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RESEARCH ARTICLE

A Copositivity-type Existence Result for Weakly Homogeneous Variational Inequalities

Mengmeng ZHENG[1], **Zhenghai HUANG**[2]

1 Department of Mathematics, National University of Defense Technology, Changsha 410073, China

2 School of Mathematics, Tianjin University, Tianjin 300350, China

Abstract Recently, Gowda and Sossa [Math. Program., 2019, 177: 149–171] studied the existence of solutions to weakly homogeneous variational inequalities. In particular, their main result, based on a degree-theoretic condition and a constraint on the corresponding cone complementarity problem, covers a majority of existence results on the subcategory problems of weakly homogeneous variational inequalities. In this paper, what we achieve is a new copositivity-type existence result for the weakly homogeneous variational inequality. The conditions we used are easier to check than the degree-theoretic condition and our result crosses each other with the main result established by Gowda and Sossa and the main result given by Ma, Zheng and Huang [SIAM J. Optim., 2020, 30(1): 132–148], respectively. Besides, we show the distinctiveness of our existence result by comparing it with the well-known coercivity result obtained for variational inequalities and a norm-coercivity result obtained for complementarity problems, respectively.

Keywords Weakly homogeneous map, variational inequality, copositivity, coercivity

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1 Introduction

Very recently, Gowda and Sossa investigated a class of continuous map, named as weakly homogeneous map, and the corresponding variational inequality (VI) over a finite dimensional real Hilbert space in [5]. As an application, they discussed the solvability of nonlinear equations with weakly homogeneous maps

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Corresponding author: Zhenghai HUANG, E-mail: huangzhenghai@tju.edu.cn
Current address: Mengmeng ZHENG: School of Mathematics, North University of China, Taiyuan 030051, China

over closed convex cones, which covers tensor equations (or multilinear systems) [2,6,8,19] as special cases. Let $\mathbb{H}$ be a finite dimensional real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, C be a closed convex cone in $\mathbb{H}$ and $\mathbb{R}_{++}$ be the set of positively real numbers. Recall that a continuous map h: $C\to\mathbb{H}$ is called positively homogeneous of degree $\gamma(\geq 0)$ if and only if $h(\lambda\mathbf{x})=\lambda^\gamma h(\mathbf{x})$ for any $\mathbf{x}\in C$ and $\lambda\in\mathbb{R}_{++}$. As a generalization of the positively homogeneous map, a weakly homogeneous map of degree $\gamma(\geq 0)$ f is defined to be a sum of a positively homogeneous of degree $\gamma(\geq 0)$ h and a remainder which is continuous on C and satisfies $\lim_{\|x\|\to\infty}\frac{f(x)-h(x)}{\|x\|^\gamma}=0$ where $x\in C$. Owing to the fact that $\lim_{\lambda\to\infty}\frac{f(\lambda x)}{\lambda^\gamma}=h(x)$ for all $x\in C$, the positively homogeneous part h is often called the "leading term" or "recession map" of the weakly homogeneous map f and denoted by f^∞.

Given a weakly homogeneous map f on a closed convex cone C and a closed convex subset K in C, the weakly homogeneous variational inequality (WHVI) [1,5,15,20], denoted by WHVI(f,K) with SOL(f,K) being its solution set, is to find a vector $x^*\in H$ such that

$$x^*\in K,\quad \langle f(x^*),y-x^*\rangle\geq 0,\quad \text{for all } y\in K. \tag{1.1}$$

When K is a cone, (1.1) is called the weakly homogeneous complementarity problem (WHCP), denoted by WHCP(f,K). WHVIs and WHCPs cover several recently researched special VIs and CPs as subcategory problems. Specifically, when f is a polynomial, WHVI(f,K) and WHCP(f,K) come back to a polynomial variational inequality (PVI) studied in [9] and polynomial complementarity problem (PCP) studied in [4, 13, 14, 21], respectively; when f is the sum of a homogeneous polynomial and a constant vector, WHVI(f,K) and WHCP(f,K) come back to a tensor variational inequality (TVI) studied in [18] and tensor complementarity problem (TCP) studied in [10,11,16], respectively.

In [5], the authors established some close connections between VIs and CPs with involved maps being weakly homogeneous map of positive degree. For instance, letting $K^\infty:=\{u\in H: u+K\subseteq K\}$ represent the recession cone [17] of K, the main result given in [5, Theorem 4.1] showed that if the corresponding recession cone complementarity problem WHCP(f^∞,K^∞) has and only has the zero solution, and the (topological) index of the natural map [3] of WHCP(f^∞,K^∞) at the origin is nonzero, then WHVI(f,K) has a nonempty, compact solution set. To the best of our knowledge, this degree-theoretic theorem is different from the famous coercivity result [3] and covers a majority of existence results on the subcategory problems of WHVIs, including the well-known Karamardian's theorem [12] for homogeneous maps on proper cones. However, an undesirable fact is that the degree-theoretic condition is often not easy to check. In addition, an interesting observation is that in all proofs of these results on the nonemptiness and compactness of solution sets to WHVIs achieved in [5], when deriving the existence of the solution, the boundedness of the solution set of WHVIs is required to hold first, because of the participate of the homotopy invariance principle of the degree. Then an important question is whether the existence of the solution can be broken away from the boundedness of the solution set or not.

Inspired by above, we aim to find some easy-verified conditions to guarantee the existence of solutions for the WHVI where the involved set contains zero, and its proof does not depend on the boundedness of the solution set. Our paper is organized as follows. In Section 2, we show an alternative theorem for WHVIs. In Section 3, we establish a copositivity-type existence result for WHVIs. In Section 4, we compare our results with existing ones, including the main result given in [5, Theorem 4.1], the main result given in [15, Theorem 3.1] and the well-known coercivity result given for VIs. Finally, we sum up the conclusions in the last section.

2 An Alternative Theorem

In this section, we give an alternative theorem for WHVIs by making use of degree theory. Let Ω be a bounded open set in $\mathbb{H}$ and map $\phi : \operatorname{cl}\Omega \to \mathbb{H}$ be continuous where $\operatorname{cl}\Omega$ denotes the closure of Ω and $p \in \mathbb{H}$. Recall that the topological degree of ϕ over Ω with respect to p is well-defined if $p \notin \phi(\partial\Omega)$ where $\partial\Omega$ denotes the boundary of Ω, which is an integer used to judge the existence of a solution to the equation $\phi(x) = p$, denoted by $\deg(\phi, \Omega, p)$. The following properties of the topological degree play important roles in our proof of alternative theorem for WHVIs.

