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RESEARCH ARTICLE

Ground States for DNLS Equation with Periodic or Asymptotically Periodic Potential

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Abstract We study the existence of ground state solutions for a class of discrete nonlinear Schrödinger equation with a sign-changing potential which is periodic or asymptotically periodic. The resulting problem engages two major difficulties: one is that the associated functional is strongly indefinite, the second is that, due to the asymptotically periodic assumption, the associated functional loses the $\mathbb{Z}$-translation invariance, and many effective methods for periodic problems cannot be applied to asymptotically periodic ones. These enables us to develop a direct approach to find ground state solutions with asymptotically periodic potential. Two types of ground state solutions are obtained with some new super-quadratic conditions on nonlinearity which are weaker that some well-known ones. Moreover, our conditions can also be used to significantly improve the well-known results of the corresponding continuous nonlinear Schrödinger equation.

Keywords Ground states, discrete NLS equation, strongly indefinite functional, Nehari–Pankov manifold

MSC2020 39A12, 39A70, 35Q51, 35Q55

1 Introduction

In this paper, we study standing waves of the following system of discrete nonlinear Schrödinger (DNLS) equation:

$$i\dot{\psi}_m = -\Delta\psi_m + (V_m + \omega)\psi_m - f_m(\psi_m), \quad m \in \mathbb{Z}, \tag{1.1}$$

where $\omega \in \mathbb{R}$, $\Delta\psi_m := \psi_{m+1} - 2\psi_m + \psi_{m-1}$ is the discrete Laplacian in one spatial dimension. The discrete potential V_m is a sequence of real numbers, f_m

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is a function sequence. Equation (1.1) may be viewed as a discrete form of the nonlinear Schrödinger equation:

$$i\dot{\psi}_t = -\Delta\psi + (V(x)+\omega)\psi - f(x,\psi), \quad x \in \mathbb{R}^N. \tag{1.2}$$

The nonlinear Schrödinger equation (1.2) has been extensively investigated analytically by many mathematicians and physicists, from the fundamental well posedness of Cauchy problem to the existence and stability of standing waves.

Assume that the nonlinearity $f(u)$ is gauge invariant, that is $f(e^{i\omega}u) = e^{i\omega}f(u)$ for any $\omega \in \mathbb{R}$. Thus, we consider the special solutions of (1.1) of the form $\psi_m = e^{-i\omega t}q_m$, these solutions are called breather solutions or standing waves due to their periodic time behavior. Inserting the ansatz of a breather solution into (1.1), we see that any breather solution satisfies the infinite nonlinear system of algebraic equations

$$-\Delta q_m + V_m q_m = f_m(q_m), \quad m \in \mathbb{Z}, \tag{1.3}$$

where q_m is a real-valued sequence.

The DNLS equation has been proven useful in describing a variety of phenomena in nonlinear physics, such as propagation of excitations in a deformable medium [7,9], dynamics of Bose–Einstein condensates inside coupled magneto-optical traps [1,18], transversal propagation of light in waveguide arrays, self-focusing and collapse of Langmuir waves in plasma physics and description of rogue waves in the ocean [8]. Its main features include the existence of localized nonlinear solutions with families of stable and unstable modes, the existence of a selftrapping transition of an initially localized excitation, and a degree of excitation mobility in 1D. All these characteristics have made the DNLS into a paradigmatic equation that describes the propagation of excitations in a nonlinear medium under a variety of different physical scenarios, see [11,13].

Discrete Schrödinger operators of the form $-\Delta + V$ appear in a wide range of fields, such as the description of random walks, the propagation of waves in crystals [12], and the theory of nonlinear integrable lattices (see [1,8,10,16] and references therein). It is worth pointing out that the existence of nontrivial solutions of (1.3) has been studied under different assumptions on the potential and the nonlinearity by using the variational method [3–6, 15, 17, 19, 24, 25, 32–34]. For instance, Pankov [19,20,22] studied the existence of a nontrivial solution problem (1.3) in the case of 0 belonging to a spectral gap of $-\Delta+V$. By using the Nehari manifold approach and the mountain pass argument, Pankov and Rothos [21] considered the existence of a nontrivial solution of (1.3) with a constant potential V and an asymptotically linear term f. Zhou and Yu [37,38] improved the classical AR superlinear condition to a general superlinear one. Later, they [39] studied the existence of nontrivial solutions of (1.3) under the strictly increasing conditions on f which is very crucial. G. Chen et al. [3,4,6] considered the nonautonomous problem of (1.3) with the potential V being periodic and f being asymptotically linear at infinity by using weak linking theorems; when either 0 is a spectral endpoint of $-\Delta + V$, or it is in a finite spectral gap of $-\Delta+V$, the authors obtained the existence of nontrivial solitons by using a generalized weak linking theorem introduced by Schechter and Zou [23]. Recently, Lin, Zhou and Yu [15] studied the existence of ground state solutions for (1.3) with a sign-changing potential V that converges at infinity and a nonlinear term being asymptotically linear at infinity.

Motivated by the interest shared by the mathematical community in this topic and the papers [3–6, 15, 17, 19, 24, 25, 32–34], the main goal of this paper is to investigate the question of existence of ground state solutions for (1.3). Based on the recent work [15, 24–30, 32, 37–39] and the non-Nehari manifold method [27,30,31] which are different from the previous work and generalize the results, this method has been proven successful, for instance, in solving Schrödinger equation and Dirac equation [2, 35, 36]. In this work, one difficulty in problem (1.3) is that the associated functional J (defined in Sect. 2) is strongly indefinite, that is, its quadratic part is respectively coercive and anti-coercive in infinitely dimensional subspaces of the energy space. To tackle this difficulty, we adapt non-Nehari method introduced by Tang [31]. It is convenient to decompose the functional space l^2 into a direct sum of two subspaces H^+ and H^- (H is defined in Sect. 2), one of which being infinite dimensional.

Another difficulty is the lack of periodic assumption. As a result, neither the periodic translation technique nor the compact inclusion method can be adapted. In this case, the functional J loses the $\mathbb{Z}$-translation invariance. For the above reasons, many effective methods for periodic problems cannot be applied to asymptotically periodic ones. To the best of our knowledge, there are no results on the existence of ground state solutions to (1.3) when V_m is asymptotically periodic. In this paper, we find new tricks to overcome the difficulties caused by the dropping of periodicity of V_m.

Now, we are ready to state the main results of the present paper as follows.

1.1 Periodic Potential

We assume that V_m and $f_m(t)$ are N-periodic sequences on m and 0 lies in a finite spectral gap of $\sigma(-\Delta + V_m)$, i.e.,

(V1) $V_{m+N} = V_m$ *and*

$$\sup[\sigma(-\Delta + V_m) \cap (-\infty, 0)] < 0 < \inf[\sigma(-\Delta + V_m) \cap (0, \infty)];$$

(f0) $f_{m+N}(t) = f_m(t)$, $f_m(t)$ *is continuous in* $t \in \mathbb{R}$ *for every* $m \in \mathbb{Z}$*;*

(f1) *for every* $m \in \mathbb{Z}$, $t \in \mathbb{R}$, $tf_m(t) \geq 0$*;*

(f2) $f_m(t) = o(|t|)$ *as* $|t| \to 0$*;*

(f3) $\lim_{|t|\to\infty} \frac{F_m(t)}{|t|^2} = \infty$ *for all* $m \in \mathbb{Z}$*, where* $F_m(t) = \int_0^t f_m(s)ds$*;*

(f4) *there exists a constant* $\eta_0 \in (0,1)$ *such that*

$$\frac{1-\eta^2}{2} t f_m(t) \geq \int_{\eta t}^{t} f_m(s)ds, \quad \forall \eta \in [0, \eta_0].$$

Theorem 1.1. *Assume that* (V1), (f0)–(f4) *hold. Then problem* (1.3) *has a ground state, i.e., a nontrivial solution* $q \in H$ *such that* $J(q) = \inf_{\mathscr{M}} J \geq \left(\frac{1}{8}\right)^{1/p-2} \Theta_0^{2/p-2} > 0$, *where*

$$\mathscr{M} = \{q \in H \setminus \{0\} : \langle J'(q), q\rangle = 0\}, \tag{1.4}$$

and

$$\Theta_0 = \left(\frac{1}{\eta_0^{p-1}} + \frac{1}{p}\right) C_{\varepsilon_0} \gamma_p^p, \quad \gamma_p = \sup_{q \in H,\, \|q\|=1} \|q\|_p, \quad p \geq 2, \tag{1.5}$$

C_{ε_0} *is defined by* (3.11) *(see Sect.* 3 *in details) with* $\varepsilon_0 = \frac{\eta_0}{2(2+\eta_0)\gamma_2^2}$.

1.2 Asymptotically Periodic Potential

In this part, we assume that V_m and $f_m(t)$ are asymptotically periodic on m.

(V1′) *For all* $m \in \mathbb{Z}$, $V_m = W_m + R_m$, $W_{m+N} = W_m$ *and*

$$\sup[\sigma(-\Delta + W_m) \cap (-\infty, 0)] < 0 < \inf[\sigma(-\Delta + W_m) \cap (0, \infty)].$$

Furthermore,

$$0 \leq -R_m \leq \sup_{\mathbb{Z}}(-R_m) < \Pi_0 := \inf[\sigma(-\Delta + W_m) \cap (0, \infty)],$$

and $\lim_{m\to\infty} R_m = 0$*;*

(f4′) $t \mapsto \frac{f_m(t)}{|t|}$ *is non-decreasing on* $(-\infty, 0) \cup (0, \infty)$*;*

(f4″) $f_m(t) = g_m(t) + h_m(t)$, $g_m(t)$ *is continuous in* $t \in \mathbb{R}$ *for every* $m \in \mathbb{Z}$, $g_{m+N}(t) = g_m(t)$, $g_m(t) = o(|t|)$, *as* $|t| \to 0$, *uniformly in* $m \in \mathbb{Z}$, $t \mapsto g_m(t)/|t|$ *is non-decreasing on* $(-\infty, 0) \cup (0, \infty)$*;* $h_m(t)$ *is continuous in* $t \in \mathbb{R}$ *for every* $m \in \mathbb{Z}$ *and satisfies that*

$$0 \leq t h_m(t) \leq a_m(|t|^2 + |t|^p), \quad \forall (m,t) \in \mathbb{Z} \times \mathbb{R}, \ p > 2$$

with $\lim_{m\to\infty} a_m = 0$*. Moreover,*

$$H_m(t) - \frac{1}{2} V_m t^2 > 0, \quad \text{for } (m,t) \in \mathbb{Z} \times \mathbb{R}.$$

Let

$$\mathscr{N}^- := \{q \in H \setminus H^- : \langle J'(q), q\rangle = \langle J'(q), h\rangle = 0, \ \forall h \in H^-\}. \tag{1.6}$$

The set $\mathscr{N}^-$ was first introduced by Pankov [20], which is a subset of the Nehari manifold (1.4). Since q_0 is a solution to the equation $J(q) = 0$ at which J has minimal energy in set $\mathscr{N}^-$, we shall call it a ground state solution of Nehari–Pankov type.

