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RESEARCH ARTICLE

Generalized $P(N)$-graded Lie superalgebras

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Abstract We generalize the $P(N)$-graded Lie superalgebras of Martinez-Zelmanov. This generalization is not so restrictive but sufficient enough so that we are able to have a classification for this generalized $P(N)$-graded Lie superalgebras. Our result is that the generalized $P(N)$-graded Lie superalgebra L is centrally isogenous to a matrix Lie superalgebra coordinated by an associative superalgebra with a super-involution. Moreover, L is $P(N)$-graded if and only if the coordinate algebra R is commutative and the super-involution is trivial. This recovers Martinez-Zelmanov's theorem for type $P(N)$. We also obtain a generalization of Kac's coordinatization via Tits-Kantor-Koecher construction. Actually, the motivation of this generalization comes from the Fermionic-Bosonic module construction.

Keywords Root system graded Lie superalgebras, associative superalgebra, quantum tori, representations

MSC2020 17B65, 17B67, 17B70

1 Introduction

Root graded Lie algebras initiated by Berman-Moody [6] played important roles in classification of extended affine Lie algebras. Thereafter, Benkart-Elduque [1–3] introduced a super-analog of the root graded Lie algebras and classified Lie superalgebras graded by the root systems of types

$$A(m;n),\quad B(m;n),\quad C(n),\quad D(m;n),\quad D(2;1;\alpha),\quad F(4),\quad G(3).$$

Martinez-Zelmanov [18] introduced and classified Lie superalgebras graded by the root systems of types $P(N)$ and $Q(N)$. Then Martinez-Shestakov-Zelmanov studied the Jordan bimodules over the superalgebras $P(N)$ and $Q(N)$. The novelty of the root system of type $P(N)$ is that there are some 'odd' negative roots without matching positive roots (so $P(N)$ has no non-degenerate invariant

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form). Martinez-Zelmanov's theorem shows that any $P(N)$-graded Lie superalgebra is isomorphic to $P(N)\otimes R$, where R is an associative commutative superalgebra. In this paper, we generalize the definition of Lie superalgebras graded by $P(N)$ by adding the 'missing' matching positive roots. This generalization is not so restrictive but sufficient enough so that we are able to have a classification for this generalized $P(N)$-graded Lie superalgebras (GPLS). Our result is that the generalized $P(N)$-graded Lie superalgebra is centrally isogenous to a matrix Lie superalgebra coordinated by an associative superalgebra with a super-involution. Moreover, the 'missing' matching positive root spaces vanish if and only if the coordinate algebra R is commutative and the super-involution is trivial. This recovers Martinez-Zelmanov's theorem for type $P(N)$. Our approach is inspired by the work of Benkart-Elduque and Martinez-Zelmanov. Actually, the motivation of this generalization comes from the Fermionic-Bosonic construction in which we do get a Lie superalgebra whose root system is close to $P(N)$ except that the 'missing' matching positive root spaces are not zero. This Lie superalgebra is a matrix Lie superalgebra coordinated by a quantum torus with a super-involution.

Following [3,4,19], and using the connection between root systems graded Lie superalgebras and their associated Jordan supersystems, we classified GPLS up to centrally isogeny.

Theorem 1.1 (i) *For any unital associative* $\mathbb{F}$*-superalgebra* A *with an anti-superinvolution* $^-$, *if* $N \geqslant 3$, *then any Lie superalgebra centrally isogenous to* $\mathscr{P}_N(A,-)$ *is generalized* $P(N-1)$*-graded.*

(ii) *Conversely, let* L *be a generalized* $P(N-1)$*-graded Lie superalgebra over the characteristic zero field* $\mathbb{F}$. *If* $N \geqslant 4$, *then there exists a unique* (*up to isomorphism*) *unital associative* $\mathbb{F}$*-superalgebra* A *which is equipped with an anti-superinvolution* $^-$ *such that* L *is centrally isogenous to the matrix Lie superalgebra* $\mathscr{P}_N(A,-)$. (*See Section* 2 *for the definition of* $\mathscr{P}_N(A,-)$.)

Theorem 1.2 *Suppose* $N \geqslant 3$. *Then the GPLS* $\mathscr{P}_N(A,-)$ *is a* $P(N-1)$*-graded Lie superalgebra if and only if the unital associative* $\mathbb{F}$*-superalgebra* $A = A_{\overline{0}} + A_{\overline{1}}$ *is supercommutative and*

$$\overline{a} = (-1)^{|a|}a, \quad \forall\, a \in A_{\overline{0}} \cup A_{\overline{1}}.$$

Then as a special case, we get the classification of $P(N-1)$-graded Lie superalgebras, and it is isomorphic to $P(N-1)\otimes_{\mathbb{F}} A$, which was given in [19].

Corollary *Suppose* $N \geqslant 4$, *and let* L *be a* $P(N-1)$*-graded Lie superalgebra over the characteristic zero field* $\mathbb{F}$. *Then there exists a unique* (*up to isomorphism*) *unital associative supercommutative* $\mathbb{F}$*-superalgebra* A *such that* L *is centrally isogenous* (*for* $N > 4$ *indeed is isomorphic*) *to*

$$\begin{aligned}&\mathscr{P}_N(A,\rho)\\ &\quad=\Big\{\begin{pmatrix} X & Y\\ Z & -\rho(X)^{\rm t}\end{pmatrix}\,\Big|\, X,Y,Z\in M_N(A),\ {\rm tr}(X)=0,\ Y=-Y^{\rm t},\ Z=Z^{\rm t}\Big\}\\ &\quad\cong P(N-1)\otimes_{\mathbb{F}} A,\end{aligned}$$

where

$$\rho(a) := (-1)^{|a|}a, \quad \forall\, a \in A_{\overline{0}} \cup A_{\overline{1}}.$$

In addition, using the connection between Lie super and Jordan super structures through Tits-Kantor-Koecher (TKK) construction (see [13]) and the Coordinatization Theorem for Jordan superalgebras (see [18]) of type $JP(n)$, we obtain immediately the characterization for GPLS when N is even. It is a generalization of a result of Kac [13] about the connection between finite dimensional simple Lie superalgebra $P(2n-1)$ and simple Jordan superalgebra $JP(n)$.

The Clifford (or Weyl) algebras have natural representations on the exterior (or symmetric) algebras of polynomials over half of generators. Those representations are important in quantum and statistical mechanics where the generators are interpreted as operators which create or annihilate particles and satisfy Fermi (or Bose) statistics. Fermionic representations for the affine Kac-Moody Lie algebras were first developed by Frenkel [10] and Kac and Peterson [14] independently. Feingold-Frenkel [9] constructed representations for all classical affine Lie algebras by using Clifford or Weyl algebras with infinitely many generators. The Bosonic and Fermionic representations for the EALA $\widetilde{gl_N(\mathbb{C}_q)}$, where $\mathbb{C}_q$ is the quantum torus in two variables, were constructed in [11]. Chen-Gao [7] constructed Fermionic representations for a class of BC_N-graded Lie algebras. Cheng-Zeng [8] constructed Bosonic and Fermionic representations for the Lie superalgebra $D(2,1;\alpha)$. Thereafter, Lau [15] gave a more general Bosonic and Fermionic representations of Lie algebra with non-trivial central extensions.

In Section 4, we give the Fermionic-Bosonic constructions to the GPLS coordinatized by quantum tori.

Throughout this paper, the base field $\mathbb{F}$ is a field of characteristic zero. 't' denotes the usual transpose of a matrix. And let $\mathbb{Z}$ be the ring of integers, $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ be the residue class ring mod 2, with the elements $\overline{0}$ and $\overline{1}$.

2 Definition and construction of GPLS

We first give some notations and definitions which will be used in the sequel. Then we construct some GPLS.

Follows the symbol in [12], let $P(N-1)$, $N \geqslant 3$, stand for the finite dimensional split simple Lie superalgebra which is a subalgebra of $sl(N,N)(\mathbb{F})$, consisted of the matrices of the form

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & -X_{11}^{\mathrm{t}} \end{pmatrix},$$

where

$$\mathrm{tr}(X_{11}) = 0, \quad X_{12} = -X_{12}^{\mathrm{t}}, \quad X_{21} = X_{21}^{\mathrm{t}}.$$

Let A_{N-1} be the special linear Lie algebra $sl_N(\mathbb{F})$.

Let

$$\mathfrak{H}=\Big\{\sum_{i=1}^{N}a_i(e_{ii}-e_{N+i,N+i})\,\Big|\,a_i\in\mathbb{F},\ \sum_{i=1}^{N}a_i=0\Big\}.$$

Then $\mathfrak{H}$ is a split Cartan subalgebra of $P(N-1)_{\overline{0}}$.

For $1\leqslant i\leqslant N$, defining $\varepsilon_i\in\mathfrak{H}^*$ by

$$\varepsilon_i\Big(\sum_{j=1}^{N}a_j(e_{jj}-e_{N+j,N+j})\Big)=a_i.$$

The root system of $P(N-1)$ with respect to the action of the Cartan subalgebra $\mathfrak{H}$ is

$$\Delta_{P(N-1)}=\{\varepsilon_i-\varepsilon_j\mid 1\leqslant i\neq j\leqslant N\}\cup\{\pm(\varepsilon_i+\varepsilon_j),\,-2\varepsilon_i\mid 1\leqslant i\neq j\leqslant N\}.$$

Let

$$\Delta:=\Delta_{P(N-1)}\cup\{2\varepsilon_i\mid 1\leqslant i\leqslant N\},$$
$$\Delta_{\overline{0}}:=\{\varepsilon_i-\varepsilon_j\mid 1\leqslant i\neq j\leqslant N\}=\Delta_{A_{N-1}}.$$

Set

$$L_\alpha=\{x\in L\mid [h,x]=\alpha(h)x,\ \forall\, h\in\mathfrak{H}\}$$

as usual.

Definition 2.1 (see [19]) A Lie superalgebra L over $\mathbb{F}$ is graded by $P(N-1)$ (or $\Delta_{P(N-1)}$), if up to isomorphism,

(i) L contains $P(N-1)$;

(ii) L has a root space decomposition

$$L=\sum_{\alpha\in\Delta_{P(N-1)}\cup\{0\}}L_\alpha$$

relating to a split Cartan subalgebra $\mathfrak{H}$ of $P(N-1)_{\overline{0}}$;

(iii)

$$L_0=\sum_{\alpha\in\Delta_{P(N-1)}}[L_{-\alpha},L_\alpha].$$

Definition 2.2 A Lie superalgebra L over $\mathbb{F}$ is generalized graded by $P(N-1)$ (GPLS), if up to isomorphism,

(i) L contains $P(N-1)$;

(ii) L has a root space decomposition

$$L=\sum_{\alpha\in\Delta\cup\{0\}}L_\alpha$$

relating to a split Cartan subalgebra $\mathfrak{H}$ of $P(N-1)_{\overline{0}}$;

(iii)

$$L_0 = \sum_{\alpha \in \Delta} [L_{-\alpha}, L_\alpha].$$

Definition 2.3 Two perfect Lie superalgebras are said to be centrally isogenous if they have the same universal covering superalgebra up to isomorphism.

Remark A Lie superalgebra graded by $P(N-1)$ is obvious a GPLS. Any GPLS is perfect. Condition (iii) is equivalent to 'L is generated by its nonzero root spaces'. Then it is reasonable to classify GPLS up to centrally isogenous (follows from Berman and Moody [6] classified Lie algebras graded by finite root systems).

Definition 2.4 (see [19]) A Lie superalgebra $L = L_{\bar{0}} + L_{\bar{1}}$ over $\mathbb{F}$ is called an A_{N-1}-graded Lie superalgebra, if up to isomorphism,

(i) $L_{\bar{0}}$ contains A_{N-1};

(ii) L has a root space decomposition

$$L = \sum_{\alpha \in \Delta_{A_{N-1}} \cup \{0\}} L_\alpha$$

relating to a split Cartan subalgebra H of A_{N-1};

(iii)

$$L_0 = \sum_{\alpha \in \Delta_{A_{N-1}}} [L_{-\alpha}, L_\alpha].$$

Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a unital associative superalgebra over $\mathbb{F}$ and $^- : A \to A$ is an anti-superinvolution on A, i.e., an $\mathbb{F}$-linear, homogenous of degree 0 map (either called $\mathbb{Z}_2$-graded or even for short) on A, satisfies for any homogeneous $a, b \in A$,

$$\bar{\bar{a}} = a, \quad \overline{ab} = (-1)^{|a|\,|b|}\bar{b}\bar{a}.$$

For arbitrary homogeneous element $a \in A$, let

$$\rho(a) := (-1)^{|a|} a, \tag{1}$$

where $|a| \in \mathbb{Z}_2$ denotes the parity of a, and extend ρ by linearity on A.

Obviously, ρ is a superalgebra automorphism of order 2 on A. And for any $\mathbb{F}$-linear homogenous of degree 0 map τ on A, we have

$$\rho\tau = \tau\rho.$$

Let

$$M_n(A) = M_n(A_{\bar{0}}) \oplus M_n(A_{\bar{1}})$$

be the associative superalgebra of $n \times n$ matrices over A. Then ρ induces a superalgebra automorphism of order 2 on $M_n(A)$, also denoted by ρ, i.e., for arbitrary homogeneous $X \in M_n(A)$,

$$\rho(X) = (-1)^{|X|} X.$$

Let $M(N,N)(A)$ be the associative superalgebra of (N,N)-*block* matrices over A, whose algebra structure is the tensor product of associative algebras $M(N,N)(\mathbb{F})$ and A over $\mathbb{F}$, i.e.,

$$(X\otimes a)\cdot(Y\otimes b)=XY\otimes ab,\quad X,Y\in M(N,N)(\mathbb{F}),\ a,b\in A,$$
$$M(N,N)(A)_\alpha=\bigoplus_{\beta+\gamma=\alpha}M(N,N)(\mathbb{F})_\beta\otimes_{\mathbb{F}}A_\gamma,\quad \alpha,\beta,\gamma\in\mathbb{Z}_2,$$

where

$$M(N,N)(\mathbb{F})_{\bar 0}=\begin{pmatrix}* & \\ & *\end{pmatrix},\quad M(N,N)(\mathbb{F})_{\bar 1}=\begin{pmatrix} & * \\ * & \end{pmatrix}.$$

The new operation

$$[X,Y]:=XY-(-1)^{|X|\,|Y|}YX$$

for homogeneous $X,Y\in M(N,N)(A)$ defines a Lie superalgebra structure on $M(N,N)(A)$, and we denote the resulting Lie superalgebra by $M(N,N)^{(-)}(A)$.

The new operation

$$X\circ Y:=\frac{1}{2}\left(XY+(-1)^{|X|\,|Y|}YX\right)$$

for homogeneous $X,Y\in M(N,N)(A)$ defines a Jordan superalgebra structure on $M(N,N)(A)$, and we denote the resulting Jordan superalgebra by $M(N,N)^{(+)}(A)$.

The supertranspose

$$\begin{pmatrix}X_{11} & X_{12}\\ X_{21} & X_{22}\end{pmatrix}^{\mathrm{st}}:=\begin{pmatrix}X_{11}^{\mathrm{t}} & X_{21}^{\mathrm{t}}\\ -X_{12}^{\mathrm{t}} & X_{22}^{\mathrm{t}}\end{pmatrix}.$$

Let

$$P:=\begin{pmatrix} & I_N\\ -I_N & \end{pmatrix}\in M(N,N)(A),$$
$$\varrho\begin{pmatrix}X_{11} & X_{12}\\ X_{21} & X_{22}\end{pmatrix}:=\begin{pmatrix}X_{11} & \rho(X_{12})\\ \rho(X_{21}) & X_{22}\end{pmatrix}.$$

By using ϱ, P, and ${}^{\mathrm{st}}$, we get the following result.

Proposition 2.1 *The map*

$$*\colon X\mapsto P^{-1}\overline{\varrho(X)}^{\mathrm{st}}P$$

is an anti-superinvolution on associative superalgebra $M(N,N)(A)$.

Proof By a direct calculation, we have

$$\begin{pmatrix}X_{11} & X_{12}\\ X_{21} & X_{22}\end{pmatrix}^{*}=\begin{pmatrix}\overline{X_{22}}^{\mathrm{t}} & \overline{\rho(X_{12})}^{\mathrm{t}}\\ \overline{-\rho(X_{21})}^{\mathrm{t}} & \overline{X_{11}}^{\mathrm{t}}\end{pmatrix}. \tag{2}$$

Then $*$ is obvious an $\mathbb{F}$-linear homogenous of degree 0 map on $M(N,N)(A)$.