Lemma 2.1 [3]**.** *Let K be a closed convex set in C and $f : C \to \mathbb{H}$ be continuous. If there exists a bounded open set U with $\operatorname{cl} U \subseteq C$ such that $\deg(F_K^{\text{nat}}, U, 0) \neq 0$ where $F_K^{\text{nat}}(x) := x - \Pi_K(x - F(x))$ is the natural map of $\mathrm{VI}(f, K)$ with F being a given continuous extension of f and $\Pi_K(x)$ meaning the orthogonal projection of an $x \in H$ onto K, then $\mathrm{VI}(f, K)$ has a solution in U.*

Lemma 2.2 [3]**.** *Let Ω be a nonempty, bounded open subset in $\mathbb{H}$. Then the (topological) degree $\deg(\mathscr{H}(\cdot, t), \Omega, p(t))$ is independent of $t \in [0, 1]$ for any two continuous maps $\mathscr{H} : \operatorname{cl}\Omega \times [0, 1] \to \mathbb{R}^n$ and $p : [0, 1] \to \mathbb{R}^n$ such that $p(t) \notin \mathscr{H}(\partial\Omega, t)$ for any $t \in [0, 1]$.*

By Lemma 2.2, we can see when the continuous map $\varphi : \operatorname{cl}\Omega \to \mathbb{H}$ satisfies $\varphi(x) = 0$ if and only if $x = 0$, then, $\deg(\varphi, \Omega', 0)$ is invariant for any bounded open set Ω' containing 0 and contained in Ω, and the common degree is written as $\operatorname{ind}(\varphi, 0)$.

In addition, we need the following result given in [5].

Lemma 2.3 [5]**.** *Let K be a closed convex set in cone C and $f : C \to \mathbb{H}$ be a weakly homogeneous map of positive degree. If f^∞ is copositive on K^∞ and*

$\mathrm{SOL}(f^\infty, K^\infty) = \{0\}$, *then* $\mathrm{WHVI}(f, K)$ *has a nonempty, compact solution set and* $\deg(F_K^{\mathrm{nat}}, \Omega, 0) \neq 0$ *for any bounded open set* Ω *containing* $\mathrm{SOL}(f, K)$.

Recall that a map $\psi : D \to \mathbb{H}$ is said to be copositive on $D \subseteq \mathbb{H}$ [7], if $\langle \psi(x) - \psi(0), x\rangle \geq 0$ holds for any $x \in D$. Now, we show an alternative theorem for WHVIs by employing copositivity of maps and above lemmas.

Theorem 2.1. *Let* K *be a closed convex set in cone* C *and* $f : C \to \mathbb{H}$ *be a weakly homogeneous map of positive degree. If* f^∞ *is copositive on* K^∞*, then either the* $\mathrm{WHVI}(f, K)$ *has a solution or there exists an unbounded sequence* $\{x_k\} \subseteq K$ *and a positive sequence* $\{t_k\} \subseteq (0, 1)$ *such that for each* k*,*

$$\langle f^\infty(x_k) + t_k x_k + (1 - t_k)(f(x_k) - f^\infty(x_k)), y - x_k\rangle \geq 0, \quad \forall y \in K. \qquad (2.1)$$

Proof. Let F and F^∞ be any given continuous extensions of f and f^∞, respectively. For the sake of contradiction, we assume that $\mathrm{SOL}(f, K) = \emptyset$ and

$$\bigcup_{0<t<1} \mathrm{SOL}(f^\infty + t\mathscr{I} + (1 - t)(f - f^\infty), K)$$

is bounded, where $\mathscr{I}$ means the identity map from $\mathbb{H}$ into $\mathbb{H}$. Then, consider the following homotopy map:

$$\mathscr{H}(x, t) = x - \Pi_K(x - (F^\infty(x) + tx + (1-t)(F(x) - F^\infty(x)))), \quad \forall (x, t) \in \mathbb{H} \times [0, 1].$$

It is easy to see that $\mathscr{H}(\cdot, t)$ is exactly the natural map of $\mathrm{WHVI}(f^\infty + t\mathscr{I} + (1 - t)(f - f^\infty), K)$ for each $t \in [0, 1]$. Denote the set of zeros of $\mathscr{H}(\cdot, t)$ by

$$\mathbb{Z} := \{x \in \mathbb{H} : \mathscr{H}(x, t) = 0 \text{ for some } t \in [0, 1]\}.$$

Since $\mathrm{SOL}(f, K) = \emptyset$, it follows that $\{x \in \mathbb{H} : \mathscr{H}(x, 0) = 0\}$ is bounded, which, together with another assumption, implies that

$$\{x \in \mathbb{H} : \mathscr{H}(x, t) = 0 \text{ for some } t \in [0, 1)\}$$

is bounded. Now, we consider the set $\{x \in \mathbb{H} : \mathscr{H}(x, 1) = 0\}$. When $t = 1$, $\mathscr{H}(x, t)$ becomes

$$\mathscr{H}(x, 1) = x - \Pi_K(x - (F^\infty(x) + x)).$$

Since f^∞ is copositive on K^∞, it is not difficult to see that

$$\mathrm{SOL}((f^\infty + \mathscr{I})^\infty, K^\infty) = \{0\}$$

and $(f^\infty + \mathscr{I})^\infty$ is copositive on K^∞. So from Lemma 2.3, it follows that $\mathrm{SOL}(f^\infty + \mathscr{I}, K)$ is nonempty and compact and $\deg((F^\infty + \mathscr{I})_K^{\mathrm{nat}}, \Omega, 0) \neq 0$

for any bounded open set Ω containing $\mathrm{SOL}(f^\infty + \mathscr{I}, K)$. Hence, the set $\mathbb{Z}$ is uniformly bounded.

Let $\Omega \subseteq \mathbb{Z}$ be a bounded open set in $\mathbb{H}$. Then for all $t \in [0,1]$, $0 \notin \mathscr{H}(\partial\Omega, t)$. By Lemma 2.2, it follows that

$$\deg(\mathscr{H}(\cdot,0),\Omega,0) = \deg(\mathscr{H}(\cdot,1),\Omega,0) = \deg((F^\infty + \mathscr{I})_K^{\mathrm{nat}},\Omega,0) \neq 0,$$

which means that $\mathrm{SOL}(f,g,K)$ is nonempty by Lemma 2.1. This contradiction means Theorem 2.1 holds. $\square$

3 An Existence Result

Recall that in [5], the authors investigated the nonemptiness and compactness of the solution set of $\mathrm{WHVI}(f,K)$ well, and established some good results based on a condition that $\mathrm{SOL}(f^\infty, K^\infty) = \{0\}$, which is an important condition for both deriving the nonemptiness and boundedness of the solution set of $\mathrm{WHVI}(f,K)$. However, it is not an easy task to judge whether the solution set of $\mathrm{WHCP}(f^\infty, K^\infty)$ contains only zero or not, since $\mathrm{WHCP}(f^\infty, K^\infty)$ is a complementarity problem. In this section, we give an existence result for $\mathrm{WHVI}(f,K)$ without adding special constraint to $\mathrm{SOL}(f^\infty, K^\infty)$, where the conditions for the existence of solutions is separate from the conditions for the boundedness of the solution set, and their proofs are also separate.