Theorem 1.2. *Assume that* (V1), (f0), (f2)–(f3), (f4′) *hold. Then problem* (1.3) *has a ground state, i.e., a nontrivial solution* $q \in H$ *such that* $J(q) = \inf_{\mathcal{N}^-} J > 0$.

Theorem 1.3. *Assume that* (V1′), (f0), (f2)–(f3), (f4″) *hold. Then problem* (1.3) *has a ground state, i.e., a nontrivial solution* $q \in H$ *such that* $J(q) = \inf_{\mathcal{N}^-} J > 0$.

The present paper is built up as follows. The variational structure and some properties of the associated functional are established in Section 2. We establish some instrumental lemmas involving our main theorem in Sections 3–5, finally the proofs of Theorems 1.1–1.3 are presented by non-Nehari method.

2 Variational Structure and Preliminaries

In order to apply variational method, we firstly state the corresponding work space, then we reduce the problem of finding solutions of (1.3) to the one of seeking the critical points of a corresponding variational functional.

Let $H = l^2(\mathbb{Z}, \mathbb{R})$. H is a Hilbert space with the usual inner product and norm:

$$(x, y)_{l^2} = \sum_{m\in\mathbb{Z}} x_m y_m, \quad |x|_{l^2} = \sum_{m\in\mathbb{Z}} |x_m|^2, \quad \forall x, y \in H.$$

Let $\mathscr{A} = -\Delta + V_m$, $\{\mathscr{E}(\lambda) : -\lambda_1 < \lambda < \lambda_2\}$ and $|\mathscr{A}|$ be the spectral family and the absolute value of $\mathscr{A}$, respectively, and $|\mathscr{A}|^{1/2}$ be the square root of $|\mathscr{A}|$. There hold

$$\mathscr{E}(-\lambda_1) = 0, \quad \mathscr{E}(\lambda_2) = \mathrm{id} = \int_{-\lambda_1}^{\lambda_2} d[\mathscr{E}(\lambda)]$$

and for any $q \in l^2$,

$$\mathscr{A}q = \int_{-\lambda_1}^{\lambda_2} \lambda d[\mathscr{E}(\lambda)q], \quad |\mathscr{A}|q = \int_{-\lambda_1}^{\lambda_2} |\lambda| d[\mathscr{E}(\lambda)q],$$
$$|\mathscr{A}|^{1/2}q = \int_{-\lambda_1}^{\lambda_2} |\lambda|^{1/2} d[\mathscr{E}(\lambda)q].$$

Set $\mathscr{U} = \mathrm{id} - \mathscr{E}(0) - \mathscr{E}(0-)$, then $\mathscr{U}$ commutes with $\mathscr{A}$, $|\mathscr{A}|$ and $|\mathscr{A}|^{1/2}$, and $\mathscr{A} = \mathscr{U}|\mathscr{A}|$ is the polar decomposition of $\mathscr{A}$. Let

$$H^- = \mathscr{E}(0-)H, \quad H^+ = [\mathrm{id} - \mathscr{E}(0)]H.$$

For any $q \in H$, it is easy to see that $q = q^- + q^+$, and

$$\mathscr{A}q^- = -|\mathscr{A}|q^-, \quad \mathscr{A}q^+ = |\mathscr{A}|q^+, \quad \forall q \in H, \tag{2.1}$$

where

$$q^- = \mathscr{E}(0-)q \in H^-, \quad q^+ = [\mathrm{id} - \mathscr{E}(0)]q \in H^+.$$

We can define an inner product

$$(q,p) = (|\mathscr{A}|^{1/2}q, |\mathscr{A}|^{1/2}p), \quad q,p \in H$$

and the corresponding norm

$$\|q\| = \||\mathscr{A}|^{1/2}q\|_2, \quad q \in H, \tag{2.2}$$

where $(\cdot,\cdot)_{l^2}$ denotes the inner product of l_2, $\|\cdot\|_s$ denotes the norm of l^s. By a standard argument, one can show that the norms $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent. Obviously, one has the orthogonal decomposition $H = H^- \oplus H^+$, where orthogonality is with respect to both $(\cdot,\cdot)_{l^2}$ and $(\cdot,\cdot)$. If $\sigma(\mathscr{A}) \subset (0,\infty)$, then $H^- = \{0\}$, otherwise H^- is infinite-dimensional.

For any $q \in H$, we define the following functional

$$J(q) = \frac{1}{2}(\mathscr{A}q,q)_{l^2} - \sum_{m\in\mathbb{Z}} F_m(q), \tag{2.3}$$

where $F_m(t) = \int_0^t f_m(s)ds$. Standard arguments show that under the assumptions of Theorems 1.1–1.3, J is well-defined and is of $C^1(E,\mathbb{R})$, and solutions of (1.3) are critical points of the functional J, and

$$\langle J'(q),p\rangle = (\mathscr{A}q,p)_{l^2} - \sum_{m\in\mathbb{Z}} f_m(q)p. \tag{2.4}$$

Let $B: H \to \mathbb{R}$,

$$B(q) = \sum_{m\in\mathbb{Z}} F_m(q). \tag{2.5}$$

Combining (2.4) with (2.5), we see that

$$J(q) = \frac{1}{2}(\mathscr{A}q,q) - B(q) \tag{2.6}$$

and

$$\langle J'(q),p\rangle = (\mathscr{A}q,p) - \sum_{m\in\mathbb{Z}} f_m(q)p. \tag{2.7}$$

In view of (2.1) and (2.2), we have

$$J(q) = \frac{1}{2}(\|q^+\|^2 - \|q^-\|^2) - \sum_{m\in\mathbb{Z}} F_m(q), \quad \forall q = q^- + q^+ \in H^- \oplus H^+ = H \tag{2.8}$$

and

$$\langle J'(q),q\rangle = \|q^+\|^2 - \|q^-\|^2 - \sum_{m\in\mathbb{Z}} f_m(q)p, \quad \forall q = q^- + q^+ \in H^- \oplus H^+ = H. \tag{2.9}$$

Let W be a real Hilbert space with $W = W^- \oplus W^+$ and $W^- \perp W^+$. For a functional $\psi \in C^1(W, \mathbb{R})$, ψ is said to be weakly sequentially lower semi-continuous if for any $u_n \rightharpoonup u$ in W one has $\psi(u) \le \liminf_{n\to\infty} \psi(u_n)$, and ψ' is said to be weakly sequentially continuous if $\lim_{n\to\infty}\langle \psi'(u_n), v\rangle = \langle \psi'(u_n), v\rangle$ for each $v \in W$.

Lemma 2.1 [14]**.** *Let W be a real Hilbert space, $W = W^- \oplus W^+$ and $W^- \perp W^+$, and $\psi \in C^1(X, \mathbb{R})$ of the form*

$$\psi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \psi(u), \quad u = u^- + u^+ \in W^- \oplus W^+.$$

Suppose that the following assumptions hold:

(A1) $\psi \in C^1(W, \mathbb{R})$ *is bounded from below and weakly sequentially lower semi-continuous;*

(A2) ψ' *is weakly sequentially continuous;*

(A3) *there exist $r > \rho > 0$, $e \in W^+$ with $\|e\| = 1$ such that*

$$\kappa := \inf \psi(S_\rho^+) > \sup \varphi(\partial Q),$$

where

$$S_\rho^+ = \{u \in X^+ : \|u\| = \rho\}, \quad Q = \{v + se : v \in X^-, \ s \ge 0, \ \|v + se\| \le r\}.$$

Then for some $c \in [\kappa, \sup \varphi(Q)]$, there exists a sequence $\{u_n\} \subset W$ satisfying

$$\psi(u_n) \to c, \quad \|\psi'(u_n)\|(1 + \|u_n\|) \to 0.$$

3 Proof of Theorem 1.1

Lemma 3.1. *Suppose that* (f0)–(f3) *are satisfied. Then $B(q)$ is nonnegative, weakly sequentially lower semi-continuous, and $B'(q)$ is weakly sequentially continuous.*

It is not difficult to verify the above lemma by means of Sobolev's imbedding theorem. The proof will be omitted.

Lemma 3.2. *Assume that* (f1), (f2) *and* (f4) *are satisfied, then for any $q \in H$, there holds*

$$\begin{aligned} J(q) \ge J(\mu q^+) + \frac{\mu^2\|q^-\|^2}{2} + \frac{1-\mu^2}{2}\langle J'(q), q\rangle + \mu^2 \langle J'(q), q^-\rangle \\ - \mu^2 \sum_{m\in\mathbb{Z}(|\mu|q^+|>\eta_0|q|)} f_m(q)q^+, \quad \forall \mu \ge 0. \end{aligned} \tag{3.1}$$

Proof. Fix $x, y \in \mathbb{R}$. Let

$$g(r) = \frac{1+r^2}{2} f_m(x)x - r^2 f_m(x)y + F_m(ry) - F_m(x).$$

If $xy \le 0$, using the assumption (f1), we have

$$\begin{aligned} g(r) &= \frac{1+r^2}{2} f_m(x)x - r^2 f_m(x)y + F_m(ry) - F_m(x) \\ &\ge \frac{1+r^2}{2} f_m(x)x - F_m(x), \quad \forall r \ge 0. \end{aligned} \tag{3.2}$$

If $xy \ge 0$, let $\eta = ry/x$. Using the assumptions (f1), (f2) and (f4), we have

$$\begin{aligned} g(r) &= \frac{1+r^2}{2} f_m(x)x - r^2 f_m(x)y + F(ry) - F(x) \\ &= \frac{1+r^2-2\eta r}{2} f_m(x)x - \int_{\eta x}^{x} f_m(s)ds \\ &= \frac{(\eta-r)^2}{2} f_m(x)x + \frac{1-\eta^2}{2} f_m(x)x - \int_{\eta x}^{x} f_m(s)ds \\ &\ge \frac{1-\eta^2}{2} f_m(x)x - \int_{\eta x}^{x} f_m(s)ds \\ &\ge 0, \quad r \ge 0, \ \frac{ry}{x} \le \eta_0. \end{aligned} \tag{3.3}$$

