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RESEARCH ARTICLE

Stochastic partial differential equations with gradient driven by space-time fractional noises

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Abstract We establish a class of stochastic partial differential equations (SPDEs) driven by space-time fractional noises, where we suppose that the drfit term contains a gradient and satisfies certain non-Lipschitz condition. We prove the strong existence and uniqueness and joint Hölder continuity of the solution to the SPDEs.

Keywords Stochastic partial differential equation (SPDE), fractional noise, uniqueness, strong solution, Hölder continuity

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1 Introduction and main results

A class of special Gaussian processes called fractional Brownian motion (fBm) has been established by many authors as random fields, due to their useful feature of preserving long term memory and a large number of interesting results from scaling invariance to the description of their laws. The study of these Gaussian processes has its historical motivation from their applications in hydrology and telecommunication, and has been applied to the mathematical finance, biotechnology, and biophysics, see, for example, [3,10,18] and references therein.

Mandelbrot and Van Ness [12] proposed a theory of stochastic calculus for the fBms as archetypal examples. Especially, Mémin et al. [13] gave an embedding theorem to estimate the moments of Wiener integrals with respect to an fBm. Then many authors were interested in the research on fractional noise instead of space time white noise. Hu [6] proposed the multiple stochastic

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integral with respect to multi-parameter fractional noise, then showed via chaos expansion the existence and uniqueness of the solutions for a class of second-order stochastic heat equations, and further estimated the Lyapunov exponents of the solutions. Nualart and Ouknine [17] discussed the existence and uniqueness of the solution to stochastic partial differential equation (SPDE) with additive fractional noise (fractional in time and white in space). Hu et al. [8] studied the central limit theorem for an additive functional of the fBm. In fact, we deal with a class of SPDEs driven by fractional noises, including the Cahn-Hillard equations and Burgers equation among others (see [1,7,9] for more details).

When the drift term in SPDE depends on the gradient, our idea originates from the Burgers equation

$$\frac{\partial u}{\partial t} = \frac{\kappa}{2}\,\Delta u + u^k \frac{\partial u}{\partial x}, \tag{1.1}$$

which describes the interaction between the diffusion part and the non-linear inertial part in fluid flow for integer $\kappa \geqslant 1$. It then comes natural to study the stochastic counterparts for such equations. Gyöngy [4] proved existence, uniqueness, and comparison theorems for a class of semilinear stochastic partial differential equations, containing special cases like the stochastic Burgers equation and the reaction-diffusion equations, driven by space-time white noise. Wang et al. [21] proposed an L_2-gradient estimate for the corresponding Galerkin approximations, and the log-Harnack inequality was established for the semigroup associated to a class of stochastic Burgers equations. Hairer and Voss [5] discussed the numerical methods of various finite-difference approximations to the stochastic Burgers equation. Mohammed and Zhang [15] proved an existence theorem for solutions of the stochastic Burgers equation on the unit interval subject to the Dirichlet boundary condition and the anticipating initial velocities. Jiang et al. [9] considered the stochastic generalized Burgers equations driven by fractional noises. Dong et al. [2] gave the irreducibility and asymptotics of stochastic Burgers equation driven by α-stable processes.

Moreover, Hu et al. [7] studied a class of SPDEs driven by space-time fractional noises, where they supposed that the drift term is Lipschitz with the gradient. Now, we extend the drift term to be satisfying a certain non-Lipschitz condition. In this paper, we consider the following SPDE:

$$u_t(x) = u_0(x) + \int_0^t \Big[\frac{1}{2}\,\Delta u_s(x) + g(u_s(x), \nabla u_s(x))\Big]\mathrm{d}s + \int_0^t W^H(\mathrm{d}s, \mathrm{d}x), \tag{1.2}$$

where W^H is the fractional white noise with Hurst parameter $H = (h_1, h_2)$ for $h_1, h_2 \in (0, 1)$. We study the strong existence and uniqueness and joint Hölder continuity of the solution.

To continue with the introduction, we state some notation. Let $C^\infty(\mathbb{R})$ be the space of functions which has derivatives of all orders and $C_c^\infty(\mathbb{R})$ be the subset of $C^\infty(\mathbb{R})$ of functions with compact supports. Let $L^2(\mathbb{R})$ be the space of quadratic integration functions and

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)g(x)\mathrm{d}x$$

whenever the integral is well defined. For $h \geqslant 1$, define

$$\|f\|_{L^h(\mathbb{R})} := \Big(\int_{\mathbb{R}} |f(x)|^h \mathrm{d}x\Big)^{1/h}$$

if it exists. For $h \geqslant 1$ and $T_1 > T_2 \geqslant 0$, define $L^h([T_1, T_2])$, $L^h([T_1, T_2] \times \mathbb{R})$, $\|f\|_{L^h([T_1,T_2])}$, and $\|f\|_{L^h([T_1,T_2]\times\mathbb{R})}$ similarly. Define

$$J(x) = \int_{\mathbb{R}} \mathrm{e}^{-|y|}\rho_0(x-y)\mathrm{d}y$$

with ρ_0 given by

$$\rho_0(x) = c_0 \exp\Big(-\frac{1}{1-x^2}\Big)1_{\{|x|<1\}},$$

where $c_0 > 0$ is a constant so that

$$\int_{\mathbb{R}} \rho_0(x)\mathrm{d}x = 1.$$

Moreover, due to [14, (2.1)], for each $n \geqslant 0$ there exist constants $\overline{C}_n, \widetilde{C}_n > 0$ so that

$$\overline{C}_n \mathrm{e}^{-|x|} \leqslant |J^{(n)}(x)| \leqslant \widetilde{C}_n \mathrm{e}^{-|x|}, \quad x \in \mathbb{R}. \tag{1.3}$$

Let $\mathscr{X}_0$ be the collection of functions f such that

$$\|f\|_0^2 := \int_{\mathbb{R}} f(x)^2 J(x)\mathrm{d}x < \infty.$$

Let $\mathscr{X}_1$ be the space consisting of all functions f such that

$$\|f\|_1^2 := \|f\|_0^2 + \|f'\|_0^2 < \infty.$$

Then, for each $i = 0, 1$, $\mathscr{X}_i$ is a Hilbert space under the norm $\|\cdot\|_i$. We use $\langle\cdot,\cdot\rangle_i$ to denote the corresponding inner product.

In this paper, we assume that all random elements are defined on a filtered complete probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbf{P})$ satisfying the usual hypotheses. We use $\mathbf{E}$ to denote the corresponding expectation.

Definition 1.1 Let $u_0 \in \mathscr{X}_1$. SPDE (1.2) has a strong $\mathscr{X}_1$-valued solution if for any fractional white noise W^H, there exists a continuous $\mathscr{X}_1$-valued process $(u_t)_{t\geqslant 0}$ so that for all $f \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned}\langle u_t, f\rangle = \langle u_0, f\rangle &+ \int_0^t \Big[\frac{1}{2}\langle u_s, f''\rangle + \langle g(u_s, \nabla u_s), f\rangle\Big]\mathrm{d}s \\ &+ \int_0^t\int_{\mathbb{R}} f(x) W^H(\mathrm{d}s, \mathrm{d}x), \quad t \geqslant 0,\end{aligned}$$

almost surely.

In this paper, we always assume that $1/2 < h_1, h_2 < 1$ and $2h_1 + h_2 > 2$, and that the continuous function g can be written into the following form:

$$g(x,y) = \overline{g}(x,y) + \widetilde{g}(x)y.$$

Recently, Xiong and Yang [23] established the strong existence and uniqueness to a class of SPDEs with this form of diffusion coefficient and Gaussian colored noises. Before state the main results, we formulate the following condition on $\overline{g}$ and $\widetilde{g}$.