Theorem 3.1. *Let $0 \in K$ be a closed convex set in cone C and $f : C \to \mathbb{H}$ be weakly homogeneous of degree $\gamma > 0$. Suppose that one of the following conditions hold:*

(a) $\lim_{x\in K,\|x\|\to\infty} \frac{f(x)-f^\infty(x)}{\|x\|} = 0$; *$f^\infty$ is copositive on K^∞ and there exists some constant $M > 0$ such that $\langle f(x), x\rangle > 0$ for any $x \in K$ satisfying $\|x\| \geq M$;*

(b) *f^∞ is copositive on K; there exists some constant $M' > 0$ such that $\|f(x) - f^\infty(x)\| \leq \|F_K^{\mathrm{nat}}(x)\|$ for any $x \in K$ satisfying $\|x\| \geq M'$ and there exists no $c > 0$ such that $-x = c(f(x) - f^\infty(x))$ for any $x \in K$ satisfying $\|x\| \geq M'$,*

where $F_K^{\mathrm{nat}}(x) := x - \Pi_K(x - F(x))$ is the natural map of $\mathrm{VI}(f,K)$ *with F being a given continuous extension of f. Then* $\mathrm{WHVI}(f,K)$ *has a nonempty solution set. If additional,* $\lim_{x\in K,\|x\|\to\infty} \|F_K^{\mathrm{nat}}(x)\| = \infty$*, then* $\mathrm{WHVI}(f,K)$ *has a nonempty and compact solution set.*

Proof. First, we show that f^∞ is copositive on K can imply that f^∞ is copositive on K^∞. From the definition of the recession cone, it follows that for any $u \in K^\infty$, there exist two sequences $\{t_k\} \subseteq \mathbb{R}_+$ and $\{x_k\} \subseteq K$ such that

$u = \lim_{k\to\infty} \frac{x_k}{t_k}$. Thus, we have that for any $u \in K^\infty$, it follows

$$\langle f^\infty(u), u\rangle = \lim_{k\to\infty} \left\langle f^\infty\left(\frac{x_k}{t_k}\right), \frac{x_k}{t_k}\right\rangle = \lim_{k\to\infty} \frac{\langle f^\infty(x_k), x_k\rangle}{t_k^{\gamma+1}}.$$

If f^∞ is copositive on K, then we can obtain that $\langle f^\infty(x_k), x_k\rangle \geq 0$, which means that $\langle f^\infty(u), u\rangle \geq 0$ for any $u \in K^\infty$. That is to say, f^∞ is copositive on K^∞.

Second, we show that $\mathrm{SOL}(f, K) \neq \emptyset$ both in conditions (a) and (b). Suppose that $\mathrm{SOL}(f, K) = \emptyset$, then from Theorem 2.1, there exists an unbounded sequence $\{x_k\}$ and a positive sequence $\{t_k\} \subseteq (0, 1)$ such that $x_k \in K$ and (2.1) holds for each k. Noting that $0 \in K$, thus we have that

$$\langle x_k, f^\infty(x_k) + t_k x_k + (1 - t_k)(f(x_k) - f^\infty(x_k))\rangle \leq 0. \tag{3.1}$$

Below, we divide the discussion into the following two cases.

Case 1: Condition (a) holds. In this case, from $\lim_{x\in K, \|x\|\to\infty} \frac{f(x)-f^\infty(x)}{\|x\|} = 0$ and $\lim_{k\to\infty} \|x_k\| = \infty$, we obtain that

$$\lim_{k\to\infty} \left\langle \frac{x_k}{\|x_k\|}, \frac{f(x_k) - f^\infty(x_k)}{\|x_k\|}\right\rangle = 0,$$

which implies that

$$\langle x_k, t_k x_k - t_k(f(x_k) - f^\infty(x_k))\rangle \geq 0,$$

when k is sufficiently large. Furthermore, from (3.1), it follows that

$$\langle x_k, f(x_k)\rangle \leq 0,$$

when k is sufficiently large with $\|x_k\| \geq M$. This contradicts condition (a). Hence, the assumption is not true and $\mathrm{SOL}(f, K) \neq \emptyset$.

Case 2: Condition (b) holds. Noting that (3.1), together with $x_k \in K$ and the condition that f^∞ is copositive on K, can imply that

$$\langle x_k, t_k x_k + (1 - t_k)(f(x_k) - f^\infty(x_k))\rangle \leq 0,$$

we have that

$$t_k\|x_k\| \leq (1 - t_k)\|f(x_k) - f^\infty(x_k)\|, \tag{3.2}$$

which implies that $f(x_k) - f^\infty(x_k) \neq 0$ for sufficiently large k. Furthermore, by $t_k > 0$, the condition that there exists no $c > 0$ such that $-x = c(f(x) - f^\infty(x))$ for any $x \in K$ satisfying $\|x\| \geq M'$, and the trigonometric inequality of the norm, we can obtain that

$$\| - t_k x_k + t_k(f(x_k) - f^\infty(x_k))\| < t_k\|x_k\| + t_k\|f(x_k) - f^\infty(x_k)\| \tag{3.3}$$

for sufficiently large k with $\|x_k\| \geq M'$. In addition, by Lemma 2.1 and (2.1), it follows that

$$x_k = \Pi_K(x_k - f^\infty(x_k) - t_k x_k - (1-t_k)(f(x_k) - f^\infty(x_k))),$$

which, together with $F(x_k) = f(x_k)$ for any $x_k \in K$, implies that for sufficiently large k with $\|x_k\| \geq M'$,

$$\begin{aligned}
&\|F_K^{\text{nat}}(x_k)\| \\
&\quad = \|x_k - \Pi_K(x_k - F(x_k))\| \\
&\quad = \|\Pi_K(x_k - f^\infty(x_k) - t_k x_k - (1-t_k)(f(x_k) - f^\infty(x_k))) - \Pi_K(x_k - f(x_k))\| \\
&\quad \leq \| - t_k x_k + t_k(f(x_k) - f^\infty(x_k))\| \\
&\quad < t_k\|x_k\| + t_k\|f(x_k) - f^\infty(x_k)\| \\
&\quad \leq (1-t_k)\|f(x_k) - f^\infty(x_k)\| + t_k\|f(x_k) - f^\infty(x_k)\| \\
&\quad = \|f(x_k) - f^\infty(x_k)\|, \qquad (3.4)
\end{aligned}$$

where the first inequality follows from non-expansiveness of Euclidean projector, the second inequality follows from (3.3) and the third inequality follows from (3.2).