Based on the above two arguments, we obtain

$$\frac{1+r^2}{2} f_m(x)x - r^2 f_m(x)y + F_m(ry) - F_m(x) \ge 0, \quad r \ge 0, \ |ry| \le \eta_0|x|. \tag{3.4}$$

Taking the assumption (f4) into consideration, we get

$$\begin{aligned} &J(q) - J(rq^+) \\ &\quad = \frac{1}{2}[(\mathscr{A}q,q) - (\mathscr{A}(rq^+, rq^+))] + \sum_{m\in\mathbb{Z}} [F_m(r(q^+)) - F_m(q)] \\ &\quad = \frac{1}{2}[(1-r^2)(\mathscr{A}q,q) + r^2(\mathscr{A}q,q^-)] + \sum_{m\in\mathbb{Z}} [F_m(r(q^+)) - F_m(q)] \\ &\quad = \frac{r^2}{2}\|q^-\|^2 + \frac{1-r^2}{2}(\mathscr{A}q,q) + r^2(\mathscr{A}q,q^-) + \sum_{m\in\mathbb{Z}} [F_m(r(q^+)) - F_m(q)] \\ &\quad = \frac{r^2}{2}\|q^-\|^2 + \frac{1-r^2}{2}\langle J'(q), q\rangle + r^2\langle J'(q), q^-\rangle \\ &\qquad + \sum_{m\in\mathbb{Z}} \left[\frac{1-r^2}{2} f_m(q)q + r^2 f_m(q)q^-\right] + \sum_{m\in\mathbb{Z}} \left[F_m(r(q^+)) - F_m(q)\right] \end{aligned}$$

$$\begin{aligned}
&= \frac{r^2}{2}\|q^-\|^2 + \frac{1-r^2}{2}\langle J'(q), q\rangle + r^2\langle J'(q), q^-\rangle \\
&\quad + \sum_{m\in\mathbb{Z}}\left[\frac{1+r^2}{2}f_m(q)q - r^2 f_m(q)q^-\right] + \sum_{m\in\mathbb{Z}}\left[F_m(r(q^+)) - F_m(q)\right] \\
&= \frac{r^2}{2}\|q^-\|^2 + \frac{1-r^2}{2}\langle J'(q), q\rangle + r^2\langle J'(q), q^-\rangle \\
&\quad + \sum_{m\in\mathbb{Z}\{\mu|q^+|\le\eta_0|q|\}}\left[\frac{1+r^2}{2}f_m(q)q - r^2 f_m(q)q^-\right] \\
&\quad + \sum_{m\in\mathbb{Z}\{\mu|q^+|\le\eta_0|q|\}}\left[F_m(r(q^+)) - F_m(q)\right] \\
&\quad + \sum_{m\in\mathbb{Z}\{\mu|q^+|\le\eta_0|q|\}}\left[\frac{1+r^2}{2}f_m(q)q - r^2 f_m(q)q^-\right] \\
&\quad + \sum_{m\in\mathbb{Z}\{\mu|q^+|\le\eta_0|q|\}}\left[F_m(rq^+) - F_m(q)\right] \\
&\ge \frac{r^2}{2}\|q^-\|^2 + \frac{1-r^2}{2}\langle J'(q), q\rangle + r^2\langle J'(q), q^-\rangle \\
&\quad - r^2 \sum_{m\in\mathbb{Z}\{\mu|q^+|\le\eta_0|q|\}} f_m(q)q^+, \quad r \ge 0.
\end{aligned}$$

□

Lemma 3.3. *Assume that* (V1), (f0)–(f2) *are satisfied. Then there is a constant* $\rho > 0$ *such that* $\kappa := \inf J(S_\rho^+) > 0$, *where* $S_\rho^+ = \partial B_\rho \cap H^+$.

Lemma 3.3 can be proved in the same way as [26].

Lemma 3.4. *Suppose that* (V1), (f0)–(f3) *are satisfied. Let* $e \in H^+$ *with* $\|e\| = 1$. *Then there is a constant* $r_0 > 0$ *such that* $\sup J(\partial Q) \le 0$, *where*

$$Q = \{q = se + q^- : q^- \in H^-,\ s \ge 0,\ \|q\| \le r_0\}.$$

Proof. From (f1) we have $F_m(t) \ge 0$ for all m, so we get $J(q) \le 0$ for any $q \in H^-$. Next, it remains to show that $J(q) \to -\infty$ as $q \in H^- \oplus \mathbb{R}e$, $\|q\| \to \infty$. The proof is by contradiction. Assume that for some sequence $\{q^{(n)}\} \subset H^- \oplus \mathbb{R}e$ with $\|q^{(n)}\| \to \infty$, there exists $M > 0$ such that $J(q^{(n)}) \ge -M$ for all $n \in \mathbb{N}$. Denote $h^{(n)} = q^{(n)}/\|q^{(n)}\| = h^{(n)^-} + s_n e$, obviously $\|h^{(n)}\| = 1$. Passing to a subsequence, we may suppose that $h^{(n)} \rightharpoonup h$ in H, thus $h^{(n)} \to h$ for all $m \in \mathbb{Z}$, $h^{(n)^-} \rightharpoonup h^-$ in H, $s_n \to \bar{s}$ and

$$-\frac{M}{\|q^{(n)}\|^2} \le \frac{J(q^{(n)})}{\|q^{(n)}\|^2} = \frac{s_n^2}{2} - \frac{1}{2}\|h^{(n)^-}\|^2 - \sum_{m\in\mathbb{Z}} \frac{F_m(q^{(n)})}{\|q^{(n)}\|^2}. \tag{3.5}$$

If $\bar{s}=0$, thanks to (3.5), it follows that

$$0 \le \frac{1}{2}\|h^{(n)^-}\|^2 + \sum_{m\in\mathbb{Z}} \frac{F_m(q^{(n)})}{\|q^{(n)}\|^2} \le \frac{s_n^2}{2} + \frac{M}{\|q^{(n)}\|^2} \to 0,$$

which leads to $\|h^{(n)^-}\| \to 0$, and so $1 = \|h^{(n)}\| \to 0$, a contradiction.

If $\bar{s} \neq 0$, then $h \neq 0$, combining (3.5), (f3) and Fatou's lemma, we see that

$$\begin{aligned}
0 &\le \lim_{n\to\infty} \sup \left[\frac{s_n^2}{2} - \frac{1}{2}\|h^{(n)^-}\|^2 - \sum_{m\in\mathbb{Z}} \frac{F_m(q^{(n)})}{\|q^{(n)}\|^2}\right] \\
&= \lim_{n\to\infty} \sup \left[\frac{s_n^2}{2} - \frac{1}{2}\|h^{(n)^-}\|^2 - \sum_{m\in\mathbb{Z}} \frac{F_m(q^{(n)})}{|q^{(n)}|^2}(h^{(n)})^2\right] \\
&\le \frac{1}{2}\lim_{n\to\infty} s_n^2 - \lim_{n\to\infty} \inf \sum_{m\in\mathbb{Z}} \frac{F_m(q^{(n)})}{|q^{(n)}|^2}(h^{(n)})^2 \\
&\le \frac{\bar{s}^2}{2} - \sum_{m\in\mathbb{Z}} \lim_{n\to\infty} \inf \frac{F_m(q^{(n)})}{|q^{(n)}|^2}(h^{(n)})^2 \\
&= -\infty,
\end{aligned} \tag{3.6}$$

which leads to a contradiction. Hence Lemma 3.4 is proved. □

Lemma 3.5. *Assume that* (V1), (f0)–(f4) *are satisfied. Then there exists a constant $c \ge \kappa$ and a sequence $\{q^{(n)}\} \subset H$ satisfying*

$$J(q^{(n)}) \to c, \quad \|J'(q^{(n)})\|(1+\|q^{(n)}\|) \to 0. \tag{3.7}$$

Proof. Lemma 3.5 is a direct corollary of Lemmas 2.1, 3.1, 3.3 and 3.4. □

Lemma 3.6. *Suppose that* (V1), (f0)–(f4) *are satisfied. Then any sequence $\{q^{(n)}\} \subset H$ satisfying*

$$J(q^{(n)}) \to c, \quad \langle J'(q^{(n)}), (q^{(n)})^{\pm}\rangle \to 0 \tag{3.8}$$

is bounded in H.

Proof. We prove boundedness of $\{q^{(n)}\}$ by contradiction. Suppose that $\|q^{(n)}\| \to \infty$. Letting $h^{(n)} = q^{(n)}/\|q^{(n)}\|$, it is easy to show that $\|h^{(n)}\| = 1$ and there exists a constant C_1 such that $\|h^{(n)}\|_2 \le C_1$. Passing to a subsequence, we may assume that $h^{(n)} \rightharpoonup h$ in H, $h^{(n)} \to h$ for all $m \in \mathbb{Z}$. Based on the concentration compactness principle of Lions, we will divide our proof into two cases: either $((h^{(n)})^+)$ is vanishing or it is nonvanishing.

Now, we assume that $((h^{(n)})^+)$ is vanishing, that is

$$\limsup_{n\to\infty} \|(h^{(n)})^+\|_\infty = 0.$$

Fix $R = [2(1+c)^{1/2}]$. It follows from (f1) and (f2) that

$$|F_m(t)| \le \frac{t^2}{4(RC_1)^2}, \quad \forall m \in \mathbb{Z},\ |t| \le \eta. \tag{3.9}$$

It follows that $\|(h^{(n)})^+\|_\infty \le \frac{\eta}{R}$, where n is sufficiently large. Hence,

$$\lim_{n\to\infty} \sup \sum_{m\in\mathbb{Z}} F_m(h^{(n)}) \le \frac{1}{4C_1^2} \lim_{n\to\infty} \|h^{(n)}\|_2^2 \le \frac{1}{4}. \tag{3.10}$$

Using (f1) and (f2), for $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f_m(t)| \le \varepsilon|t| + C_\varepsilon |t|^{p-1} \tag{3.11}$$

and

$$|F_m(t)| \le \varepsilon|t|^2 + C_\varepsilon |t|^p \tag{3.12}$$

for any $t \in \mathbb{R}$ and $m \in \mathbb{Z}$, where $p > 2$. Then,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{R^2}{\|q^{(n)}\|} \sum_{m\in\mathbb{Z}\{R|h^+|>\eta_0|q|\}} f_m(q)|(h^{(n)})^+| \\
&\le \lim_{n\to\infty} \frac{R^2}{\|q^{(n)}\|} \sum_{m\in\mathbb{Z}\{R|h^+|>\eta_0|q|\}} (\varepsilon|q| + C_\varepsilon|q|^{p-1})|(h^{(n)})^+| \\
&\le \lim_{n\to\infty} \frac{R^2}{\|q^{(n)}\|} \sum_{m\in\mathbb{Z}\{R|(h^{(n)})^+|>\eta_0|q|\}} (\varepsilon R\eta_0^{-1}|(h^{(n)})^+|^2 + C_\varepsilon R^{p-1}\eta_0^{1-p}|(h^{(n)})^+|^p) \\
&\le \lim_{n\to\infty} \frac{\varepsilon R^3\eta_0^{-1}\|(h^{(n)})^+\|_2^2 + C_\varepsilon R^{p+1}\eta_0^{1-p}\|(h^{(n)})^+\|_p^p}{\|q^{(n)}\|} = 0.
\end{aligned} \tag{3.13}$$