Condition 1.2 There exists a constant $C > 0$ so that

$$|\overline{g}(x_1,y_1) - \overline{g}(x_2,y_2)| \leqslant C(|x_1 - x_2| + |y_1 - y_2|), \quad |\widetilde{g}(x)| \leqslant C,$$

for all $x, x_1, x_2, y_1, y_2 \in \mathbb{R}$.

Theorem 1.3 *Suppose that Condition* 1.2 *holds and* $u_0 \in \mathscr{X}_1$. *Then SPDE* (1.2) *has a unique strong* $\mathscr{X}_1$*-valued solution* $(u_t)_{t\geqslant 0}$.

Theorem 1.4 (Joint Hölder continuity) *Suppose that Condition* 1.2 *holds. Let* $(u_t)_{t\geqslant 0}$ *be an* $\mathscr{X}_1$*-valued solution to* (1.2). *Then, for any* $T > T_1 > 0$, $a > 0$, *and* $\theta \in (0,1)$, *there exists a random variable* $K_{T,T_1,a,\theta} \geqslant 0$ *with* $\mathbf{E}[K_{T,T_1,a,\theta}] < \infty$ *so that almost surely for all* $t_1, t_2 \in [T_1, T]$ *and* $|x_1|, |x_2| \leqslant a$,

$$|u_{t_1}(x_1) - u_{t_2}(x_2)| \leqslant K_{T,T_1,a,\theta}(|t_1 - t_2|^{1/2} + |x_1 - x_2|)^{\theta}.$$

Furthermore, if u_0 *is also Hölder continuous with exponent* $\gamma \geqslant 1/2$, *i.e.*,

$$\sup_{x_1,x_2\in\mathbb{R}} \frac{|u_0(x_1) - u_0(x_2)|}{|x_1 - x_2|^{\gamma}} < \infty,$$

then, for some random variable $K_{T,a,\theta} \geqslant 0$ *with* $\mathbf{E}[K_{T,a,\theta}] < \infty$, *we have, almost surely, for all* $t_1, t_2 \in [T_1, T]$ *and* $|x_1|, |x_2| \leqslant a$,

$$|u_{t_1}(x_1) - u_{t_2}(x_2)| \leqslant K_{T,a,\theta}(|t_1 - t_2|^{1/2} + |x_1 - x_2|)^{\theta}.$$

Remark 1.5 If $g(x,y) = y(x^{\alpha} \wedge 1)$ for some $\alpha \in (0,1)$, then Condition 1.2 is satisfied.

The rest of this paper is organized as follows. In Section 2, we introduce fractional noise and state some assertions on fractional noise and heat kernel. Theorem 1.3 is proved in Section 3. In Section 4, we establish the proof of Theorem 1.4.

Notation Let ∇ and Δ be the first and the second order spatial differential operators, respectively. Let C denote a positive constant whose value might change from line to line.

2 Preliminaries

2.1 Fractional noise

A one-dimensional fBm $W^h = (W^h_t)_{0\leqslant t\leqslant T}$ with Hurst parameter $h \in (0,1)$ on $[0,T]$ is a centered Gaussian process with its covariance function given by

$$\mathbf{E}[W^h_t W^h_s] = \frac{1}{2}\,(t^{2h} + s^{2h} - |t-s|^{2h}).$$

The existence of such a Gaussian process and the regularity of its sample paths are well documented. We may generalize the definition to fractional noises with two parameters.

Definition 2.1 A one-dimensional double-parameter fractional Brownian sheet

$$W^H = \{W^H(t,x),\ (t,x) \in [0,T]\times\mathbb{R}\},$$

with Hurst parameter $H = (h_1, h_2)$ for $h_i \in (0,1)$ and $i = 1,2$, is a centered Gaussian field defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with covariance

$$\begin{aligned} R(t,s;x,y) &:= \mathbf{E}[W^H(t,x)W^H(s,y)] \\ &= \frac{1}{4}\,(t^{2h_1} + s^{2h_1} - |t-s|^{2h_1})(|x|^{2h_2} + |y|^{2h_2} - |x-y|^{2h_2}), \end{aligned} \tag{2.1}$$

for all $t, s \in [0,T]$ and $x, y \in \mathbb{R}$.

Let $\mathscr{E}$ denote the collection of all step functions defined on $[0,T]\times\mathbb{R}$ and L^2_H denote the Hilbert space of the closure of $\mathscr{E}$ under scalar product

$$R(t,s;x,y) := \langle 1_{[0,t]\times[0,x]}, 1_{[0,s]\times[0,y]}\rangle_H.$$

In the above formula, if $x < 0$, then we assume, by convention, that

$$1_{[0,x]} = -1_{[-x,0]}.$$

Then the mapping $1_{[0,t]\times[0,x]} \to W^H(t,x)$ can be extended to an isometry between L^2_H and the Gaussian space $\mathscr{H}$ associated with W^H.

Introduce the square integrable kernel

$$K_H(t,s;x,y) = c_H s^{\frac{1}{2}-h_1} y^{\frac{1}{2}-h_2} \int_s^t \mathrm{d}z \int_y^x (u-s)^{h_1-\frac{3}{2}} u^{h_1-\frac{1}{2}} (z-y)^{h_2-\frac{3}{2}} z^{h_2-\frac{1}{2}} \mathrm{d}u$$

and its derivative

$$\frac{\partial^2}{\partial t\partial x}\,K_H(t,s;x,y) = c_H (t-s)^{h_1-\frac{3}{2}} \Big(\frac{t}{s}\Big)^{h_1-\frac{1}{2}} (x-y)^{h_2-\frac{3}{2}} \Big(\frac{x}{y}\Big)^{h_2-\frac{1}{2}},$$

where $c_H > 0$ is a constant depending only on H. Define the operator K^*_H from $\mathscr{E}$ to $L^2([0,T]\times\mathbb{R})$ by

$$(K^*_H\phi)(s,y) = \int_s^T \mathrm{d}t \int_y^\infty \phi(t,x) \frac{\partial^2}{\partial t\partial x}\,K_H(t,s;x,y)\mathrm{d}x.$$

It is easy to check that

$$(K_H^*1_{[0,t]\times[0,x]})(s,y)=K_H(t,s;x,y)1_{[0,t]\times[0,x]}(s,y)$$

and

$$\langle K_H^*1_{[0,t]\times[0,x]},K_H^*1_{[0,s]\times[0,y]}\rangle=R_H(t,s;x,y)=\langle 1_{[0,t]\times[0,x]},1_{[0,s]\times[0,y]}\rangle_H.$$

Hence, the operator K_H^* is an isometry between $\mathscr{E}$ and $L^2([0,T]\times\mathbb{R})$ which can be extended to L_H^2. Let ${K_H^*}^{-1}$ denote the inverse mapping of K_H^*. By definition,

$$B(t,x)=W^H({K_H^*}^{-1}(1_{[0,t]\times[0,x]})),\quad (t,x)\in[0,T]\times\mathbb{R},$$

is a Brownian sheet, and in turn the fractional noise has a representation

$$W^H(t,x)=\int_0^t\int_0^x K_H(t,s;x,y)B(\mathrm{d}s,\mathrm{d}y).$$

Then the integral $\int_0^t\int_0^x\phi(s,y)W^H(\mathrm{d}s,\mathrm{d}y)$ is defined by

$$\int_0^t\int_0^x\phi(s,y)W^H(\mathrm{d}s,\mathrm{d}y)=\int_0^t\int_0^x(K_H^*\phi)(s,y)B(\mathrm{d}s,\mathrm{d}y).\tag{2.2}$$

For $0\leqslant s<t\leqslant T$ and $x,y\in\mathbb{R}$, define

$$\Psi_h(t,s,x,y):=4h_1h_2(2h_1-1)(2h_2-1)|t-s|^{2h_1-2}\,|x-y|^{2h_2-2}.$$

A routine calculation shows the equivalence of the stochastic integrals defined in [7] and those in this section for functions in L_H^2.