On one hand, it follows from (3.4) that

$$\|F_K^{\text{nat}}(x_k)\| < \|f(x_k) - f^\infty(x_k)\|$$

for sufficiently large k with $\|x_k\| \geq M'$. On the other hand, by the condition that $\|f(x) - f^\infty(x)\| \leq \|F_K^{\text{nat}}(x)\|$ for any $x \in K$ with $\|x_k\| \geq M'$, we have

$$\|f(x_k) - f^\infty(x_k)\| \leq \|F_K^{\text{nat}}(x_k)\|$$

for sufficiently large k with $\|x_k\| \geq M'$. This contradiction means $\text{SOL}(f, K) \neq \emptyset$.

Last, from $\lim_{x\in K, \|x\|\to\infty} \|F_K^{\text{nat}}(x)\| = \infty$, it follows that $\text{SOL}(f, K)$ is bounded. Therefore, we obtain that $\text{SOL}(f, K)$ is nonempty and compact. □

Since the condition that "$\langle f(x), x\rangle > 0$ for any $x \in K$ satisfying $\|x\| \geq M$" can imply the boundedness of the solution set of $\text{WHCP}(f, K)$, the following corollary of Theorem 3.1 holds immediately.

Corollary 3.1. *Let $0 \in K$ be a closed convex cone and $f : K \to \mathbb{H}$ be weakly homogeneous of degree $\gamma > 0$ with $\lim_{x\in K, \|x\|\to\infty} \frac{f(x) - f^\infty(x)}{\|x\|} = 0$. Suppose that f^∞ is copositive on K^∞ and there exists some constant $M > 0$ such that $\langle f(x), x\rangle > 0$ for any $x \in K$ satisfying $\|x\| \geq M$. Then* $\text{WHCP}(f, K)$ *has a nonempty and compact solution set.*

Remark 3.1. In [5, Theorem 6.1], a copositivity result was given. However, the condition that "$\mathrm{SOL}(f^\infty, K^\infty) = \{0\}$" in [5, Theorem 6.1 (a)] is relatively complicated to verify, while the condition that "f^∞ is strictly copositive on K^∞" in [5, Theorem 6.1 (b)] is relatively easy-verified but a little strict. Theorem 3.1 (a) shows that in the case of $0 \in K$, the condition that "$\mathrm{SOL}(f^\infty, K^\infty) = \{0\}$" in [5, Theorem 6.1 (a)] can be replaced by "$\langle f(x), x\rangle > 0$ for any $x \in K$ satisfying $\|x\| \geq M$" and $\lim_{x\in K, \|x\|\to\infty} \frac{f(x)-f^\infty(x)}{\|x\|} = 0$ to derive the existence of solutions to WHVIs, which are both relatively easy to check. Besides, both of conditions "$\langle f(x), x\rangle > 0$ for any $x \in K$ satisfying $\|x\| \geq M$" and "f^∞ is copositive on K^∞" in Theorem 3.1 (a) are weaker than the condition that "f^∞ is strictly copositive on K^∞" in [5, Theorem 6.1 (b)].

Below, we show that conditions (a) and (b) in Theorem 3.1 are different via the following examples:

Example 3.1. Consider $\mathrm{WHCP}(f, K)$ where $K = \{x \in \mathbb{R}^2 : x_1 = 0,\, x_2 \geq 0\}$, and for any $x \in \mathbb{R}^2$,

$$f(x) = (\sqrt{x_2} - \sqrt[4]{x_1}, \sqrt{x_1} - \sqrt[4]{x_2})^{\mathrm{T}}.$$

Obviously, K is a convex cone, and f is weakly homogeneous of degree $\frac{1}{2}$.

- First, for any $x \in K$, we have that $\langle f^\infty(x), x\rangle = x_1\sqrt{x_2} + x_2\sqrt{x_1} = 0$, thus f^∞ is copositive on K.
- Second, take $M' = 2$. It is easy to see that for any $x \in K$ with $\|x\| \geq M'$, there exists no $c > 0$ such that

$$-x = c(f(x) - f^\infty(x)) = c(-\sqrt[4]{x_1}, -\sqrt[4]{x_2})^{\mathrm{T}}.$$

- Furthermore, for any $x \in K$ with $\|x\| \geq M' = 2$, it follows that

$$\begin{cases} (x - f(x))_1 = x_1 - (\sqrt{x_2} - \sqrt[4]{x_1}) < 0, \\ (x - f(x))_2 = x_2 - (\sqrt{x_1} - \sqrt[4]{x_2}) > 0, \end{cases}$$

  which implies that

$$\Pi_K(x - f(x)) = (0, x_2 + \sqrt[4]{x_2})^{\mathrm{T}}.$$

  Thus, we have that

$$\|f_K^{\mathrm{nat}}(x)\| = \|x - \Pi_K(x - f(x))\| = \sqrt[4]{x_2}$$

  and

$$\frac{\|f_K^{\mathrm{nat}}(x)\|}{\|f(x) - f^\infty(x)\|} = \frac{\sqrt[4]{x_2}}{\sqrt[4]{x_2}} = 1$$

  for any $x \in K$ with $\|x\| \geq M'$.

So, condition (b) in Theorem 3.1 holds.

However, it is easy to see $\langle f(x), x\rangle = x_2(-\sqrt[4]{x_2}) < 0$ for any $x \in K \setminus \{0\}$, which means that condition (a) in Theorem 3.1 does not hold.

Example 3.2. Consider WHVI(f, K) where $K := \{x \in \mathbb{R}^2 : x_1 \geq -1, x_2 \geq 0\}$ and for any $x \in \mathbb{R}^2$,

$$f(x) = \left(\sqrt[3]{x_1^2} + \sqrt[5]{(x_1+x_2)^2}, \sqrt[3]{x_1^2} + \sqrt[5]{(x_1+x_2)^2}\right)^{\mathrm{T}}.$$

Obviously, $0 \in K$ is a convex subset in $\mathbb{R}^2$ with $K^\infty = \mathbb{R}^2_+$, and f is weakly homogeneous of degree $\frac{2}{3}$.