Letting $\mu_n = R/\|q^{(n)}\|$, it follows from (3.8), (3.10), (3.13) and Lemma 3.2 that

$$\begin{aligned}
c + o(1) &= J(q^{(n)}) \\
&\ge J(\mu_n(q^{(n)})^+) + \frac{\mu_n^2\|(q^{(n)})^-\|^2}{2} \\
&\quad + \frac{1-\mu_n^2}{2}\langle J'(q^{(n)}), q^{(n)}\rangle + \mu_n^2\langle J'(q^{(n)}), (q^{(n)})^-\rangle \\
&\quad - \mu_n^2 \sum_{m\in\mathbb{Z}\{\mu_n|(h^{(n)})^+|>\eta_0|q|\}} f_m(q)(h^{(n)})^+ \\
&= J(R(h^{(n)})^+) + \frac{R^2\|(h^{(n)})^-\|}{2} + \left(\frac{1}{2} - \frac{R^2}{2\|q^{(n)})\|^2}\right)\langle J'(q^{(n)}), q^{(n)}\rangle \\
&\quad + \frac{R^2}{\|q^{(n)})\|^2}\langle J'(q^{(n)}), (q^{(n)})^-\rangle - \frac{R^2}{\|q^{(n)})\|} \sum_{m\in\mathbb{Z}\{R|(h^{(n)})^+|>\eta_0|q|\}} f_m(q)|(h^{(n)})^+|
\end{aligned}$$

$$\begin{aligned}
&= \frac{R^2}{2}(\|(h^{(n)})^+\|^2 + \|(h^{(n)})^-\|^2) - \frac{R^2}{\|q^{(n)})\|} \sum_{m\in\mathbb{Z}\{R|(h^{(n)})^+|>\eta_0|q|\}} f_m(q)|(h^{(n)})^+| \\
&\quad - \sum_{m\in\mathbb{Z}} F_m(R(h^{(n)})^+) + \left(\frac{1}{2} - \frac{R^2}{2\|q^{(n)})\|^2}\right)\langle J'(q^{(n)}), q^{(n)}\rangle \\
&\quad + \frac{R^2}{\|q^{(n)})\|^2}\langle J'(q^{(n)}), (q^{(n)})^-\rangle \\
&\geq \frac{R^2}{2} - \sum_{m\in\mathbb{Z}} F_m(R((h^{(n)})^+)) \\
&\quad - \frac{R^2}{\|q^{(n)})\|} \sum_{m\in\mathbb{Z}\{R|(h^{(n)})^+|>\eta_0|q|\}} f_m(q)|(h^{(n)})^+| + o(1) \\
&\geq \frac{R^2}{2} - \frac{1}{4} + o(1) \\
&> c + \frac{3}{4} + o(1),
\end{aligned}$$

which leads to a contradiction.

Going if necessary to a subsequence, we may assume the existence of $m_k \in \mathbb{Z}$ such that

$$|(h_{m_k}^{(n)})^+| = \|(h_m^{(n)})^+\|_\infty > \frac{\delta}{2}.$$

Choose integers i_k and n_k with $0 \leq n_k \leq N-1$ such that $m_k = i_kN + n_k$. Let $w_m^{(n)} = h_{m+ikN}^{(n)}$. Then

$$|(w_{n_k}^{(n)})^+| > \frac{\delta}{2}, \quad \forall k \in \mathbb{N}. \tag{3.14}$$

Now we define $\tilde{q}_m^{(n)} = q_{m+i_kN}^{(n)}$. Then $\tilde{q}_m^{(n)}/\|\tilde{q}_m^{(n)}\| = w_m^{(n)}$ and $\|w_m^{(n)}\|_2 = \|h_m^{(n)}\|_2 \leq C_1$. Passing to a subsequence, we have $w_m^{(n)} \rightharpoonup w_m$ in H, then $w_m^{(n)} \to w_m$ for all $m \in \mathbb{Z}$. Thus, (3.14) implies that $w_m \neq 0$ for some $m \in \{0, 1, \ldots, N-1\}$. It is obvious that $w_m \neq 0$ implies $\lim_{n\to\infty} |\tilde{q}_m^{(n)}| = \infty$.

Hence, it follows from (2.3), (f1) and (f3) that

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c + o(1)}{\|q^{(n)}\|^2} \\
&= \lim_{n\to\infty} \frac{J(q^{(n)})}{\|q^{(n)}\|^2} \\
&= \lim_{n\to\infty} \left[\frac{1}{2}(\|(q^{(n)})^+\|^2 - \|(q^{(n)})^-\|^2) - \sum_{m\in\mathbb{Z}} \frac{F_m(q^{(n)})}{|q^{(n)}|^2}|h^{(n)}|^2\right] \\
&= \lim_{n\to\infty} \left[\frac{1}{2}(\|(q^{(n)})^+\|^2 - \|(q^{(n)})^-\|^2) - \sum_{m\in\mathbb{Z}} \frac{F_m\big(q_{m+i_kN}^{(n)}\big)}{\big|q_{m+i_kN}^{(n)}\big|^2}\big|h_{m+i_kN}^{(n)}\big|^2\right]
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\left[\frac{1}{2}(\|(q^{(n)})^+\|^2 - \|(q^{(n)})^-\|^2) - \sum_{m\in\mathbb{Z}}\frac{F_m(\tilde{q}_m^{(n)})}{|\tilde{q}_m^{(n)}|^2}|w_m^{(n)}|^2\right] \\
&\le \frac{1}{2} - \liminf_{n\to\infty}\sum_{m\in\mathbb{Z}}\frac{F_m(\tilde{q}_m^{(n)})}{|\tilde{q}_m^{(n)}|^2}|w_m^{(n)}|^2 \\
&\le \frac{1}{2} - \liminf_{n\to\infty}\sum_{m=0}^{N-1}\frac{F_m(\tilde{q}_m^{(n)})}{|\tilde{q}_m^{(n)}|^2}|w_m^{(n)}|^2 \\
&= -\infty,
\end{aligned}
\tag{3.15}
$$

which is a contradiction. Hence the statement of Lemma 3.6 is proved. □

Lemma 3.7. *Suppose that* (V), (f0)–(f4) *are satisfied. Then problem* (1.3) *has a nontrivial solution, i.e.,* $\mathscr{M} \neq \emptyset$.

Proof. Lemma 3.5 implies the existence of a sequence $\{q^{(n)}\} \subset H$ satisfying (3.3). By Lemma 3.6, $\{q^{(n)}\}$ is bounded in H. If $\delta := \limsup_{n\to\infty}\|(h^{(n)})^+\|_\infty = 0$, by virtue of (f1) and (f2), one can get that

$$
\limsup_{n\to\infty}\sum_{m\in\mathbb{Z}}\left[\frac{1}{2}f_m(q^{(n)}) - F_m(q^{(n)})\right] = 0. \tag{3.16}
$$

From (2.3), (2.4), (3.7) and (3.16), one has

$$
\begin{aligned}
c &= J(q^{(n)}) - \frac{1}{2}\langle J'(q^{(n)}), q^{(n)}\rangle + o(1) \\
&= \sum_{m\in\mathbb{Z}}\left[\frac{1}{2}f_m(q^{(n)}) - F_m(q^{(n)})\right] + o(1) = o(1),
\end{aligned}
$$

which is a contradiction. Thus $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $m_k \in \mathbb{Z}$ such that

$$
|h_{m_k}^{(n)}| = \|h_m^{(n)}\|_\infty > \frac{\delta}{2}.
$$

Choose integers i_k and n_k with $0 \le n_k \le N-1$ such that $m_k = i_k N + n_k$. Let $w_m^{(n)} = h_{m+ikN}^{(n)}$. Then

$$
|w_{n_k}^{(n)}| > \frac{\delta}{2}, \quad \forall k \in \mathbb{N}. \tag{3.17}
$$

Since V_m and $f_m(t)$ are N-periodic on m, then $\|w_m^{(n)}\| = \|h_m^{(n)}\|$ and

$$
J(w_m^{(n)}) \to c_*, \quad \|J'(w_m^{(n)})\|(1 + \|w_m^{(n)}\|) \to 0. \tag{3.18}
$$

Passing to a subsequence, we have $w_m^{(n)} \rightharpoonup w$ in H, and $w_m^{(n)} \to w_m$ for all $m \in \mathbb{Z}$. Hence, it follows from (3.17) and (3.18) that $J'(w) = 0$ and $w \neq 0$. This shows that $w \in \mathscr{M}$ is a nontrivial of problem (1.3). □

Proof of Theorem 1.1. Lemma 3.7 shows that $\mathscr{M}$ is not an empty set. Let $c_0 = \inf_{\mathscr{M}} J$. By Lemma 2.3, one has $J(q) \geq J(0) = 0$ for all $q \in \mathscr{M}$. Thus $c_0 \geq 0$. Let $\{q^{(n)}\} \subset \mathscr{M}$ such that $J(q^{(n)}) \to c_0$. The $\langle J'(q^{(n)}), p\rangle = 0$ for any $p \in H$. According to the proof of Lemma 3.6 ($c > 0$ is not necessary), we can certify that $\{q^{(n)}\}$ is bounded in H and

$$0 = \langle J'(q^{(n)}), (q^{(n)})^+\rangle = \|(q^{(n)})^+\|^2 - \sum_{m\in\mathbb{Z}} f_m(q^{(n)})(q^{(n)})^+ \tag{3.19}$$

and

$$0 = \langle J'(q^{(n)}), (q^{(n)})^-\rangle = \|(q^{(n)})^-\|^2 - \sum_{m\in\mathbb{Z}} f_m(q^{(n)})(q^{(n)})^-. \tag{3.20}$$

