int_0^t g(X_s)\mathrm{d}s.$$

For a CTMP, Poisson's equation (see Glynn et al. [9]) is defined by

$$-\widetilde{\mathscr{A}}\,\breve{g} = \bar{g}. \tag{10}$$

Choi and Li [7] established Hoeffding's inequality for uniformly ergodic one-dimensional reflective diffusion processes. Actually, as shown in the following, their arguments can be extended to general Markov processes, which are analogous to the results for DTMCs.

Theorem 2 *Let g be a bounded real-valued function. If Poisson's equation* (10) *admits a solution $\breve{g}$ such that $\|\breve{g}\| \leqslant c$ for some finite positive constant c, then for any $\varepsilon > 0$ and $t > 2c/\varepsilon$, we have*

$$\mathbb{P}_x\Big(\frac{1}{t}\,S_t(g) - \pi(g) \geqslant \varepsilon\Big) \leqslant \exp\Big\{\frac{-(t\varepsilon - 2c)^2}{2(t+1)(2c + \|g\| + |\pi(g)|)^2}\Big\}. \tag{11}$$

Proof By the assumption that g is bounded, we have $\pi(|g|) < \infty$. Following the arguments of Choi and Li [7], we construct a bounded martingale sequence through Poisson's equation (10) as follows:

$$M_t := \breve{g}(X_t) - \breve{g}(X_0) - \int_0^t \widetilde{\mathscr{A}}\,\breve{g}(X_s)\mathrm{d}s. \tag{12}$$

M_t can be rewritten as

$$M_t = \sum_{s=1}^{\lfloor t \rfloor}(M_s - M_{s-1}) + M_t - M_{\lfloor t \rfloor}.$$

Since $\|\breve{g}\| \leqslant c$, by (10) and (12), we know that $(M_s - M_{s-1})$ and $(M_t - M_{\lfloor t \rfloor})$ lie almost surely in an interval of length $2(2c + \|g\| + |\pi(g)|)$.

Using Markov's inequality, (10), and (12), for any $\theta > 0$, we have

$$\mathbb{P}_x\Big(\frac{1}{t}\,S_t(g) - \pi(g) \geqslant \varepsilon\Big) \leqslant \exp\{2\theta\|\breve{g}\| - \varepsilon t\theta\}\mathbb{E}_x\left[\exp\{\theta M_t\}\right]. \tag{13}$$

Moreover, from (13) and [8, Lemma 8.1], we obtain

$$\mathbb{P}_x\Big(\frac{1}{t}\,S_t(g)-\pi(g) \geqslant \varepsilon\Big) \leqslant \exp\Big\{\frac{(t+1)(2c+\|g\|+|\pi(g)|)^2\theta^2}{2}-\varepsilon t\theta+2c\theta\Big\}. \quad (14)$$

The right-hand side of (14) is minimized at

$$\theta = \frac{(t\varepsilon - 2c)}{(t+1)(2c+\|g\|+|\pi(g)|)^2}. \quad (15)$$

Substituting (15) into (14) yields the assertion immediately. □

Remark 2 (i) A ψ-irreducible CTMP X_t is called aperiodic, if for some small set $C \in \mathscr{B}(E)$, there exists a constant $T > 0$ such that $P^t(x, C) > 0$ for all $t \geqslant T$ and all $x \in C$. Like for DTMCs, Theorem 2 shows that the aperiodicity is not required for a Hoeffding's inequality to hold for CTMPs.

(ii) The essential arguments of Theorem 2 are taken from Choi and Li [7]. We mention that there is a small error with the presentation of Hoeffding's inequality in [7, Theorem 1.1], where the term $|\pi(g)|$ is missing in the upper bound of the deviation probability.

We now establish bounds on the solution of Poisson's equation for the CTMP. Since the ergodicity coefficient cannot be modified directly, we focus on deriving the bound in terms of the first hitting times and the drift condition.