- First, it is easy to see

$$\begin{aligned}&\lim_{x\in K, \|x\|\to\infty} \frac{f(x) - f^\infty(x)}{\|x\|}\\ &= \lim_{x\in K, \|x\|\to\infty} \left(\frac{\sqrt[5]{(x_1+x_2)^2}}{\sqrt{(x_1+x_2)^2}}, \frac{\sqrt[5]{(x_1+x_2)^2}}{\sqrt{(x_1+x_2)^2}}\right)^{\mathrm{T}} = 0.\end{aligned}$$

- Second, for any $x \in K^\infty$, we have that $\langle f^\infty(x), x\rangle = \sqrt[3]{x_1^2}(x_1 + x_2) \geq 0$, thus f^∞ is copositive on K^∞.

- Third, take $M = 2$, then for any $x \in K$ with $\|x\| \geq M$, it follows that $x_1 + x_2 > 0$, which means that

$$\langle f(x), x\rangle = \sqrt[3]{x_1^2}(x_1 + x_2) + \sqrt[5]{(x_1+x_2)^2}(x_1 + x_2) > 0.$$

So, condition (a) in Theorem 3.1 holds.

However, it is easy to see $\langle f^\infty(\hat{x}), \hat{x}\rangle = \sqrt[3]{\hat{x}_1^2}(\hat{x}_1 + \hat{x}_2) = -1 < 0$ by taking $\hat{x} = (-1, 0) \in K$, which means that f^∞ is not copositive on K, i.e., condition (b) in Theorem 3.1 does not hold.

Besides, we can see that $\mathrm{SOL}(f^\infty, K^\infty) = \{x \in \mathbb{R}^2_+ : x_1 = 0\} \neq \{0\}$, which implies that the condition "$\mathrm{SOL}(f^\infty, K^\infty) = \{0\}$" in [5, Theorem 6.1 (a)] does not hold, either. This example also demonstrates that Theorem 3.1 (a) is a different copositivity result from the corresponding one given in [5, Theorem 6.1].

In [5], an extension of the Z-property for nonlinear maps was given as follows:

Lemma 3.1. *Suppose C is a closed convex cone with the dual cone being $C^* := \{y \in \mathbb{H} : \langle y, x\rangle \geq 0, \forall x \in C\}$ and $f : C \to \mathbb{H}$ satisfies:*

$$x \in C,\ y \in C^* \text{ and } \langle x, y\rangle = 0 \implies \langle f(x), y\rangle \leq 0. \tag{3.5}$$

Then, $f(x^*) = q$ if and only if $q \in C$ and $x^* \in \mathrm{SOL}(\bar{f}, C)$ where $\bar{f}(x) := f(x) - q$.

With Lemma 3.1, we can obtain the following corollary of Theorem 3.1.

Corollary 3.2. *Suppose conditions of Theorem* 3.1 *hold with* $K = K^\infty = C$ *and* f *satisfies* (3.5). *Then for all* $q \in C$, *the equation* $f(x) = q$ *has a solution in* C.

4 Some Comparisons Between Our Existence Result and Four Related Results

As an important subclass of VIs, which covers PVIs and TVIs as special cases, the WHVI has received extensive attention recently, and the existence of solutions for WHVIs has been well studied. In this section, we compare our existence results with several main related existence results for WHVIs or VIs in the setting of weakly homogeneous situation.

4.1 Comparison with the Main Result in [5]

In this section, we compare our result with the main result given in [5]:

Theorem 4.1 [5]**.** *Let* K *be a closed convex set in cone* C *and* $f : C \to \mathbb{H}$ *be a weakly homogeneous map of positive degree. If* $\mathrm{SOL}(f^\infty, K^\infty) = \{0\}$ *and* $\mathrm{ind}(G_{K^\infty}, 0) \neq 0$ *where* $G_{K^\infty}(x) := x - \Pi_{K^\infty}(x - G(x))$ *with* G *being a given continuous extension of* f^∞, *then* $\mathrm{WHVI}(f, K)$ *has a nonempty, compact solution set.*

Below, we construct three examples to show that conditions of Theorem 3.1 and Theorem 4.1 cross each other.

Example 4.1. Consider $\mathrm{WHVI}(f, K)$ where $K := \{x \in \mathbb{R}^2 : x_1 = x_2 \geq -1\}$ and for any $x \in \mathbb{R}^2$,

$$f(x) = ((x_1 - x_2)^2 + x_1, (x_1 - x_2)^2 + x_2)^{\mathrm{T}}.$$

Obviously, K is a convex subset in $\mathbb{R}^2$ with $0 \in K$, and f is weakly homogeneous of degree 2.

- First, for any $x \in K$, we have that $\langle f^\infty(x), x\rangle = (x_1 - x_2)^2(x_1 + x_2) = 0$, thus f^∞ is copositive on K.
- Second, take $M' = 2$, then for any $x \in K$ with $\|x\| \geq M'$, there exists no $c > 0$ such that
$$-x = c(f(x) - f^\infty(x)) = cx.$$
- Third, for any $x \in K$ with $\|x\| \geq M' = 2$, it follows that $x - f(x) = 0$. Thus, for any $x \in K$ with $\|x\| \geq M'$, we have that
$$\|f(x) - f^\infty(x)\| = \|f_K^{\mathrm{nat}}(x)\| = \|x\|$$

and

$$\lim_{x\in K,\|x\|\to\infty}\|f_K^{\mathrm{nat}}(x)\| = \lim_{x\in K,\|x\|\to\infty}\|x\| = \infty.$$

Thus, the norm coercivity condition $\lim_{x\in K,\|x\|\to\infty}\|f_K^{\mathrm{nat}}(x)\| = \infty$ and all conditions of Theorem 3.1 (b) are satisfied, which means that $\mathrm{SOL}(f,K)$ is nonempty and compact.

However, noting that

$$\mathrm{SOL}(f^\infty, K^\infty) = K^\infty = \{x \in \mathbb{R}^2 : x_1 = x_2 \geq 0\} \neq \{0\},$$

thus at least there is one condition of the main result given in Theorem 4.1 that does not hold. That is to say, conditions in Theorem 3.1 cannot be covered by those of Theorem 4.1. In addition, we can see that conditions in Theorem 3.1 are easy to check, while it is not clear whether $\deg(G_{K^\infty}, \Omega, 0) \neq 0$ holds or not where $\Omega \supseteq \mathrm{SOL}(f^\infty, K^\infty)$ is a bounded open set and $G_{K^\infty}(x) = x - \Pi_{K^\infty}(x - G(x))$ with G being a given continuous extension of $f^\infty(x) = ((x_1 - x_2)^2, (x_1 - x_2)^2)^{\mathrm{T}}$.