By (3.11) with $\varepsilon_0 = \frac{\eta_0}{2(2+\eta_0)\gamma_2^2}$, (3.19), (3.20) and the Hölder inequality, one has

$$\begin{aligned}
\|q^{(n)}\|^2 &= \|(q^{(n)})^+\|^2 + \|(q^{(n)})^-\|^2 \\
&= \sum_{m\in\mathbb{Z}(q^{(n)}\neq 0)} \frac{f_m(q^{(n)})}{q^{(n)}} \left[|(q^{(n)})^+|^2 - |(q^{(n)})^-|^2\right] \\
&\leq \varepsilon_0 \sum_{m\in\mathbb{Z}(q_m^{(n)}\neq 0)} |(q^{(n)})^+|^2 + C_{\varepsilon_0} \sum_{m\in\mathbb{Z}(q_m^{(n)}\neq 0)} \left[|(q^{(n)})^+|^{p-2}|(q^{(n)})^+|^2\right] \\
&\leq \varepsilon_0\|(q^{(n)})^+\|_2^2 + C_{\varepsilon_0}\|q^{(n)}\|_p^{p-2}\|q^{(n)}\|_p^2 \\
&\leq \varepsilon_0\gamma_2^2\|q^{(n)}\|^2 + C_{\varepsilon_0}\gamma_p^p\|q^{(n)}\|^p \\
&\leq \frac{1}{2}\|q^{(n)}\|^2 + C_{\varepsilon_0}\gamma_p^p\|q^{(n)}\|^p,
\end{aligned} \tag{3.21}$$

which, together with (3.21), shows that

$$8\Theta_0\|q^{(n)}\|^{p-2} \geq 2C_{\varepsilon_0}\gamma_p^p\|q^{(n)}\|^{p-2} > 1. \tag{3.22}$$

Let $t_n = \frac{1}{\|q^{(n)}\|}\left(\frac{1}{8\Theta_0}\right)^{1/p-1}$. Then (3.22) implies that $0 < t_n < 1$. Since $q^{(n)} \in \mathscr{M}$, it follows from (1.5), (3.1), (3.11), (3.12) that

$$\begin{aligned}
J(q^{(n)}) &\geq \frac{t_n^2\|q^{(n)}\|^2}{2} - t_n^2 \sum_{m\in\mathbb{Z}(\mu|q^+|>\eta_0|q|)} f_m(q^{(n)})(q^{(n)})^+ - \sum_{m\in\mathbb{Z}} F_m(t_n(q^{(n)})^+) \\
&\geq \frac{t_n^2\|q^{(n)}\|^2}{2} - \left(\frac{t_n}{\eta_0} + \frac{1}{2}\right)\varepsilon_0 t_n^2\|(q^{(n)})^+\|_2^2 - \left(\frac{t_n}{\eta_0^{p-1}} + \frac{1}{p}\right)C_{\varepsilon_0}t_n^p\|(q^{(n)})^+\|_p^p \\
&\geq \frac{t_n^2\|q^{(n)}\|^2}{2} - \left(\frac{1}{\eta_0} + \frac{1}{2}\right)\varepsilon_0\gamma_2^2 t_n^2\|q^{(n)}\|^2 - \left(\frac{1}{\eta_0^{p-1}} + \frac{1}{p}\right)C_{\varepsilon_0}\gamma_p^p t_n^p\|q^{(n)}\|^p \\
&= \frac{t_n^2\|q^{(n)}\|^2}{4} - \Theta_0 t_n^p\|q^{(n)}\|^p = \left(\frac{1}{8}\right)^{1/p-2}\Theta_0^{2/p-2}.
\end{aligned}$$

This shows that $c_0 \geq \left(\frac{1}{8}\right)^{1/p-2}\Theta_0^{2/p-2}$. By a standard argument, we can demonstrate that $q \in \mathscr{M}$ such that $J(q) = c_0 = \inf_{\mathscr{M}} J$. □

4 Proof of Theorem 1.2

Lemma 4.1. *Suppose that* (V1), (f0) *and* (f2) *hold, then* $B(q)$ *is non-negative and weakly sequentially lower semi-continuous, and* $B'(q)$ *is weakly sequentially continuous.*

Using Sobolev's imbedding theorem, one can check the above lemma easily. So we omit the proof.

Lemma 4.2. *Suppose that* (V1), (f0), (f2), (f4′) *hold. Then* $\forall q \in H$, $\mu \geq 0$, $p \in H^-$, *there holds*

$$J(q) \geq J(\mu q + p) + \frac{1}{2}\|p\|^2 + \frac{1-\mu^2}{2}\langle J'(q), q\rangle - \mu\langle J'(q), p\rangle. \tag{4.1}$$

Proof. For any $m \in \mathbb{Z}$ and $\tau \neq 0$, we have

$$f_m(s) \leq \frac{f_m(\tau)}{|\tau|}|s|, \quad s \leq \tau, \quad f_m(s) \geq \frac{f_m(\tau)}{|\tau|}|s|, \quad s \geq \tau. \tag{4.2}$$

It is easy to prove that

$$\left(\frac{1-\mu^2}{2}\tau - \mu\sigma\right) f_m(\tau) \geq \int_{\mu\tau+\sigma}^{\tau} f_m(s)ds, \quad \mu \geq 0,\ \sigma \in \mathbb{R}. \tag{4.3}$$

Thus, by (2.1), (2.2), (2.8), (2.9) and (4.3), one has

$$\begin{aligned}
&J(q) - J(\mu q + p)\\
&= \frac{1-\mu^2}{2}(\mathscr{A}q, q) - \mu(\mathscr{A}q, p) - \frac{1}{2}(\mathscr{A}p, p) + \sum_{m\in\mathbb{Z}}[F_m(\mu q) - F_m(q)]\\
&= -\frac{1}{2}(\mathscr{A}p, p) + \frac{1-\mu^2}{2}\langle J'(q), q\rangle - \mu\langle J'(q), p\rangle - \sum_{m\in\mathbb{Z}}\int_{\mu q}^{q} f_m(s)ds\\
&\quad + \sum_{m\in\mathbb{Z}}\left[\frac{1-\mu^2}{2}f_m(q)q - \mu f_m(q)p\right]\\
&= \frac{1}{2}\|p\|^2 + \frac{1-\mu^2}{2}\langle J'(q), q\rangle - \mu\langle J'(q), p\rangle - \sum_{m\in\mathbb{Z}}\int_{\mu q+p}^{q} f_m(s)ds\\
&\quad + \sum_{m\in\mathbb{Z}}\left[\frac{1-\mu^2}{2}f_m(q)q - \mu f_m(q)p\right]\\
&\geq \frac{1}{2}\|p\|^2 + \frac{1-\mu^2}{2}\langle J'(q), q\rangle - \mu\langle J'(q), p\rangle.
\end{aligned}$$

□

From Lemma 4.2, we have the following corollaries.

Corollary 4.1. *Suppose that* (V1), (f0), (f2), (f4′) *are satisfied. Then*

$$J(q) \geq J(\mu q + p) + \frac{1}{2}\|q\|^2, \quad \forall \mu \geq 0,\ p \in H^-,\ q \in \mathcal{N}^-. \tag{4.4}$$

Corollary 4.2. *Suppose that* (V1), (f0), (f2), (f4′) *are satisfied. Then for* $q \in H,\ \mu \geq 0$,

$$J(q) \geq \frac{\mu^2}{2}\|q\|^2 + \frac{1-\mu^2}{2}\langle J'(q), q\rangle + \mu^2\langle J'(q), q^-\rangle - \sum_{m\in\mathbb{Z}} F_m(\mu q^+). \tag{4.5}$$

Lemma 4.3. *Suppose that* (V1′), (f0), (f2), (f4′) *are satisfied. Then*

(i) *there exists* $\rho > 0$ *such that*

$$b := \inf_{\mathcal{N}^-} J \geq \kappa := \inf\{J(q) : q \in H^+,\ \|q\| = \rho\} > 0;$$

(ii) *for all* $q \in \mathcal{N}^-$,

$$\|q^+\| \geq \max\{\|q^-\|, \sqrt{2b}\}.$$

Lemma 4.4. *Suppose that* (V1), (f0), (f2)–(f3), (f4′) *are satisfied. Then for any* $e \in H^+$, $\sup J(H^- \oplus \mathbb{R}^+ e) < \infty$*, and there is* $R_e > 0$ *such that*

$$J(q) \leq 0, \quad \forall q \in H^- \oplus \mathbb{R}^+ e,\ \|q\| \geq R_e. \tag{4.6}$$

Proof. Notice that $F_m(t) \geq 0$ for any $(m,t) \in \mathbb{Z}\times\mathbb{R}$. Then we have $J(q) \leq 0$ as $q \in H^-$. To prove this lemma, it suffices to show that $J(q) \to -\infty$ as $q \in H^- \oplus \mathbb{R}e$, $\|q\| \to \infty$. Arguing indirectly, assume that for some sequence $(q^{(n)}) \subset H^- \oplus \mathbb{R}e$ with $\|q^{(n)}\| \to \infty$ such that $J(q^{(n)}) \geq 0$ for all $n \in \mathbb{N}$, set $h^{(n)} = q^{(n)}/\|q^{(n)}\| = h^{(n)^-} + s_n e$, then $\|h^{(n)}\| = 1$. Passing to a subsequence, we may assume that $s_n \to \bar{s}$, $h^{(n)} \rightharpoonup h$, $(h^{(n)})^- \rightharpoonup h^-$ in H^-, and $(h^{(n)})^+ \rightharpoonup h^+$ in H^+. Hence

$$0 \leq \frac{J(q^{(n)})}{\|q^{(n)}\|^2} = \frac{s_n^2}{2}\|e\|^2 - \frac{1}{2}\|h^{(n)^-}\|^2 - \sum_{m\in\mathbb{Z}} \frac{F_m(q^{(n)})}{\|q^{(n)}\|^2}. \tag{4.7}$$

If $\bar{s} = 0$, then by (4.7), we have

$$0 \leq \frac{1}{2}\|h^{(n)^-}\|^2 + \sum_{m\in\mathbb{Z}} \frac{F_m(q^{(n)})}{\|q^{(n)}\|^2} \leq \frac{s_n^2}{2}\|e\|^2 \to 0,$$

which yields $\|h^{(n)^-}\| \to 0$, and so $1 = \|h^{(n)^-} + s_n e\|^2 \to 0$, a contradiction.