Proposition 2.2 *For $f,g\in L_H^2$, we have*

$$\mathbf{E}\Big[\int_0^t\int_{\mathbb{R}}f(s,x)W^H(\mathrm{d}x,\mathrm{d}s)\Big]=0$$

and

$$\begin{aligned}&\mathbf{E}\Big[\int_0^t\int_{\mathbb{R}}f(s,x)W^H(\mathrm{d}x,\mathrm{d}s)\int_0^t\int_{\mathbb{R}}g(s,x)W^H(\mathrm{d}x,\mathrm{d}s)\Big]\\&\quad=\int_{[0,t]^2}\int_{\mathbb{R}^2}\Psi_h(u,v,x,y)f(u,x)g(v,y)\mathrm{d}y\mathrm{d}x\mathrm{d}v\mathrm{d}u.\end{aligned}$$

Note that we have the following inequality (see [13] for more details).

Lemma 2.3 *If $h\in(1/2,1)$ and $f,g\in L^{1/h}([a,b])$, then*

$$\int_a^b\int_a^b f(u)g(v)|u-v|^{2h-2}\mathrm{d}u\mathrm{d}v\leqslant C\|f\|_{L^{1/h}([a,b])}\,\|g\|_{L^{1/h}([a,b])},$$

where $C=C(h)>0$ is a constant depending only on h.

Next, we will obtain the embedding property which enables us to define the integral for $f \in L_H^2$ with respect to W^H.

Proposition 2.4 *We have* $L^{1/H}([0,T] \times \mathbb{R}) \subset L_H^2$.

Proof In fact,

$$\begin{aligned}
\|f\|_H^2 &= \int_{[0,t]^2} \int_{\mathbb{R}^2} \Psi_h(s_1, s_2, y_1, y_2) f(s_1, y_1) f(s_2, y_2) \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}s_1 \mathrm{d}s_2 \\
&= C \int_{[0,t]^2} \int_{\mathbb{R}^2} |s_1 - s_2|^{2h_1 - 2} |y_1 - y_2|^{2h_2 - 2} f(s_1, y_1) f(s_2, y_2) \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}s_1 \mathrm{d}s_2 \\
&\leqslant C \int_{[0,t]^2} |s_1 - s_2|^{2h_1 - 2} \|f(s_1, \cdot)\|_{L^{1/h_2}(\mathbb{R})} \|f(s_2, \cdot)\|_{L^{1/h_2}(\mathbb{R})} \mathrm{d}s_1 \mathrm{d}s_2 \\
&\leqslant C \bigg(\int_0^t (\|f(s, \cdot)\|_{L^{1/h_2}(\mathbb{R})})^{1/h_1} \mathrm{d}s \bigg)^{2h_1} \\
&= C \|f\|_{L^{1/H}([0,t] \times \mathbb{R})}^2,
\end{aligned}$$

which ends the proof. □

Moreover, we end this subsection with the following estimation, which will be used in our later derivation.

Lemma 2.5 *Suppose that* $f(t,x) \in L_H^2$ *and* $p \geqslant 1$. *Then*

$$\mathbf{E}\bigg[\Big| \int_0^t \int_{\mathbb{R}} f(s,x) W^H(\mathrm{d}x, \mathrm{d}s) \Big|^{2p} \bigg] \leqslant C \bigg(\int_0^t (\|f(s, \cdot)\|_{L^{1/h_2}(\mathbb{R})})^{1/h_1} \mathrm{d}s \bigg)^{2ph_1}.$$

Proof By Proposition 2.4, we have

$$\begin{aligned}
\mathbf{E}\bigg[\Big| \int_0^t \int_{\mathbb{R}} f(s,x) W^H(\mathrm{d}x, \mathrm{d}s) \Big|^{2p} \bigg] &= \mathbf{E}\bigg[\Big| \int_0^t \int_{\mathbb{R}} [K_H^* f(\cdot, \cdot)](x,s) W(\mathrm{d}x, \mathrm{d}s) \Big|^{2p} \bigg] \\
&\leqslant C \Big| \int_0^t \int_{\mathbb{R}} [K_H^* f(\cdot, \cdot)]^2(x,s) \mathrm{d}x \mathrm{d}s \Big|^p \\
&= C \|f\|_{L_H^2}^{2p} \\
&\leqslant C \|f\|_{L^{1/H}([0,t] \times \mathbb{R})}^{2p}.
\end{aligned}$$

Then the proof of this lemma is complete. □

2.2 Estimation of heat kernel

Lemma 2.6 *Let* $\alpha \in [0,1]$ *be fixed. Then*

$$p_t(x_1) \leqslant \frac{C}{t^{1/2}}, \quad |p_t(x_1) - p_t(x_2)| \leqslant C \frac{|x_1 - x_2|^\alpha}{t^{\alpha/2}} \Big[p_t\Big(\frac{x_1}{2}\Big) + p_t\Big(\frac{x_2}{2}\Big) \Big], \quad t > 0,\ x_1, x_2 \in \mathbb{R}. \tag{2.3}$$

Moreover,

$$|\nabla p_t(x)| \leqslant \frac{C}{t^{1/2}}\, p_t\Big(\frac{x}{2}\Big) \leqslant \frac{C}{t}, \quad t>0,\ x\in\mathbb{R}, \tag{2.4}$$

and for $\theta\in[0,1]$,

$$|\nabla p_t(x)-\nabla p_t(y)| \leqslant C\frac{|x-y|^\theta}{t^{(1+\theta)/2}}\Big[p_t\Big(\frac{x}{2}\Big)+p_t\Big(\frac{y}{2}\Big)\Big], \quad t>0,\ x,y\in\mathbb{R}. \tag{2.5}$$

Proof The estimation (2.3) is given by [19, (2.4e)] and the rests follow from [16, Lemma 2.2]. □

Lemma 2.7 *Let* $T>0$ *and* $p\geqslant 1$ *be fixed. Then*

$$\sup_{0<t\leqslant T,\, x\in\mathbb{R}} \mathbf{E}\Big[\Big|\int_0^t\int_{\mathbb{R}} p_{t-s}(x,y)W^H(\mathrm{d}s,\mathrm{d}y)\Big|^{2p} + \Big|\int_0^t\int_{\mathbb{R}} \nabla p_{t-s}(x,y)W^H(\mathrm{d}s,\mathrm{d}y)\Big|^{2p}\Big] < \infty.$$

Proof In view of (2.3), we have

$$\begin{aligned}\|p_s(x,\cdot)\|_{L^{1/h_2}(\mathbb{R})} &= \Big(\int_{\mathbb{R}} |p_s(x,y)|^{1/h_2}\mathrm{d}y\Big)^{h_2}\\ &= \Big(\int_{\mathbb{R}} |p_s(x,y)|^{\frac{1}{h_2}-1}\cdot p_s(x,y)\mathrm{d}y\Big)^{h_2}\\ &\leqslant Cs^{(h_2-1)/2}\end{aligned} \tag{2.6}$$

and

$$\|\nabla p_s(x,\cdot)\|_{L^{1/h_2}(\mathbb{R})} = \Big(\int_{\mathbb{R}} |\nabla p_s(x,y)|^{1/h_2}\mathrm{d}y\Big)^{h_2} \leqslant Cs^{-(2-h_2)/2}.$$

