Front. Math. 2023, 18(4): 751–795
https://doi.org/10.1007/s11464-021-0209-1

RESEARCH ARTICLE

Very Regular Solution to Landau–Lifshitz–Gilbert System with Spin-polarized Transport

Bo CHEN[1,3,4], **Youde WANG**[2,4,5]

1 Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China
2 School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China
3 Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
4 School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
5 Hua Loo-Keng Key Laboratory of Mathematics, Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

Abstract In this paper, we provide a precise description of the compatibility conditions for the initial data so that one can show the existence and uniqueness of regular short-time solution to the Neumann initial-boundary problem of a class of Landau–Lifshitz–Gilbert system with spin-polarized transport, which is a strong nonlinear coupled parabolic system with non-local energy.

Keywords Landau–Lifshitz system with spin-polarized transport, very regular solution, compatibility conditions of the initial data, Galerkin approximation method, auxiliary approximation equation

MSC2020 35K59, 35Q60, 78A25

1 Introduction

1.1 Background

In physics, the Landau–Lifshtiz (LL) equation is a fundamental evolution equation for the ferromagnetic spin chain and was proposed on the phenomenological background in studying the dispersive theory of magnetization of ferromagnets. It was first deduced by Landau and Lifshitz in [20], and then proposed by Gilbert in [15] with dissipation. In fact, this equation describes the Hamiltonian dynamics corresponding to the Landau–Lifshitz energy, which is defined as follows.

Received May 23, 2021; accepted June 16, 2021
Corresponding author: Youde WANG, E-mail: wyd@math.ac.cn

Let Ω be a smooth bounded domain in the Euclidean space $\mathbb{R}^3$, whose coordinates are denoted by $x = (x^1, x^2, x^3)$. We assume that a ferromagnetic material occupies the domain $\Omega \subset \mathbb{R}^3$. Let u, denoting magnetization vector, be a mapping from Ω into a unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. The Landau–Lifshitz energy of map u is defined by

$$\mathscr{E}(u) := \int_\Omega \Phi(u)\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{2}\int_\Omega h_d \cdot u\,dx.$$

Here ∇ denotes the gradient operator and dx is the volume element of $\mathbb{R}^3$.

In the above Landau–Lifshitz functional, the first and second terms are the anisotropy and exchange energies, respectively. $\Phi(u)$ is a real function on $\mathbb{S}^2$. The last term is the self-induced energy, and $h_d = -\nabla w$ is the demagnetizing field. In fact, the magnetostatic potential w has precise formula

$$w(x) = \int_\Omega \nabla N(x-y)u(y)dy,$$

where $N(x) = -\frac{1}{4\pi|x|}$ is the Newtonian potential in $\mathbb{R}^3$.

The LL equation with dissipation can be written as

$$u_t - \alpha u \times u_t = -u \times h,$$

where "$\times$" denotes the cross production in $\mathbb{R}^3$ and the local field h of $\mathscr{E}(u)$ can be derived as

$$h := -\frac{\delta\mathscr{E}(u)}{\delta u} = \Delta u + h_d - \nabla_u \Phi.$$

Here, the constant α is the damping parameter, which is characteristic of the material, and is usually called the Gilbert damping coefficient. Hence the Landau–Lifshitz equation with damping term is also called the Landau–Lifshitz–Gilbert (LLG) equation in the literature.

On the other hand, another new physical model for the spin-magnetization system, which takes into account the diffusion process of the spin accumulation through the multilayer, has attracted considerable attention of many mathematicians. Especially, it was originally presented by Zhang et al. [25, 33], and later be extended by [14] to three dimensions of the model for spin-polarized transport. In this paper, we call the model as the Landau–Lifschitz–Gilbert equation with spin-polarized transport (LLGSP), which is given in below

$$\begin{cases} \partial_t u - \alpha u \times \partial_t u = -u \times (h+s), & (x,t) \in \Omega \times \mathbb{R}^+, \\ \partial_t s = -\mathrm{div} J_s - D_0(x)s - D_0(x)s \times u, & (x,t) \in \Omega_0 \times \mathbb{R}^+, \end{cases} \tag{1.1}$$

with initial-boundary condition

$$\begin{cases} u(\cdot,0) = u_0 : \Omega \to \mathbb{S}^2, & \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0, \\ s(\cdot,0) = s_0 : \Omega_0 \to \mathbb{R}^3, & \left.\dfrac{\partial s}{\partial \nu}\right|_{\partial\Omega_0} = 0. \end{cases} \tag{1.2}$$