If $\bar{s} \neq 0$, similar to the proof in Lemma 3.4, we can get (3.6), which is a contradiction. □

Corollary 4.3. *Suppose that* (V1), (f0), (f2)–(f3) *and* (f4′) *are satisfied. Let* $e \in H^+$ *with* $\|e\| = 1$. *Then there exists* $r_0 > \rho$ *such that* $\sup J(\partial Q) \le 0$ *for* $r \ge r_0$, *where*

$$Q = \{p + se : p \in H^-,\ s \ge 0,\ \|p + se\| \le r\}. \tag{4.8}$$

Lemma 4.5. *Suppose that* (V1), (f0), (f2)–(f3) *and* (f4′) *are satisfied. Then there exist a constant* $c \in [\kappa, \sup J(Q)]$ *and a sequence* $\{q^{(n)}\} \subset H$ *satisfying*

$$J(q^{(n)}) \to c, \quad \|J'(q^{(n)})\|(1 + \|q^{(n)}\|) \to 0,$$

where Q *is defined by* (4.8).

Proof. Lemma 4.5 is a direct corollary of Lemmas 2.1, 4.1, 4.5 (i) and Corollary 4.3. □

Lemma 4.6. *Suppose that* (V1), (f0), (f2)–(f3) *and* (f4′) *are satisfied. Then there exist a constant* $c_* \in [\kappa, b]$ *and a sequence* $\{q^{(n)}\} \subset H$ *such that*

$$J(q^{(n)}) \to c_*, \quad \|J'(q^{(n)})\|(1 + \|q^{(n)}\|) \to 0. \tag{4.9}$$

Proof. Choose $w^{(k)} \in \mathscr{N}^-$ such that

$$b \le J(w^{(k)}) < b + \frac{1}{k}, \quad k \in \mathbb{N}. \tag{4.10}$$

By Lemma 4.3, $\|(w^{(k)})^+\| \ge \sqrt{2b} > 0$. Set $e_k = w^{(k)}/\|w^{(k)}\|$, then $e_k \in H^+$ and $\|e_k\| = 1$. There exists a sequence $(q^{(k,n)})_{n\in\mathbb{N}} \subset H$ satisfying

$$J(q^{(k,n)}) \to c_k, \quad \|J'(q^{(k,n)})\|(1 + \|q^{(k,n)}\|) \to 0, \quad k \in \mathbb{N}. \tag{4.11}$$

By Corollary 4.3, one can get that

$$J(w^{(k)}) \ge J(\eta w^{(k)} + p), \quad \forall \eta \ge 0,\ p \in H^-. \tag{4.12}$$

Since $w^{(k)} \in Q_k$, it follows that from Corollary 4.3, there exists $r_k > \max\{\rho, \|w^{(k)}\|\}$ such that $\sup J(\partial Q_k) \le 0$, where

$$Q_k = \{p + se_k : p \in H^-,\ s \ge 0,\ \|p + se_k\| \le r_k\}, \quad k \in \mathbb{N}. \tag{4.13}$$

Hence, applying Lemma 4.5 to the above set Q_k, there exists a constant $c_k \in [\kappa, \sup J(Q_k)]$ and from (4.13) and (4.12) we have $J(w^{(k)}) = \sup J(Q_k)$. Hence, by virtue of (4.10) and (4.11), one has

$$J(q^{(k,n)}) \to c_k < b + \frac{1}{k}, \quad \|J'(q^{(k,n)})\|(1 + \|q^{(k,n)}\|) \to 0, \quad k \in \mathbb{N}.$$

Now, we can choose a sequence $\{n_k\} \subset \mathbb{N}$ such that

$$J(q^{(k,n_k)}) < b + \frac{1}{k}, \quad \|J'(q^{(k,n_k)})\|(1 + \|q^{(k,n_k)}\|) < \frac{1}{k}, \quad k \in \mathbb{N}.$$

Let $q^{(k)} = q^{(k,n_k)}$. Then going if necessary to a subsequence, we have

$$J(q^{(n)}) \to c^* \in [\kappa, b], \quad \|J'(q^{(n)})\|(1 + \|q^{(n)}\|) \to 0.$$

The proof of Lemma 4.6 is complete. □

Lemma 4.7. *Suppose that* (V1), (f0), (f2)–(f3) *and* (f4′) *are satisfied. Then for any* $q \in H \setminus H^-$, *there exist* $s(q) > 0$ *and* $\omega(q) \in H^-$ *such that* $s(q)q + w(q) \in \mathscr{N}^-$.

Proof. Since $H^- \oplus \mathbb{R}^+ q = H^- \oplus \mathbb{R}^+ q^+$, we may assume that $q \in H^+$. By Lemma 4.4, there exists $R > 0$ such that $J(q) \leq 0$ for any $q \in (H^- \oplus \mathbb{R}^+ q) \setminus B_R(0)$. By Lemma 4.3 (i), $J(sq) > 0$ for small $s \geq 0$. Thus, $0 < \sup J(H^- \oplus \mathbb{R}^+ q) < \infty$. It is easy to see that J is weakly upper semi-continuous on $H^- \oplus \mathbb{R}^+ q$, therefore, $J(\bar{q}) = \sup J(H^- \oplus \mathbb{R}^+ q)$ for some $\bar{q} \in H^- \oplus \mathbb{R}^+ q$. This $\bar{q}$ is a critical point of $J|_{H^- \oplus \mathbb{R}q}$, so $\langle J'(\bar{q}), \bar{q}\rangle = \langle J'(\bar{q}), p\rangle$ for all $p \in H^- \oplus \mathbb{R}q$. Consequently, $\bar{q} \in \mathscr{N}^- \cap (H^- \oplus \mathbb{R}^+ q)$. □

Lemma 4.8. *Suppose that* (V1), (f0), (f2)–(f3) *and* (f4′) *are satisfied. Then any* $\{q^{(n)}\} \subset H$ *satisfying*

$$J'(q^{(n)}) \to c \geq 0, \quad \langle J'(q^{(n)}), (q^{(n)})^{\pm}\rangle \to 0 \tag{4.14}$$

is bounded in H.

Proof. We prove the boundedness of $\{q^{(n)}\}$ by contradiction. If the assertion would not hold, then $\|q^{(n)}\| \to \infty$. Denoting $h^{(n)} = q^{(n)}/\|q^{(n)}\|$, we have $\|h^{(n)}\| = 1$ and there exists a constant $C_1 > 0$ such that $\|h^{(n)}\|_2 \leq C_1$. Passing to a subsequence, we may assume that $h^{(n)} \rightharpoonup h$ in l^2, then $h^{(n)} \to h$ for all $n \in \mathbb{Z}$. Fixing $\Lambda = [2(1+c)]^{1/2}$, by (f2), there exists $\omega > 0$ such that

$$|F_m(t)| \leq \frac{|t|^2}{4(\Lambda C_1)^2}, \quad \forall m \in \mathbb{Z},\ |t| \leq \omega. \tag{4.15}$$

Let

$$\delta := \limsup_{n\to\infty} \|(h^{(n)})^+\|_\infty.$$

If $\delta = 0$, then $\|(h^{(n)})^+\|_\infty < \frac{\omega}{\Lambda}$ for large n. Thus, it follows from (4.15) that

$$\limsup_{n\to\infty} \sum_{m\in\mathbb{Z}} F_m\big(\Lambda(h^{(n)})^+\big) \leq \frac{1}{4C_1^2} \limsup_{n\to\infty} \|(h^{(n)})^+\|_2^2 \leq \frac{1}{4}. \tag{4.16}$$

Set $\theta^{(n)} = \Lambda/\|(h^{(n)})^+\|$. Combining (4.14) and (4.16), we have, in light of

Corollary 4.14,

$$\begin{aligned}
c+o(1) &= J((h^{(n)})^+) \\
&\geq \frac{\left(\theta^{(n)}\right)^2}{2}\|h^{(n)}\|^2 - \sum_{m\in\mathbb{Z}} F_m\big(\theta^{(n)}(h^{(n)})^+\big) \\
&\quad + \frac{1-(\theta^{(n)})^2}{2}\langle J'(h^{(n)}), h^{(n)}\rangle + (\theta^{(n)})^2\langle J'(h^{(n)}), h^{(n)-}\rangle \\
&= \frac{\Lambda^2}{2} - \sum_{m\in\mathbb{Z}} F_m\left(\frac{\Lambda(h^{(n)})^+}{\|h^{(n)}\|}\right) + \left(\frac{1}{2} - \frac{\Lambda^2}{2\|h^{(n)}\|^2}\right)\langle J'(h^{(n)}), h^{(n)}\rangle \\
&\quad + \frac{\Lambda^2}{\|h^{(n)}\|^2}\langle J'(h^{(n)}), h^{(n)-}\rangle \\
&= \frac{\Lambda^2}{2} - \sum_{m\in\mathbb{Z}} F_m\left(\frac{\Lambda h^{(n)+}}{\|h^{(n)}\|}\right) + o(1) \\
&\geq \frac{\Lambda^2}{2} - \frac{1}{4} + o(1) \\
&> \frac{3}{4} + c + o(1).
\end{aligned}$$

This leads to a contradiction, so $\delta > 0$. Similar to the proof of Lemma 3.6, we can verify (3.14) holds, which is a contradiction. Hence the statement of Lemma 4.8 is proved. □

Proof of Theorem 1.2. Lemmas 4.6 and 4.8 imply the existence of a bounded sequence $\{q^{(n)}\} \subset H$ satisfying (4.9). Thus there exist constants $C_2, C_3 > 0$ such that

$$\|q^{(n)}\| \leq \|q^{(n)}\|_2 \leq C_2\|q^{(n)}\| \leq C_3, \quad \forall n \in \mathbb{Z}. \tag{4.17}$$

Hence, by (f2) and (f3), there exists a constant $C_4 > 0$ such that

$$|f_m(t)t - 2F_m(t)| \leq \frac{c^*}{2C_3^2}|t|^2 + C_4|t|^3, \quad \forall (m,t) \in \mathbb{Z}\times\mathbb{R},\ |t| \leq C_3. \tag{4.18}$$

If

$$\delta := \limsup_{n\to\infty} \|q^{(n)}\|_\infty = 0,$$

then for $p > 2$,

$$\sum_{m\in\mathbb{Z}} |q^{(n)}|^p \leq \|q^{(n)}\|_\infty^{p-2} \sum_{m\in\mathbb{Z}} |q^{(n)}|^2 \leq C_3\|q^{(n)}\|_\infty^{p-2} \to 0, \quad n\to\infty. \tag{4.19}$$

Thus

$$\begin{aligned}\lim_{n\to\infty} \sup \sum_{m\in\mathbb{Z}} \left[\frac{1}{2} f_m(q^{(n)})q^{(n)} - F_m(q^{(n)})\right] \\ \leq \frac{3\varepsilon}{2}C_2^2 + \frac{3\varepsilon}{2}C_\varepsilon \lim_{n\to\infty} |q^{(n)}|^p \\ = \frac{3c_*}{8}.\end{aligned}$$