Example 4.2. Consider $\mathrm{WHCP}(f,K)$ where $K := \mathbb{R}^2_+$ and for any $x \in \mathbb{R}^2$,

$$f(x) = ((x_1 + x_2)^2 + x_1, (x_1 + x_2)^2 + x_2)^{\mathrm{T}}.$$

Obviously, K is a convex cone, and f is weakly homogeneous of degree 2.

- First, for any $x \in K$, we have $\langle f^\infty(x), x\rangle = (x_1 + x_2)^3 \geq 0$, thus f^∞ is copositive on K.
- Second, for any $x \in K \setminus \{0\}$, there exists no $c > 0$ such that $-x = c(f(x) - f^\infty(x)) = cx$.
- Third, for any $x \in K$, it follows that

  $$(x - f(x))_1 = (x - f(x))_2 = -(x_1 + x_2)^2 < 0,$$

  which implies that $\Pi_K(x - f(x)) = 0$. So, for any $x \in K$, we can obtain that

  $$\|f(x) - f^\infty(x)\| = \|f_K^{\mathrm{nat}}(x)\| = \|x\|$$

  and

  $$\lim_{x\in K,\|x\|\to\infty}\|f_K^{\mathrm{nat}}(x)\| = \lim_{x\in K,\|x\|\to\infty}\|x\| = \infty.$$

Thus, the norm coercivity condition $\lim_{x\in K,\|x\|\to\infty}\|f_K^{\mathrm{nat}}(x)\| = \infty$ and all conditions of Theorem 3.1 (b) are satisfied.

In addition, it is not difficult to see that $\mathrm{SOL}(f^\infty, K^\infty) = \{0\}$, which, together with f^∞ is copositive on K^∞, implies that conditions in Theorem 4.1 hold by Lemma 2.3.

Example 4.3. Consider WHVI(f, K) where

$$K = \{x \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \geq 1\},$$

and for any $x \in \mathbb{R}^2$,

$$f(x) = \left((x_1 + x_2)^2 - \sqrt[4]{x_1^2}, (x_1 + x_2)^2 - \sqrt[4]{x_2^2}\right)^{\mathrm{T}}.$$

Obviously, K is a convex subset in $\mathbb{R}^2$ with $K^\infty = \mathbb{R}^2_+$, and f is weakly homogeneous of degree 2.

By a similar discussion as Example 4.2, we can see that all conditions of Theorem 4.1 are satisfied. However, noting that $0 \notin K$, those conditions in Theorem 3.1 do not hold.

From Examples 4.1–4.3, we can see that Theorem 4.1 and Theorem 3.1 cannot contain each other. Hence, our result enriches the theory for not only WHVIs, but also its subcategory problems such as WHCPs, PVIs, PCPs, TVIs and TCPs, in the sense that the main result established by Gowda and Sossa covers a majority of existence results on the subcategory problems of WHVIs.

4.2 Comparison with the Main Result in [15]

In this subsection, we compare our result with the main result given in [15]:

Theorem 4.2 [15]**.** *Let K be a nonempty closed convex subset of $\mathbb{H}$, $f : K \to \mathbb{H}$ be a weakly homogeneous mapping of degree γ, and $p \in \mathbb{H}$. Suppose that the following conditions hold:*

(a) *f is q-copositive on K, i.e., $\langle f(x) - q, x\rangle \geq 0$ for any $x \in K$;*

(b) *there exists a vector $\hat{x} \in K$ such that $\langle f(x), \hat{x}\rangle \leq 0$ for all $x \in K$;*

(c) *$\mathscr{S} := \mathrm{SOL}(f^\infty, K^\infty)$ and $p + q \in \mathrm{int}(\mathscr{S}^*)$,*

then $\mathrm{VI}(f, K, p)$ *has a nonempty compact solution set.*

Here, we construct three examples to show that conditions of Theorem 3.1 and Theorem 4.2 cross each other.

Example 4.4. Consider WHVI(f, K) where $K := \{x \in \mathbb{R}^2 : x_1 \geq -1,\ x_2 \geq 0\}$ and for any $x \in \mathbb{R}^2$,

$$f(x) = ((x_1 + x_2)^2 + \sqrt[5]{(x_1 + x_2)^2}, (x_1 + x_2)^2 + \sqrt[5]{(x_1 + x_2)^2})^{\mathrm{T}}.$$

Obviously, $0 \in K$ is a convex subset in $\mathbb{R}^2$ with $K^\infty = \mathbb{R}^2_+$, and f is weakly homogeneous of degree 2.

- First, it is easy to see

$$\begin{aligned}&\lim_{x\in K,\|x\|\to\infty}\frac{f(x)-f^{\infty}(x)}{\|x\|}\\&=\lim_{x\in K,\|x\|\to\infty}\left(\frac{\sqrt[5]{(x_1+x_2)^2}}{\sqrt{(x_1+x_2)^2}},\frac{\sqrt[5]{(x_1+x_2)^2}}{\sqrt{(x_1+x_2)^2}}\right)^{\mathrm{T}}=0.\end{aligned}$$

- Second, for any $x\in K^{\infty}$, we have that $\langle f^{\infty}(x),x\rangle=(x_1+x_2)^3\geq 0$, thus f^{∞} is copositive on K^{∞}.

- Third, take $M=16$, then for any $x\in K$ with $\|x\|\geq M$, it follows that $x_1+x_2\geq 4$, which means that

$$\langle f(x),x\rangle=(x_1+x_2)^3+\sqrt[5]{(x_1+x_2)^2}(x_1+x_2)>0.$$

- Furthermore, for any $x\in K$ with $\|x\|\geq M=16$, it follows that

$$\begin{cases}(x-f(x))_1=x_1-(x_1+x_2)^2-\sqrt[5]{(x_1+x_2)^2}<-1,\\(x-f(x))_2=x_2-(x_1+x_2)^2-\sqrt[5]{(x_1+x_2)^2}<0,\end{cases}$$

  which implies that $\Pi_K(x-f(x))=(-1,0)^{\mathrm{T}}$. So, for any $x\in K$, we can obtain that

$$\lim_{x\in K,\|x\|\to\infty}\|f_K^{\mathrm{nat}}(x)\|=\lim_{x\in K,\|x\|\to\infty}\sqrt{(x_1+1)^2+x_2^2}=\infty.$$

So, the norm coercivity condition $\lim_{x\in K,\|x\|\to\infty}\|f_K^{\mathrm{nat}}(x)\|=\infty$ and all conditions of Theorem 3.1 (a) are satisfied, which means that $\mathrm{SOL}(f,K)$ is nonempty and compact.