For homogeneous

$$X = \begin{pmatrix} & X_{12} \\ X_{21} & \end{pmatrix}, \quad Y = \begin{pmatrix} & Y_{12} \\ Y_{21} & \end{pmatrix},$$

since $^{-}$ is an anti-superinvolution on A, we get

$$\begin{aligned}
(XY)^* &= \mathrm{diag}(\overline{X_{21}Y_{12}}^{\mathrm{t}}, \overline{X_{12}Y_{21}}^{\mathrm{t}}) \\
&= (-1)^{(|X|+1)(|Y|+1)}\mathrm{diag}(\overline{Y_{12}}^{\mathrm{t}}\overline{X_{21}}^{\mathrm{t}}, \overline{Y_{21}}^{\mathrm{t}}\overline{X_{12}}^{\mathrm{t}}) \\
&= (-1)^{|X|\,|Y|}\left((-1)^{|Y|+1}\begin{pmatrix} & \overline{Y_{12}}^{\mathrm{t}} \\ -\overline{Y_{21}}^{\mathrm{t}} & \end{pmatrix}\right)\left((-1)^{|X|+1}\begin{pmatrix} & \overline{X_{12}}^{\mathrm{t}} \\ -\overline{X_{21}}^{\mathrm{t}} & \end{pmatrix}\right) \\
&= (-1)^{|X|\,|Y|}\begin{pmatrix} & \overline{\rho(Y_{12})}^{\mathrm{t}} \\ -\overline{\rho(Y_{21})}^{\mathrm{t}} & \end{pmatrix}\begin{pmatrix} & \overline{\rho(X_{12})}^{\mathrm{t}} \\ -\overline{\rho(X_{21})}^{\mathrm{t}} & \end{pmatrix} \\
&= (-1)^{|X|\,|Y|}(Y)^*(X)^*.
\end{aligned}$$

For homogeneous

$$X = \begin{pmatrix} X_{11} & \\ & X_{22} \end{pmatrix}, \quad Y = \begin{pmatrix} & Y_{12} \\ Y_{21} & \end{pmatrix},$$

we have

$$\begin{aligned}
(XY)^* &= \begin{pmatrix} & \overline{\rho(X_{11}Y_{12})}^{\mathrm{t}} \\ -\overline{\rho(X_{22}Y_{21})}^{\mathrm{t}} & \end{pmatrix} \\
&= (-1)^{(|X|+|Y|+1)}\begin{pmatrix} & \overline{X_{11}Y_{12}}^{\mathrm{t}} \\ -\overline{X_{22}Y_{21}}^{\mathrm{t}} & \end{pmatrix} \\
&= (-1)^{(|X|+|Y|+1+|X|(|Y|+1))}\begin{pmatrix} & \overline{Y_{12}}^{\mathrm{t}}\overline{X_{11}}^{\mathrm{t}} \\ -\overline{Y_{21}}^{\mathrm{t}}\overline{X_{22}}^{\mathrm{t}} & \end{pmatrix} \\
&= (-1)^{|X|\,|Y|}\left((-1)^{|Y|+1}\begin{pmatrix} & \overline{Y_{12}}^{\mathrm{t}} \\ -\overline{Y_{21}}^{\mathrm{t}} & \end{pmatrix}\right)\mathrm{diag}(\overline{X_{22}}^{\mathrm{t}}, \overline{X_{11}}^{\mathrm{t}}) \\
&= (-1)^{|X|\,|Y|}\begin{pmatrix} & \overline{\rho(Y_{12})}^{\mathrm{t}} \\ -\overline{\rho(Y_{21})}^{\mathrm{t}} & \end{pmatrix}\mathrm{diag}(\overline{X_{22}}^{\mathrm{t}}, \overline{X_{11}}^{\mathrm{t}}) \\
&= (-1)^{|X|\,|Y|}(Y)^*(X)^*.
\end{aligned}$$

The proof of the remaining cases is similar, so we omit the detailed calculation.

Since

$$(X^*)^* = \begin{pmatrix} \overline{\overline{X_{11}}} & \overline{\overline{X_{12}}} \\ \overline{\overline{X_{21}}} & \overline{\overline{X_{22}}} \end{pmatrix},$$

$^{-}$ is an anti-superinvolution, and then $*$ is an anti-superinvolution too. □

Remark If $M(N,N)(A)$ is considered as the graded tensor product $M(N,N)(\mathbb{F})\overline{\otimes}A$ of associative superalgebras $M(N,N)(\mathbb{F})$ and A over $\mathbb{F}$, i.e.,

$$(X\otimes a)\cdot(Y\otimes b)=(-1)^{|a||Y|}XY\otimes ab$$

for homogenous $X,Y\in M(N,N)(\mathbb{F})$, $a,b\in A$, then

$$*\colon X\mapsto P^{-1}\overline{X}^{\mathrm{st}}P$$

is an anti-superinvolution on $M(N,N)(A)$. There is no essential difference between these two choices of algebra structures of $M(N,N)(A)$ in our investigation below.

The following result is well known.

Proposition 2.2 *Let L be an associative superalgebra with an anti-superinvolution $*$. Then*

$$L_{-}=\{a\in L\mid *(a)=-a\}$$

is a Lie subsuperalgebra of L whose Lie superalgebra structure induced naturally by associativity;

$$L_{+}=\{a\in L\mid *(a)=a\}$$

is a Jordan subsuperalgebra of L whose Jordan superalgebra structure induced naturally by associativity.

Due to Propositions 2.1, 2.2, and formula (2), we have the following result.

Proposition 2.3

$$\widetilde{\mathscr{P}}:=\{X\in M(N,N)(A)\mid X^{*}=-X\}$$

is a Lie subsuperalgebra of $M(N,N)^{(-)}(A)$, and

$$\widetilde{\mathscr{P}}=\left\{\begin{pmatrix}X_{11} & X_{12}\\ X_{21} & -\overline{X_{11}}^{\mathrm{t}}\end{pmatrix}\,\middle|\, X_{11},X_{12},X_{21}\in M_N(A),\right.$$
$$\left. X_{12}=-\overline{\rho(X_{12})}^{\mathrm{t}},\ X_{21}=\overline{\rho(X_{21})}^{\mathrm{t}}\right\}.$$

Proposition 2.4

$$\widetilde{J}_N(A,-):=\{X\in M(N,N)(A)\mid X^{*}=X\}$$

is a Jordan subsuperalgebra of $M(N,N)^{(+)}(A)$, and

$$\widetilde{J}_N(A,-)=\left\{\begin{pmatrix}X_{11} & X_{12}\\ X_{21} & \overline{X_{11}}^{\mathrm{t}}\end{pmatrix}\,\middle|\, X_{11},X_{12},X_{21}\in M_N(A),\right.$$
$$\left. X_{12}=\overline{\rho(X_{12})}^{\mathrm{t}},\ X_{21}=-\overline{\rho(X_{21})}^{\mathrm{t}}\right\}.$$

As Jordan subsuperalgebra of $M(N,N)^{(+)}(A)$, $\widetilde{J}_N(A,-)$ *is isomorphic to*

$$J_N(A,-) := \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & \overline{X_{11}}^{\mathrm{t}} \end{pmatrix} \,\middle|\, X_{11}, X_{12}, X_{21} \in M_N(A), \right.$$
$$\left. X_{12} = -\overline{\rho(X_{12})}^{\mathrm{t}},\ X_{21} = \overline{\rho(X_{21})}^{\mathrm{t}} \right\}.$$

Remark $J_N(\mathbb{F}, id_{\mathbb{F}})$ is simple Jordan superalgebra of type $JP(N)$ (see [13]) for $N > 1$.

Let $A = A^+ \oplus A^-$, where

$$A^+ = \{a \in A \mid \overline{a} = a\}, \quad A^- = \{a \in A \mid \overline{a} = -a\}.$$

Notice that $^-$ is a homogeneous of degree 0 map on A. Then

$$A = A^+ \oplus A^-, \quad A = A_{\overline{0}} \oplus A_{\overline{1}},$$

are compatible gradings, and

$$A = A^+_{\overline{0}} \oplus A^-_{\overline{0}} \oplus A^+_{\overline{1}} \oplus A^-_{\overline{1}},$$

where

$$A^+_{\overline{0}} = A_{\overline{0}} \cap A^+, \quad A^-_{\overline{0}} = A_{\overline{0}} \cap A^-, \quad A^+_{\overline{1}} = A_{\overline{1}} \cap A^+, \quad A^-_{\overline{1}} = A_{\overline{1}} \cap A^-.$$

Let

$$\Theta := \sum_{\alpha\in\Delta} [\widetilde{\mathscr{P}}_{-\alpha}, \widetilde{\mathscr{P}}_{\alpha}],$$
$$\widetilde{f}_{ij}(a) := a e_{ij} - \overline{a} e_{N+j,N+i},$$
$$\widetilde{g}_{ij}(a) := a e_{i,N+j} - \overline{\rho(a)} e_{j,N+i},$$
$$\widetilde{h}_{ij}(a) := a e_{N+i,j} + \overline{\rho(a)} e_{N+j,i}$$

where $a \in A$.

Proposition 2.5 *For* $N \geqslant 3$, *we have*

$$\widetilde{f}_{NN}([A,A]) \subseteq \Theta.$$

Proof For $N \geqslant 3$ and arbitrary homogeneous elements $a, b \in A$, we have

$$\begin{aligned}
[\widetilde{g}_{1N}(\overline{a}), \widetilde{h}_{1N}(b)] &= \overline{a}\rho(\overline{b})e_{11} - (-1)^{(|a|+1)(1+|b|)}\rho(\overline{b})\overline{a}e_{2N,2N} \\
&\quad - \rho(a)b e_{NN} + (-1)^{(|a|+1)(1+|b|)} b\rho(a) e_{N+1,N+1} \\
&= \overline{a}\rho(\overline{b})e_{11} - (-1)^{|a|\,|b|}\rho(b)a e_{N+1,N+1} \\
&\quad - \rho(a)b e_{NN} + (-1)^{|a|\,|b|}\overline{b}\rho(\overline{a})e_{2N,2N} \\
&= \widetilde{f}_{11}(\overline{a}\rho(\overline{b})) - \widetilde{f}_{NN}(\rho(a)b) \\
&\in \Theta.
\end{aligned}$$

Replacing b and a by 1 and $(-1)^{|a|\,|b|}\rho(b)a$, respectively, we have

$$\widetilde{f}_{11}(\overline{a}\rho(\overline{b})) - (-1)^{|a|\,|b|}\widetilde{f}_{NN}(b\rho(a)) \in \Theta.$$

Then we get

$$\widetilde{f}_{NN}(\rho(a)b - (-1)^{|a|\,|b|}b\rho(a)) = \widetilde{f}_{NN}([\rho(a), b]) \in \Theta.$$

By the arbitrariness of a and b, and ρ is an automorphism on A, we get

$$\widetilde{f}_{NN}([A, A]) \subseteq \Theta. \qquad \square$$

Proposition 2.6 *For $N \geqslant 3$,*

$$\begin{aligned}\mathscr{P}_0 := &\{\widetilde{f}_{ii}(a) - \widetilde{f}_{NN}(a) \mid 1 \leqslant i \leqslant N-1,\ a \in A\} \\ &\oplus \{\widetilde{f}_{NN}(c) \mid c + \rho(\overline{c}) \in [A, A]\} \\ \subseteq\ &\Theta.\end{aligned}$$

Proof Notice that for arbitrary homogeneous elements $a, b \in A$, $1 \leqslant i \neq j \leqslant N$,

$$\begin{aligned}[\widetilde{f}_{ij}(a), \widetilde{f}_{ji}(b)] &= abe_{ii} - (-1)^{|a|\,|b|}bae_{jj} + \overline{a}\overline{b}e_{N+j,N+j} - (-1)^{|a|\,|b|}\overline{b}\overline{a}e_{N+i,N+i} \\ &= abe_{ii} - \overline{ab}e_{N+i,N+i} - (-1)^{|a|\,|b|}(bae_{jj} - \overline{ba}e_{N+j,N+j}) \\ &= \widetilde{f}_{ii}(ab) - (-1)^{|a|\,|b|}\widetilde{f}_{jj}(ba).\end{aligned}$$

Then, for $b = 1$, $j = N$, we get

$$\widetilde{f}_{ii}(a) - \widetilde{f}_{NN}(a) \in \Theta, \quad i = 1, 2, \ldots, N-1,$$

$$\begin{aligned}[\widetilde{g}_{NN}(a), \widetilde{h}_{NN}(b)] = &\ abe_{NN} - (-1)^{(|a|+1)(1+|b|)}bae_{2N,2N} \\ &- \rho(\overline{a})be_{NN} + (-1)^{(|a|+1)(1+|b|)}b\rho(\overline{a})e_{2N,2N} \\ &+ a\rho(\overline{b})e_{NN} - (-1)^{(|a|+1)(1+|b|)}\rho(\overline{b})ae_{2N,2N} \\ &- \rho(\overline{a})\rho(\overline{b})e_{NN} + (-1)^{(|a|+1)(1+|b|)}\rho(\overline{b})\rho(\overline{a})e_{2N,2N} \\ = &\ \widetilde{f}_{NN}(ab - \rho(\overline{a})b + a\rho(\overline{b}) - \rho(\overline{a})\rho(\overline{b})).\end{aligned}$$

Let $b = 1$ and $a \in A_{\overline{1}}^{+}$. Then we have

$$\widetilde{f}_{NN}(a - \overline{\rho(a)} + a - \overline{\rho(a)}\,) = 4\widetilde{f}_{NN}(a) \in \Theta. \tag{3}$$

By the arbitrariness of a, we get $\widetilde{f}_{NN}(A_{\overline{1}}^{+}) \subseteq \Theta$.

Let $b = 1$ and $a \in A_{\overline{0}}^{-}$. Then we also have (3). By the arbitrariness of a, we get $\widetilde{f}_{NN}(A_{\overline{0}}^{-}) \subseteq \Theta$.

For any $c \in A$, let

$$c = c_{\overline{0}}^{+} + c_{\overline{0}}^{-} + c_{\overline{1}}^{+} + c_{\overline{1}}^{-},$$

where

$$c_0^+ \in A_{\overline{0}}^+, \quad c_0^- \in A_{\overline{0}}^-, \quad c_1^+ \in A_{\overline{1}}^+, \quad c_1^- \in A_{\overline{1}}^-.$$

Since

$$c_0^+ + c_1^- = \frac{1}{2}\,(c + \rho(\overline{c})) \in [A, A],$$

from Proposition 2.5, we have $\widetilde{f}_{NN}(c_0^+ + c_1^-) \in \Theta$.

Notice that

$$\widetilde{f}_{NN}(c) = \widetilde{f}_{NN}(c_0^+ + c_1^-) + \widetilde{f}_{NN}(c_0^-) + \widetilde{f}_{NN}(c_1^+).$$

Then, summarizing the above discussion, we get $\mathscr{P}_0 \subseteq \Theta$ at once. □

Lemma 2.7 *For $N \geqslant 3$, let*

$$\mathscr{P}_N(A, -) := \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & -\overline{X_{11}}^{\mathrm{t}} \end{pmatrix} \in \widetilde{\mathscr{P}} \,\middle|\, \mathrm{tr}(X_{11} + \overline{\rho(X_{11})}) \in [A, A] \right\}.$$

Then

$$\mathscr{P}_N(A, -) = \sum_{\alpha \in \Delta} \widetilde{\mathscr{P}}_\alpha \oplus \mathscr{P}_0 = [\widetilde{\mathscr{P}}, \widetilde{\mathscr{P}}] = \sum_{\alpha \in \Delta} \widetilde{\mathscr{P}}_\alpha \oplus \Theta.$$

Proof First, we check

$$[\widetilde{\mathscr{P}}, \widetilde{\mathscr{P}}] \subseteq \mathscr{P}_N(A, -).$$

Indeed, for arbitrary homogeneous elements $a, b \in A$ and $1 \leqslant i, j, k, l \leqslant N$, we have

$$\begin{aligned}
&[\widetilde{g}_{ij}(a), \widetilde{h}_{kl}(b)] \\
&\quad = \delta_{jk} ab e_{il} + (-1)^{(|b|+1)(|a|+1)} \delta_{jk} \rho(\overline{b}) \rho(\overline{a}) e_{l+N,i+N} \\
&\qquad - \delta_{ik} \rho(\overline{a}) b e_{jl} - (-1)^{(|b|+1)(|a|+1)} \delta_{ik} \rho(\overline{b}) a e_{l+N,j+N} \\
&\qquad + \delta_{jl} a \rho(\overline{b}) e_{ik} + (-1)^{(|b|+1)(|a|+1)} \delta_{jl} b \rho(\overline{a}) e_{k+N,i+N} \\
&\qquad - \delta_{il} \rho(\overline{a}) \rho(\overline{b}) e_{jk} - (-1)^{(|b|+1)(|a|+1)} \delta_{il} ba e_{k+N,j+N} \\
&\quad = \delta_{jk} ab e_{il} - (-1)^{|b|\,|a|} \delta_{jk} \overline{b}\,\overline{a} e_{l+N,i+N} \\
&\qquad - \delta_{ik} \rho(\overline{a}) b e_{jl} + (-1)^{|b|\,|a|} \delta_{ik} \overline{b} \rho(a) e_{l+N,j+N} \\
&\qquad + \delta_{jl} a \rho(\overline{b}) e_{ik} - (-1)^{|b|\,|a|} \delta_{jl} \rho(b) \overline{a} e_{k+N,i+N} \\
&\qquad - \delta_{il} \rho(\overline{a}) \rho(\overline{b}) e_{jk} + (-1)^{|b|\,|a|} \delta_{il} \rho(b) \rho(a) e_{k+N,j+N} \\
&\quad = (-\delta_{ik} \widetilde{f}_{jl}(\rho(\overline{a}) b) + \delta_{jl} \widetilde{f}_{ik}(a \rho(\overline{b}))) + (\delta_{jk} \widetilde{f}_{il}(ab) - \delta_{il} \widetilde{f}_{jk}(\rho(\overline{a}) \rho(\overline{b}))).
\end{aligned}$$

Notice that

$$\begin{aligned}
&- \delta_{ik} \delta_{jl} (\rho(\overline{a}) b - a \rho(\overline{b}) + \overline{\rho(\rho(\overline{a}) b - a \rho(\overline{b}))}\,) \\
&\quad = - \delta_{ik} \delta_{jl} (\rho(\overline{a}) b - a \rho(\overline{b}) + (-1)^{|a|\,|b|} \rho(\overline{b}) a - (-1)^{|a|\,|b|} b \rho(\overline{a})) \\
&\quad = - \delta_{ik} \delta_{jl} (\rho(\overline{a}) b - (-1)^{|a|\,|b|} b \rho(\overline{a}) - a \rho(\overline{b}) + (-1)^{|a|\,|b|} \rho(\overline{b}) a) \\
&\quad = - \delta_{ik} \delta_{jl} ([\rho(\overline{a}), b] - [a, \rho(\overline{b})]),
\end{aligned}$$

and similarly,

$$\delta_{jk}\delta_{il}(ab-\rho(\overline{a})\rho(\overline{b})+\overline{\rho(ab-\rho(\overline{a})\rho(\overline{b}))}\,)=\delta_{jk}\delta_{il}([a,b]-[\rho(\overline{a}),\rho(\overline{b})]).$$

Then

$$[\widetilde{g}_{ij}(a),\widetilde{h}_{kl}(b)]\in\mathscr{P}_N(A,-).$$

The remainder of cases is similar, so we omit the detailed calculation.