Then

$$\begin{aligned}c_* &= J(q^{(n)}) - \frac{1}{2}\langle J'(q^{(n)}), q^{(n)}\rangle + o(1) \\ &= \sum_{m\in\mathbb{Z}} \left[\frac{1}{2} f_m(q^{(n)})q^{(n)} - F_m(q^{(n)})\right] + o(1) \\ &\leq \frac{3c_*}{8} + o(1),\end{aligned}$$

which is a contradiction. Then $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $m_k \in \mathbb{Z}$ such that

$$|h_{m_k}^{(n)}| = \|h_m^{(n)}\|_\infty > \frac{\delta}{2}.$$

Choose integers i_k and n_k with $0 \leq n_k \leq N-1$ such that $m_k = i_k N + n_k$. Let $w_m^{(n)} = h_{m+ikN}^{(n)}$. Then

$$|w_{n_k}^{(n)}| > \frac{\delta}{2}, \quad \forall k \in \mathbb{N}. \tag{4.20}$$

Since V_m and $f_m(t)$ are N-periodic on m, then $\|w_m^{(n)}\| = \|h_m^{(n)}\|$ and

$$J(w_m^{(n)}) \to c_*, \quad \|J'(w_m^{(n)})\|(1 + \|w_m^{(n)}\|) \to 0. \tag{4.21}$$

Passing to a subsequence, we have $w^{(n)} \rightharpoonup w$ in l^2, $w^{(n)} \to w$ for all $m \in \mathbb{Z}$. Thus, (4.20) implies that $w \neq 0$. Let

$$l_0 = \{q_m \in l_2 : m \in \mathbb{Z},\ |q_m| > 0 \text{ is a finite set}\}.$$

Then for every $\varphi \in l_0$, there exists an $m_0 \in \mathbb{Z}$ such that $\varphi_m = 0$ for all $|m| > m_0$. Hence, it follows from (2.4) and (4.21) that

$$\begin{aligned}\langle J'(w), \varphi\rangle &= (\mathscr{A} w, \varphi)_{l^2(\mathbb{Z})} - \sum_{m\in\mathbb{Z}} f_m(w)\varphi \\ &= (\mathscr{A} w, \varphi)_{l^2(|m|\leq m_0)} - \sum_{|m|\leq m_0} f_m(w)\varphi \\ &= \lim_{n\to\infty}\left[(\mathscr{A} w^{(n)}, \varphi)_{l^2(|m|\leq m_0)} - \sum_{|m|\leq m_0} f_m(w^{(n)})\varphi\right]\end{aligned}$$

$$= \lim_{n\to\infty}\langle J'(w^{(n)}),\varphi\rangle$$
$$= 0.$$

Since l_0 is dense in l_2, then $J'(w) = 0$. This shows that $w \in \mathscr{N}^-$ and so $J(w) \geq b$. On the other way, it follows from (2.8), (2.9), (4.9), (f4$'$) and Fatou Lemma that

$$\begin{aligned}
b &\geq c_* \\
&= \lim_{n\to\infty}\left[J(w_m) - \frac{1}{2}\langle J'(w_m^{(n)}), w_m^{(n)}\rangle\right] \\
&= \lim_{n\to\infty}\sum_{m\in\mathbb{Z}}\left[\frac{1}{2}f_m(w_m^{(n)})w_m^{(n)} - F_m(w_m^{(n)})\right] \\
&\geq \sum_{m\in\mathbb{Z}}\lim_{n\to\infty}\left[\frac{1}{2}f_m(w_m^{(n)})w_m^{(n)} - F_m(w_m^{(n)})\right] \\
&= \sum_{m\in\mathbb{Z}}\left[\frac{1}{2}f_m(w)w_m - F_m(w)\right] \\
&= J(w) - \frac{1}{2}\langle J'(w), w\rangle \\
&= J(w),
\end{aligned}$$

which implies that $J(w) \leq b$. So $J(w) = b = \inf_{\mathscr{N}^-} J > 0$. The proof is completed. □

5 Proof of Theorem 1.3

In this section, we always assume that V satisfies (V1$'$) and define functionals J, J_0 as follows:

$$\begin{aligned}
J(q) &= \frac{1}{2}((-\Delta + W_m + R_m)q, q)_{l^2} - \sum_{m\in\mathbb{Z}}[G_m(q) + H_m(q)], \\
J_0(q) &= \frac{1}{2}((-\Delta + W_m)q, q)_{l^2} - \sum_{m\in\mathbb{Z}} G_m(q),
\end{aligned}$$

where $G_m(t) := \int_0^t g_m(s)ds$, $H_m(t) := \int_0^t h_m(s)ds$.

Then (V1$'$), (f0), (f2) and (f4$''$) imply that $J \in C^1(H,\mathbb{R})$ and

$$\begin{aligned}
\langle J'(q), w\rangle &= ((-\Delta + W_m + R_m)q, w)_{l^2} - \sum_{m\in\mathbb{Z}}[g_m(q) + h_m(q)]w, \\
\langle J_0'(q), w\rangle &= ((-\Delta + W_m)q, w)_{l^2} - \sum_{m\in\mathbb{Z}} g_m(q)w.
\end{aligned}$$

Lemma 5.1. *Suppose that* (V1$'$), (f0) *and* (f2) *hold. Then* $B(q)$ *is non-negative and weakly sequentially lower semi-continuous, and* $B'(q)$ *is weakly sequentially continuous.*

Lemma 5.2. *Suppose that* (V1$'$), (f0), (f2)–(f3) *and* (f4$''$) *hold. Then for* $q \in H$, $\mu \geq 0$, $p \in H^-$, *there holds*

$$J(q) \geq J(\mu q + p) + \frac{1}{2}\|p\|^2 - \frac{1}{2}\sum_{m\in\mathbb{Z}} R_m p^2 + \frac{1-\mu^2}{2}\langle J'(q), q\rangle - \mu\langle J'(q), p\rangle. \quad (5.1)$$

From Lemma 5.2, we have the following corollaries.

Corollary 5.1. *Suppose that* (V1$'$), (f0), (f2)–(f3) *and* (f4$''$) *are satisfied. Then*

$$J(q) \geq J(\mu q + p) + \frac{1}{2}\|p\|^2 - \frac{1}{2}\sum_{m\in\mathbb{Z}} R_m p^2, \quad \forall \mu \geq 0,\ p \in H^-,\ q \in \mathscr{N}^-. \quad (5.2)$$

Corollary 5.2. *Suppose that* (V1$'$), (f0), (f2)–(f3) *and* (f4$''$) *are satisfied. Then for* $q \in H, \mu \geq 0$

$$\begin{aligned} J(q) \geq\ & \frac{\mu^2}{2}\|q\|^2 + \frac{1-\mu^2}{2}\langle J'(q), q\rangle + \mu^2\langle J'(q), q^-\rangle \\ & - \sum_{m\in\mathbb{Z}} F_m(\mu q^+) + \frac{\mu^2}{2}\sum_{m\in\mathbb{Z}} R_m[(q^+)^2 - (q^-)^2]. \end{aligned} \quad (5.3)$$

Lemma 5.3. *Suppose that* (V1$'$), (f0), (f2) *are satisfied. Then*

(i) *there exists* $\rho > 0$ *such that*

$$b := \inf_{\mathscr{N}^-} J \geq \kappa := \inf\{J(q) : q \in H^+ : \|q\| = \rho\} > 0;$$

(ii) *for all* $q \in \mathscr{N}^-$,

$$\|q^+\| \geq \max\{\|q^-\|, \sqrt{2b}\}.$$

Proof. Set $\gamma_0 = \sup_{\mathbb{Z}}(-R_m)$. Then (f2) and (f3) imply that there exists a constant $C_{\varepsilon_0} > 0$ such that

$$F_m(q) \leq \varepsilon_0|q|^2 + C_{\varepsilon_0}|q|^p, \quad \forall m \in \mathbb{Z}.$$

Then we have for $q \in \mathscr{N}^-$,

$$\begin{aligned}J(q) &= \frac{1}{2}\|q\|^2 + \frac{1}{2}\sum_{m\in\mathbb{Z}} R_m q^2 - \sum_{m\in\mathbb{Z}} F_m(q)\\ &\geq \frac{1}{2}\|q\|^2 - \frac{\gamma_0}{2}\|q\|_2^2 - \varepsilon_0\|q\|_2^2 - C_{\varepsilon_0}\|q\|_p^p\\ &\geq \frac{1}{2}\left(1 - \frac{\gamma_0 + 2\varepsilon_0}{\Pi_0}\right)\|q\|^2 - C_{\varepsilon_0}\gamma^p\|q\|^p > 0, \quad \forall q \in H^+.\end{aligned}$$

This shows that there exists a $\rho > 0$ such that (i) holds.

On the other hand, we have for $q \in \mathscr{N}^-$,

$$\begin{aligned}b &\leq \frac{1}{2}\|q^+\|^2 - \frac{1}{2}\|q^-\|^2 + \frac{1}{2}\sum_{m\in\mathbb{Z}} R_m q^2 - \sum_{m\in\mathbb{Z}} F_m(q)\\ &\leq \frac{1}{2}\|q^+\|^2 - \frac{1}{2}\|q^-\|^2 \leq \frac{1}{2}\|q^+\|^2,\end{aligned}$$

which implies that $\|q^+\| \geq \max\{\|q^-\|, \sqrt{2b}\}$. □

Lemma 5.4. *Suppose that* (V1′), (f0)–(f3) *are satisfied. Then for any* $e \in H^+$, $\sup J(H^- \oplus \mathbb{R}^+ e) < \infty$, *and there is* $R_e > 0$ *such that*

$$J(q) \leq 0, \quad \forall q \in H^- \oplus \mathbb{R}^+ e, \ \|q\| \geq R_e. \tag{5.4}$$

Proof. Notice that $F_m(t) \geq 0$ for any $(m,t) \in \mathbb{Z}\times\mathbb{R}$. Then we have $J(q) \leq 0$ as $q \in H^-$. To prove this lemma, it suffices to show that $J(q) \to -\infty$ as $q \in H^- \oplus \mathbb{R}e$, $\|q\| \to \infty$. Arguing indirectly, assume that for some sequence $(q^{(n)}) \subset H^- \oplus \mathbb{R}e$ with $\|q^{(n)}\| \to \infty$ such that $J(q^{(n)}) \geq 0$ for all $n \in \mathbb{N}$, set $h^{(n)} = q^{(n)}/\|q^{(n)}\| = h^{(n)^-} + s_n e$, then $\|h^{(n)}\| = 1$. Passing to a subsequence, we may assume that $s_n \to \bar{s}$, $h^{(n)} \rightharpoonup h$, $h^{(n)^-} \rightharpoonup h^-$ in H^-, and $h^{(n)^+} \rightharpoonup h^+$ in H^+. Hence

$$0 \leq \frac{J(q^{(n)})}{\|q^{(n)}\|^2} = \frac{s_n^2}{2}\|e\|^2 - \frac{1}{2}\|h^{(n)^-}\|^2 + \sum_{m\in\mathbb{Z}} R_m (h^{(n)})^2 + \sum_{m\in\mathbb{Z}} \frac{F_m(q^{(n)})}{\|q^{(n)}\|^2}. \tag{5.5}$$

If $\bar{s} = 0$, then by (5.5), we have

$$0 \leq \frac{1}{2}\|h^{(n)^-}\|^2 + \sum_{m\in\mathbb{Z}} \frac{F_m(q^{(n)})}{\|q^{(n)}\|^2} \leq \frac{s_n^2}{2}\|e\|^2 \to 0,$$

which yields $\|h^{(n)^-}\| \to 0$, and so $1 = \|h^{(n)^-} + s_n e\|^2 \to 0$, a contradiction.