For continuous-time process X_t, a non-empty set $C \in E$ is called a υ_a-petite set, if there exists a probability measure a on $\mathbb{R}_+$ and a nontrivial measure υ_a on $\mathscr{B}(E)$ such that

$$K_a(i, B) := \int_0^\infty P^t(i, B)a(\mathrm{d}t) \geqslant \upsilon_a(B), \quad \forall\, i \in C, \forall\, B \in \mathscr{B}(\mathbb{E}).$$

A non-empty set C is called a υ_r-small set, if there exists a constant $r > 0$ and a nontrivial measure υ_r on $\mathscr{B}(E)$ such that

$$P^r(i, B) \geqslant \upsilon_r(B), \quad \forall\, i \in C, \forall\, B \in \mathscr{B}(\mathbb{E}).$$

Moreover, a small set α is called an atom, if there exists a measure υ on $\mathscr{B}(E)$ such that $P^t(i, B) = \upsilon(B)$ for any $i \in \alpha$, $t > 0$, and $B \in \mathscr{B}(E)$. For any $\eta > 0$, define

$$\sigma_C(\eta) := \inf\{t \geqslant \eta\colon X_t \in C\}.$$

We denote $\sigma_C(0)$ by σ_C simply.

We now present the continuous-time analogue to the drift condition in Section 2.

$\boldsymbol{\mathrm{D1}'(V, b, C)}$ Suppose that there exists a set C, a positive constant $b < \infty$, and a non-negative bounded function $V \geqslant 1$, such that

$$\widetilde{\mathscr{A}}\, V(x) \leqslant -1 + b\mathbb{I}_C(x), \quad x \in E. \quad (16)$$

Lemma 1 (Comparison theorem) *Suppose that X_t is ψ-irreducible (not necessarily positive recurrent). If there exist nonnegative functions V, f, and s such that*

$$\widetilde{\mathscr{A}}\, V(x) \leqslant -f(x) + s(x)$$

for each $x \in E$, then for any stopping time τ,

$$\mathbb{E}_x\bigg[\int_0^\tau f(X_t)\mathrm{d}t\bigg] \leqslant V(x) + \mathbb{E}_x\bigg[\int_0^\tau s(X_t)\mathrm{d}t\bigg].$$

Proof The proof is similar to [19, Lemma B.1], which is omitted here to save space. □

Assumption 2 Let $C \in \mathscr{B}(E)$, and let $\alpha \subseteq C$ be an atom. Assume that C and α satisfy one of the following relations:

(i) the set $C = \alpha$ is an atom;

(ii) the set C is a υ_t-small set and $\upsilon_t(\alpha) > 0$.

Theorem 3 *Let g be a bounded real-valued function.*

(i) *If $\sup_{x\in E}\mathbb{E}_x[\sigma_\alpha] < \infty$, then Poisson's equation* (10) *admits a solution $\breve{g}$ such that*

$$\|\breve{g}\| \leqslant (\|g\| + |\pi(g)|)\big(\eta + \sup_{x\in E}\mathbb{E}_x[\sigma_\alpha]\big),$$

and Hoeffding's inequality (11) *holds for*

$$c = (\|g\| + |\pi(g)|)\sup_{i\in E}\mathbb{E}_i[\sigma_\alpha].$$

(ii) *If Assumption* 2 *and* D1$'(V, b, C)$ *hold, then Poisson's equation* (10) *admits a solution $\breve{g}$ such that*

$$\|\breve{g}\| \leqslant (\|g\| + |\pi(g)|)(\|V\| + d(\eta)),$$

where the constant $d(\eta)$ is specified as follows:

$$d(\eta) = \begin{cases} b\eta, & \text{if (i) of Assumption 2 holds,} \\ \dfrac{b(r+\eta)}{\upsilon_r(\alpha)}, & \text{if (ii) of Assumption 2 holds,} \end{cases}$$

and Hoeffding's inequality (11) *holds for*

$$c = (\|g\| + |\pi(g)|)(\|V\| + d(0)).$$

Proof For any $x \in E$ and any $\eta > 0$, let

$$\breve{g}(x) = \mathbb{E}_x\bigg[\int_0^{\sigma_\alpha(\eta)} \bar{g}(X_t)\mathrm{d}t\bigg] = \mathbb{E}_x\bigg[\int_0^{\sigma_\alpha(\eta)} g(X_t)\mathrm{d}t\bigg] - \pi(g)\mathbb{E}_x[\sigma_\alpha(\eta)].$$

According to [2, Theorem 3.1], we know that $\breve{g}(x)$ is a solution of Poisson's equation (10).