It then follows from (2.5) that

$$\begin{aligned}\mathbf{E}\Big[\Big|\int_0^t\int_{\mathbb{R}} p_{t-s}(x,y)W^H(\mathrm{d}s,\mathrm{d}y)\Big|^{2p}\Big] &\leqslant C\Big(\int_0^t (\|p_{t-s}(x,\cdot)\|_{L^{1/h_2}(\mathbb{R})})^{1/h_1}\mathrm{d}s\Big)^{2ph_1}\\ &\leqslant C\Big(\int_0^t (t-s)^{(h_2-1)/(2h_1)}\mathrm{d}s\Big)^{2ph_1}\\ &\leqslant Ct^{p(2h_1+h_2-1)}\end{aligned}$$

and

$$\mathbf{E}\Big[\Big|\int_0^t\int_{\mathbb{R}} \nabla p_{t-s}(x,y)W^H(\mathrm{d}s,\mathrm{d}y)\Big|^{2p}\Big] \leqslant Ct^{p(2h_1+h_2-2)}.$$

This completes the proof. □

Lemma 2.8 *Let $T > 0$ and $p \geqslant 1$ be fixed. Then*

$$\mathbf{E}\Big[\Big|\int_0^{t_1}\int_{\mathbb{R}} p_{t_1-s}(x_1,y)W^H(\mathrm{d}s,\mathrm{d}y) - \int_0^{t_2}\int_{\mathbb{R}} p_{t_2-s}(x_2,y)W^H(\mathrm{d}s,\mathrm{d}y)\Big|^{2p}\Big] \leqslant C(|t_1-t_2|^{\mu} + |x_1-x_2|^{\nu})^{2p}$$

for $x_1, x_2 \in \mathbb{R}$, $0 < t_2 < t_1 < T$, *where* $\mu \in [0, 1/2)$ *and* $\nu \in [0,1)$.

Proof By virtue of (2.3) and (2.6), we obtain

$$\begin{aligned}
&\|p_s(x_1,\cdot) - p_s(x_2,\cdot)\|_{L^{1/h_2}(\mathbb{R})}\\
&\leqslant C s^{-\nu/2}|x_1-x_2|^{\nu}\Big(\int_{\mathbb{R}}\Big|p_s\Big(\frac{x_1-y}{2}\Big) + p_s\Big(\frac{x_2-y}{2}\Big)\Big|^{1/h_2}\mathrm{d}y\Big)^{h_2}\\
&\leqslant C s^{(h_2-1-\nu)/2}|x_1-x_2|^{\nu}.
\end{aligned}$$

Then, applying Lemma 2.5, we get

$$\begin{aligned}
&\mathbf{E}\Big[\Big|\int_0^t\int_{\mathbb{R}}(p_{t-s}(x_1,y) - p_{t-s}(x_2,y))W^H(\mathrm{d}s,\mathrm{d}y)\Big|^{2p}\Big]\\
&\leqslant C\Big(\int_0^t \|p_{t-s}(x_1,\cdot) - p_{t-s}(x_2,\cdot)\|_{L^{1/h_2}(\mathbb{R})}^{1/h_1}\mathrm{d}s\Big)^{2ph_1}\\
&\leqslant C|x_1-x_2|^{2p\nu}\Big(\int_0^t (t-s)^{-(1+\nu-h_2)/(2h_1)}\mathrm{d}s\Big)^{2ph_1}\\
&\leqslant C|x_1-x_2|^{2p\nu}, \quad t\in(0,T],\ x_1,x_2\in\mathbb{R}.
\end{aligned}$$

By a similar argument, one can deduce that

$$\mathbf{E}\Big[\Big|\int_0^{t_2}\int_{\mathbb{R}}(p_{t_1-s}(x,y) - p_{t_2-s}(x,y))W^H(\mathrm{d}s,\mathrm{d}y)\Big|^{2p}\Big] \leqslant C|t_1-t_2|^{2p\mu}$$

and

$$\mathbf{E}\Big[\Big|\int_{t_2}^{t_1}\int_{\mathbb{R}} p_{t_1-s}(x,y)W^H(\mathrm{d}s,\mathrm{d}y)\Big|^{2p}\Big] \leqslant C|t_1-t_2|^{2p\mu}, \tag{2.7}$$

which ends the proof. □

3 Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. We first establish the following assertion.

Lemma 3.1 *The definition of solution to* (1.2) *is equivalent to the following mild formulation*:

$$\langle u_t, f\rangle = \langle u_0, P_t f\rangle + \int_0^t \langle g(u_s, \nabla u_s), P_{t-s}f\rangle\mathrm{d}s + \int_0^t\int_{\mathbb{R}} P_{t-s}f(x)W^H(\mathrm{d}s,\mathrm{d}x), \quad t>0, \tag{3.1}$$

where f is bounded function on $\mathbb{R}$. Moreover, for all $t>0$ and $x\in\mathbb{R}$,

$$u_t(x)=\langle u_0,p_t(x-\cdot)\rangle+\int_0^t\langle g(u_s,\nabla u_s),p_{t-s}(x-\cdot)\rangle\mathrm{d}s \\ +\int_0^t\int_{\mathbb{R}}p_{t-s}(x-y)W^H(\mathrm{d}s,\mathrm{d}y) \tag{3.2}$$

almost surely.

Proof Suppose that $(u_t)_{t\geqslant 0}$ is a solution to (1.2). We first assume that $f\in C_c^\infty(\mathbb{R})$. Let $t_i=i/n$ for $0\leqslant i\leqslant n$ and $n\geqslant 1$. By (1.2), we have

$$\begin{aligned}&\langle u_{t_i},P_{t-t_{i-1}}f\rangle-\langle u_{t_{i-1}},P_{t-t_{i-1}}f\rangle\\ &\quad=\frac{1}{2}\int_{t_{i-1}}^{t_i}\langle u_s(x),P_{t-t_{i-1}}f''\rangle\mathrm{d}s+\int_{t_{i-1}}^{t_i}\langle g(u_s,\nabla u_s),P_{t-t_{i-1}}f\rangle\mathrm{d}s\\ &\qquad+\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}}P_{t-t_{i-1}}f(x)W^H(\mathrm{d}s,\mathrm{d}x)\end{aligned}$$

and

$$\begin{aligned}P_{t-t_i}f(x)-P_{t-t_{i-1}}f(x)&=\int_{t-t_{i-1}}^{t-t_i}\partial_sP_sf(x)\mathrm{d}s\\ &=\frac{1}{2}\int_{t-t_{i-1}}^{t-t_i}P_sf''(x)\mathrm{d}s\\ &=-\frac{1}{2}\int_{t_{i-1}}^{t_i}P_{t-s}f''(x)\mathrm{d}s.\end{aligned}$$