On the other hand, notice that

$$\mathscr{P}_N(A,-)=\sum_{\alpha\in\Delta}\widetilde{\mathscr{P}_\alpha}\oplus\mathscr{P}_0.$$

Then from Proposition 2.6, we have

$$\mathscr{P}_N(A,-)\subseteq\sum_{\alpha\in\Delta}\widetilde{\mathscr{P}_\alpha}\oplus\Theta.$$

Now, the lemma holds at once since

$$\sum_{\alpha\in\Delta}\widetilde{\mathscr{P}_\alpha}\oplus\Theta\subseteq[\widetilde{\mathscr{P}},\widetilde{\mathscr{P}}].$$

□

Theorem 2.1 *Suppose $N\geqslant 3$. Then, for any unital associative $\mathbb{F}$-superalgebra A with an anti-superinvolution $^-$, we have any Lie superalgebra centrally isogenous to $\mathscr{P}_N(A,-)$ is a generalized $P(N-1)$-graded Lie superalgebra. Furthermore, $\mathscr{P}_N(A,-)$ is a $P(N-1)$-graded Lie superalgebra if and only if A is supercommutative, and $\overline{a}=\rho(a)$, $\forall\, a\in A$.*

Proof Since A is unital, we have $P(N-1)\subseteq\mathscr{P}_N(A,-)$.

Noticing that

$$\Theta=\sum_{\alpha\in\Delta}[\widetilde{\mathscr{P}}_{-\alpha},\widetilde{\mathscr{P}_\alpha}],$$

and from Lemma 2.7,

$$\mathscr{P}_N(A,-)=\sum_{\alpha\in\Delta}\widetilde{\mathscr{P}_\alpha}\oplus\Theta,$$

we get $\mathscr{P}_N(A,-)$ is a GPLS at once.

Following [3, Lemma 2.4], any universal covering superalgebra of $\mathscr{P}_N(A,-)$ is a GPLS, so is its any central quotients.

Furthermore, $\mathscr{P}_N(A,-)$ is a Lie superalgebra graded by $P(N-1)$ if and only if

$$\mathscr{P}_{2\varepsilon_i}=\{\widetilde{g}_{ii}(a)=(a-\overline{\rho(a)}\,)e_{i,N+i}\mid a\in A\}=0,\quad 1\leqslant i\leqslant N,$$

$$\Longleftrightarrow\quad a-\overline{\rho(a)}=0,\quad\forall\, a\in A,$$

$$\Longleftrightarrow\quad \overline{a}=\rho(a),\quad\forall\, a\in A.$$

Moreover, for arbitrary homogeneous elements $a, b \in A$, notice that $^-$ is an anti-superinvolution, ρ is a superalgebra automorphism on A. Then we get

$$ab = \overline{\rho(ab)} = \overline{\rho(a)\rho(b)} = (-1)^{|a|\,|b|}(\overline{\rho(b)}\,)(\overline{\rho(a)}\,) = (-1)^{|a|\,|b|}ba. \qquad \square$$

Later, we say that the GPLS $\mathscr{P}_N(A,-)$ is coordinatized by A.

Remark If $M(N,N)(A)$ is considered as the graded tensor product of $M(N,N)(\mathbb{F})$ and A over $\mathbb{F}$, then any Lie superalgebra centrally isogenous to

$$\mathscr{P}'_N(A,-) := \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & -\overline{X_{11}}^{\mathrm{t}} \end{pmatrix} \Big| \, X_{11}, X_{12}, X_{21} \in M_N(A), \right.$$
$$\left. X_{12} = -\overline{X_{12}}^{\mathrm{t}},\ X_{21} = \overline{X_{21}}^{\mathrm{t}},\ \mathrm{tr}(X_{11} + \overline{X_{11}}\,) \in [A,A] \right\}$$

is a GPLS. Furthermore, $\mathscr{P}'_N(A,-)$ is a $P(N-1)$-graded Lie superalgebra if and only if A is supercommutative, and $\overline{a} = a$, $\forall\, a \in A$.

About the structure of $\mathscr{P}_N(A,-)$, from Lemma 2.7, we get the following root space decomposition at once. And the nontrivial Lie superbrackets between the root vectors of $\mathscr{P}_N(A,-)$ are as below.

Theorem 2.2

$$\mathscr{P}_N(A,-) = \mathscr{P}_0 \oplus \sum_{1\leqslant i\neq j\leqslant N} \mathscr{P}_{\varepsilon_i-\varepsilon_j} \oplus \sum_{1\leqslant i\leqslant j\leqslant N} \mathscr{P}_{\varepsilon_i+\varepsilon_j} \oplus \sum_{1\leqslant i\leqslant j\leqslant N} \mathscr{P}_{-\varepsilon_i-\varepsilon_j},$$

$$\mathscr{P}_{\varepsilon_i-\varepsilon_j} = \{\widetilde{f}_{ij}(a) \mid a \in A\},$$

$$\mathscr{P}_{\varepsilon_i+\varepsilon_j} = \{\widetilde{g}_{ij}(a) \mid a \in A\},$$

$$\mathscr{P}_{-\varepsilon_i-\varepsilon_j} = \{\widetilde{h}_{ij}(a) \mid a \in A\},$$

$$\mathscr{P}_0 = \{\widetilde{f}_{ii}(a) - \widetilde{f}_{NN}(a) \mid 1 \leqslant i \leqslant N-1,\ a \in A\} \oplus \{\widetilde{f}_{NN}(c) \mid c + \rho(\overline{c}) \in [A,A]\}.$$

For arbitrary homogeneous $a, b \in A$,

$$[\widetilde{g}_{ij}(a), \widetilde{f}_{kl}(b)] = -(-1)^{|b|\,|a|}\delta_{il}\widetilde{g}_{kj}(\rho(b)a) + (-1)^{|b|\,|a|}\delta_{jl}\widetilde{g}_{ki}(\rho(b)\rho(\overline{a})),$$

$$[\widetilde{f}_{ij}(a), \widetilde{h}_{kl}(b)] = -\delta_{ik}\widetilde{h}_{jl}(\overline{a}b) - (-1)^{|b|\,|a|}\delta_{il}\widetilde{h}_{kj}(b\rho(a)),$$

$$[\widetilde{f}_{ij}(a), \widetilde{f}_{kl}(b)] = \delta_{jk}\widetilde{f}_{il}(ab) - (-1)^{|b|\,|a|}\delta_{il}\widetilde{f}_{kj}(ba),$$

$$[\widetilde{g}_{ij}(a), \widetilde{h}_{kl}(b)] = \delta_{jk}\widetilde{f}_{il}(ab) - \delta_{ik}\widetilde{f}_{jl}(\rho(\overline{a})b) - \delta_{il}\widetilde{f}_{jk}(\rho(\overline{a})\rho(\overline{b})) + \delta_{jl}\widetilde{f}_{ik}(a\rho(\overline{b})).$$

Proof

$$\begin{aligned}
[\widetilde{g}_{ij}(a), \widetilde{f}_{kl}(b)] &= [ae_{i,N+j} - \rho(\overline{a})e_{j,N+i}, be_{kl} - \overline{b}e_{N+l,N+k}] \\
&= \rho(\overline{a})\overline{b}\delta_{il}e_{j,k+N} - a\overline{b}\delta_{jl}e_{i,k+N} \\
&\quad - (-1)^{|b|(|a|+1)}\delta_{il}bae_{k,j+N} + (-1)^{|b|(|a|+1)}\delta_{jl}b\rho(\overline{a})e_{k,i+N} \\
&= -(-1)^{|b|\,|a|}\delta_{il}\widetilde{g}_{kj}(\rho(b)a) + (-1)^{|b|\,|a|}\delta_{jl}\widetilde{g}_{ki}(\rho(b)\rho(\overline{a})).
\end{aligned}$$

The proofs of others are similar. $\square$

3 Classification of GPLS

In this section, we follow the symbols given in Section 2, and all the root spaces decompositions were with respect to the action of the Cartan subalgebra $\mathfrak{H}$ which was given in Section 2.

In this section, we assume $N \geqslant 4$ unless otherwise stated.

Let

$$f_{ij} := e_{ij} - e_{N+j,N+i}, \quad g_{ij} := e_{i,N+j} - e_{j,N+i}, \quad h_{ij} := e_{N+i,j} + e_{N+j,i},$$

for short.

Following [3,4,19], and using the connection between root systems graded Lie superalgebras and their associated Jordan supersystems, next, we classify GPLS up to centrally isogenous.

Due to Benkart et al. [3, Sect. 3], we have the following result.

Lemma 3.1 *Let*

$$L = \sum_{\gamma\in\Delta\cup\{0\}} L_\gamma, \quad L' = \sum_{\gamma\in\Delta\cup\{0\}} L'_\gamma,$$

be two GPLS. If there exists a family of homogenous of degree 0, $\mathbb{F}$*-linear isomorphisms*

$$\eta = (\eta_\gamma, \gamma \in \Delta), \quad \eta_\gamma\colon L_\gamma \to L'_\gamma,$$

such that

$$\eta_{\alpha+\beta}([x_\alpha, x_\beta]) = [\eta_\alpha(x_\alpha), \eta_\beta(x_\beta)], \quad \forall\, \alpha, \beta, \alpha+\beta \in \Delta,\ x_\alpha \in L_\alpha,\ x_\beta \in L_\beta,$$

and for any $\alpha \in \Delta$,

$$L'_\alpha = \sum_{\delta,\gamma\in\Delta,\delta,\gamma\neq\pm\alpha,\delta+\gamma=\alpha} [L'_\delta, L'_\gamma],$$

then L *and* L' *are centrally isogenous.*

Due to Martinez and Zelmanov [19, Sect. 3], we have the following result.

Proposition 3.2 *Let* $L = L_{\bar 0} + L_{\bar 1}$ *be an* A_{N-1}*-graded Lie superalgebra over* $\mathbb{F}$, $N \geqslant 4$. *Then there exists a unital associative superalgebra* $A = A_{\bar 0} + A_{\bar 1}$ *such that* L *is centrally isogenous with* $sl_N(A)$; *and furthermore, there exists a family of homogenous of degree* 0, $\mathbb{F}$*-linear isomorphisms*

$$\eta = (\eta_{\varepsilon_i-\varepsilon_j}, 1 \leqslant i \neq j \leqslant N), \quad \eta_{\varepsilon_i-\varepsilon_j}\colon L_{\varepsilon_i-\varepsilon_j} \to sl_N(A)_{\varepsilon_i-\varepsilon_j},$$

such that

$$\eta_{\varepsilon_i-\varepsilon_j+\varepsilon_k-\varepsilon_l}([x_{\varepsilon_i-\varepsilon_j}, x_{\varepsilon_k-\varepsilon_l}]) = [\eta_{\varepsilon_i-\varepsilon_j}(x_{\varepsilon_i-\varepsilon_j}), \eta_{\varepsilon_k-\varepsilon_l}(x_{\varepsilon_k-\varepsilon_l})],$$

$$\eta(e_{ij}) = e_{ij}(1),$$

for arbitrary $i, j, k, l = 1, 2, \ldots, N$, $i \neq j$, $k \neq l$, $x_{\varepsilon_i-\varepsilon_j} \in L_{\varepsilon_i-\varepsilon_j}$, $x_{\varepsilon_k-\varepsilon_l} \in L_{\varepsilon_k-\varepsilon_l}$.

In addition, following the Berman-Moody proof of [6, Proposition 1.29], we have the unital associative $\mathbb{F}$-superalgebra A which satisfies the above conditions is unique up to isomorphism.

Following [19], for a GPLS L, let

$$L_{\Delta_{\bar{0}}} := \sum_{\alpha\in\Delta_{\bar{0}}} L_\alpha \oplus \sum_{\alpha\in\Delta_{\bar{0}}} [L_\alpha, L_{-\alpha}],$$

and notice that $P(N-1)_{\bar{0}} \cong A_{N-1}$. Then we have the following result.

Lemma 3.3 *Let L be a GPLS. Then $L_{\Delta_{\bar{0}}}$ is an A_{N-1}-graded Lie superalgebra, and there exists a unique* (*up to isomorphism*) *unital associative $\mathbb{F}$-superalgebra A, such that $L_{\Delta_{\bar{0}}}$ is centrally isogenous with $sl_N(A)$; and furthermore, there exists a family of homogenous of degree* 0, *$\mathbb{F}$-linear isomorphisms*

$$\theta = (\theta_{\varepsilon_i-\varepsilon_j}, 1 \leqslant i \neq j \leqslant N), \quad \theta_{\varepsilon_i-\varepsilon_j} \colon L_{\varepsilon_i-\varepsilon_j} \to sl_N(A)_{\varepsilon_i-\varepsilon_j},$$

such that

$$\theta_{\varepsilon_i-\varepsilon_j+\varepsilon_k-\varepsilon_l}([x_{\varepsilon_i-\varepsilon_j}, x_{\varepsilon_k-\varepsilon_l}]) = [\theta_{\varepsilon_i-\varepsilon_j}(x_{\varepsilon_i-\varepsilon_j}), \theta_{\varepsilon_k-\varepsilon_l}(x_{\varepsilon_k-\varepsilon_l})],$$

$$\theta(f_{ij}) = e_{ij}(1)$$

for arbitrary $i, j, k, l = 1, 2, \ldots, N$, $i \neq j$, $k \neq l$, $x_{\varepsilon_i-\varepsilon_j} \in L_{\varepsilon_i-\varepsilon_j}$, $x_{\varepsilon_k-\varepsilon_l} \in L_{\varepsilon_k-\varepsilon_l}$.

Next, we call A the superalgebra associated with GPLS L, and we denote

$$\theta^{-1}(e_{ij}(a)) =: f_{ij}(a), \quad 1 \leqslant i \neq j \leqslant N, \forall\, a \in A.$$

The structures of the even-root spaces of L are clear, and about the odd-root spaces of L, we have the following result.

Lemma 3.4 *Let L be a GPLS. Then*

$$L_{\varepsilon_k+\varepsilon_l} = [g_{kt}, L_{\varepsilon_l-\varepsilon_t}] = [g_{lt}, L_{\varepsilon_k-\varepsilon_t}],$$

$$L_{-\varepsilon_k-\varepsilon_l} = [h_{kt}, L_{\varepsilon_t-\varepsilon_l}] = [h_{lt}, L_{\varepsilon_t-\varepsilon_k}],$$

for arbitrary $k, l, t = 1, 2, \ldots, N$, $t \neq k, l$.

Proof For arbitrary $k, l, t = 1, 2, \ldots, N$, $t \neq k, l$, we have

$$\begin{aligned}[g_{kt}, L_{\varepsilon_l-\varepsilon_t}] &\supseteq [g_{kt}, [h_{tk}, L_{\varepsilon_k+\varepsilon_l}]] \\ &= [[g_{kt}, h_{tk}], L_{\varepsilon_k+\varepsilon_l}] \\ &= [f_{kk} - f_{tt}, L_{\varepsilon_k+\varepsilon_l}] \\ &= (\varepsilon_k + \varepsilon_l)(f_{kk} - f_{tt}) L_{\varepsilon_k+\varepsilon_l} \\ &= L_{\varepsilon_k+\varepsilon_l}.\end{aligned}$$

Obviously, $[g_{kt}, L_{\varepsilon_l-\varepsilon_t}] \subseteq L_{\varepsilon_k+\varepsilon_l}$, and then

$$L_{\varepsilon_k+\varepsilon_l} = [g_{kt}, L_{\varepsilon_l-\varepsilon_t}].$$

Similarly, we have

$$L_{\varepsilon_k+\varepsilon_l} = [g_{lt}, L_{\varepsilon_k-\varepsilon_t}],$$
$$L_{-\varepsilon_k-\varepsilon_l} = [h_{kt}, L_{\varepsilon_t-\varepsilon_l}] = [h_{lt}, L_{\varepsilon_t-\varepsilon_k}]. \qquad \square$$

Proposition 3.5 *Let L be a GPLS. Then for any fixed distinct i, j, k, $1 \leqslant i, j, k \leqslant N$, respectively, there exists a unique pair of maps G and G' on the superalgebra A associated with L such that*

$$\begin{aligned}[h_{ik}, f_{kj}(a)] &= [h_{jk}, f_{ki}(G(\rho(a)))],\\ [g_{ik}, f_{jk}(a)] &= -[g_{jk}, f_{ik}(G'(\rho(a)))],\end{aligned} \quad \forall\, a \in A.$$

(*The definition of ρ for a superalgebra A, see formula* (1).)

Furthermore, $G(1) = 1$, and G is an $\mathbb{F}$-linear homogeneous of degree 0 map:

$$\rho(G(a)) = G(\rho(a)).$$

Proof For any distinct $i, k, j = 1, 2, \ldots, N$, and any $a \in A$, we have

$$[g_{kj}, [h_{jk}, f_{ki}(a)]] = [f_{kk} - f_{jj}, f_{ki}(a)] - [h_{jk}, [g_{kj}, f_{ki}(a)]] = f_{ki}(a),$$

and then $[h_{jk}, f_{ki}(a)] = [h_{jk}, f_{ki}(a')]$ if and only if $a = a'$.

From Lemma 3.4, we have

$$[h_{ik}, L_{\varepsilon_k-\varepsilon_j}] = L_{-\varepsilon_i-\varepsilon_j} = [h_{jk}, L_{\varepsilon_k-\varepsilon_i}],$$

and then, $\forall\, a \in A$, there exists unique $G(a) \in A$ such that

$$[h_{ik}, f_{kj}(\rho(a))] = [h_{jk}, f_{ki}(G(a))].$$

Noticing that ρ is an automorphism of order 2 on A, and replacing a by $\rho(a)$, we have

$$[h_{ik}, f_{kj}(a)] = [h_{jk}, f_{ki}(G(\rho(a)))]. \tag{4}$$

Obviously, G is $\mathbb{F}$-linear, $G(1) = 1$. For any homogenous element $a \in A$, notice that ρ is homogeneous of degree 0. Then, from formula (4), we have

$$1 + |a| = 1 + |G(\rho(a))| = 1 + |G(a)|,$$

and then G is homogeneous of degree 0 too, so

$$\rho(G(a)) = G(\rho(a)).$$

Similarly, we can get the claim for G'. $\square$

Lemma 3.6 *Let L be a GPLS. Then $[h_{ij},]$ is injective on $L_{\varepsilon_i-\varepsilon_k}, L_{\varepsilon_j-\varepsilon_k}$ and $L_{\varepsilon_i+\varepsilon_k}, L_{\varepsilon_j+\varepsilon_k}$; $[g_{ij},]$ is injective on $L_{\varepsilon_k-\varepsilon_i}, L_{\varepsilon_k-\varepsilon_j}$ and $L_{-\varepsilon_i-\varepsilon_k}, L_{-\varepsilon_j-\varepsilon_k}$, for any distinct $i,k,j=1,2,\ldots,N$.*

Proof Notice that for arbitrary distinct $i,k,j = 1,2,\ldots,N$, and arbitrary $[g_{ij}, f_{kj}(a)] \in L_{\varepsilon_i+\varepsilon_k}$, we have

$$[h_{ij},[g_{ij},f_{kj}(a)]] = [f_{ii}-f_{jj}, f_{kj}(a)] - [g_{ij},[h_{ij},f_{kj}(a)]] = f_{kj}(a).$$

Then

$$[h_{ij},[g_{ij},f_{kj}(a)]] = 0 \iff a = 0 \iff [g_{ij},f_{kj}(a)] = 0,$$

and we get that $[g_{ij},]$ is injective on $L_{\varepsilon_k-\varepsilon_j}$, $[h_{ij},]$ is injective on $L_{\varepsilon_i+\varepsilon_k}$.