If $\bar{s} \neq 0$, similar to the proof of Lemma 3.4, we can get (3.6), which is a contradiction. □

Lemma 5.5. *Suppose that* (V1′), (f0), (f2), (f3) *and* (f4″) *are satisfied. Then any* $\{q^{(n)}\} \subset H$ *satisfying*

$$J'(q^{(n)}) \to c \geq 0, \quad \langle J'(q^{(n)}), (q^{(n)})^{\pm}\rangle \to 0 \tag{5.6}$$

is bounded in H.

Proof. To prove the boundedness of $\{q^{(n)}\}$, arguing by contradiction, suppose that $\|q^{(n)}\| \to \infty$. Let $h^{(n)} = q^{(n)}/\|q^{(n)}\|$. Then $1 = \|h^{(n)}\|$. By Sobolev imbedding theorem, there exists a constant $C_4 > 0$ such that $\|h^{(n)}\|_2 \leq C_4$. Passing to a subsequence, we have $h^{(n)} \rightharpoonup h$ in H. There are two possible cases: (i) $h = 0$ and (ii) $h \neq 0$.

Case (i): $h = 0$, i.e., $h^{(n)} \rightharpoonup 0$ in H. Then $(h^{(n)})^+ \to 0$ and $(h^{(n)})^- \to 0$ for all $m \in \mathbb{Z}$. By (V1′), it is easy to show that

$$\lim_{n\to\infty} \sum_{m\in\mathbb{Z}} R_m((h^{(n)})^+)^2 = \lim_{n\to\infty} \sum_{m\in\mathbb{Z}} R_m((h^{(n)})^-)^2 = 0. \tag{5.7}$$

Letting

$$\delta := \limsup_{n\to\infty} \|(h^{(n)})^+\|_\infty,$$

if $\delta = 0$, then $\|(h^{(n)})^+\|_\infty < \frac{\omega}{\Lambda}$ for large n. Thus,

$$\limsup_{n\to\infty} \sum_{m\in\mathbb{Z}} F_m(\Lambda(h^{(n)})^+) \leq \frac{1}{4C_1^2} \limsup_{n\to\infty} \|(h^{(n)})^+\|_2^2 \leq \frac{1}{4}. \tag{5.8}$$

Set $\theta^{(n)} = \Lambda/\|(h^{(n)})^+\|$. Combining (5.7) and (5.8), we have, in light of Corollary 5.2

$$\begin{aligned}
c + o(1) &= J((h^{(n)})^+) \\
&\geq \frac{\left(\theta^{(n)}\right)^2}{2}\|h^{(n)}\|^2 - \sum_{m\in\mathbb{Z}} F_m(\theta^{(n)}(h^{(n)})^+) + \frac{1-(\theta^{(n)})^2}{2}\langle J'(h^{(n)}), h^{(n)}\rangle \\
&\quad +(\theta^{(n)})^2\langle J'(h^{(n)}), h^{(n)-}\rangle + \frac{(\theta^{(n)})^2}{2}\sum_{m\in\mathbb{Z}} R_m\big[((h^{(n)})^+)^2 - ((h^{(n)})^-)^2\big] \\
&= \frac{\Lambda^2}{2} - \sum_{m\in\mathbb{Z}} F_m\left(\frac{\Lambda(h^{(n)})^+}{\|h^{(n)}\|}\right) + \left(\frac{1}{2} - \frac{\Lambda^2}{2\|h^{(n)}\|^2}\right)\langle J'(h^{(n)}), h^{(n)}\rangle \\
&\quad + \frac{\Lambda^2}{\|h^{(n)}\|^2}\langle J'(h^{(n)}), h^{(n)-}\rangle + \frac{(\theta^{(n)})^2}{2}\sum_{m\in\mathbb{Z}} R_m\big[((h^{(n)})^+)^2 - ((h^{(n)})^-)^2\big] \\
&= \frac{\Lambda^2}{2} - \sum_{m\in\mathbb{Z}} F_m\left(\frac{\Lambda h^{(n)+}}{\|h^{(n)}\|}\right) + o(1) \\
&\geq \frac{\Lambda^2}{2} - \frac{1}{4} + o(1) > \frac{3}{4} + c + o(1).
\end{aligned}$$

This leads to a contradiction, so $\delta > 0$.

Similar to the proof in Lemma 3.6, we can verify (3.14) holds, which is a contradiction. Hence the statement of Lemma 5.5 are proved.

Case (ii): $h \neq 0$. In this case, we can also deduce a contradiction by a standard argument.

Cases (i) and (ii) show that $\{h^{(n)}\}$ is bounded in H. □

Proof of Theorem 1.3. Applying Lemmas 4.9 and 5.7, we deduce that there exists a bounded sequence $\{q^{(n)}\} \subset H$ satisfying (4.9). Passing to a subsequence, we have $q^{(n)} \rightharpoonup q$ in H. Next, we prove $q \neq 0$.

Arguing by contradiction, suppose that $q = 0$, i.e., $q^{(n)} \rightharpoonup 0$ in H, and so $q^{(n)} \rightharpoonup 0$ in l^2 and $u_n \to 0$ for all $m \in \mathbb{Z}$. By (V1′) and (f4″), it is easy to show that

$$\lim_{n\to\infty} \sum_{m\in\mathbb{Z}} R_m(q^{(n)})(q^{(n)})^2 = 0, \quad \lim_{n\to\infty} \sum_{m\in\mathbb{Z}} R_m(q^{(n)})(q^{(n)})p = 0 \tag{5.9}$$

and

$$\lim_{n\to\infty} H_m(q^{(n)}) = 0, \quad \lim_{n\to\infty} h_m(q^{(n)})p = 0, \quad \forall p \in H. \tag{5.10}$$

Note that

$$J_0(q) = J(q) - \frac{1}{2}\sum_{m\in\mathbb{Z}} R_m q^2 + \sum_{m\in\mathbb{Z}} H_m(q), \quad \forall q \in H \tag{5.11}$$

and

$$\langle J_0'(q), p\rangle = \langle J'(q), p\rangle - \sum_{m\in\mathbb{Z}} R_m(q)qp + \sum_{m\in\mathbb{Z}} h_m(q)p, \quad \forall q, p \in H. \tag{5.12}$$

From (3.4), (5.5)–(5.8), one can get that

$$J_0(q^{(n)}) \to c, \quad \|J_0'(q^{(n)})\|(1 + \|q^{(n)}\|) \to 0. \tag{5.13}$$

Going if necessary to a subsequence, we may assume the existence of $m_k \in \mathbb{Z}$ such that

$$|h_{m_k}^{(n)}| = \|h_m^{(n)}\|_\infty > \frac{\delta}{2}.$$

Choose integers i_k and n_k with $0 \le n_k \le N-1$ such that $m_k = i_k N + n_k$. Let $w_m^{(n)} = h_{m+ikN}^{(n)}$. Then

$$|(w_{n_k}^{(n)})^+| > \frac{\delta}{2}, \quad \forall k \in \mathbb{N}. \tag{5.14}$$

Since V_m and $f_m(t)$ are N-periodic on m, then $\|w_m^{(n)}\| = \|h_m^{(n)}\|$ and

$$J_0(w_m^{(n)}) \to c_*, \quad \|J_0'(w_m^{(n)})\|(1 + \|w_m^{(n)}\|) \to 0. \tag{5.15}$$

In the same way as the last part of the proof of Theorem 1.2, we can prove that $J_0'(w) = 0$ and $J_0(w) \le c_*$.

It follows from $J_0'(w) = 0$ and (5.12) that $w^+ \neq 0$. By Lemma 4.7, there exist $s_0 = s(w) > 0$ and $w_0 = \omega(w) \in H^-$ such that $s_0 w + w_0 \in \mathscr{N}^-$, and so $J(s_0w+w_0)>b$. By virtue of (f4$''$), $g_m(t)/|t|$ is non-decreasing on $t\in(-\infty,0)\cup(0,\infty)$, similar to (4.3), we have

$$\frac{1-s_0^2}{2}g_m(w)w-s_0g_m(w)w_0-\int_{s_0w+w_0}^{w}g_m(s)ds\geq 0.$$

Hence, from the fact that $H_m(t)-\frac{1}{2}V_mt^2>0$, for $(m,t)\in\mathbb{Z}\times\mathbb{R}$, we have

$$\begin{aligned}
b\geq c_*\geq J_0(w)\\
&=J_0(s_0w+w_0)+\frac{1}{2}\|w_0\|^2+\frac{1-s_0^2}{2}\langle J_0'(w),w\rangle-s_0\langle J_0'(w),w_0\rangle\\
&\quad+\sum_{m\in\mathbb{Z}}\left[\frac{1-t_0^2}{2}g_m(w)w-t_0g_m(w)w_0-\int_{t_0w+w_0}^{w}g_m(s)ds\right]\\
&\geq J_0(s_0w+w_0)+\frac{1}{2}\|w_0\|^2\\
&=\frac{1}{2}\|w_0\|^2+J(s_0w+w_0)-\frac{1}{2}\sum_{m\in\mathbb{Z}}V_m(s_0w+w_0)^2+\sum_{m\in\mathbb{Z}}H_m(s_0w+w_0)\\
&>J(s_0w+w_0)\geq b.
\end{aligned}$$

This contradiction implies that $q\neq 0$. In the same way as the last part of the proof of Theorem 1.2, we can certify that $J'(q)=0$ and $J(q)=b=\inf_{\mathcal{N}^-}J$. This shows that $q\in H$ is a solution to (1.3) with $J(q)=\inf_{\mathcal{N}^-}J>0$. The proof is completed. □

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