Then

$$\begin{aligned}&\langle u_t,f\rangle-\langle u_0,P_tf\rangle\\ &\quad=\sum_{i=1}^n\langle u_{t_i},P_{t-t_i}f-P_{t-t_{i-1}}f\rangle+\sum_{i=1}^n[\langle u_{t_i},P_{t-t_{i-1}}f\rangle-\langle u_{t_{i-1}},P_{t-t_{i-1}}f\rangle]\\ &\quad=-\frac{1}{2}\int_0^t\sum_{i=1}^n 1_{(t_{i-1},t_i]}\langle u_{t_i},P_{t-s}f''\rangle\mathrm{d}s+\frac{1}{2}\int_0^t\sum_{i=1}^n 1_{(t_{i-1},t_i]}\langle u_s,P_{t-t_{i-1}}f''\rangle\mathrm{d}s\\ &\qquad+\int_0^t\sum_{i=1}^n 1_{(t_{i-1},t_i]}\langle g(u_s,\nabla u_s),P_{t-t_{i-1}}f\rangle\mathrm{d}s\\ &\qquad+\int_0^t\int_{\mathbb{R}}\sum_{i=1}^n 1_{(t_{i-1},t_i]}P_{t-t_{i-1}}f(x)W^H(\mathrm{d}s,\mathrm{d}x).\end{aligned}$$

Letting $n\to\infty$, we get (3.1) for $f\in C_c^\infty(\mathbb{R})$. By the dominated convergence, one can get (3.1) for bounded function f. Conversely, if $(u_t)_{t\geqslant 0}$ satisfies (3.1), then similar to the argument in the proof of [20, Theorem 2.1], we can show that $(u_t)_{t\geqslant 0}$ is the solution to (1.2). Let $M_t(x)$ denote the right-hand side of

(3.2). Then by (3.1) and the Fubini theorem, $\langle u_t, f\rangle = \langle H_t, f\rangle$ for all bounded function f, which ends the proof. □

Proof of Theorem 1.3 (*Uniqueness*) Let $(u_t^{(1)})_{t\geqslant 0}$ and $(u_t^{(2)})_{t\geqslant 0}$ be two strong $\mathscr{X}_1$-valued solutions to (1.2) with $u_0^{(1)} = u_0^{(2)}$. For $s, \delta > 0$ and $x \in \mathbb{R}$, define

$$\overline{u}_s(x) := u_s^{(1)}(x) - u_s^{(2)}(x), \quad \overline{g}_s(x) := g(u_s^{(1)}(x), \nabla u_s^{(1)}(x)) - g(u_s^{(2)}(x), \nabla u_s^{(2)}(x)),$$

$$\overline{u}_s^\delta(x) := \int_{\mathbb{R}} \overline{u}_s(y) p_\delta(x-y)\mathrm{d}y, \quad \overline{g}_s^\delta(x) := \int_{\mathbb{R}} \overline{g}_s(y) p_\delta(x-y)\mathrm{d}y.$$

Then $\overline{u}_s^\delta \in \mathscr{X}_0 \cap C^\infty(\mathbb{R})$ for each $s, \delta > 0$. It follows from (1.2) that

$$\overline{u}_t^\delta(x) = \frac{1}{2}\int_0^t \Delta \overline{u}_s^\delta(x)\mathrm{d}s + \int_0^t \overline{g}_s^\delta(x)\mathrm{d}s, \quad x \in \mathbb{R}, \tag{3.3}$$

which deduces that for all $f \in C_c^\infty(\mathbb{R})$,

$$\langle \overline{u}_t^\delta, f\rangle_0 = \frac{1}{2}\int_0^t \mathrm{d}s \int_{\mathbb{R}} (\Delta \overline{u}_s^\delta(x)) f(x) J(x)\mathrm{d}x + \int_0^t \langle \overline{g}_s^\delta, f\rangle_0 \mathrm{d}s.$$

Using Itô's formula, we obtain

$$\langle \overline{u}_t^\delta, f\rangle_0^2 = \int_0^t \langle \overline{u}_t^\delta, f\rangle_0 \, \langle \Delta \overline{u}_s^\delta, f\rangle_0 \mathrm{d}s + 2\int_0^t \langle \overline{u}_t^\delta, f\rangle_0 \, \langle \overline{g}_s^\delta, f\rangle_0 \mathrm{d}s.$$

Summing on f over a complete orthonormal system of $\mathscr{X}_0$, we get

$$\|\overline{u}_t^\delta\|_0^2 = \int_0^t \langle \overline{u}_s^\delta, \Delta \overline{u}_s^\delta\rangle_0 \mathrm{d}s + 2\int_0^t \langle \overline{u}_s^\delta, \overline{g}_s^\delta\rangle_0 \mathrm{d}s. \tag{3.4}$$

By integration by parts, for each $u, \widetilde{u} \in \mathscr{X}_1$, we have

$$\int_{\mathbb{R}} u(x)\widetilde{u}'(x)J(x)\mathrm{d}x = -\int_{\mathbb{R}} u'(x)\widetilde{u}(x)J(x)\mathrm{d}x - \int_{\mathbb{R}} u(x)\widetilde{u}(x)J'(x)\mathrm{d}x, \tag{3.5}$$

which implies that

$$\begin{aligned}
\langle \overline{u}_s^\delta, \Delta \overline{u}_s^\delta\rangle_0 &= -\,\|\nabla \overline{u}_s^\delta\|^2 - \int_{\mathbb{R}} \overline{u}_s^\delta(x)\nabla \overline{u}_s^\delta(x) J'(x)\mathrm{d}x \\
&= -\,\|\nabla \overline{u}_s^\delta\|^2 - \frac{1}{2}\int_{\mathbb{R}} \nabla((\overline{u}_s^\delta(x))^2) J'(x)\mathrm{d}x \\
&= -\,\|\nabla \overline{u}_s^\delta\|^2 - \frac{1}{2}\int_{\mathbb{R}} (\overline{u}_s^\delta(x))^2 J''(x)\mathrm{d}x \\
&\leqslant -\,\|\nabla \overline{u}_s^\delta\|^2 + C\|\overline{u}_s^\delta\|^2,
\end{aligned}$$

where we used (1.3) in the last inequality. Combining this with (3.4), we obtain

$$\|\overline{u}_t^\delta\|_0^2 \leqslant -\int_0^t \|\nabla \overline{u}_s^\delta\|^2 \mathrm{d}s + C\int_0^t \|\overline{u}_s^\delta\|^2 \mathrm{d}s + 2\int_0^t \langle \overline{u}_s^\delta, \overline{g}_s^\delta\rangle_0 \mathrm{d}s.$$

Now, letting $\delta \to 0$ and using [22, Lemma 2.1], we get

$$\|\overline{u}_t\|_0^2 \leqslant -\int_0^t \|\nabla \overline{u}_s\|_0^2 \mathrm{d}s + C\int_0^t \|\overline{u}_s\|^2 \mathrm{d}s + 2\int_0^t \langle \overline{u}_s, \overline{g}_s\rangle_0 \mathrm{d}s. \tag{3.6}$$

Observe that

$$\widetilde{g}(u_s^{(1)}(x))\nabla u_s^{(1)}(x) - \widetilde{g}(u_s^{(2)}(x))\nabla u_s^{(2)}(x) = \nabla \int_{u_s^{(2)}(x)}^{u_s^{(1)}(x)} \widetilde{g}(v)\mathrm{d}v.$$