The proof of others is similar. □

Furthermore, about G and G', we have the following result.

Proposition 3.7 $G^2 = id_A$, $G = G'$. *The definition of G does not depend on the choice of i,j,k, in particular,*

$$[h_{ik}, f_{kj}(a)] = [h_{il}, f_{lj}(a)], \quad [g_{ik}, f_{jk}(a)] = [g_{il}, f_{jl}(a)],$$

for any distinct $i,j,k,l = 1,2,\ldots,N$, and $a \in A$.

Proof For any fixed distinct $i,j,k,1 = 1,2,\ldots,N$, and $a \in A$, from Proposition 3.5, there exists a unique map G on the superalgebra A such that

$$[h_{ik}, f_{kj}(a)] = [h_{jk}, f_{ki}(G(\rho(a)))].$$

Notice that for distinct i,j,k,l, we have

$$[h_{ik}, f_{kj}(a)] = [h_{ik},[f_{kl}, f_{lj}(a)]] = [[h_{ik}, f_{kl}], f_{lj}(a)] = [h_{il}, f_{lj}(a)],$$

and then G does not depend on the choice of k.

Next, we check that G does not depend on the choice of j:

$$\begin{aligned}
[h_{ik}, f_{kl}(a)] &= [h_{ik},[f_{kj}(a), f_{jl}]] \\
&= [[h_{ik}, f_{kj}(a)], f_{jl}] + (-1)^{|a|}[f_{kj}(a),[h_{ik}, f_{jl}]] \\
&= [[h_{ik}, f_{kj}(a)], f_{jl}] \\
&= [[h_{jk}, f_{ki}(G(\rho(a)))], f_{jl}] \\
&= [h_{jk},[f_{ki}(G(\rho(a))), f_{jl}]] + [[h_{jk}, f_{jl}], f_{ki}(G(\rho(a)))] \\
&= [h_{lk}, f_{ki}(G(\rho(a)))].
\end{aligned}$$

Notice that the map G defined in Proposition 3.5 for arbitrary fixed distinct i,k,l is unique. Then we get the definition of G does not depend on the choice of j.

Similarly, we have

$$[h_{lk}, f_{kj}(a)] = [h_{jk}, f_{kl}(G(\rho(a)))],$$

and G does not depend on the choice of i too.

In particular, we have

$$[h_{ik}, f_{kj}(a)] = [h_{jk}, f_{ki}(G(\rho(a)))] = [h_{ik}, f_{kj}(G(\rho(G(\rho(a)))))].$$

From Lemma 3.6, h_{ik} is injective on $L_{\varepsilon_k-\varepsilon_j}$, and then we get

$$G(\rho(G(\rho(a)))) = a, \quad \forall\, a \in A.$$

From Proposition 3.5, $G\rho = \rho G$, and noticing that $\rho^2 = id$, we get

$$G(\rho(G(\rho(a)))) = G(G(a)) = a, \quad \forall\, a \in A.$$

Next, we check $G = G'$. We have

$$[h_{il}, [g_{ik}, f_{jk}(a)]] = [[h_{il}, g_{ik}], f_{jk}(a)] - [g_{ik}, [h_{il}, f_{jk}(a)]] = [-f_{kl}, f_{jk}(a)] = f_{jl}(a),$$

$$\begin{aligned}[h_{il}, -[g_{jk}, f_{ik}(G(\rho(a)))]] &= -[[h_{il}, g_{jk}], f_{ik}(G(\rho(a)))] + [g_{jk}, [h_{li}, f_{ik}(G(\rho(a)))]] \\ &= [g_{jk}, [h_{ki}, f_{il}(G(\rho(G(\rho(a)))))]] \\ &= [[g_{jk}, h_{ki}], f_{il}(a)] \\ &= f_{jl}(a).\end{aligned}$$

From Lemma 3.6, h_{il} is injective on $L_{\varepsilon_j+\varepsilon_i}$. Then have

$$[g_{ik}, f_{jk}(a)] = -[g_{jk}, f_{ik}(G(\rho(a)))].$$

By the uniqueness of G', we get $G = G'$. □

About G, summarizing the above discussion in Propositions 3.5 and 3.7, we have the following result.

Lemma 3.8 *Let L be a GPLS. Then there exists a unique map G on the superalgebra A associated with L such that*

$$[h_{it}, f_{tj}(a)] = [h_{jt}, f_{ti}(G(\rho(a)))], [g_{it}, f_{jt}(a)] = -[g_{jt}, f_{it}(G(\rho(a)))],$$

for any distinct $i, t, j = 1, 2, \ldots, N$, $\forall\, a \in A$.

Furthermore, G is an $\mathbb{F}$-linear homogeneous of degree 0 map on A, and

$$G(1) = 1, \quad G^2 = id, \quad |a| = |G(a)|, \quad \rho(G(a)) = G(\rho(a)).$$

Proposition 3.9 *Let L be a GPLS, and let G be the map associated with L defined as above. Then*

$$\begin{aligned}[[h_{it}, f_{ti}(a)], f_{ik}(b)] &= [h_{kl}, f_{li}(G(\rho(ab)) + (-1)^{|a|\,|b|}G(\rho(b))a)], \\ [[g_{it}, f_{it}(a)], f_{ki}(b)] &= [g_{kl}, f_{il}((-1)^{|a|\,|b|}G(\rho(ba)) - aG(\rho(b)))],\end{aligned}$$

for distinct $i, t, k, l = 1, 2, \ldots, N$, and any homogeneous $a, b \in A$, which is the superalgebra associated with L.

Proof Noticing that i, t, k, l are distinct, and using Lemma 3.8, we have

$$\begin{aligned}
[[h_{it}, f_{ti}(a)], f_{ik}(b)] &= [h_{it}, [f_{ti}(a), f_{ik}(b)]] + (-1)^{|a|\,|b|}[[h_{ti}, f_{ik}(b)], f_{ti}(a)] \\
&= [h_{it}, f_{tk}(ab)] + (-1)^{|a|\,|b|}[[h_{tl}, f_{lk}(b)], f_{ti}(a)] \\
&= [h_{kt}, f_{ti}(G(\rho(ab)))] + (-1)^{|a|\,|b|}[[h_{kl}, f_{lt}(G(\rho(b)))], f_{ti}(a)] \\
&= [h_{kt}, f_{ti}(G(\rho(ab)))] + (-1)^{|a|\,|b|}[h_{kl}, [f_{lt}(G(\rho(b))), f_{ti}(a)]] \\
&= [h_{kl}, f_{li}(G(\rho(ab)))] + (-1)^{|a|\,|b|}[h_{kl}, f_{li}(G(\rho(b))a)] \\
&= [h_{kl}, f_{li}(G(\rho(ab)) + (-1)^{|a|\,|b|}G(\rho(b))a)].
\end{aligned}$$

The proof of $[[g_{it}, f_{it}(a)], f_{ki}(b)]$ is similar. □

Lemma 3.10 *Let L be a GPLS, let A be the superalgebra associated with L, and let G be the map associated with L defined as above. Then we have*

$$[h_{it}, f_{ti}(a)] = [h_{it}, f_{ti}(G(\rho(a)))], [g_{it}, -f_{it}(G(a))] = [g_{it}, f_{it}(\rho(a))],$$

and

$$\forall\, [h_{it}, f_{ti}(a)] \in L_{-2\varepsilon_i},\ [h_{it}, f_{ti}(a)] = 0 \quad \textit{if and only if} \quad a + G(\rho(a)) = 0,$$

$$\forall\, [g_{it}, -f_{it}(G(a))] \in L_{2\varepsilon_i},\ [g_{it}, -f_{it}(G(a))] = 0 \quad \textit{if and only if} \quad a - G(\rho(a)) = 0,$$

where $1 \leqslant i \neq t \leqslant N$, $a \in A$.

Proof Assume that $i, j, k, t = 1, 2, \ldots, N$ are distinct, and $a \in A$ is homogeneous.

Using Proposition 3.9, letting $b = 1$, and then using Lemma 3.8, we have

$$\begin{aligned}
[[h_{it}, f_{ti}(a)], f_{ik}] &= [h_{kl}, f_{li}(G(\rho(a)) + a)], \\
[[g_{it}, -f_{it}(G(a))], f_{ki}] &= [g_{kl}, f_{il}(G(a) - \rho(a))].
\end{aligned}$$

From Lemma 3.6, h_{kl} is injective on $L_{\varepsilon_l - \varepsilon_i}$ and g_{kl} is injective on $L_{\varepsilon_i - \varepsilon_l}$. Then we get $[h_{it}, f_{ti}(a)] \neq 0$ if $G(\rho(a)) + a \neq 0$, and $[g_{it}, -f_{it}(G(a))] \neq 0$ if $G(a) - \rho(a) \neq 0$. Notice that $G^2 = id$. Then we get $[g_{it}, -f_{it}(G(a))] \neq 0$ if $a - G(\rho(a)) \neq 0$.

Again using Lemma 3.8, and noticing that i, j, k, t are distinct, we have

$$\begin{aligned}
[[g_{jk}, [h_{it}, f_{tk}(a)]], h_{ij}] &= -\,[[h_{it}, [g_{jk}, f_{tk}(a)]], h_{ij}] \\
&= [[h_{it}, [g_{tk}, f_{jk}(G(\rho(a)))]], h_{ij}] \\
&= [[[h_{it}, g_{tk}], f_{jk}(G(\rho(a)))], h_{ij}] \\
&= -\,[[f_{ki}, f_{jk}(G(\rho(a)))], h_{ij}] \\
&= [f_{ji}(G(\rho(a))), h_{ij}] \\
&= -\,(-1)^{|a|}[h_{ij}, f_{ji}(G(\rho(a)))].
\end{aligned}$$

On the other hand, noticing that $-\varepsilon_i-\varepsilon_k-\varepsilon_i-\varepsilon_j$ is not a root, we have

$$\begin{aligned}[[g_{jk},[h_{it},f_{tk}(a)]],h_{ij}] &= [g_{jk},[[h_{it},f_{tk}(a)],h_{ij}]]+(-1)^{|a|+1}[[g_{jk},h_{ij}],[h_{it},f_{tk}(a)]]\\ &= (-1)^{|a|}[f_{ki},[h_{it},f_{tk}(a)]]\\ &= (-1)^{|a|}[[f_{ki},h_{it}],f_{tk}(a)]+(-1)^{|a|}[h_{it},[f_{ki},f_{tk}(a)]]\\ &= -(-1)^{|a|}[h_{it},f_{ti}(a)].\end{aligned}$$

Then

$$[h_{it},f_{ti}(G(\rho(a)))]=[h_{ij},f_{ji}(G(\rho(a)))]=[h_{it},f_{ti}(a)],$$

so

$$[h_{it},f_{ti}(a)]=0 \iff a+G(\rho(a))=0.$$

The proof of $[g_{it},-f_{it}(G(a))]$ is similar. □

Lemma 3.11 *Let L be a GPLS, and let G be the map associated with L defined as above. Then G is an anti-superinvolution on the superalgebra A associated with L.*

Proof Assume that $a,b\in A$ are arbitrary homogeneous elements, $1\leqslant i,t,k,l\leqslant N$, and i,t,k,l are distinct.

From Lemma 3.8, we have $\rho G=G\rho$, $G^2=id$, and G is homogeneous of degree 0. Notice that $\rho^2=id$ and ρ is a superalgebra automorphism on A. Then, from Proposition 3.9, we have

$$[[h_{it},f_{ti}(a)],f_{ik}(b)]=[h_{kl},f_{li}(G(\rho(ab))+(-1)^{|a|\,|b|}G(\rho(b))a)],$$

$$[[h_{it},f_{ti}(G(\rho(a)))],f_{ik}(b)]=[h_{kl},f_{li}(G(G(a)\rho(b))+(-1)^{|a|\,|b|}G(\rho(b))G(\rho(a)))],$$

$$[[g_{it},-f_{it}(G(a))],f_{ki}(b)]=[g_{kl},f_{il}(-(-1)^{|a|\,|b|}G(\rho(bG(a)))+G(a)G(\rho(b)))],$$

$$[[g_{it},f_{it}(\rho(a))],f_{ki}(b)]=[g_{kl},f_{il}((-1)^{|a|\,|b|}G(\rho(b)a)-\rho(a)G(\rho(b)))].$$

From Lemma 3.10,

$$[h_{it},f_{ti}(a)]=[h_{it},f_{ti}(G(\rho(a)))],\quad [g_{it},-f_{it}(G(a))]=[g_{it},f_{it}(\rho(a))],$$

and from Lemma 3.6, h_{kl} is injective on $L_{\varepsilon_l-\varepsilon_i}$, g_{kl} is injective on $L_{\varepsilon_i-\varepsilon_l}$. Then we get

$$G(\rho(ab))+(-1)^{|a|\,|b|}G(\rho(b))a=G(G(a)\rho(b))+(-1)^{|a|\,|b|}G(\rho(b))G(\rho(a)),\quad (5)$$

$$-(-1)^{|a|\,|b|}G(\rho(bG(a)))+G(a)G(\rho(b))=(-1)^{|a|\,|b|}G(\rho(b)a)-\rho(a)G(\rho(b)).\quad (6)$$

Acting (5) by ρ, acting (6) by $\rho\circ G$, and keeping in mind $\rho G=G\rho$, $G^2=\rho^2=id$, and ρ being a superalgebra automorphism on A, we have

$$G(ab)+(-1)^{|a|\,|b|}G(b)\rho(a)=G(G(\rho(a))b)+(-1)^{|a|\,|b|}G(b)G(a),\quad (7)$$

$$-(-1)^{|a|\,|b|}bG(a)+G(G(\rho(a))G(b))=(-1)^{|a|\,|b|}b\rho(a)-G(aG(b)). \tag{8}$$

For (8), by using the arbitrariness of b, we replace b by $G(b)$, and notice that G is homogeneous of degree 0, $G^2=id$. Then

$$-(-1)^{|a|\,|b|}G(b)G(a)+G(G(\rho(a))b)=(-1)^{|a|\,|b|}G(b)\rho(a)-G(ab).$$

Exchanging of left- and right-hand sides, we get

$$G(ab)-(-1)^{|a|\,|b|}G(b)\rho(a)=-G(G(\rho(a))b)+(-1)^{|a|\,|b|}G(b)G(a). \tag{9}$$

Adding (7) and (9), we get

$$G(ab)=(-1)^{|a|\,|b|}G(b)G(a),$$

i.e., G is a superalgebra anti-endomorphism on the superalgebra A associated with L.

Furthermore, G is an anti-superinvolution from Lemma 3.8. □

Theorem 3.1 *Let L be a generalized $P(N-1)$-graded Lie superalgebra over the characteristic zero field $\mathbb{F}$. If $N\geqslant 4$, then there exists a unique (up to isomorphism) unital associative $\mathbb{F}$-superalgebra A; furthermore, it exists an anti-superinvolution G on A such that L is centrally isogenous with the matrix Lie superalgebra $\mathscr{P}_N(A,G)$.*

Proof From Lemma 3.3, there exists a unique (up to isomorphism) unital associative $\mathbb{F}$-superalgebra A associated with GPLS L such that $L_{\Delta_{\bar 0}}$ is centrally isogenous with $sl_N(A)$.

From Lemma 3.11, we get an anti-superinvolution G on the unital associative superalgebra A. Through the given construction in Section 2, we obtain a GPLS $\mathscr{P}_N(A,G)$.

The root vectors of $\mathscr{P}_N(A,G)$ denote by

$$\begin{aligned}\widetilde{f}_{ij}(a)&=ae_{ij}-G(a)e_{N+j,N+i},\\ \widetilde{g}_{ij}(a)&=ae_{i,N+j}-G(\rho(a))e_{j,N+i},\\ \widetilde{h}_{ij}(a)&=ae_{N+i,j}+G(\rho(a))e_{N+j,i}.\end{aligned} \tag{10}$$

Notice that $\mathscr{P}_N(A,G)_{\Delta_{\bar 0}}$ is centrally isogenous with $sl_N(A)$ too. Then, by Lemma 3.3, there exists a family of homogenous of degree 0, $\mathbb{F}$-linear isomorphisms

$$\eta=(\eta_{\varepsilon_m-\varepsilon_n},\,1\leqslant m\neq n\leqslant N),\quad \eta_{\varepsilon_m-\varepsilon_n}\colon L_{\varepsilon_m-\varepsilon_n}\to\mathscr{P}_N(A,G)_{\varepsilon_m-\varepsilon_n},$$

such that

$$\eta_{\varepsilon_m-\varepsilon_n+\varepsilon_p-\varepsilon_q}([x_{\varepsilon_m-\varepsilon_n},x_{\varepsilon_p-\varepsilon_q}])=[\eta_{\varepsilon_m-\varepsilon_n}(x_{\varepsilon_m-\varepsilon_n}),\eta_{\varepsilon_p-\varepsilon_q}(x_{\varepsilon_p-\varepsilon_q})],$$

$$\eta(f_{mn})=\widetilde{f}_{mn}(1),$$

for arbitrary $m,n,p,q=1,2,\ldots,N$, $m\neq n$, $p\neq q$, $x_{\varepsilon_m-\varepsilon_n}\in L_{\varepsilon_m-\varepsilon_n}$, $x_{\varepsilon_p-\varepsilon_q}\in L_{\varepsilon_p-\varepsilon_q}$.

For any $a\in A$, we denote $\eta^{-1}(\widetilde{f}_{ij}(a))$ by $f_{ij}(a)$.