First, we show assertion (i). Since $\|g\| < \infty$, we have $\pi(|g|) < \infty$ and

$$|\breve{g}(x)| \leqslant (\|g\| + |\pi(g)|)\mathbb{E}_x[\sigma_\alpha(\eta)]. \tag{17}$$

Since $\sup_{x\in E} \mathbb{E}_x[\sigma_\alpha] < \infty$, we obtain

$$\begin{aligned}|\breve{g}(x)| &\leqslant (\|g\| + |\pi(g)|)\Big(\eta + \int_E P^\eta(x, \mathrm{d}y)\mathbb{E}_y[\sigma_\alpha]\Big)\\ &\leqslant (\|g\| + |\pi(g)|)\big(\eta + \sup_{x\in E} \mathbb{E}_x[\sigma_\alpha]\big).\end{aligned}$$

By letting $\eta \downarrow 0$, we derive the first assertion.

Then we show assertion (ii). Since $\sigma_\alpha(\eta)$ is a stopping time, from Lemma 1, we obtain

$$\mathbb{E}_x[\sigma_\alpha(\eta)] \leqslant V(x) + b\mathbb{E}_x\Big[\int_0^{\sigma_\alpha(\eta)} \mathbb{I}_C(X_t)\mathrm{d}t\Big].$$

Then, from (17), we derive that

$$|\breve{g}(x)| \leqslant (\|g\| + |\pi(g)|)\bigg(V(x) + b\mathbb{E}_x\Big[\int_0^{\sigma_\alpha(\eta)} \mathbb{I}_C(X_t)\mathrm{d}t\Big]\bigg). \tag{18}$$

First, we suppose that Assumption 3.1 (i) holds, i.e., $C = \alpha$. If $x \notin \alpha$, then

$$\begin{aligned}\mathbb{E}_x\Big[\int_0^{\sigma_\alpha(\eta)} \mathbb{I}_\alpha(X_t)\mathrm{d}t\Big] &\leqslant \mathbb{E}_x\Big[\int_0^{\eta} \mathbb{I}_\alpha(X_t)\mathrm{d}t\Big] + \int_E P^\eta(x, \mathrm{d}y)\mathbb{E}_y\Big[\int_0^{\sigma_\alpha} \mathbb{I}_\alpha(X_t)\mathrm{d}t\Big]\\ &\leqslant \eta.\end{aligned} \tag{19}$$

For $x \in \alpha$, from Asmussen [2], we have

$$\breve{g}(x) = \mathbb{E}_\alpha\Big[\int_0^{\sigma_\alpha(\eta)} g(X_t)\mathrm{d}t\Big] - \pi(g)\mathbb{E}_\alpha[\sigma_\alpha(\eta)] = 0. \tag{20}$$

From (18)–(20), we obtain

$$|\breve{g}(x)| \leqslant (\|g\| + |\pi(g)|)(V(x) + b\eta). \tag{21}$$

Now, we consider the case where C is a υ_r-small set. It is easy to check that a υ_r-small set is a δ_r-petite set, where δ_r is the Dirac probability measure defined by

$$\delta_r(A) = \begin{cases} 1, & \text{if } r \in A,\\ 0, & \text{if } r \notin A,\end{cases} \tag{22}$$

for any $A \in \mathscr{B}(\mathbb{R}_+)$.

Then, by the strong Markov property and (22), for any $x \in E$, we have

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^{\sigma_\alpha(\eta)} \mathbb{I}_C(X_t)\mathrm{d}t\right] \\
&\leqslant \mathbb{E}_x\left[\int_0^{\sigma_\alpha(\eta)} \nu_{\delta_r}^{-1}(\alpha)K_{\delta_r}(X_t,\alpha)\mathrm{d}t\right] \\
&= \nu_r^{-1}(\alpha)\mathbb{E}_x\left[\int_0^{\sigma_\alpha(\eta)}\int_0^\infty P^s(X_t,\alpha)\delta_r(\mathrm{d}s)\mathrm{d}t\right] \\
&= \nu_r^{-1}(\alpha)\int_0^\infty \mathbb{E}_x\left[\int_0^{\sigma_\alpha(\eta)} P^s(X_t,\alpha)\mathrm{d}t\right]\delta_r(\mathrm{d}s) \\
&\leqslant \nu_r^{-1}(\alpha)\int_0^\infty \left(\eta + \int_E P^\eta(x,\mathrm{d}y)\mathbb{E}_y\left[\int_0^{\sigma_\alpha} P^s(X_t,\alpha)\mathrm{d}t\right]\right)\delta_r(\mathrm{d}s) \\
&= \nu_r^{-1}(\alpha)\left(\eta + \int_0^\infty\int_E P^\eta(x,\mathrm{d}y)\mathbb{E}_y\left[\int_0^{\sigma_\alpha} \mathbb{I}_\alpha(X_{t+s})\mathrm{d}t\right]\delta_r(\mathrm{d}s)\right) \\
&\leqslant \nu_r^{-1}(\alpha)\left(\eta + \int_0^\infty\int_E P^\eta(x,\mathrm{d}y)s\delta_r(\mathrm{d}s)\right) \\
&= \nu_r^{-1}(\alpha)(\eta + r),
\end{aligned} \tag{23}$$