Now, we show that condition (a) in Theorem 4.2 does not hold. Suppose that there exists a vector $q\in\mathbb{R}^n$ such that f is q-copositive on K. Since $(-1,0)^{\mathrm{T}}\in K$, we can obtain that $q_1\geq 2$. But $(\frac{1}{2},0)^{\mathrm{T}}\in K$, so it must follow that $q_1<2$. This contradiction means condition (a) in Theorem 4.2 does not hold.

Example 4.5 [15, Example 3.1]**.** Consider $\mathrm{WHVI}(f,K)$ where

$$K:=\left\{(x_1,x_2)^{\mathrm{T}}\in\mathbb{R}^2:x_1\geq 0,x_2=\frac{1}{2}\right\}$$

and for any $x\in\mathbb{R}^2$,

$$f(x)=(x_2^2+x_1,-x_2^2-x_1-x_2)^{\mathrm{T}}.$$

Obviously, f is weakly homogeneous of degree 2.

In [15, Example 3.1], it has been shown that all conditions of Theorem 4.2 are satisfied. However, noting that $0\notin K$, those conditions in Theorem 3.1 do not hold.

Example 4.6. Consider WHVI(F, K) where $K := \mathbb{R}^2_+$ and for any $x \in \mathbb{R}^2$, $F(x) := f(x) + p$ with $p := (2, 2)^{\mathrm{T}}$ and

$$f(x) = \left(x_1 + \sqrt[5]{(x_1 + x_2)^2}, x_2 + \sqrt[5]{(x_1 + x_2)^2}\right)^{\mathrm{T}}.$$

Obviously, $0 \in K$ is a convex cone in $\mathbb{R}^2$, and f is weakly homogeneous of degree 1.

- First, it is easy to see

$$\begin{aligned}&\lim_{x \in K, \|x\| \to \infty} \frac{F(x) - F^\infty(x)}{\|x\|} \\ &= \lim_{x \in K, \|x\| \to \infty} \left(\frac{\sqrt[5]{(x_1 + x_2)^2} + 2}{\sqrt{(x_1 + x_2)^2}}, \frac{\sqrt[5]{(x_1 + x_2)^2} + 2}{\sqrt{(x_1 + x_2)^2}}\right)^{\mathrm{T}} = 0.\end{aligned}$$

- Second, for any $x \in K^\infty$, we have that $\langle F^\infty(x), x\rangle = x_1^2 + x_2^2 \geq 0$, thus F^∞ is copositive on K^∞.

- Third, for any $x \in K \setminus \{0\}$, it follows that

$$\langle F(x), x\rangle = x_1^2 + x_2^2 + \sqrt[5]{(x_1 + x_2)^2}(x_1 + x_2) + 2(x_1 + x_2) > 0.$$

- Furthermore, for any $x \in K$, it follows that

$$\begin{cases} (x - f(x))_1 = -\sqrt[5]{(x_1 + x_2)^2} \leq 0, \\ (x - f(x))_2 = -\sqrt[5]{(x_1 + x_2)^2} \leq 0, \end{cases}$$

  which implies that $\Pi_K(x - f(x)) = 0$, and

$$\lim_{x \in K, \|x\| \to \infty} \|f_K^{\mathrm{nat}}(x)\| = \lim_{x \in K, \|x\| \to \infty} \|x\| = \infty.$$

So, the norm coercivity condition $\lim_{x \in K, \|x\| \to \infty} \|f_K^{\mathrm{nat}}(x)\| = \infty$ and all conditions of Theorem 3.1 (a) are satisfied, which means that SOL(f, K) is nonempty and compact.

Noting that $0 \in K$, f is copositive on K, i.e., f is q-copositive on K with $q = 0$, $S = \mathrm{SOL}(f^\infty, K^\infty) = \{0\}$ and $p + q \in \mathrm{int}(S^*) = \mathbb{R}^n$, it follows that all conditions of Theorem 4.2 hold.

From Examples 4.4–4.6, we can see that Theorem 4.2 and Theorem 3.1 cannot contain each other. In addition, since Theorem 4.2 is a genuine generalization of another copositivity-type result given in [5, Theorem 6.2], our existence result cannot be covered by [5, Theorem 6.2].

4.3 Comparison with Two Coercivity-type Results

In this subsection, we compare our copositivity-type existence result with two coercivity-type results. It was shown in [3, 7] that for a general $\mathrm{VI}(f, K)$, employing the coercivity property of f to establish existence is a common approach. Last, we compare our result with the following well-known coercivity result for VIs.

Theorem 4.3 [3]**.** *Let K be a closed convex set in $\mathbb{R}^n$ and $f : K \to \mathbb{R}^n$ be continuous. If f is coercive on K, which is to say that there exists some $x^{\mathrm{ref}} \in K$, $c > 0$ and $\xi \geq 0$ such that $\langle f(x), x - x^{\mathrm{ref}} \rangle \geq c\|x\|^{\xi}$ for any $x \in K$ with $\|x\| \to \infty$, then $\mathrm{VI}(f, K)$ has a nonempty, compact solution set.*

The following example illustrates that the conditions in Theorem 3.1 cannot be deduced by the above coercive one.

Example 4.7. Consider $\mathrm{WHCP}(f, K)$ where $K := \{x \in \mathbb{R}^2 : x_1 = x_2\}$, and for any $x \in \mathbb{R}^2$,

$$f(x) = (x_2^3 - 2x_1, -x_1^3)^{\mathrm{T}}.$$

Obviously, K is a convex cone, and f is weakly homogeneous of degree 3.

First, we show that all conditions of Theorem 3.1 are satisfied.

- First, for any $x \in K$, i.e., $x_1 = x_2$, we have that

$$\langle f^{\infty}(x), x \rangle = x_1(x_2^3) + x_2(-x_1^3) = 0,$$

thus f^{∞} is copositive on K.