Next, we assume that $1\leqslant i,j,k,l,t\leqslant N$, $t\neq i,j,k$, and $a,b\in A$ are any homogeneous elements.

From Lemma 3.4, we have

$$L_{\varepsilon_i+\varepsilon_j}=[g_{it},L_{\varepsilon_j-\varepsilon_t}],\quad L_{-\varepsilon_i-\varepsilon_j}=[h_{it},L_{\varepsilon_t-\varepsilon_j}],$$

Define an extension of η as follows, which is denoted by $\widehat{\eta}$:

$$\widehat{\eta}(f_{ij}(a))=\widetilde{f}_{ij}(a),\quad \widehat{\eta}([h_{it},f_{tj}(a)])=\widetilde{h}_{ij}(a),\quad \widehat{\eta}([g_{it},-f_{jt}(G(a))])=\widetilde{g}_{ij}(a).$$

From Lemma 3.6, for any odd root $\pm(\varepsilon_i+\varepsilon_j)$, $i\neq j$, we have

$$[g_{it},-f_{jt}(G(a))]=0\iff a=0\iff [h_{it},f_{tj}(a)]=0.$$

From Lemma 3.10, for any odd root $\pm 2\varepsilon_i$, $1\leqslant i\leqslant N$, we have

$$[h_{it},f_{ti}(a)]=0\iff a+G(\rho_(a))=0,$$

$$[g_{it},-f_{it}(G(a))]=0\iff a-G(\rho(a))=0,$$

Comparing these with the root vectors of $\mathscr{P}_N(A,G)$ which are given in formula (10), we get that $\widehat{\eta}$ is a collection of well-defined bijective mappings.

Observing the definition of $\widehat{\eta}$, we get that $\widehat{\eta}$ is a collection of homogenous of degree 0, $\mathbb{F}$-linear mappings.

For arbitrary homogeneous $a,b\in A$, from Theorem 2.2, we have

$$[\widetilde{g}_{ij}(a),\widetilde{f}_{kl}(b)]=-(-1)^{|b|\,|a|}\delta_{il}\widetilde{g}_{kj}(\rho(b)a)+(-1)^{|b|\,|a|}\delta_{jl}\widetilde{g}_{ki}(\rho(b)G(\rho(a))),\quad (11)$$

$$[\widetilde{f}_{ij}(a),\widetilde{f}_{kl}(b)]=\delta_{jk}\widetilde{f}_{il}(ab)-(-1)^{|b|\,|a|}\delta_{il}\widetilde{f}_{kj}(ba),$$

$$\begin{aligned}[\widetilde{g}_{ij}(a),\widetilde{h}_{kl}(b)]=&-\delta_{ik}\widetilde{f}_{jl}(G(\rho(a))b)+\delta_{jk}\widetilde{f}_{il}(ab)\\&-\delta_{il}\widetilde{f}_{jk}(G(\rho(a))G(\rho(b)))+\delta_{jl}\widetilde{f}_{ik}(aG(\rho(b))),\end{aligned}\quad (12)$$

$$[\widetilde{f}_{ij}(a),\widetilde{h}_{kl}(b)]=-\delta_{ik}\widetilde{h}_{jl}(G(a)b)-(-1)^{|b|\,|a|}\delta_{il}\widetilde{h}_{kj}(b\rho(a)).\quad (13)$$

Obviously, for any $\alpha\in\Delta$, $\mathscr{P}_N(A,G)$ satisfies

$$\mathscr{P}_N(A,G)_\alpha=\sum_{\delta,\gamma\in\Delta,\,\delta,\gamma\neq\pm\alpha,\,\delta+\gamma=\alpha}[\mathscr{P}_N(A,G)_\delta,\mathscr{P}_N(A,G)_\gamma].$$

Now, we only remain to check that for arbitrary

$$\alpha,\beta,\alpha+\beta\in\Delta,\quad x_\alpha\in L_\alpha,\quad x_\beta\in L_\beta,$$

$\widehat{\eta}$ satisfies

$$\widehat{\eta}_{\alpha+\beta}([x_\alpha,x_\beta])=[\widehat{\eta}_\alpha(x_\alpha),\widehat{\eta}_\beta(x_\beta)].$$

By using Lemma 3.8, G is an anti-superinvolution and keep in mind $t \neq i, j, k$. We have

$$\begin{aligned}
&[[g_{it}, -f_{jt}(G(a))], f_{ki}(b)] \\
&\quad = [g_{it}, [-f_{jt}(G(a)), f_{ki}(b)]] + (-1)^{|a|\,|b|}[[g_{it}, f_{ki}(b)], -f_{jt}(G(a))] \\
&\quad = [g_{it}, (-1)^{|a|\,|b|}\delta_{ij} f_{kt}(bG(a))] + (-1)^{|a|\,|b|}[[g_{ki}, f_{ti}(G(\rho(b)))], -f_{jt}(G(a))] \\
&\quad = (-1)^{|a|\,|b|}\delta_{ij}[g_{it}, f_{kt}(bG(a))] - (-1)^{|a|\,|b|}[g_{ki}, [f_{ti}(G(\rho(b))), f_{jt}(G(a))]] \\
&\quad = (-1)^{|a|\,|b|}\delta_{ij}[g_{kt}, -f_{it}(G(\rho(bG(a))))] \\
&\qquad - (-1)^{|a|\,|b|}[g_{ki}, \delta_{ij} f_{tt}(G(\rho(b))G(a))] - [g_{ki}, -f_{ji}(G(a)G(\rho(b)))] \\
&\quad = (-1)^{|a|\,|b|}\delta_{ij}[g_{kt}, -f_{it}(G(\rho(bG(a))))] - (-1)^{|a|\,|b|}[g_{ki}, -f_{ji}(G(\rho(b)a))].
\end{aligned}$$

Notice that $\rho G = G\rho$ and ρ is a superalgebra automorphism. Then we get

$$\begin{aligned}
&\widehat{\eta}([[g_{it}, -f_{jt}(G(a))], f_{ki}(b)]) \\
&\quad = -(-1)^{|b|\,|a|}\widetilde{g}_{kj}(\rho(b)a) + (-1)^{|b|\,|a|}\delta_{ji}\widetilde{g}_{ki}(\rho(b)G(\rho(a))).
\end{aligned}$$

Comparing it with (11), they are consistent.

Similarly, we can get that it holds when $l = j$.

Again using Lemma 3.8, $\rho(a) = (-1)^{|a|}a$, and noticing that $t \neq i, j, k$, we have

$$\begin{aligned}
&[f_{ij}(a), [h_{kt}, f_{ti}(b)]] \\
&\quad = [[f_{ij}(a), h_{kt}], f_{ti}(b)] + (-1)^{|a|}[h_{kt}, [f_{ij}(a), f_{ti}(b)]] \\
&\quad = (-1)^{|a|+1}\delta_{ik}[[h_{it}, f_{ij}(a)], f_{ti}(b)] - (-1)^{|a|+|a|\,|b|}[h_{kt}, f_{tj}(ba)] \\
&\quad = -\delta_{ik}[[h_{ji}, f_{it}(G(a))], f_{ti}(b)] - (-1)^{|a|+|a|\,|b|}[h_{kt}, f_{tj}(ba)] \\
&\quad = -\delta_{ik}[h_{ji}, [f_{it}(G(a)), f_{ti}(b)]] - (-1)^{|a|\,|b|}[h_{kt}, f_{tj}(b\rho(a))] \\
&\quad = -\delta_{ik}[h_{ji}, f_{ii}(G(a)b)] - (-1)^{|a|\,|b|}[h_{kt}, f_{tj}(b\rho(a))] \\
&\quad = -\delta_{ik}[h_{ji}, f_{ii}(G(a)b)] - (-1)^{|a|\,|b|}[h_{kt}, f_{tj}(b\rho(a))].
\end{aligned}$$

Then we get

$$\widehat{\eta}([f_{ij}(a), [h_{kt}, f_{ti}(b)]]) = -\delta_{ik}\widetilde{h}_{ji}(G(a)b) - (-1)^{|a|\,|b|}\widetilde{h}_{kj}(b\rho(a)).$$

Comparing it with (13), they are consistent.

Similarly, we can get that it holds when $k = i$.

When i, j, k are distinct, let $1 \leqslant t' \leqslant N$, $t' \neq k, i$. Using Lemma 3.8 and $\rho(a) = (-1)^{|a|}a$, we have

$$\begin{aligned}
&[[g_{it}, -f_{jt}(G(a))], [h_{kt'}, f_{t'i}(b)]] \\
&\quad = [g_{it}, [-f_{jt}(G(a)), [h_{kt'}, f_{t'i}(b)]]] \\
&\qquad + (-1)^{|a|(|b|+1)}[[g_{it}, [h_{kt'}, f_{t'i}(b)]], -f_{jt}(G(a))]
\end{aligned}$$

$$
\begin{aligned}
&= [g_{it}, [-f_{jt}(G(a)), [h_{kt'}, f_{t'i}(b)]]] \\
&= (-1)^{|a|(|b|+1)}[[g_{it}, [h_{kj}, f_{ji}(b)]], -f_{jt}(G(a))] \\
&= (-1)^{|a|(|b|+1)}[[h_{kj}, [g_{ti}, f_{ji}(b)]], -f_{jt}(G(a))] \\
&= (-1)^{|a|(|b|+1)}[[h_{kj}, [g_{ji}, f_{ti}(G(\rho(b)))]], f_{jt}(G(a))] \\
&= (-1)^{|a|(|b|+1)+|b|}[[[h_{kj}, g_{ji}], f_{ti}(G(b))], f_{jt}(G(a))] \\
&= (-1)^{|a|(|b|+1)+|b|}[[-f_{ik}, f_{ti}(G(b))], f_{jt}(G(a))] \\
&= (-1)^{|a|+|b|+1} f_{jk}(G(a)G(b)) \\
&= -\, f_{jk}(G(\rho(a))G(\rho(b))).
\end{aligned}
$$

Then we get

$$\widehat{\eta}([[g_{it}, -f_{jt}(G(a))], [h_{kt'}, f_{t'i}(b)]]) = -\widetilde{f}_{jk}(G(\rho(a))G(\rho(b))).$$

Comparing it with (12), they are consistent.

When $i = k \neq j$, let $1 \leqslant t' \leqslant N, t' \neq i$. Then we have

$$
\begin{aligned}
&[[g_{it}, -f_{jt}(G(a))], [h_{it'}, f_{t'i}(b)]] \\
&\quad = [g_{it}, [-f_{jt}(G(a)), [h_{it'}, f_{t'i}(b)]]] + (-1)^{|a|(|b|+1)}[[g_{it}, [h_{it'}, f_{t'i}(b)]], -f_{jt}(G(a))] \\
&\quad = (-1)^{|a|(|b|+1)}[[g_{it}, [h_{ij}, f_{ji}(b)]], -f_{jt}(G(a))] \\
&\quad = (-1)^{|a|(|b|+1)}([[[g_{it}, h_{ij}], f_{ji}(b)], -f_{jt}(G(a))] - [[h_{ij}, [g_{it}, f_{ji}(b)]], -f_{jt}(G(a))]) \\
&\quad = (-1)^{|a|(|b|+1)}([[f_{tj}, f_{ji}(b)], f_{jt}(G(a))] - [[h_{ij}, [g_{ji}, f_{ti}(G(\rho(b)))]], -f_{jt}(G(a))]) \\
&\quad = -\, f_{ji}(G(\rho(a))b) - (-1)^{|a|(|b|+1)}[[[h_{ij}, g_{ji}], f_{ti}(G(\rho(b)))], -f_{jt}(G(a))] \\
&\quad = -\, f_{ji}(G(\rho(a))b) - (-1)^{|a|\,|b|}[[f_{jj} - f_{ii}, f_{ti}(G(\rho(b)))], -f_{jt}(G(\rho(a)))] \\
&\quad = -\, f_{ji}(G(\rho(a))b) - f_{ji}(G(\rho(a))G(\rho(b))).
\end{aligned}
$$

Thus,

$$\widehat{\eta}([[g_{it}, -f_{jt}(G(a))], [h_{it'}, f_{t'i}(b)]]) = -\widetilde{f}_{ji}(G(\rho(a))b) - \widetilde{f}_{ji}(G(\rho(a))G(\rho(b))).$$

Comparing it with (12), they are consistent.

The proofs of other cases are similar.

Now, summarizing the above discussion, from Lemma 3.1, we get the theorem holds at once. □

Then, by using Theorems 2.1 and 3.1, we get the classification of $P(N-1)$-graded Lie superalgebras at once, which is isomorphic to $P(N-1) \otimes_{\mathbb{F}} A$ given in [19].

Corollary *Let L be a $P(N-1)$-graded Lie superalgebra over the characteristic zero field $\mathbb{F}$ with $N \geqslant 4$. Then there exists a unique (up to isomorphism) unital associative supercommutative $\mathbb{F}$-superalgebra A such that L is centrally* *isogenous* (*for* $N > 4$ *indeed is isomorphic*) *to*

$$\begin{aligned}&\mathscr{P}_N(A,\rho)\\&=\left\{\begin{pmatrix}X & Y\\ Z & -\rho(X)^{\mathrm{t}}\end{pmatrix}\,\middle|\, X,Y,Z\in M_N(A),\ \mathrm{tr}(X)=0,\ Y=-Y^{\mathrm{t}},\ Z=Z^{\mathrm{t}}\right\}\\&\cong P(N-1)\otimes_{\mathbb{F}} A.\end{aligned}$$

Proof Assume that $a, b \in A$ are arbitrary homogeneous elements.

Notice that A is supercommutative. From Theorem 2.2, we get the non-trivial Lie superbrackets between the root vectors of $\mathscr{P}_N(A,\rho)$ as follows:

$$\begin{aligned}[\widetilde{g}_{ij}(a),\widetilde{f}_{kl}(b)] &= -\delta_{il}\widetilde{g}_{kj}(a\rho(b))+\delta_{jl}\widetilde{g}_{ki}(a\rho(b)),\\ [\widetilde{g}_{ij}(a),\widetilde{h}_{kl}(b)] &= -\delta_{ik}\widetilde{f}_{jl}(ab)+\delta_{jk}\widetilde{f}_{il}(ab)-\delta_{il}\widetilde{f}_{jk}(ab)+\delta_{jl}\widetilde{f}_{ik}(ab),\\ [\widetilde{f}_{ij}(a),\widetilde{h}_{kl}(b)] &= -\delta_{ik}\widetilde{h}_{jl}(\rho(a)b)-\delta_{il}\widetilde{h}_{kj}(\rho(a)b),\\ [\widetilde{f}_{ij}(a),\widetilde{f}_{kl}(b)] &= \delta_{jk}\widetilde{f}_{il}(ab)-\delta_{il}\widetilde{f}_{kj}(ab).\end{aligned}$$

We denote the root vectors of $P(N-1)\otimes_{\mathbb{F}} A$ by

$$F_{ij}(a) := af_{ij}, \quad G_{ij}(a) := ag_{ij}, \quad H_{ij}(a) := ah_{ij}.$$

The Lie superbracket of $P(N-1)\otimes_{\mathbb{F}} A$ is defined by

$$[X\otimes a, Y\otimes b] = (-1)^{|a||Y|}[X,Y]\otimes ab, \quad X,Y\in P(N-1),\ a,b\in A,$$

Then

$$\begin{aligned}[G_{ij}(\rho(a)),F_{kl}(b)] &= [g_{ij},f_{kl}]\otimes\rho(a)b = -\delta_{il}G_{kj}(\rho(a\rho(b)))+\delta_{jl}G_{ki}(\rho(a\rho(b))),\\ [G_{ij}(\rho(a)),H_{kl}(b)] &= (-1)^{|a|}[g_{ij},h_{kl}]\otimes\rho(a)b\\ &= -\delta_{ik}F_{jl}(ab)+\delta_{jk}F_{il}(ab)-\delta_{il}F_{jk}(ab)+\delta_{jl}F_{ik}(ab),\\ [F_{ij}(a),H_{kl}(b)] &= (-1)^{|a|}[f_{ij},h_{kl}]\otimes ab-\delta_{ik}H_{jl}(\rho(a)b)-\delta_{il}H_{kj}(\rho(a)b),\\ [F_{ij}(a),F_{kl}(b)] &= \delta_{jk}F_{il}(ab)-\delta_{il}F_{kj}(ab).\end{aligned}$$

Now, we get a isomorphism

$$\widetilde{f}_{ij}(a)\mapsto F_{ij}(a), \quad \widetilde{g}_{ij}(a)\mapsto G_{ij}(\rho(a)), \quad \widetilde{h}_{ij}(a)\mapsto H_{ij}(a),$$

between $\mathscr{P}_N(A,\rho)$ and $P(N-1)\otimes_{\mathbb{F}} A$.

Further, A is an $\mathbb{F}$-superextension (see [20, Sect. 1.1]) and $P(N-1)$, $N>4$, is centrally closed. Then applying [20, Lemma 1.12], we have

$$\mathfrak{uce}(A\otimes_{\mathbb{F}} P(N-1)) \cong A\otimes_{\mathbb{F}} \mathfrak{uce}(P(N-1)) \cong A\otimes_{\mathbb{F}} P(N-1).$$

On the other hand, $P(N-1)\otimes_{\mathbb{F}} A$ is centerless. Then, for any $P(N-1)$-graded Lie superalgebra L over $\mathbb{F}$, there exists a unique unital associative supercommutative $\mathbb{F}$-superalgebra A such that $L\cong P(N-1)\otimes_{\mathbb{F}} A$ if $N>4$. (Or see [19, Sect. 6].) □

Remark In addition, by using the connection between Lie super and Jordan super structures through Tits-Kantor-Koecher TKK construction (see [13]) and the Coordinatization Theorem for Jordan superalgebras (see [18]) of type $JP(n)$, we can obtain the characterization for GPLS when N is even as follows immediately. It is a generalization of Kac's result [13] about the connection between the finite dimensional simple Lie superalgebra $P(2n-1)$ and simple Jordan superalgebra $J_n(\mathbb{F}, id_{\mathbb{F}})$.

Proposition 3.12 *Suppose* $n\geqslant 4$.