where we have used

$$\int_0^\infty f(s)\delta_r(\mathrm{d}s) = f(r)$$

to derive the last equality.

From (18) and (23), we obtain

$$|\breve{g}(x)| \leqslant (\|g\| + |\pi(g)|)\left(V(x) + \frac{b(\eta + r)}{\nu_r(\alpha)}\right). \tag{24}$$

Since Hoeffding's inequality (11) is independent of η, letting $\eta \downarrow 0$ in both (21) and (24) derives the remainder of the assertion directly. □

4 Applications

In this section, we apply our results to several processes.

4.1 Discrete-time single-birth processes

The discrete-time single-birth process (see, e.g., [4,5]) has a special transition matrix $P = (p_{ij})_{i,j\in\mathbb{Z}_+}$ with $p_{i,i+1} > 0$ and $p_{i,i+k} = 0$ for all $i \geqslant 0$ and $k \geqslant 2$. Here, we consider two particular cases in order to investigate Hoeffding's inequality. First, we consider a finite state single-birth process on the state space $E_n = \{0, 1, \ldots, n\}$, with the irreducible and stochastic transition matrix

$$P=\begin{pmatrix} p_{00} & p_{01} & & & & \\ p_{10} & p_{11} & p_{12} & & \mathbf{0} & \\ p_{20} & p_{21} & p_{22} & \ddots & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ p_{n-1,0} & p_{n-1,1} & p_{n-1,2} & \cdots & p_{n-1,n-1} & p_{n-1,n} \\ p_{n0} & p_{n1} & p_{n2} & \cdots & p_{n,n-1} & p_{nn} \end{pmatrix}.$$

In order to solve Poisson's equation, we need the following notations. Define

$$p_m^{(k)}=\sum_{j=0}^{k}p_{mj},\quad 0\leqslant k<m,$$

and

$$F_i^{(i)}=1,\quad F_m^{(i)}=\frac{1}{p_{m,m+1}}\sum_{k=i}^{m-1}p_m^{(k)}F_k^{(i)},\ 0\leqslant i<m.$$

Similar to [12, Theorem 2.1], for any bounded real-valued function g, Poisson's equation (2) admits a unique solution $\breve{g}$ with a fixed $j\in E_n$, given by

$$\breve{g}(i)=\begin{cases}\displaystyle\sum_{m=i}^{j-1}\sum_{k=0}^{m}\frac{F_m^{(k)}\overline{g}(k)}{p_{k,k+1}}, & \text{if } i<j\leqslant n,\\ 0, & \text{if } i=j\leqslant n,\\ \displaystyle-\sum_{m=j}^{i-1}\sum_{k=0}^{m}\frac{F_m^{(k)}\overline{g}(k)}{p_{k,k+1}}, & \text{if } j<i\leqslant n.\end{cases}$$

Let $j=0$. Then it follows from Theorem 1 that Hoeffding's inequality (3) holds with

$$c=(\|g\|+|\pi(g)|)\sum_{m=0}^{n-1}\sum_{k=0}^{m}\frac{F_m^{(k)}}{p_{k,k+1}}.$$

Note that if $p_{ij}=0$ for all $i>j+1$ or $i<j-1$, and $p_{ii}=0$ for any $i\in E$, then the single-birth process becomes a periodic birth-and-death process with periodicity $d=2$.