- Second, for any $x \in K \setminus \{0\}$, there exists no $c > 0$ such that

$$(-x_1, -x_1)^{\mathrm{T}} = -x = c(f(x) - f^{\infty}(x)) = (-2cx_1, 0).$$

- Third, for any $x \in K$, it follows that $x - f(x) = (-x_1^3 + 3x_1, x_1^3 + x_1)^{\mathrm{T}}$. Thus, we have that $\Pi_K(x - f(x)) = (2x_1, 2x_1)$, for any $x \in K$,

$$\|f(x) - f^{\infty}(x)\| = \|f_K^{\mathrm{nat}}(x)\| = \|x\|$$

and

$$\lim_{x \in K, \|x\| \to \infty} \|f_K^{\mathrm{nat}}(x)\| = \lim_{x \in K, \|x\| \to \infty} \| - x\| = \infty.$$

Thus, the norm coercivity condition $\lim_{x \in K, \|x\| \to \infty} \|f_K^{\mathrm{nat}}(x)\| = \infty$ and all conditions of Theorem 3.1 (b) are satisfied, which means that $\mathrm{SOL}(f, K)$ is nonempty and compact.

Second, we show the coercivity condition of Theorem 4.3 does not hold. For any $x^{\mathrm{ref}} \in K$, we have that $x_1^{\mathrm{ref}} = x_2^{\mathrm{ref}}$. Then for any $x \in K$, i.e., $x_1 = x_2$, we have

$$\langle f(x), x - x^{\mathrm{ref}} \rangle = -2x_1^2 - 2x_1 x_1^{\mathrm{ref}} \to -\infty \quad \text{as } \|x\| \to \infty.$$

Thus, we cannot find some $x^{\mathrm{ref}} \in K$, $c > 0$ and $\xi \geq 0$ such that $\langle f(x), x - x^{\mathrm{ref}} \rangle \geq c\|x\|^{\xi}$ for any $x \in K$ satisfying $\|x\| \to \infty$, which implies that f is not coercive on K.

Recall that for CPs, a related existence result which requires the norm-coercivity of the natural map is as follows:

Theorem 4.4 [3]. *Let K be a closed convex cone in $\mathbb{R}^n$ and $f : K \to \mathbb{R}^n$ be a continuous map. Suppose that for any $x \in K$, $\lim_{\|x\|\to\infty} \|f_K^{\mathrm{nat}}(x)\| = \infty$ and $\langle x, f(x) - f(0)\rangle \geq 0$, then* $\mathrm{CP}(f, K)$ *has a nonempty, compact solution set.*

Actually, the condition that "$\langle x, f(x) - f(0)\rangle \geq 0$" implies that f is a q-copositive map on cone K with $q = f(0)$. From [15, Theorem 5.1], it follows that if f is a q-copositive map on cone K, then f^{∞} is copositive on cone K. Furthermore, when K is a closed convex cone and the involved map $f(x) = f^{\infty}(x) + p$ with p being a vector in $\mathbb{H}$, $\|f(x) - f^{\infty}(x)\| = \|p\|$ is a constant, which implies that Theorem 3.1 (b) together with the norm-coercivity condition "$\lim_{\|x\|\to\infty} \|f_K^{\mathrm{nat}}(x)\| = \infty$" coincides with Theorem 4.4 in this case. However, for the case where $f - f^{\infty}$ is not a constant vector in $\mathbb{H}$, Theorem 3.1 is different from Theorem 4.4, which can be illustrated from the following example, where $\mathrm{WHCP}(f, K)$ satisfies all the conditions in Theorem 3.1 (b) and the norm-coercivity condition "$\lim_{\|x\|\to\infty} \|f_K^{\mathrm{nat}}(x)\| = \infty$", but it does not satisfy the conditions of Theorem 4.4.

Example 4.8. Consider $\mathrm{WHCP}(f, K)$ where $K = \{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 \leq 0\}$, and for any $x \in \mathbb{R}^2$,

$$f(x) = \left(x_1^2 + x_2^2 - \sqrt[4]{x_1^2} + 1, -(x_1^2 + x_2^2) - \sqrt[4]{x_2^2} + 2\right)^{\mathrm{T}}.$$

Obviously, K is a convex cone, and f is weakly homogeneous of degree 2.

- For any $x \in K$, we have that $\langle f^{\infty}(x), x\rangle = (x_1^2 + x_2^2)(x_1 - x_2) \geq 0$, thus f^{∞} is copositive on K.
- For any $x \in K$ with $\|x\| \to \infty$, since $-x_2 \geq 0$ and $-\sqrt{-x_2} \leq 0$, it is easy to see that there exists no $c > 0$ such that

$$-x = c(f(x) - f^{\infty}(x)) = c(-\sqrt{x_1} + 1, -\sqrt{-x_2} + 2)^{\mathrm{T}}.$$

- For any $x \in K$, as $\|x\| \to \infty$, it follows that

$$\begin{cases} (x - f(x))_1 = x_1 - (x_1^2 + x_2^2) + \sqrt{x_1} - 1 \to -\infty, \\ (x - f(x))_2 = x_2 + (x_1^2 + x_2^2) + \sqrt{-x_2} - 2 \to +\infty. \end{cases}$$

So, $\Pi_K(x - f(x)) = 0$,

$$\lim_{x\in K, \|x\|\to\infty} \|f_K^{\mathrm{nat}}(x)\| = \lim_{x\in K, \|x\|\to\infty} \|x - \Pi_K(x - f(x))\| = \infty$$

and

$$\lim_{x\in K,\|x\|\to\infty} \frac{\|f_K^{\text{nat}}(x)\|}{\|f(x)-f^\infty(x)\|}$$
$$= \lim_{x\in K,\|x\|\to\infty} \frac{\sqrt{x_1^2+x_2^2}}{\sqrt{(1-\sqrt{x_1})^2+(-\sqrt{-x_2}+2)^2}} = \infty.$$

Thus, the norm coercivity condition $\lim_{x\in K,\|x\|\to\infty} \|f_K^{\text{nat}}(x)\| = \infty$ and all conditions of Theorem 3.1 (b) are satisfied, which means that $\text{SOL}(f,K)$ is nonempty and compact.

By taking $x \in K$ with $x_1 = \frac{1}{4}$ and $x_2 = 0$, we have that $\langle x, f(x) - f(0)\rangle = \frac{1}{64} - \frac{1}{8} < 0$, so the conditions in Theorem 4.4 do not hold.

5 Conclusions

In this paper, we established an existence result for $\text{WHVI}(f,K)$ under the copositivity of f^∞ and some additional easy-verified conditions. Some examples were constructed to demonstrate that our conditions cannot be deduced by the existing ones, especially the wide degree-theoretic theorem given for WHVIs in the main result of [5], the main result in [15] and the well-known coercivity result given for general VIs in [3]. Our result also provided a supplement to the existence theory for those subclasses of WHVIs.

Besides, for the case where K is a set with $0 \notin K$, whether can easy-verified conditions be found to guarantee the existence of solutions for $\text{WHVI}(f,K)$ or not? It is an interesting issue which deserves further study.

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