(i) *Any Lie superalgebra centrally isogenous with a TKK Lie superalgebras* $\mathscr{K}(J)$ *of a Jordan superalgebra* J *which contains* $J_n(\mathbb{F}, id_{\mathbb{F}})$ *as a unital sub-superalgebra is generalized* $P(2n-1)$*-graded.*

(ii) *Let* L *be a generalized* $P(2n-1)$*-graded Lie superalgebra over* $\mathbb{F}$*. Then there exists a unital associative* $\mathbb{F}$*-superalgebra* A *with an anti-superinvolution* $^-$*, such that* L *is centrally isogenous to* $\mathscr{K}(J_n(A,-))$*, which is centrally isogenous with the matrix Lie superalgebra* $\mathscr{P}_{2n}(A,-)$ *(furthermore, if* A *is supercommutative, then they are isomorphic indeed).*

Proof For a Lie superalgebra L, it is well known that (see [17]) if L contains an sl_2-triple

$$sl_2 = Fe+Fh+Ff,$$

with

$$[h,e]=2e,\quad [h,f]=-2f,\quad [e,f]=h,$$

such that $\operatorname{ad} h\colon L\to L$ is diagonalizable which only having eigenvalues $-2,0,2$, then L_2 becomes a Jordan superalgebra with the product $x\circ y:=\frac{1}{2}\,[[x,f],y]$.

Let L be a generalized $P(2n-1)$-graded Lie superalgebra over the characteristic zero field $\mathbb{F}$. Then

$$h=\sum_{i=1}^{n} f_{ii}-\sum_{j=n+1}^{2n} f_{jj},\quad e=\sum_{i=1}^{n} f_{i,n+i},\quad f=\sum_{i=1}^{n} f_{n+i,i},$$

form an sl_2-triple in L, the operator $\operatorname{ad} h$ acts on L which only having eigenvalues $-2,0,2$, and

$$L_2=\sum_{1\leqslant i\leqslant n,\, n+1\leqslant j\leqslant 2n} L_{\varepsilon_i-\varepsilon_j}+\sum_{1\leqslant i,j\leqslant n} L_{\varepsilon_i+\varepsilon_j}+\sum_{n+1\leqslant i,j\leqslant 2n} L_{-\varepsilon_i-\varepsilon_j}$$

with product $X\circ Y=\frac{1}{2}\,[[X,f],Y]$ is a Jordan superalgebra.

From [13], we know that $P(2n-1)_2$ is isomorphic to the Jordan superalgebra $J_n(\mathbb{F}, id_{\mathbb{F}})$. Then up to isomorphism, L_2 contains $J_n(\mathbb{F}, id_{\mathbb{F}})$ as a unital subsuperalgebra. Using the Coordinatization Theorem for Jordan superalgebras of type $JP(n)$, $n \geqslant 4$, which was given in [18], we have L_2 is isomorphic to some Jordan matrix superalgebra $J_n(A,G)$, where A is a unital associative $\mathbb{F}$-superalgebra with an anti-superinvolution G.

For the generalized $P(2n-1)$-graded Lie superalgebra $\mathscr{P}_{2n}(A,G)$, we claim that $\mathscr{P}_{2n}(A,G)_2$ is isomorphic to $J_n(A,G)$ as Jordan superalgebras.

We denote the root vectors of $\mathscr{P}_{2n}(A,G)$ by

$$\begin{aligned}\widetilde{f}_{ij}(a) &= ae_{ij} - G(a)e_{2n+j,2n+i},\\ \widetilde{g}_{ij}(a) &= ae_{i,2n+j} - G(\rho(a))e_{j,2n+i},\\ \widetilde{h}_{ij}(a) &= ae_{2n+i,j} + G(\rho(a))e_{2n+j,i},\end{aligned}$$

where $1 \leqslant i,j \leqslant 2n$, $a \in A$.

Let

$$\begin{aligned}F_{kl}(a) &:= ae_{kl} + G(a)e_{n+l,n+k},\\ G_{kl}(a) &:= ae_{k,n+l} - G(\rho(a))e_{l,n+k},\\ H_{kl}(a) &:= ae_{n+k,l} + G(\rho(a))e_{n+l,k},\end{aligned}$$

where $a \in A$, $1 \leqslant k,l \leqslant n$. Then

$$\begin{aligned}\widetilde{f}_{ij}(a) &\mapsto F_{i,j-n}(a), \quad 1 \leqslant i \leqslant n < j \leqslant 2n,\\ \widetilde{g}_{ij}(a) &\mapsto -G_{i,j}(a), \quad 1 \leqslant i,j \leqslant n,\\ \widetilde{h}_{ij}(a) &\mapsto H_{i-n,j-n}, \quad 1+n \leqslant i,j \leqslant 2n,\end{aligned}$$

give an isomorphism between $\mathscr{P}_{2n}(A,G)_2$ and $J_n(A,G)$.

Indeed, for $1 \leqslant i,k \leqslant n < j,l \leqslant 2n$,

$$\begin{aligned}\widetilde{f}_{ij}(a) \circ \widetilde{f}_{kl}(b) &= \frac{1}{2}\,[[\widetilde{f}_{ij}(a), f], \widetilde{f}_{kl}(b)]\\ &= \frac{1}{2}\,[\widetilde{f}_{i,j-n}(a) - \widetilde{f}_{n+i,j}(a), \widetilde{f}_{kl}(b)]\\ &= \frac{1}{2}\,\delta_{j-n,k}\widetilde{f}_{il}(ab) + (-1)^{|a|\,|b|}\frac{1}{2}\,\delta_{n+i,l}\widetilde{f}_{kj}(ba).\end{aligned}$$

$$F_{i,j-n}(a) \circ F_{k,l-n}(b) = \frac{1}{2}\,\delta_{j-n,k}F_{i,l-n}(ab) + (-1)^{|a|\,|b|}\frac{1}{2}\,\delta_{l-n,i}F_{k,j-n}(ba).$$

For $1 \leqslant i,k,l \leqslant n < j \leqslant 2n$,

$$\begin{aligned}\widetilde{f}_{ij}(a) \circ \widetilde{g}_{kl}(b) &= \frac{1}{2}\,[\widetilde{f}_{i,j-n}(a) - \widetilde{f}_{n+i,j}(a), \widetilde{g}_{kl}(b)]\\ &= \frac{1}{2}\,\delta_{j-n,k}\widetilde{g}_{i,l}(ab) - \frac{1}{2}\,\delta_{j-n,l}\widetilde{g}_{i,k}(aG(\rho(b))),\end{aligned}$$

$$
\begin{aligned}
&F_{i,j-n}(a)\circ(-G_{k,l}(b))\\
&\quad= -\frac{1}{2}\,\delta_{j-n,k}abe_{i,n+l}-(-1)^{|a|(|b|+1)}\frac{1}{2}\,bG(a)\delta_{n+l,j}e_{k,i+n}\\
&\qquad+\frac{1}{2}\,(-1)^{|a|(|b|+1)}\delta_{n+k,j}G(\rho(b))G(a)e_{l,n+i}+\frac{1}{2}\,\delta_{j-n,l}aG(\rho(b))e_{i,n+k}\\
&\quad=\frac{1}{2}\,\delta_{j-n,k}(-G_{i,l}(ab))-\frac{1}{2}\,\delta_{j-n,l}(-G_{i,k}(aG(\rho(b)))).
\end{aligned}
$$

For $1\leqslant i,j\leqslant n<k,l\leqslant 2n$,

$$
\begin{aligned}
\widetilde{g}_{ij}(a)\circ\widetilde{h}_{k,l}(b)&=\frac{1}{2}\,[[\widetilde{g}_{ij}(a),f],\widetilde{h}_{k,l}(b)]\\
&=\frac{1}{2}\,[\widetilde{g}_{j,n+i}(G(\rho(a)))-\widetilde{g}_{i,n+j}(a),\widetilde{h}_{k,l}(b)]\\
&=-\frac{1}{2}\,\delta_{j+n,k}\widetilde{f}_{i,l}(ab)+\frac{1}{2}\,\delta_{i+n,k}\widetilde{f}_{j,l}(G(\rho(a))b)\\
&\quad-\frac{1}{2}\,\delta_{j+n,l}\widetilde{f}_{i,k}(aG(\rho(b)))+\frac{1}{2}\,\delta_{i+n,l}\widetilde{f}_{j,k}(G(\rho(a))G(\rho(b))),
\end{aligned}
$$

$$
\begin{aligned}
&-G_{i,j}(a)\circ H_{k-n,l-n}(b)\\
&\quad=-\frac{1}{2}\,\delta_{j+n,k}abe_{i,l-n}-(-1)^{(|a|+1)(|b|+1)}\frac{1}{2}\,\delta_{l-n,i}bae_{k,j+n}\\
&\qquad+\frac{1}{2}\,\delta_{i+n,k}G(\rho(a))be_{j,l-n}+(-1)^{(|a|+1)(|b|+1)}\frac{1}{2}\,\delta_{l-n,j}bG(\rho(a))e_{k,i+n}\\
&\qquad-\frac{1}{2}\,\delta_{j+n,l}aG(\rho(b))e_{i,k-n}-(-1)^{(|a|+1)(|b|+1)}\frac{1}{2}\,\delta_{k-n,i}G(\rho(b))ae_{l,j+n}\\
&\qquad+\frac{1}{2}\,\delta_{i+n,l}G(\rho(a))G(\rho(b))e_{j,k-n}\\
&\qquad+(-1)^{(|a|+1)(|b|+1)}\frac{1}{2}\,\delta_{k-n,j}G(\rho(b))G(\rho(a))e_{l,i+n}\\
&\quad=-\frac{1}{2}\,\delta_{j+n,k}F_{i,l-n}(ab)+\frac{1}{2}\,\delta_{i+n,k}F_{j,l-n}(G(\rho(a))b)\\
&\qquad-\frac{1}{2}\,\delta_{j+n,l}F_{i,k-n}(aG(\rho(b)))+\frac{1}{2}\,\delta_{i+n,l}F_{j,k-n}(G(\rho(a))G(\rho(b))).
\end{aligned}
$$

Similarly, we can check that the remaining products are consistent too. Thus, we get

$$\mathscr{P}_{2n}(A,G)_2\cong J_n(A,G).$$

We denote the TKK construction for a Jordan superalgebra J by $\mathscr{K}(J)$, and [4, 1.11–1.14] obviously can be extended to the super case. Then $\mathscr{P}_{2n}(A,G)$ is centrally isogenous with the centerless TKK Lie superalgebra $\mathscr{K}(J_n(A,G))$. Since $L_2\cong J_n(A,G)$, L is centrally isogenous with $\mathscr{K}(J_n(A,G))$ too.

(i) follows from the above discussion and Theorem 2.1 immediately. □

4 GPLS coordinatized by quantum tori

4.1 Structure of GPLS coordinatized by quantum tori

We denote the field of complex numbers by $\mathbb{C}$. We first recall some basic facts on quantum tori.

Let $0 \neq q \in \mathbb{C}$. A quantum torus associated to q (see [16]) is a unital associative $\mathbb{C}$-algebra $\mathbb{C}_q[x^{\pm 1}, y^{\pm 1}]$ (or simply, $\mathbb{C}_q$) with generators $x^{\pm 1}$, $y^{\pm 1}$ and relations

$$xx^{-1} = x^{-1}x = yy^{-1} = y^{-1}y = 1, \quad yx = qxy.$$

Then

$$x^m y^n x^p y^s = q^{np} x^{m+p} y^{n+s},$$
$$\mathbb{C}_q = \sum_{m,n\in\mathbb{Z}} \oplus \mathbb{C} x^m y^n.$$

Set $\Lambda(q) = \{u \in \mathbb{Z} \mid q^u = 1\}$. q is said to be generic if $\Lambda(q) = \{0\}$.

From [5], we see that $[\mathbb{C}_q, \mathbb{C}_q]$ has a basis consisting of monomials $x^m y^n$ for $m \notin \Lambda(q)$ or $n \notin \Lambda(q)$.

Let $^-$ be the anti-involution on $\mathbb{C}_q$ given by

$$\overline{x} = x, \quad \overline{y} = y^{-1}.$$

Then

$$\mathbb{C}_q = \mathbb{C}_q^+ \oplus \mathbb{C}_q^-,$$

where

$$\mathbb{C}_q^{\pm} = \{s \in \mathbb{C}_q \mid \overline{s} = \pm s\},$$

and

$$\mathbb{C}_q^+ = \operatorname{span}\{x^m y^n + \overline{x^m y^n} \mid m \in \mathbb{Z},\ n \geqslant 0\},$$
$$\mathbb{C}_q^- = \operatorname{span}\{x^m y^n - \overline{x^m y^n} \mid m \in \mathbb{Z},\ n > 0\}.$$

We get a GPLS $\mathscr{P}_N(\mathbb{C}_q, -)$ through $\mathbb{C}_q$ with the anti-involution $^-$.

Let $0 < M, N \in \mathbb{Z}$. As done in [11], we get a central extension of the Lie superalgebra $gl(M,N)(\mathbb{C}_q)$:

$$\widehat{gl(M,N)}(\mathbb{C}_q) = gl(M,N)(\mathbb{C}_q) \oplus \Big(\sum_{u\in\Lambda(q)} \oplus \mathbb{C}c(u)\Big) \oplus \mathbb{C}c_y$$

with Lie superbracket

$$\begin{aligned}&[A(x^m y^n), B(x^p y^s)]\\ &\quad = A(x^m y^n)B(x^p y^s) - (-1)^{\deg A \deg B} B(x^p y^s)A(x^m y^n)\\ &\qquad + mq^{np}\operatorname{str}(AB)\delta_{m+p,0}\delta_{\overline{n+s},\overline{0}}c(n+s) + nq^{np}\operatorname{str}(AB)\delta_{m+p,0}\delta_{n+s,0}c_y \end{aligned} \tag{14}$$

for $m, p, n, s \in \mathbb{Z}$, $A, B \in gl(M,N)_\alpha$, and $\alpha \in \mathbb{Z}_2$, where str is the supertrace of $gl(M,N)$, $c(u)$ with $u \in \Lambda(q)$, c_y are central elements of $\widehat{gl(M,N)}(\mathbb{C}_q)$, and $\overline{t}$ means $\overline{t} \in \mathbb{Z}/\Lambda(q)$ for $t \in \mathbb{Z}$.

Then we get a nontrivial central extension of $\mathscr{P}_N(\mathbb{C}_q,-)$:

$$\widehat{\mathscr{P}_N(\mathbb{C}_q,-)} = \mathscr{P}_N(\mathbb{C}_q,-) \oplus \Big(\sum_{u\in\Lambda(q)} \oplus \mathbb{C}c(u)\Big) \oplus \mathbb{C}c_y$$

with Lie superbracket as in (14).

Let

$$\begin{aligned} \widetilde{f}_{ij}(m,n) &:= x^m y^n e_{ij} - \overline{x^m y^n}\, e_{N+j,N+i}, \\ \widetilde{g}_{ij}(m,n) &:= x^m y^n e_{i,N+j} - \overline{x^m y^n}\, e_{j,N+i}, \\ \widetilde{h}_{ij}(m,n) &:= x^m y^n e_{N+i,j} + \overline{x^m y^n}\, e_{N+j,i}. \end{aligned}$$

According to Theorem 2.2 and the properties of $\mathbb{C}_q$ discussed above, we get the following root space decomposition of $\mathscr{P}_N(\mathbb{C}_q,-)$.

Proposition 4.1

$$\mathscr{P}_N(\mathbb{C}_q,-) = \mathscr{P}_0 \oplus \sum_{1\leqslant i\neq j\leqslant N} \mathscr{P}_{\varepsilon_i-\varepsilon_j} \oplus \sum_{1\leqslant i\leqslant j\leqslant N} \mathscr{P}_{\varepsilon_i+\varepsilon_j} \oplus \sum_{1\leqslant i\leqslant j\leqslant N} \mathscr{P}_{-\varepsilon_i-\varepsilon_j},$$

where

$$\begin{aligned} \mathscr{P}_{\varepsilon_i-\varepsilon_j} &= \mathrm{span}_{\mathbb{C}}\{\widetilde{f}_{ij}(m,n) \mid m,n\in\mathbb{Z}\}, \\ \mathscr{P}_{\varepsilon_i+\varepsilon_j} &= \mathrm{span}_{\mathbb{C}}\{\widetilde{g}_{ij}(m,n) \mid m,n\in\mathbb{Z}\}, \\ \mathscr{P}_{-\varepsilon_i-\varepsilon_j} &= \mathrm{span}_{\mathbb{C}}\{\widetilde{h}_{ij}(m,n) \mid m,n\in\mathbb{Z}\}, \end{aligned}$$

and

$$\begin{aligned} \mathscr{P}_0 = {} & \mathrm{span}_{\mathbb{C}}\{\widetilde{f}_{ii}(m,n) - \widetilde{f}_{NN}(m,n) \mid 1\leqslant i\leqslant N-1,\ m,n\in\mathbb{Z}\} \\ & \oplus \mathrm{span}_{\mathbb{C}}\{\widetilde{f}_{NN}(m,n) \mid m,n \in (\mathbb{Z}\times\mathbb{Z})\backslash(\Lambda(q)\times\Lambda(q))\}. \end{aligned}$$

About the root vectors of $\widehat{\mathscr{P}_N(\mathbb{C}_q,-)}$, we have the following result.