Now, we consider another single-birth process (see, Liu [15]) on $E=\mathbb{Z}_+$ with the following transition matrix elements:

$$p_{ij}=\begin{cases}1, & \text{if } j=1,\ i=0,\\ q, & \text{if } j=i+1,\ i\geqslant 1,\\ p, & \text{if } j=0,\ i \text{ odd or } j=1,\ i\geqslant 1,\ i \text{ even},\\ 0, & \text{otherwise},\end{cases}\tag{25}$$

where p and q are positive numbers such that $p + q = 1$. Clearly, (25) is irreducible and periodic with periodicity $d = 2$. Let

$$V(0) = 0, \quad V(2i-1) = \frac{2-p}{p}, \quad V(2i) = \frac{2}{p}, \quad i \geqslant 1.$$

It is easy to check that

$$\sum_{j=0}^{\infty} p_{ij} V(j) \leqslant V(i) - 1 + \frac{2}{p}\,\mathbb{I}_{\{0\}}(i), \quad i \in E.$$

Then for any bounded real-valued function g, it follows from Proposition 1 that Hoeffding's inequality (3) holds with $c = 2(\|g\| + |\pi(g)|)/p$.

4.2 GI/G/1-type Markov chain

Consider the DTMC on $E = \mathbb{Z}_+$ with the following transition matrix:

$$P = \begin{pmatrix} b_0 & a_0 & a_1 & a_2 & \cdots \\ b_{-1} & a_{-1} & a_0 & a_1 & \cdots \\ b_{-2} & a_{-2} & a_{-1} & a_0 & \cdots \\ b_{-3} & a_{-3} & a_{-2} & a_{-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{26}$$

Suppose that (26) is stochastic and irreducible such that $\sum_{k=-\infty}^{\infty} a_k < 1$. When $a_k = 0$ for $k \geqslant 1$, the transition matrix is of the lower-Hessenberg form, which contains the classical embedded GI/M/1 queue as special cases.

Since $\sum_{k=-\infty}^{\infty} a_k < 1$, we have

$$\inf_{i\in E} p_{i0} \geqslant 1 - \sum_{k=-\infty}^{\infty} a_k > 0.$$

For any bounded real-valued function g, from the proof of [15, Theorem 2.1] and Proposition 2 (ii), we obtain

$$\|\check{g}\| \leqslant \frac{2\|g\|}{1 - \sum_{k=-\infty}^{\infty} a_k}.$$

Thus, according to Theorem 1, Hoeffding's inequality (3) holds with $c = 2\|g\|/(1 - \sum_{k=-\infty}^{\infty} a_k)$.

4.3 A Markov modulated fluid queue

Let $\{J_t\colon t \geqslant 0\}$ be an irreducible CTMC on a countable state space $E = \mathbb{Z}_+$, with the totally stable and regular q-matrix $Q = (q_{ij})_{i,j\in E}$. A fluid queue (V_t, J_t) (see, e.g., [17,21]) is a two-dimensional CTMP, where the first component V_t, called the level, takes values continuously and represents the content of the fluid buffer at time t, and the second component J_t, called the phase, takes discrete values and corresponds to the state of an underling CTMC at time t. The level is controlled by the phase in the following way:

$$V_t := \int_0^t g(J_s)\mathrm{d}s,$$

where g, called the rate function, is a real-valued function on $\mathbb{Z}_+$. Fluid flows have been widely used for modelling information flows in performance analysis of packet telecommunication systems.

When Q is finite, J_t is automatically ergodic with invariant distribution π and Poisson's equation $Q\breve{g} = -\bar{g}$ admits a unique solution $\breve{g} = Dg$, where D is the deviation matrix of J_t. In this case, $\breve{g}$ can be calculated explicitly. When Q is infinite, the case is more complicated. Here we consider two specifical cases.

First, consider a fluid queue driven by a infinitely countable continuous-time birth-death process J_t, whose q-matrix Q is given by

$$q_{i,i+1} = b_i,\ i \in \mathbb{Z}_+; \quad q_{i,i-1} = a_i,\ i \in \mathbb{N}; \quad q_{ij} = 0,\ |i-j| \geqslant 2,$$

where $a_i > 0$ for $i \in \mathbb{N}$ and $b_i > 0$ for $i \in \mathbb{Z}_+$. It is well known that Q is regular if and only if Q is conservative and

$$R := \sum_{n=1}^{\infty} \Big(\frac{1}{b_n} + \frac{a_n}{b_n b_{n-1}} + \cdots + \frac{a_n \cdots a_2}{b_n \cdots b_1}\Big) = \infty.$$