It then follows from (3.5) that

$$\begin{aligned} H_1(s) :=& \int_{\mathbb{R}} \overline{u}_s(x)[\widetilde{g}(u_s^{(1)}(x))\nabla u_s^{(1)}(x) - \widetilde{g}(u_s^{(2)}(x))\nabla u_s^{(2)}(x)]J(x)\mathrm{d}x \\ =& -\int_{\mathbb{R}} (\nabla \overline{u}_s(x))\Big[\int_{u_s^{(2)}(x)}^{u_s^{(1)}(x)} \widetilde{g}(v)\mathrm{d}v\Big]J(x)\mathrm{d}x \\ & -\int_{\mathbb{R}} \overline{u}_s(x)\Big[\int_{u_s^{(2)}(x)}^{u_s^{(1)}(x)} \widetilde{g}(v)\mathrm{d}v\Big]J'(x)\mathrm{d}x. \end{aligned}$$

Using (1.3), we get

$$H_1(s) \leqslant C\int_{\mathbb{R}} |\nabla \overline{u}_s(x)|\,|\overline{u}_s(x)|J(x)\mathrm{d}x + C\|\overline{u}_s\|_0^2 \leqslant \frac{1}{8}\,\|\nabla \overline{u}_s\|_0^2 + C\|\overline{u}_s\|_0^2. \tag{3.7}$$

Observe that under Condition 1.2,

$$|x_1 - x_2|\,|\overline{g}(x_1, y_1) - \overline{g}(x_2, y_2)| \leqslant C|x_1 - x_2|^2 + \frac{1}{8}\,|y_1 - y_2|^2.$$

Then, together this with (3.7), we get

$$2\langle \overline{u}_s, \overline{g}_s\rangle_0 \leqslant \frac{1}{2}\,\|\nabla \overline{u}_s\|_0^2 + C\|\overline{u}_s\|_0^2.$$

Combining the above inequality with (3.6), we obtain

$$\|\overline{u}_t\|_0^2 + \frac{1}{2}\int_0^t \|\nabla \overline{u}_s\|_0^2 \mathrm{d}s \leqslant C\int_0^t \|\overline{u}_s\|_0^2 \mathrm{d}s.$$

Now, by Gronwall's lemma, we have $\|\overline{u}_t\|_1 = 0$ for all $t \geqslant 0$, which ends the proof. □

Proof of Theorem 1.3 (*Existence*) Recall that

$$p_t(x) = \frac{1}{\sqrt{2\pi t}}\,\mathrm{e}^{-x^2/(2t)}, \quad P_t f(x) = \int_{\mathbb{R}} p_t(x-y)f(y)\mathrm{d}y.$$

We consider the mild form (3.2) for (1.2).

For $t, \delta > 0$ and $x \in \mathbb{R}$, define $u_t^n(x)$ as

$$u_t^1(x) = P_t u_0(x),$$

and for $n \geqslant 1$,

$$u_t^{n+1}(x) = P_t u_0(x) + \int_0^t P_{t-s} g(u_s^n, \nabla u_s^n)(x) \mathrm{d}s + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) W^H(\mathrm{d}s, \mathrm{d}y). \tag{3.8}$$

By Lemma 2.7, one can see that $u_t^n \in \mathscr{X}_1$. Let

$$u_t^{n,\delta}(x) = P_\delta u_t^n(x).$$

Then $u_t^{n,\delta} \in C^\infty(\mathbb{R})$. It follows from (3.8) that

$$\begin{aligned} u_t^{n+1,\delta}(x) = P_{t+\delta} u_0(x) &+ \int_0^t P_{t-s+\delta} g(u_s^n, \nabla u_s^n)(x) \mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}} p_{t-s+\delta}(x-y) W^H(\mathrm{d}s, \mathrm{d}y), \end{aligned} \tag{3.9}$$

which can be written into the form

$$\begin{aligned} \langle u_t^{n+1,\delta}, f\rangle_0 = \langle u_0, P_\delta f\rangle_0 + \frac{1}{2}\int_0^t \langle \Delta u_s^{n+1,\delta}, f\rangle_0 \mathrm{d}s + \int_0^t \langle P_\delta g(u_s^n, \nabla u_s^n), f\rangle_0 \mathrm{d}s \\ + \int_0^t \int_{\mathbb{R}} P_\delta(fJ)(y) W^H(\mathrm{d}s, \mathrm{d}y), \quad f \in C_c^\infty(\mathbb{R}). \end{aligned} \tag{3.10}$$

By [22, Lemma 2.1], $u_t^n \in \mathscr{X}_1$. For $x \in \mathbb{R}$, $s, \delta > 0$, and $n \geqslant 1$ let

$$\overline{u}_s^{n+1,\delta} := u_s^{n+1,\delta} - u_s^{n,\delta},$$
$$\overline{g}_s^{n+1}(x) := g(u_s^{n+1}(x), \nabla u_s^{n+1}(x)) - g(u_s^n(x), \nabla u_s^n(x)).$$

It then follows from (3.10) that

$$\langle \overline{u}_t^{n+1,\delta}, f\rangle_0 = \frac{1}{2}\int_0^t \langle \Delta \overline{u}_s^{n+1,\delta}, f\rangle_0 \mathrm{d}s + \int_0^t \langle P_\delta \overline{g}_s^n, f\rangle_0 \mathrm{d}s.$$

By Itô's formula, we have

$$\langle \overline{u}_t^{n+1,\delta}, f\rangle_0^2 = \int_0^t \langle \overline{u}_t^{n+1,\delta}, f\rangle_0 \langle \Delta \overline{u}_s^{n+1,\delta}, f\rangle_0 \mathrm{d}s + 2\int_0^t \langle \overline{u}_t^{n+1,\delta}, f\rangle_0 \langle P_\delta \overline{g}_s^n, f\rangle_0 \mathrm{d}s.$$

Summing on f over a complete orthonormal system of $\mathscr{X}_0$, we get

$$\|\overline{u}_t^{n+1,\delta}\|_0^2 = \int_0^t \langle \Delta \overline{u}_s^{n+1,\delta}, \overline{u}_s^{n+1,\delta}\rangle_0^2 \mathrm{d}s + 2\int_0^t \langle P_\delta \overline{g}_s^n, \overline{u}_s^{n+1,\delta}\rangle_0 \mathrm{d}s.$$

Then, by integration by parts again and (3.5), we get

$$\begin{aligned}\|\overline{u}_t^{n+1,\delta}\|_0^2\mathrm{e}^{-\lambda t} &= \int_0^t \langle \Delta\overline{u}_s^{n+1,\delta}, \overline{u}_s^{n+1,\delta}\rangle_0^2\,\mathrm{e}^{-\lambda s}\mathrm{d}s - \lambda\int_0^t \|\overline{u}_s^{n+1,\delta}\|_0^2\,\mathrm{e}^{-\lambda s}\mathrm{d}s \\ &\quad + 2\int_0^t \langle P_\delta \overline{g}_s^n, \overline{u}_s^{n+1,\delta}\rangle_0\,\mathrm{e}^{-\lambda s}\mathrm{d}s \\ &\leqslant -\int_0^t \|\nabla\overline{u}_s^{n+1,\delta}\|_0^2\,\mathrm{e}^{-\lambda s}\mathrm{d}s + (C-\lambda)\int_0^t \|\overline{u}_s^{n+1,\delta}\|_0^2\,\mathrm{e}^{-\lambda s}\mathrm{d}s \\ &\quad + 2\int_0^t \langle \overline{g}_s^n, \overline{u}_s^{n+1,\delta}\rangle_0\,\mathrm{e}^{-\lambda s}\mathrm{d}s.\end{aligned}$$