Proposition 4.2

$$\begin{aligned} [\widetilde{g}_{ij}(m,n), \widetilde{g}_{kl}(p,t)] &= 0, \\ [\widetilde{h}_{ij}(m,n), \widetilde{h}_{kl}(p,t)] &= 0, \end{aligned}$$

$$\begin{aligned} & [\widetilde{g}_{ij}(m,n), \widetilde{h}_{kl}(p,t)] \\ & \quad = -\delta_{ik} q^{-n(m+p)} \widetilde{f}_{jl}(m+p,t-n) + \delta_{jk} q^{np} \widetilde{f}_{il}(m+p,n+t) \\ & \qquad - \delta_{il} q^{-(mn+np+pt)} \widetilde{f}_{jk}(m+p,-(n+t)) + \delta_{jl} q^{(n-t)p} \widetilde{f}_{ik}(m+p,n-t) \\ & \qquad + m q^{np} \delta_{jk}\delta_{il}\delta_{m+p,0}\delta_{\overline{n+t},\overline{0}}(c(n+t)-c(-n-t)) \\ & \qquad + 2n q^{np} \delta_{jk}\delta_{il}\delta_{m+p,0}\delta_{n+t,0} c_y \\ & \qquad + m\delta_{ik}\delta_{jl}\delta_{m+p,0}\delta_{\overline{n-t},\overline{0}}(c(n-t)-c(t-n)) + 2n\delta_{ik}\delta_{jl}\delta_{m+p,0}\delta_{n-t,0}c_y, \end{aligned}$$

$$[\widetilde{g}_{ij}(m,n), \widetilde{f}_{kl}(p,t)] = -\delta_{il} q^{mt} \widetilde{g}_{kj}(m+p,n+t) + \delta_{jl} q^{(t-n)m} \widetilde{g}_{ki}(m+p,t-n),$$

$$\begin{aligned}[\widetilde{f}_{ij}(m,n),\widetilde{f}_{kl}(p,t)] &= \delta_{jk}q^{np}\widetilde{f}_{il}(m+p,n+t)-\delta_{il}q^{tm}\widetilde{f}_{kj}(m+p,n+t)\\ &\quad - mq^{np}\delta_{jk}\delta_{il}\delta_{m+p,0}\delta_{\overline{n+t},\overline{0}}(c(-n-t)-c(n+t))\\ &\quad + 2nq^{np}\delta_{jk}\delta_{il}\delta_{m+p,0}\delta_{n+t,0}c_y,\end{aligned}$$

$$[\widetilde{f}_{ij}(m,n),\widetilde{h}_{kl}(p,t)] = -\delta_{ik}q^{-n(m+p)}\widetilde{h}_{jl}(m+p,t-n)-\delta_{il}q^{mt}\widetilde{h}_{kj}(m+p,n+t),$$

for all $m,p,n,t\in\mathbb{Z}$ *and* $1\leqslant i,j,k,l\leqslant N$.

Proof Since

$$\delta_{\overline{n-t},\overline{0}}q^{np-tp}=\delta_{m+p,0}q^{-mn-np}=\delta_{\overline{n+t},\overline{0}}q^{-tp-np}=1,$$

we have

$$\begin{aligned}
&[\widetilde{g}_{ij}(m,n),\widetilde{h}_{kl}(p,t)]\\
&= [x^my^ne_{i,N+j},x^py^te_{N+k,l}]+[x^my^ne_{i,N+j},\overline{x^py^t}e_{N+l,k}]\\
&\quad -[\overline{x^my^n}e_{j,N+i},x^py^te_{N+k,l}]-[\overline{x^my^n}e_{j,N+i},\overline{x^py^t}e_{N+l,k}]\\
&= (\delta_{jk}x^my^nx^py^te_{il}+\delta_{il}x^py^tx^my^ne_{N+k,N+j}\\
&\quad + mq^{np}\delta_{jk}\delta_{il}\delta_{m+p,0}\delta_{\overline{n+t},\overline{0}}c(n+t)+nq^{np}\delta_{jk}\delta_{il}\delta_{m+p,0}\delta_{n+t,0}c_y)\\
&\quad + (\delta_{jl}x^my^n\overline{x^py^t}e_{ik}+\delta_{ki}\overline{x^py^t}x^my^ne_{N+l,N+j}\\
&\quad + m\delta_{jl}\delta_{ik}\delta_{m+p,0}\delta_{\overline{n-t},\overline{0}}c(n-t)+n\delta_{jl}\delta_{ik}\delta_{m+p,0}\delta_{n-t,0}c_y)\\
&\quad - (\delta_{ik}\overline{x^my^n}x^py^te_{jl}+\delta_{lj}x^py^t\overline{x^my^n}e_{N+k,N+i}\\
&\quad + m\delta_{jl}\delta_{ik}\delta_{m+p,0}\delta_{\overline{n-t},\overline{0}}c(t-n)-n\delta_{jl}\delta_{ik}\delta_{m+p,0}\delta_{n-t,0}c_y)\\
&\quad - (\delta_{il}\overline{x^py^tx^my^n}e_{jk}+\delta_{kj}\overline{x^my^nx^py^t}e_{N+l,N+i}\\
&\quad + mq^{np}\delta_{il}\delta_{jk}\delta_{m+p,0}\delta_{\overline{n+t},\overline{0}}c(-n-t)-nq^{np}\delta_{il}\delta_{jk}\delta_{m+p,0}\delta_{n+t,0}c_y)\\
&= -\delta_{ik}q^{-n(m+p)}\widetilde{f}_{jl}(m+p,t-n)+\delta_{jk}q^{np}\widetilde{f}_{il}(m+p,n+t)\\
&\quad -\delta_{il}q^{-(mn+np+pt)}\widetilde{f}_{jk}(m+p,-(n+t))+\delta_{jl}q^{(n-t)p}\widetilde{f}_{ik}(m+p,n-t)\\
&\quad + mq^{np}\delta_{jk}\delta_{il}\delta_{m+p,0}\delta_{\overline{n+t},\overline{0}}(c(n+t)-c(-n-t))\\
&\quad + 2nq^{np}\delta_{jk}\delta_{il}\delta_{m+p,0}\delta_{n+t,0}c_y\\
&\quad + m\delta_{ik}\delta_{jl}\delta_{m+p,0}\delta_{\overline{n-t},\overline{0}}(c(n-t)-c(t-n))+2n\delta_{ik}\delta_{jl}\delta_{m+p,0}\delta_{n-t,0}c_y.
\end{aligned}$$

The proof of others is similar. □

Remark The subsuperalgebra

$$\widetilde{\mathscr{P}_N(\mathbb{C}_q,-)} := \mathscr{P}_N(\mathbb{C}_q,-)\oplus\left(\sum_{u\in\Lambda(q)^+}\mathbb{C}(c(u)-c(-u))\right)\oplus c_y \qquad (15)$$

of $\widehat{\mathscr{P}_N(\mathbb{C}_q,-)}$ is perfect, which is generalized $P(N-1)$-graded.

4.2 Representation of GPLS coordinatized by quantum tori

In this subsection, we use the Fermionic-Bosonic operators to obtain a class of generalized $P(N-1)$-graded Lie superalgebras coordinatized by quantum tori.

Let $\mathscr{R}$ be an arbitrary associative algebra, $\tau = \pm 1$, and define a τ-bracket on $\mathscr{R}$ as follow:

$$\{a, b\}_\tau = ab + \tau ba, \quad a, b \in \mathscr{R}.$$

Let $\mathfrak{a}$ be a unital associative algebra with generators a_i, a_i^*, $1 \leqslant i \leqslant N$, subject to relations

$$\{a_i, a_j\}_\tau = \{a_i^*, a_j^*\}_\tau = 0, \quad \{a_i, a_j^*\}_\tau = \delta_{ij}.$$

Let the associative algebra $\alpha(N, \tau)$ be generated by

$$\left\{ u(m) \,\middle|\, u \in \bigoplus_{i=1}^{N} (\mathbb{C}a_i \oplus \mathbb{C}a_i^*),\ m \in \mathbb{Z} \right\}$$

subject to relations

$$\{u(m), v(n)\}_\tau = \{u, v\}_\tau \delta_{m+n,0}.$$

Then we define the normal ordering as in [9]:

$$\begin{aligned} : u(m)v(n) : &= \begin{cases} u(m)v(n), & n > m, \\ \dfrac{1}{2}\left(u(m)v(n) - \tau v(n)u(m)\right), & m = n, \\ -\tau v(n)u(m), & m > n, \end{cases} \\ &= -\tau : v(n)u(m) : \end{aligned}$$

for $n, m \in \mathbb{Z}$, $u, v \in \mathfrak{a}$. Set

$$\theta(n) = \begin{cases} 1, & n > 0, \\ \dfrac{1}{2}, & n = 0, \\ 0, & n < 0. \end{cases}$$

Then $1 - \theta(n) = \theta(-n)$. Thus, we have

$$\begin{aligned} : a_i(m)a_j(n) : &= a_i(m)a_j(n) = -\tau a_j(n)a_i(m), \\ : a_i^*(m)a_j^*(n) : &= a_i^*(m)a_j^*(n) = -\tau a_j^*(n)a_i^*(m), \\ a_i(m)a_j^*(n) &= : a_i(m)a_j^*(n) : + \delta_{ij}\delta_{m+n,0}\theta(m-n), \\ a_j^*(n)a_i(m) &= : a_i(m)a_j^*(n) : - \delta_{ij}\delta_{m+n,0}\theta(n-m). \end{aligned} \tag{16}$$

For the Fermionic and Bosonic quadratic operators, by a direct computation, we have the following result.

Proposition 4.3 *The subspace of Clifford algebra $\alpha(N,+1)$ consisted of quadratic operators are closed under the Lie bracket $[\cdot\,,\cdot]_-$. Furthermore, the Lie-commutators of Fermionic quadratic basis operators $a_i(m)a_j(n)$, $a_i(m)a_j^*(n)$, $a_i^*(m)a_j^*(n)$ are as follows*:

$$\begin{aligned}
&[a_i(m)a_j(n), a_k(p)a_l(t)]_- = 0,\\
&[a_i(m)a_j(n), a_k(p)a_l^*(t)]_- = \delta_{jl}\delta_{n,-t}a_k(p)a_i(m) - \delta_{il}\delta_{m,-t}a_k(p)a_j(n),\\
&[a_i(m)a_j^*(n), a_k(p)a_l^*(t)]_- = \delta_{jk}\delta_{n,-p}a_i(m)a_l^*(t) - \delta_{il}\delta_{m,-t}a_k(p)a_j^*(n),\\
&[a_i(m)a_j^*(n), a_k^*(p)a_l^*(t)]_- = -\delta_{il}\delta_{m,-t}a_k^*(p)a_j^*(n) - \delta_{ik}\delta_{m,-p}a_j^*(n)a_l^*(t),\\
&[a_i^*(m)a_j^*(n), a_k^*(p)a_l^*(t)]_- = 0,\\
&[a_i(m)a_j(n), a_k^*(p)a_l^*(t)]_- = -\delta_{il}\delta_{m,-t}a_k^*(p)a_j(n) + \delta_{ik}\delta_{m,-p}a_l^*(t)a_j(n)\\
&\qquad + \delta_{jk}\delta_{n,-p}a_i(m)a_l^*(t) - \delta_{jl}\delta_{n,-t}a_i(m)a_k^*(p).
\end{aligned} \tag{17}$$

Proposition 4.4 *The subspace of Weyl algebra $\alpha(N,-1)$ consisted of quadratic operators are closed under the Lie bracket $[\cdot\,,\cdot]_-$. Furthermore, the Lie-commutators of Bosonic quadratic basis operators $a_i(m)a_j(n)$, $a_i(m)a_j^*(n)$, $a_i^*(m)a_j^*(n)$ are as follows*:

$$\begin{aligned}
&[a_i(m)a_j(n), a_k(p)a_l(t)]_- = 0,\\
&[a_i(m)a_j(n), a_k(p)a_l^*(t)]_- = \delta_{il}\delta_{m,-t}a_k(p)a_j(n) + \delta_{jl}\delta_{n,-t}a_k(p)a_i(m),\\
&[a_i(m)a_j^*(n), a_k(p)a_l^*(t)]_- = \delta_{il}\delta_{m,-t}a_k(p)a_j^*(n) - \delta_{jk}\delta_{n,-p}a_i(m)a_l^*(t),\\
&[a_i(m)a_j^*(n), a_k^*(p)a_l^*(t)]_- = \delta_{il}\delta_{m,-t}a_k^*(p)a_j^*(n) + \delta_{ik}\delta_{m,-p}a_j^*(n)a_l^*(t),\\
&[a_i^*(m)a_j^*(n), a_k^*(p)a_l^*(t)]_- = 0,\\
&[a_i(m)a_j(n), a_k^*(p)a_l^*(t)]_- = \delta_{il}\delta_{m,-t}a_j(n)a_k^*(p) + \delta_{ik}\delta_{m,-p}a_l^*(t)a_j(n)\\
&\qquad + \delta_{jl}\delta_{n,-t}a_i(m)a_k^*(p) + \delta_{jk}\delta_{n,-p}a_l^*(t)a_i(m).
\end{aligned} \tag{18}$$

Proposition 4.5 *The subspace of the tensor product algebra $\alpha(N,+1)\otimes\alpha(N,-1)$ consisted of Fermionic-Bosonic quadratic operators are closed under the Jordan bracket $[\cdot\,,\cdot]_+$. Moreover, if we denote the generators of $\alpha(N,-1)$ by $e_i(m), e_j^*(n)$, and identify $u(m)\otimes 1$ and $1\otimes v(n)$ with $u(m)$ and $v(n)$, respectively, in tensor algebra, then*

$$u(m)\otimes v(n) = u(m)v(n) = v(n)u(m).$$

The Jordan-commutators of Fermionic-Bosonic quadratic basis operators $a_i(m)e_j(n)$, $a_i(m)e_j^*(n)$, $a_i^*(m)e_j(n)$, $a_i^*(m)e_j^*(n)$ are as follows:

$$\begin{aligned}
&[a_i(m)e_j(n), a_k(p)e_l(t)]_+ = 0,\\
&[a_i(m)e_j(n), a_k^*(p)e_l(t)]_+ = \delta_{ik}\delta_{m,-p}e_j(n)e_l(t),\\
&[a_i(m)e_j(n), a_k(p)e_l^*(t)]_+ = \delta_{jl}\delta_{n,-t}a_i(m)a_k(p),
\end{aligned}$$

$$\begin{aligned}&[a_i(m)e_j(n), a_k^*(p)e_l^*(t)]_+\\&\quad = \delta_{ik}\delta_{m,-p}e_l^*(t)e_j(n) + \delta_{jl}\delta_{n,-t}a_i(m)a_k^*(p)\\&\quad = \delta_{ik}\delta_{m,-p}e_j(n)e_l^*(t) + \delta_{jl}\delta_{n,-t}a_i(m)a_k^*(p) - \delta_{ik}\delta_{jl}\delta_{m,-p}\delta_{n,-t},\end{aligned} \tag{19}$$

$$[a_i(m)e_j^*(n), a_k(p)e_l^*(t)]_+ = 0, \tag{20}$$

$$[a_i(m)e_j^*(n), a_k^*(p)e_l(t)]_+ = \delta_{ik}\delta_{m,-p}e_l(t)e_j^*(n) - \delta_{jl}\delta_{n,-t}a_i(m)a_k^*(p), \tag{21}$$

$$\begin{aligned}[a_i(m)e_j^*(n), a_k^*(p)e_l^*(t)]_+ &= \delta_{ik}\delta_{m,-p}e_j^*(n)e_l^*(t),\\ [a_i^*(m)e_j(n), a_k^*(p)e_l(t)]_+ &= 0,\\ [a_i^*(m)e_j(n), a_k^*(p)e_l^*(t)]_+ &= \delta_{jl}\delta_{n,-t}a_i^*(m)a_k^*(p),\\ [a_i^*(m)e_j^*(n), a_k^*(p)e_l^*(t)]_+ &= 0.\end{aligned} \tag{22}$$

Proof We have

$$\begin{aligned}&[a_i(m)e_j(n), a_k^*(p)e_l^*(t)]_+\\&\quad = a_i(m)e_j(n)a_k^*(p)e_l^*(t) + a_k^*(p)e_l^*(t)a_i(m)e_j(n)\\&\quad = a_i(m)e_j(n)a_k^*(p)e_l(t) + \delta_{ik}\delta_{m,-p}e_l^*(t)e_j(n) - a_i(m)a_k^*(p)e_l^*(t)e_j(n)\\&\quad = a_i(m)e_j(n)a_k^*(p)e_l^*(t) + \delta_{ik}\delta_{m,-p}e_l^*(t)e_j(n)\\&\qquad - a_i(m)a_k^*(p)e_j(n)e_l^*(t) + \delta_{jl}\delta_{n,-t}a_i(m)a_k^*(p)\\&\quad = \delta_{ik}\delta_{m,-p}e_j(n)e_l^*(t) + \delta_{jl}\delta_{n,-t}a_i(m)a_k^*(p) - \delta_{ik}\delta_{jl}\delta_{m,-p}\delta_{n,-t}.\end{aligned}$$

Then (19) holds, and the proof of others is similar. □

As in [9,11], let $\alpha(N,\tau)^+$ be the subalgebra generated by $a_i(n), a_j^*(m)$, $a_k^*(0)$ for $n, m > 0$ and $1 \leqslant i, j, k \leqslant N$. Let $\alpha(N,\tau)^-$ be the subalgebra generated by $a_i(n)$, $a_j^*(m)$, $a_k(0)$ for $n, m < 0$ and $1 \leqslant i, j, k \leqslant N$. Those generators in $\alpha(N,\tau)^+$ are called annihilation operators while those in $\alpha(N,\tau)^-$ are called creation operators.

Let $V(N,\tau)$ be a simple $\alpha(N,\tau)$-module containing an element v_0^τ, called a 'vacuum vector' which satisfies

$$\alpha(N,\tau)^+ v_0^\tau = 0.$$

So all annihilation operators kill v_0^τ and

$$V(N,\tau) = \alpha(N,\tau)^- v_0^\tau.$$

We define the normal orderings of the mixed quadratic elements as follows:

$$\begin{aligned}&: a_i(m)e_j(n) := a_i(m)e_j(n), \quad : a_i(m)e_j^*(n) := a_i(m)e_j^*(n),\\&: a_i^*(m)e_j(n) := a_i^*(m)e_j(n), \quad : a_i^*(m)e_j^*(m) := a_i^*(m)e_j^*(m).\end{aligned} \tag{23}$$

Obviously, the $\alpha(N,+1) \otimes \alpha(N,-1)$-module

$$V(N) := V(N,+1) \otimes V(N,-1) = \alpha(N,+1) \otimes \alpha(N,-1)v_0^{+1} \otimes v_0^{-1}$$

is simple.

By comparing Proposition 4.2 with Proposition 4.3–4.5, let

$$g_{ij}(m,n) = -\sum_{s\in\mathbb{Z}} q^{-ns} : a_i(m-s)e_j^*(s) : + \sum_{s\in\mathbb{Z}} q^{-ns} : a_j(s)e_i^*(m-s) : , \quad (24)$$

$$h_{ij}(m,n) = \sum_{s\in\mathbb{Z}} q^{-ns} : a_i^*(m-s)e_j(s) : + \sum_{s\in\mathbb{Z}} q^{-ns} : a_j^*(s)e_i(m-s) : .$$

Then we have the following result.