From [1, Chapter 8], we know that

$$\mathbb{E}_i[\tau_0] = \sum_{k=0}^{i-1} \bigg(\frac{1}{a_{k+1}} + \sum_{j=k+1}^{\infty} \frac{b_{k+1} \cdots b_j}{a_{k+1} \cdots a_{j+1}}\bigg), \quad i \geqslant 1.$$

If $R = \infty$ and

$$S := \sum_{k=0}^{\infty} \bigg(\frac{1}{a_{k+1}} + \sum_{j=k+1}^{\infty} \frac{b_{k+1} \cdots b_j}{a_{k+1} \cdots a_{j+1}}\bigg) < \infty,$$

then for any bounded real-valued function g, it follows from (i) of Theorem 3 that Hoeffding's inequality (11) holds with $c = (\|g\| + |\pi(g)|)S$.

Now, consider the fluid queue driven by a generalized Markov branching processes, i.e., J_t is of the following q-matrix Q:

$$q_{ij} = \begin{cases} q_{ij}, & \text{if } j > i = 0, \\ -\sum_{k=1}^{\infty} q_{0k}, & \text{if } j = i = 0, \\ r_i p_{j-i+1}, & \text{if } j \geqslant i-1 \geqslant 0,\, j \neq i, \\ -r_i(1-p_1), & \text{if } j = i \geqslant 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $r_i \geqslant 0$. This extended model was first introduced and investigated by Chen [6]. Assume that

$$M_1 := \sum_{k=1}^{\infty} kp_k \leqslant 1, \quad \sum_{i=1}^{\infty} \frac{1}{r_i} < \infty,$$

where $\{p_k \colon k \geqslant 0\}$ forms a probability distribution on $\mathbb{Z}_+$. Construct a function V as follows:

$$V(0) = 0, \quad V(n) = \frac{1}{p_0 - \Gamma} \sum_{i=1}^{n} \frac{1}{r_i}, \ n \geqslant 1,$$

where $\Gamma = \sum_{k=1}^{\infty} kp_{k+1}$. Then by Liu et al. [18], we know that V satisfies $\mathrm{D1}'(V, b, C)$ with $b = -q_{00} \sum_{i=1}^{\infty} \frac{1}{r_i} < \infty$ and $C = \{0\}$. Then for any bounded real-valued function g, it follows from (ii) of Theorem 3 that Hoeffding's inequality (11) holds with $c = (\|g\| + |\pi(g)|) \frac{1}{p_0 - \Gamma} \sum_{i=1}^{\infty} \frac{1}{r_i}$.

4.4 A reflective diffusion process

Let X_t be a one-dimensional reflective diffusion process on $\mathbb{R}_+$ with 0 as the reflecting barrier. The diffusion operator is given by

$$L = \mu(x) \frac{\partial L}{\partial x} + \frac{1}{2} \sigma^2(x) \frac{\partial^2 L}{\partial x^2}.$$

The parameters $\mu(x)$ and $\sigma^2(x)$ are continuous functions of x and $\sigma^2(x) > 0$, which are said to be the drift and diffusion coefficients of X_t, respectively.

For any $x \in [0, \infty)$, let

$$s(x) = \exp\bigg\{ -\int_0^x \frac{2\mu(t)}{\sigma^2(t)}\, \mathrm{d}t \bigg\}, \quad m(x) = \frac{1}{\sigma^2(x) s(x)},$$

be the scale density and the speed density, respectively. Correspondingly, define $S(x) = \int_0^x s(t) \mathrm{d}t$ and $M(x) = \int_0^x m(t) \mathrm{d}t$. Suppose that X_t is non-explosive:

$$\int_0^{\infty} M(x) dS(x) = \int_0^{\infty} s(x) \mathrm{d}x \int_0^x m(y) \mathrm{d}y = \infty.$$

Let $S_x = 2 \int_0^x s(\eta) [\int_\eta^\infty m(\xi) \mathrm{d}\xi] \mathrm{d}\eta$, $x \geqslant 0$. It follows from [13, Chapter 15] that

$$E_x[\sigma_0] = S_x, \quad x \geqslant 0.$$

Applying (i) of Theorem 3 yields that if $S_\infty < \infty$, then for any bounded real-valued function g, Hoeffding's inequality (11) holds with $c = (\|g\| + |\pi(g)|) S_\infty$.

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