As the same argument in the proof of Theorem 1.3 (uniqueness), we have

$$2\langle \overline{g}_s^{n,\delta}, \overline{u}_s^{n+1,\delta}\rangle_0 \leqslant C[\|\overline{u}_s^{n+1,\delta}\|_0^2 + \|\overline{u}_s^n\|_0^2] + \frac{1}{4}\,[\|\nabla\overline{u}_s^{n+1,\delta}\|_0^2 + \|\nabla\overline{u}_s^n\|_0^2].$$

Then, for $\lambda > 0$ large enough, we can see that

$$\begin{aligned}&\|\overline{u}_t^{n+1,\delta}\|_0^2\,\mathrm{e}^{-\lambda t} + \lambda\int_0^t \|\overline{u}_s^{n+1,\delta}\|_0^2\,\mathrm{e}^{-\lambda s}\mathrm{d}s + \frac{3}{4}\int_0^t \|\nabla\overline{u}_s^{n+1,\delta}\|_0^2\,\mathrm{e}^{-\lambda s}\mathrm{d}s \\ &\leqslant \frac{1}{2}\Big[\lambda\int_0^t \|\overline{u}_s^n\|_0^2\,\mathrm{e}^{-\lambda s}\mathrm{d}s + \frac{3}{4}\int_0^t \|\nabla\overline{u}_s^n\|_0^2\,\mathrm{e}^{-\lambda s}\mathrm{d}s\Big]. \end{aligned}\tag{3.11}$$

It then follows from Fatou's lemma that

$$\begin{aligned}&\int_0^t \Big[\lambda\|\overline{u}_s^{n+1}\|^2 + \frac{3}{4}\,\|\nabla\overline{u}_s^{n+1}\|^2\Big]\mathrm{e}^{-\lambda s}\mathrm{d}s \\ &= \int_0^t \Big[\|\lim_{\delta\to 0}\overline{u}_s^{n+1,\delta}\|^2 + \frac{3}{4}\,\|\lim_{\delta\to 0}\nabla\overline{u}_s^{n+1,\delta}\|^2\Big]\mathrm{e}^{-\lambda s}\mathrm{d}s \\ &\leqslant \liminf_{\delta\to 0}\int_0^t \Big[\lambda\|\overline{u}_s^{n+1,\delta}\|^2 + \frac{3}{4}\,\|\nabla\overline{u}_s^{n+1,\delta}\|^2\Big]\mathrm{e}^{-\lambda s}\mathrm{d}s \\ &\leqslant \frac{1}{2^n}\int_0^t \Big[\lambda\|\overline{u}_s^1\|^2 + \frac{3}{4}\,\|\nabla\overline{u}_s^1\|^2\Big]\mathrm{e}^{-\lambda s}\mathrm{d}s,\end{aligned}$$

which means that for each $T > 0$,

$$\int_0^T [\|\overline{u}_s^{n+1}\|^2 + \|\nabla\overline{u}_s^{n+1}\|^2]\mathrm{d}s \leqslant \frac{C}{2^n}.$$

Thus, $(u_t^n)_{0\leqslant t\leqslant T}$ is a Cauchy sequence in $\mathscr{X}_1 \times [0,T]$ and $\mathscr{X}_1$-valued process $(u_t)_{t\geqslant 0}$ denotes the limit. Letting $n \to \infty$ in (3.8), we obtain that $(u_t)_{t\geqslant 0}$ satisfies (3.2), which ends the proof. □

4 Proof of Theorem 1.4

To establish the proof of Theorem 1.4, we first state two lemmas.

Lemma 4.1 *For each $T > 0$ and $p \geqslant 1$,*

$$\sup_{0<t\leqslant T} \mathbf{E}\left[\|u_t\|_0^{2p} + \Big|\int_0^t [\|u_s\|_0^2 + \|\nabla u_s\|_0^2]\mathrm{d}s\Big|^p\right] < \infty.$$

Proof Let

$$v_t(x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) W^H(\mathrm{d}s, \mathrm{d}y), \quad \overline{u}_t = u_t - v_t.$$

Then, by Lemma 2.7, we have $v_t, \overline{u}_t \in \mathscr{X}_1$ and

$$\overline{u}_t(x) = \langle u_0, p_t(x-\cdot)\rangle + \int_0^t \langle g(u_s, \nabla u_s), p_{t-s}(x-\cdot)\rangle \mathrm{d}s,$$

which can be written into the form

$$\langle \overline{u}_t, f\rangle = \langle u_0, f\rangle + \frac{1}{2}\int_0^t \langle \overline{u}_s, f''\rangle \mathrm{d}s + \int_0^t \langle g(u_s, \nabla u_s), f\rangle \mathrm{d}s.$$

As the same argument in (3.11), for $\lambda > 0$ large enough, we have

$$\begin{aligned}
&\|\overline{u}_t\|_0^2\,\mathrm{e}^{-\lambda t} + \int_0^t \Big(\lambda\|\overline{u}_s\|_0^2 + \frac{3}{4}\,\|\nabla \overline{u}_s\|_0^2\Big)\mathrm{e}^{-\lambda s}\mathrm{d}s \\
&\quad \leqslant \|u_0\|_0^2 + \frac{1}{2}\int_0^t \Big(\lambda\|u_s\|_0^2 + \frac{3}{4}\,\|\nabla u_s\|_0^2\Big)\mathrm{e}^{-\lambda s}\mathrm{d}s,
\end{aligned}$$

which deduces that

$$\begin{aligned}
&\|u_t\|_0^2\,\mathrm{e}^{-\lambda t} + \int_0^t (\|u_s\|_0^2 + \|\nabla u_s\|_0^2)\mathrm{e}^{-\lambda s}\mathrm{d}s \\
&\quad \leqslant C(\|u_0\|_0^2 + \|v_t\|_0^2) + C\int_0^t (\|v_s\|_0^2 + \|\nabla v_s\|_0^2)\mathrm{e}^{-\lambda s}\mathrm{d}s.
\end{aligned}$$

Now, by Lemma 2.7, one can conclude the assertion. □

Lemma 4.2 *Let $0 < \theta < 1 < p$ and $T, a > 0$ be fixed. Then there is a constant $C > 0$ so that*

$$\mathbf{E}[|u_t(x_1) - u_t(x_2)|^{2p}] \leqslant C\frac{|x_1 - x_2|^{p\theta}}{t^{p(1+\theta)/2}}, \quad 0 < t \leqslant T,\ |x_1|, |x_2| \leqslant a, \tag{4.1}$$

and

$$\mathbf{E}[|u_{t_1}(x) - u_{t_2}(x)|^{2p}] \leqslant C\frac{|t_1 - t_2|^{p\theta/2}}{t_2^{p(1+\theta)/2}}, \quad 0 < t_1 \leqslant t_2 \leqslant T,\ |x| \leqslant a.$$

Moreover, if u_0 is also Hölder continuous with exponent $\gamma \geqslant 1/2$, then

$$\mathbf{E}[|u_t(x_1) - u_t(x_2)|^{2p}] \leqslant C|x_1 - x_2|^{p\theta}, \quad 0 < t \leqslant T,\ |x_1|, |x_2| \leqslant a, \tag{4.2}$$

and

$$\mathbf{E}[|u_{t_1}(x) - u_{t_2}(x)|^{2p}] \leqslant C|t_1 - t_2|^{p\theta/2}, \quad 0 < t_1, t_2 \leqslant T,\ |x| \leqslant a.$$