Lemma 4.6

$$[g_{ij}(m,n), g_{kl}(p,t)] = 0, \quad (25)$$

$$[h_{ij}(m,n), h_{kl}(p,t)] = 0, \quad (26)$$

$$\begin{aligned}
&[g_{ij}(m,n), h_{kl}(p,t)] \\
&= -\delta_{ik}q^{-n(m+p)} \sum_{s\in\mathbb{Z}} q^{-(t-n)s}\{: a_j(m+p-s)a_l^*(s) : + : e_l(s)e_j^*(m+p-s) :\} \\
&\quad + \delta_{jk}q^{np} \sum_{s\in\mathbb{Z}} q^{-(n+t)s}\{: a_i(m+p-s)a_l^*(s) : + : e_l(s)e_i^*(m+p-s) :\} \\
&\quad - \delta_{il}q^{-(mn+np+pt)} \sum_{s\in\mathbb{Z}} q^{-[-(n+t)]s}\{: a_j(m+p-s)a_k^*(s) : \\
&\quad + \, : e_k(s)e_j^*(m+p-s) :\} + \delta_{jl}q^{(n-t)p} \\
&\quad \cdot \sum_{s\in\mathbb{Z}} q^{-(n-t)s}\{: a_i(m+p-s)a_k^*(s) : + : e_k(s)e_i^*(m+p-s) :\}. \quad (27)
\end{aligned}$$

Set

$$f_{ij}(m,n) = \sum_{s\in\mathbb{Z}} q^{-ns} : a_i(m-s)a_j^*(s) : + \sum_{s\in\mathbb{Z}} q^{-ns} : e_j(s)e_i^*(m-s) : .$$

Then

$$\begin{aligned}
&[g_{ij}(m,n), h_{kl}(p,t)] \\
&\quad = -\delta_{ik}q^{-n(m+p)} f_{jl}(m+p, t-n) + \delta_{jk}q^{np} f_{il}(m+p, n+t) \\
&\qquad - \delta_{il}q^{-(mn+np+pt)} f_{jk}(m+p, -(n+t)) + \delta_{jl}q^{(n-t)p} f_{ik}(m+p, n-t).
\end{aligned}$$

Proof Notice that (25) and (26) come from (20) and (22) in Proposition 4.5 immediately. We need to check (27).

By using the definition of the normal ordering (23), we have

$$\begin{aligned}
[g_{ij}(m,n), h_{kl}(p,t)] = \sum_{s_1,s_2\in\mathbb{Z}} & q^{-ns_1-ts_2}[-a_i(m-s_1)e_j^*(s_1) + a_j(s_1)e_i^*(m-s_1), \\
& a_k^*(p-s_2)e_l(s_2) + a_l^*(s_2)e_k(p-s_2)]_+.
\end{aligned}$$

Then by using (21) in Proposition 4.5, we get

$$\begin{aligned}&\sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}[-a_i(m-s_1)e_j^*(s_1), a_k^*(p-s_2)e_l(s_2)]_+\\&\quad=\sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}(-\delta_{ik}\delta_{m-s_1,s_2-p}e_l(s_2)e_j^*(s_1)\\&\qquad+\delta_{jl}\delta_{s_1,-s_2}a_i(m-s_1)a_k^*(p-s_2)).\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}[a_j(s_1)e_i^*(m-s_1), a_l^*(s_2)e_k(p-s_2)]_+\\&\quad=\sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}(-\delta_{ik}\delta_{m-s_1,s_2-p}a_j(s_1)a_l^*(s_2)\\&\qquad+\delta_{jl}\delta_{s_1,-s_2}e_k(p-s_2)e_i^*(m-s_1)).\end{aligned}$$

From (16), we have

$$a_j(s_1)a_l^*(s_2) =: a_j(s_1)a_l^*(s_2) : +\delta_{jl}\delta_{s_1,-s_2}\theta(s_1-s_2),$$

$$e_l(s_2)e_j^*(s_1) =: e_l(s_2)e_j^*(s_1) : +\delta_{jl}\delta_{s_1,-s_2}\theta(s_2-s_1),$$

$$\begin{aligned}&a_i(m-s_1)a_k^*(p-s_2)\\&\quad= \ : a_i(m-s_1)a_k^*(p-s_2) : +\delta_{ik}\delta_{m-s_1,s_2-p}\theta(m-p+s_2-s_1),\\&e_k(p-s_2)e_i^*(m-s_1)\\&\quad= \ : e_k(p-s_2)e_i^*(m-s_1) : +\delta_{ik}\delta_{m-s_1,s_2-p}\theta(p-m-s_2+s_1),\end{aligned}$$

and then, when we turn to normal ordering series, the additional scalar terms are cancelled out in the sum. We have

$$\begin{aligned}&\sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}\{[-a_i(m-s_1)e_j^*(s_1), a_k^*(p-s_2)e_l(s_2)]_+\\&\qquad+[a_j(s_1)e_i^*(m-s_1), a_l^*(s_2)e_k(p-s_2)]_+\}\\&\quad=\sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}\{-\delta_{ik}\delta_{m-s_1,s_2-p}(: a_j(s_1)a_l^*(s_2) : + : e_l(s_2)e_j^*(s_1) :)\\&\qquad+\delta_{jl}\delta_{s_1,-s_2}(: a_i(m-s_1)a_k^*(p-s_2) : + : e_k(p-s_2)e_i^*(m-s_1) :)\}\\&\quad=-\delta_{ik}q^{-n(m+p)}\sum_{s\in\mathbb{Z}} q^{-(t-n)s}\{: a_j(m+p-s)a_l^*(s) : + : e_l(s)e_j^*(m+p-s) :\}\\&\qquad+\delta_{jl}q^{(n-t)p}\sum_{s\in\mathbb{Z}} q^{-(n-t)s}\{: a_i(m+p-s)a_k^*(s) : + : e_k(s)e_i^*(m+p-s) :\}.\end{aligned}$$

Similarly, for the remainder terms, we have

$$\begin{aligned}
&\sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}\{[-a_i(m-s_1)e_j^*(s_1), a_l^*(s_2)e_k(p-s_2)]_+ \\
&\quad + [a_j(s_1)e_i^*(m-s_1), a_k^*(p-s_2)e_l(s_2)]_+\} \\
&= \sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}\{-\delta_{il}\delta_{m-s_1,-s_2}(: a_j(s_1)a_k^*(p-s_2) : + : e_k(p-s_2)e_j^*(s_1) :) \\
&\quad + \delta_{jk}\delta_{s_1,s_2-p}(: a_i(m-s_1)a_l^*(s_2) : + : e_l(s_2)e_i^*(m-s_1) :)\} \\
&= +\delta_{jk}q^{np}\sum_{s\in\mathbb{Z}} q^{-(n+t)s}\{: a_i(m+p-s)a_l^*(s) : + : e_l(s)e_i^*(m+p-s) :\} \\
&\quad - \delta_{il}q^{-(mn+np+pt)}\sum_{s\in\mathbb{Z}} q^{-[-(n+t)]s}\{: a_j(m+p-s)a_k^*(s) : \\
&\quad + : e_k(s)e_j^*(m+p-s) :\}.
\end{aligned}$$

Adding these terms, the (27) follows. □

About the remainder of Lie superbrackets, we have the following result.

Lemma 4.7 *For all $m,p,n,t\in\mathbb{Z}$ and $1\leqslant i,j,k,l\leqslant N$,*

$$[g_{ij}(m,n), f_{kl}(p,t)] = \delta_{jl}q^{(t-n)m}g_{ki}(m+p,t-n) - \delta_{il}q^{mt}g_{kj}(m+p,n+t), \quad (28)$$

$$[f_{ij}(m,n), f_{kl}(p,t)] = \delta_{jk}q^{np}f_{il}(m+p,n+t) - \delta_{il}q^{tm}f_{kj}(m+p,n+t),$$

$$\begin{aligned}
[f_{ij}(m,n), h_{kl}(p,t)] = &-\delta_{ik}q^{-n(m+p)}h_{jl}(m+p,t-n) \\
&-\delta_{il}q^{mt}h_{kj}(m+p,n+t). \qquad (29)
\end{aligned}$$

Proof Notice that the influence of removing normal ordering is at most a scalar element which has no effect in Lie bracket. Then we have

$$\begin{aligned}
[g_{ij}(m,n), f_{kl}(p,t)] = \sum_{s_1,s_2\in\mathbb{Z}} & q^{-ns_1-ts_2}[-a_i(m-s_1)e_j^*(s_1) + a_j(s_1)e_i^*(m-s_1), \\
& a_k(p-s_2)a_l^*(s_2) + e_l(s_2)e_k^*(p-s_2)]_-.
\end{aligned}$$

Since

$$-a_l^*(s_2)a_i(m-s_1) = a_i(m-s_1)a_l^*(s_2) - \delta_{il}\delta_{m-s_1,-s_2},$$

we get

$$\begin{aligned}
&\sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}[-a_i(m-s_1)e_j^*(s_1), a_k(p-s_2)a_l^*(s_2)]_- \\
&= \sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}\delta_{il}\delta_{m-s_1,-s_2}a_k(p-s_2)e_j^*(s_1) \\
&= \delta_{il}q^{tm}\sum_{s\in\mathbb{Z}} q^{-(n+t)s}a_k(m+p-s)e_j^*(s).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}[-a_i(m-s_1)e_j^*(s_1), e_l(s_2)e_k^*(p-s_2)]_-\\&\quad=\sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}\delta_{jl}\delta_{s_1,-s_2}a_i(m-s_1)e_k^*(p-s_2)\\&\quad=\delta_{jl}q^{m(t-n)}\sum_{s\in\mathbb{Z}} q^{-(t-n)s}a_i(s)e_k^*(m+p-s),\end{aligned}$$

$$\begin{aligned}&\sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}[a_j(s_1)e_i^*(m-s_1), a_k(p-s_2)a_l^*(s_2)]_-\\&\quad=-\sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}\delta_{jl}\delta_{s_1,-s_2}a_k(p-s_2)e_i^*(m-s_1)\\&\quad=-\delta_{jl}q^{m(t-n)}\sum_{s\in\mathbb{Z}} q^{-(t-n)s}a_k(m+p-s)e_i^*(s),\end{aligned}$$

$$\begin{aligned}&\sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}[a_j(s_1)e_i^*(m-s_1), e_l(s_2)e_k^*(p-s_2)]_-\\&\quad=-\sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}\delta_{il}\delta_{m-s_1,-s_2}a_j(s_1)e_k^*(p-s_2)\\&\quad=-\delta_{il}q^{tm}\sum_{s\in\mathbb{Z}} q^{-(n+t)s}a_j(s)e_k^*(m+p-s).\end{aligned}$$

Adding the four terms and noticing that (23) and (24), we get (28) at once.

Notice again that the influence of removing normal ordering is at most a scalar element which has no effect in Lie bracket. Then we have

$$\begin{aligned}[f_{ij}(m,n), f_{kl}(p,t)] = \sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}&[a_i(m-s_1)a_j^*(s_1)+e_j(s_1)e_i^*(m-s_1),\\&a_k(p-s_2)a_l^*(s_2)+e_l(s_2)e_k^*(p-s_2)]_-.\end{aligned}$$

By using (17) in Proposition 4.3, we get

$$\begin{aligned}&\sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}[a_i(m-s_1)a_j^*(s_1), a_k(p-s_2)a_l^*(s_2)]_-\\&\quad=\sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}(-\delta_{il}\delta_{m-s_1,-s_2}a_k(p-s_2)a_j^*(s_1)\\&\qquad+\delta_{jk}\delta_{s_1,s_2-p}a_i(m-s_1)a_l^*(s_2)).\end{aligned}$$

Then, by using (18) in Proposition 4.4, we get

$$\begin{aligned}&\sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}[e_j(s_1)e_i^*(m-s_1), e_l(s_2)e_k^*(p-s_2)]_-\\&\quad=\sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}(-\delta_{il}\delta_{m-s_1,-s_2}e_j(s_1)e_k^*(p-s_2)\\&\qquad+\delta_{jk}\delta_{s_1,s_2-p}e_l(s_2)e_i^*(m-s_1)),\end{aligned}$$

$$\sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}[e_j(s_1)e_i^*(m-s_1), a_k(p-s_2)a_l^*(s_2)]_-$$
$$= \sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}[a_i(m-s_1)a_j^*(s_1), e_l(s_2)e_k^*(p-s_2)]_-$$
$$= 0.$$

About the normal ordering, from (16), we have

$$a_k(p-s_2)a_j^*(s_1) =: a_k(p-s_2)a_j^*(s_1) : + \delta_{kj}\delta_{p-s_2,-s_1}\theta(p-s_2-s_1),$$
$$a_i(m-s_1)a_l^*(s_2) =: a_i(m-s_1)a_l^*(s_2) : + \delta_{il}\delta_{m-s_1,-s_2}\theta(m-s_1-s_2),$$
$$e_j(s_1)e_k^*(p-s_2) =: e_j(s_1)e_k^*(p-s_2) : + \delta_{jk}\delta_{s_1,s_2-p}\theta(s_1-p+s_2),$$
$$e_l(s_2)e_i^*(m-s_1) =: e_l(s_2)e_i^*(m-s_1) : + \delta_{li}\delta_{s_2,s_1-m}\theta(s_2-m+s_1),$$

and then, when we turn to normal ordering series, the additional scalar terms are cancel out in the sum. We have

$$[f_{ij}(m,n), f_{kl}(p,t)]$$
$$= \sum_{s_1,s_2\in\mathbb{Z}} q^{-ns_1-ts_2}\{-\delta_{il}\delta_{m-s_1,-s_2}(: a_k(p-s_2)a_j^*(s_1) : + : e_j(s_1)e_k^*(p-s_2) :)$$
$$+ \delta_{jk}\delta_{s_1,s_2-p}(: a_i(m-s_1)a_l^*(s_2) : + : e_l(s_2)e_i^*(m-s_1) :)\}$$
$$= -\delta_{il}q^{tm}\sum_{s\in\mathbb{Z}} q^{-(n+t)s}(: a_k(m+p-s)a_j^*(s) : + : e_j(s)e_k^*(m+p-s) :)$$
$$+ \delta_{jk}q^{np}\sum_{s\in\mathbb{Z}} q^{-(n+t)s}(: a_i(m+p-s)a_l^*(s) : + : e_l(s)e_i^*(m+p-s) :)$$
$$= \delta_{jk}q^{np}f_{il}(m+p,n+t) - \delta_{il}q^{tm}f_{kj}(m+p,n+t).$$

The proof of (29) is similar, so we omit the detailed calculation. □

Although $g_{ij}(m,n)$, $h_{ij}(m,n)$, $f_{ij}(m,n)$ are infinite sums, they are well defined as operators on $V(N)$ since at most finitely many terms can make a nontrivial action to $\forall\, v \in V(N) = \alpha(N,+1)\otimes\alpha(N,-1)v_0^{+1}\otimes v_0^{-1}$.

Then, from Lemmas 4.6 and 4.7, we have the following result.

Theorem 4.1 *The correspondence*

$$\pi(\widetilde{g}_{ij}(m,n)) = g_{ij}(m,n),$$
$$\pi(\widetilde{h}_{ij}(m,n)) = h_{ij}(m,n),$$
$$\pi(\widetilde{f}_{ij}(m,n)) = f_{ij}(m,n),$$

gives rise to a representation for the GPLS $\mathscr{P}_N(\mathbb{C}_q,-)$. Furthermore, π is faithful if q is generic.

Moreover, let

$$\pi(c(u)) = 1,\ u\in\Lambda(q), \quad \pi(c_y) = 0.$$

Then $(V(N),\pi)$ *is also a representation for* $\widetilde{\mathscr{P}_N(\mathbb{C}_q,-)}$ *which is a nontrivial central extension of* $\mathscr{P}_N(\mathbb{C}_q,-)$ *given in* (15).

Proof We only remain to check that π is faithful for $\mathscr{P}_N(\mathbb{C}_q,-)$ when q is generic.

If q is generic, then

$$\mathscr{P}_0=\mathrm{span}_{\mathbb{C}}\{\widetilde{f}_{ii}(m,n)\mid 1\leqslant i\leqslant N,\ m,n\in(\mathbb{Z}\times\mathbb{Z})\backslash\{0\}\times\{0\}\}.$$

Due to Proposition 4.1, and noticing that $V(N)$ is faithful for $\alpha(N,+1)\otimes\alpha(N,-1)$, it is sufficient to check for any summation

$$\sum_{i,j,m,n}a_{ij}(m,n)f_{ij}(m,n)+\sum_{k\leqslant l,p,t}(b_{kl}(p,t)g_{kl}(p,t)+c_{kl}(p,t)h_{kl}(p,t)),$$

which contains at most finitely many nonzero terms vanishing implies that

$$a_{ij}(m,n)=b_{kl}(p,t)=c_{kl}(p,t)=0$$

for all i,j,m,n,k,l,p,t.

Notice that (16), (23), and that the linearly dependent quadratic operators only come from the same form root vectors such as $f_{i,j}(m,-)$, $g_{kl}(p,-)$, or $h_{k'l'}(p',-)$. Then, for any fixed i,j,m,k,l,p, we have

$$\sum_n q^{-ns}a_{ij}(m,n)=0,\quad \sum_t q^{-ts}b_{kl}(p,t)=0,\quad \sum_t q^{-ts}c_{kl}(p,t)=0,\quad \forall\, s\in\mathbb{Z}.$$

In $\sum_n q^{-ns}a_{ij}(m,n)$, we assume that all distinct nonzero terms are $a_{ij}(m,n_1),a_{ij}(m,n_2),\ldots,a_{ij}(m,n_r)$. Let $s=0,1,\ldots,r-1$. Then we get a homogeneous linearity equations with Vandermonde type coefficient matrix for $a_{ij}(m,n_1),a_{ij}(m,n_2),\ldots,a_{ij}(m,n_r)$.

Notice that $q^{-n_1},q^{-n_2},\ldots,q^{-n_r}$ are distinct since q is not a root of unity. Then the determinant of the Vandermonde coefficient matrix is nonzero, and we get $a_{ij}(m,n)=0$ for all i,j,m,n.

The proofs of $b_{kl}(p,t)$ and $c_{kl}(p,t)$ are similar. □

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