Proof Since the proofs are similar, we only state those of (4.1) and (4.2). By (3.2), we have

$$\begin{aligned} u_t(x_1) - u_t(x_2) &= \langle u_0, M_t^{x_1,x_2}\rangle + \int_0^t \langle \overline{g}(u_s, \nabla u_s), M_{t-s}^{x_1,x_2}\rangle \mathrm{d}s \\ &\quad + \int_0^t \langle \widetilde{g}(u_s)\nabla u_s, M_{t-s}^{x_1,x_2}\rangle \mathrm{d}s + \int_0^t \int_{\mathbb{R}} M_{t-s}^{x_1,x_2}(y) W^H(\mathrm{d}s, \mathrm{d}y) \\ &=: I_t^1(x_1,x_2) + I_t^2(x_1,x_2) + I_t^3(x_1,x_2) + I_t^4(x_1,x_2), \end{aligned} \tag{4.3}$$

where

$$M_t^{x_1,x_2}(y) := p_t(x_1 - y) - p_t(x_2 - y).$$

Observe that

$$\int_{\mathbb{R}} \frac{p_t((x-y)/2)}{J(y)}\,\mathrm{d}y \leqslant C\mathrm{e}^{|x|}\int_{\mathbb{R}} p_t\Big(\frac{x-y}{2}\Big)\mathrm{e}^{|x-y|}\mathrm{d}y \leqslant C\mathrm{e}^{a}\int_{\mathbb{R}} p_1\Big(\frac{z}{2}\Big)\mathrm{e}^{\sqrt{T}|z|}\mathrm{d}z. \tag{4.4}$$

Then, by (2.3) and Condition 1.2, for $0 < t \leqslant T$ and $|x_1|, |x_2| \leqslant a$, we have

$$\begin{aligned} &\int_{\mathbb{R}} \frac{|M_t^{x_1,x_2}(y)|^2}{J(y)}\,\mathrm{d}y \\ &\quad \leqslant Ct^{-(1+\theta)/2}|x_1 - x_2|^\theta \int_{\mathbb{R}} \frac{p_t((x_1-y)/2) + p_t((x_2-y)/2)}{J(y)}\,\mathrm{d}y \\ &\quad \leqslant 4Ct^{-(1+\theta)/2}|x_1 - x_2|^\theta \end{aligned}$$

and

$$|\overline{g}(x,y)| \leqslant C(|x| + |y| + 1).$$

It follows from Hölder's inequality that

$$|I_t^1(x_1,x_2)|^{2p} \leqslant \Big|\|u_0\|_0^2 \int_{\mathbb{R}} \frac{|M_t^{x_1,x_2}(y)|^2}{J(y)}\,\mathrm{d}y\Big|^p \leqslant C\frac{\|u_0\|_0^{2p}\,|x_1 - x_2|^{p\theta}}{t^{p(1+\theta)/2}} \tag{4.5}$$

and

$$\begin{aligned} |I_t^2(x_1,x_2)|^2 &\leqslant \int_0^t \|\overline{g}(u_s, \nabla u_s)\|_0^2 \mathrm{d}s \int_0^t \mathrm{d}s \int_{\mathbb{R}} \frac{|M_{t-s}^{x_1,x_2}(y)|^2}{J(y)}\,\mathrm{d}y \\ &\leqslant C|x_1 - x_2|^\theta \int_0^t (\|u_s\|_0^2 + \|\nabla u_s\|_0^2 + 1)\mathrm{d}s \end{aligned}$$

for all $t \in (0,T]$ and $|x_1|, |x_2| \leqslant a$. Using Lemma 4.1, we obtain

$$\mathbf{E}[|I_t^2(x_1,x_2)|^{2p}] \leqslant C|x_1-x_2|^{p\theta}, \quad 0<t\leqslant T,\ |x_1|,|x_2|\leqslant a. \tag{4.6}$$

If

$$|u_0(x_1)-u_0(x_2)| \leqslant C|x_1-x_2|^{\gamma}, \quad x_1,x_2\in\mathbb{R},$$

then, by a change of variable, we have

$$\begin{aligned}|I_t^1(x_1,x_2)| &= \Big|\int_{\mathbb{R}}[u_0(x_1-y)-u_0(x_2-y)]p_t(y)\mathrm{d}y\Big| \\ &\leqslant C|x_1-x_2|^{\gamma}\int_{\mathbb{R}}p_t(y)\mathrm{d}y \\ &= C|x_1-x_2|^{\gamma}\end{aligned}$$

for all $t \in (0,T]$ and $|x_1|, |x_2| \leqslant a$, which implies

$$|I_t^1(x_1,x_2)|^{2p} \leqslant C|x_1-x_2|^{2p\gamma}, \quad 0<t_1,t_2\leqslant T,\ |x|\leqslant a. \tag{4.7}$$

By (2.4), (2.5), and (4.4), for all $t \in (0,T]$ and $|x_1|, |x_2| \leqslant a$, we have

$$\begin{aligned}\int_{\mathbb{R}}\frac{|\nabla M_t^{x_1,x_2}(y)|^2}{J(y)}\,\mathrm{d}y &\leqslant \frac{C}{t}\int_{\mathbb{R}}\frac{|\nabla M_t^{x_1,x_2}(y)|}{J(y)}\,\mathrm{d}y \\ &\leqslant C\frac{|x_1-x_2|^{\theta}}{t^{(3+\theta)/2}}\int_{\mathbb{R}}\frac{p_t((x_1+y)/2)+p_t((x_2+y)/2)}{J(y)}\,\mathrm{d}y \\ &\leqslant C\frac{|x_1-x_2|^{\theta}}{t^{(3+\theta)/2}}.\end{aligned}$$

It then follows from Condition 1.2, Hölder's inequality, and (4.4) that

$$\begin{aligned}|I_t^3(x_1,x_2)| &\leqslant C\int_0^t\langle|u_s|,|\nabla M_{t-s}^{x_1,x_2}|\rangle\mathrm{d}s \\ &\leqslant C\int_0^t\|u_s\|_0\bigg[\int_{\mathbb{R}}\frac{|\nabla M_{t-s}^{x_1,x_2}(y)|^2}{J(y)}\,\mathrm{d}y\bigg]^{1/2}\mathrm{d}s \\ &\leqslant C\int_0^t\|u_s\|_0\bigg[\int_{\mathbb{R}}|\nabla M_{t-s}^{x_1,x_2}(y)|^2\mathrm{e}^{|y|}\mathrm{d}y\bigg]^{1/2}\mathrm{d}s \\ &\leqslant C|x_1-x_2|^{\theta/2}\int_0^t\frac{\|u_s\|_0}{(t-s)^{(3+\theta)/4}}\,\mathrm{d}s.\end{aligned}$$

Combining the above inequality with Lemma 4.1, we obtain

$$\mathbf{E}[|I_t^3(x_1,x_2)|^{2p}] \leqslant C|x_1-x_2|^{p\theta}\int_0^t\frac{\mathbf{E}[\|u_s\|_0^{2p}]}{(t-s)^{(3+\theta)/4}}\,\mathrm{d}s \leqslant C|x_1-x_2|^{p\theta} \tag{4.8}$$

for all $t \in (0,T]$ and $|x_1|, |x_2| \leqslant a$. By Lemma 2.8, we have

$$\mathbf{E}[|I_t^4(x_1,x_2)|^{2p}] \leqslant C|x_1-x_2|^{2p\theta}, \quad 0<t\leqslant T,\ x_1,x_2\in\mathbb{R}.$$

Together this with (4.3) and (4.5)–(4.8), one ends the proof. □

Now, we are ready to prove Theorem 1.4.

Proof of Theorem 1.4 By Kolmogorov's continuity criteria (see, e.g., [11, p. 31]) and Lemma 4.2, the assertion follows immediately. □

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