Here, $\Omega_0 \subset \mathbb{R}^3$ is a bounded domain and $\Omega \subset \Omega_0$, $u(x,t) \in \mathbb{S}^2$ is the magnetized field, $s(x,t) \in \mathbb{R}^3$ is the spin accumulation, $D_0(x)$ is a positive measurable function, which represents the diffusion coefficient of the spin accumulation, $\alpha > 0$ is the Gilbert damping parameter, and J_s is the spin current given by

$$J_s = u \otimes J_e - D_0\{\nabla s - \theta u \otimes (\nabla s \cdot u)\} = u \otimes J_e + A(u)\nabla s,$$

where J_e is the applied electric current, $\theta \in (0,1)$ is the spin polarization parameter and the coefficient matrix $A(u)$ is expressed as

$$A(u) = -D_0 \begin{pmatrix} 1-\theta u_1^2 & -\theta u_1 u_2 & -\theta u_1 u_3 \\ -\theta u_2 u_1 & 1-\theta u_2^2 & -\theta u_2 u_3 \\ -\theta u_3 u_1 & -\theta u_3 u_2 & 1-\theta u_3^2 \end{pmatrix}.$$

The spin accumulation s is defined on Ω_0 and the magnetization u is defined on the magnetic domain Ω and extended as zero outside. The additional term in the LLG equation is induced by the interaction $F[u,s] = -\int_\Omega u \cdot s\, dx$.

However, we are concerned in this paper with the existence of regular solution to system (1.1), so it is natural to assume that $\Omega = \Omega_0$ and the boundary of Ω is smooth. For simplicity, we also assume that Φ is a smooth function on $\mathbb{S}^2$.

First, we note a fact that for $u : \Omega \times \mathbb{R}^+ \to \mathbb{S}^2$, the first equation of system (1.1) is equivalent to

$$\partial_t u = \frac{\alpha}{\alpha^2+1}(\Delta u + |\nabla u|^2 u - u \times (u \times (\tilde{h} + s))) - \frac{1}{1+\alpha^2} u \times (h+s),$$

where

$$\tilde{h} = h_d(u) - \nabla_u \Phi.$$

Without loss of generality, we consider the following equivalent system

$$\begin{cases} \partial_t u = \alpha(\Delta u + |\nabla u|^2 u - u \times (u \times (\tilde{h}+s))) - u \times (h+s), & (x,t) \in \Omega \times \mathbb{R}^+, \\ \partial_t s = -\mathrm{div}(A(u)\nabla s + u \otimes J_e) - D_0(x)s - D_0(x)s \times u, & (x,t) \in \Omega \times \mathbb{R}^+, \end{cases} \tag{1.3}$$

with the initial-boundary condition (1.2) and $\alpha > 0$.

1.2 Related Work

The LL equation is an important topic in both mathematics and physics, not only because it is a hybrid of the heat flow of harmonic maps and the Schrödinger flow on the sphere, but also because it has concrete physics background in the study of the magnetization in ferromagnets. In recent years, there has been tremendous interest in developing the well-posedness of LL equation and its related topics [9–11, 14, 18, 19, 21, 24, 28, 30]. Here, we list only a few of results that are closely related to our work in the present paper.

First we recall some results on the weak solutions to the LL equation. The existence of weak solutions to LLG equation, also in the presence of magnetostrictive effects, was established by Visintin [29] in 1985. P.-L. Sulem, C. Sulem and C. Bardos in [27] employed difference method to prove that the LL equation without dissipation term (that is Schrödinger flow for maps into $\mathbb{S}^2$) defined on $\mathbb{R}^n$ admits a global weak solution and a smooth local solution. For a bounded domain $\Omega \subset \mathbb{R}^3$, Alouges and Soyeur [1] showed the nonuniqueness of weak solutions to LLG equation.

Later, Y. D. Wang [31] obtained the existence of weak solution to Schrödinger flow for maps from a closed Riemannian manifold into $\mathbb{S}^2$ by adopting a more effective approximation equation than the Ginzburg–Landau penalized equation used in [1], Galerkin method and then choosing suitable test functions to derive a priori estimates of L^∞ on the approximate solutions. Tilioua [28] (also see [6]) also used the penalized method to show the global weak solution to the LLG equation with spin-polarized current. Recently, Z. L. Jia and Y. D. Wang [18] (also see [10]) employed a method originated from [31] to achieve the global weak solutions to a large class of LL flows in more general setting, where the base manifold is a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ or a compact Riemannian manifold M^n and the target space is $\mathbb{S}^2$ or the unit sphere $\mathbb{S}^n_{\mathfrak{g}}$ in a compact Lie algebra $\mathfrak{g}$.

The first global well-posedness result for LL equation on $\mathbb{R}^n$ in critical spaces (precisely, global well-posedness for small data in the critical Besov spaces in dimensions $n \geq 3$) was proved by Ionescu and Kenig in [17], and independently by Bejanaru in [2]. This was later improved to global regularity for small data in the critical Sobolev spaces in dimensions $n \geq 4$ in [4]. Finally, in [5] the global well-posedness result for LL equation for small data in the critical Sobolev spaces in dimensions $n \geq 2$ was addressed. In $\mathbb{R}^n$ with $n \geq 3$, Melcher [21] proved the existence, uniqueness and asymptotics of global smooth solutions for the LLG equation, valid under a smallness condition of initial gradients in the L^n norm. His argument is based on deriving a covariant complex Ginzburg–Landau equation and using the Coulomb gauge.

Next, we retrospect some of the work related to local regular solutions of LL equation on bounded domains or compact manifolds. For the case LL equation without Gilbert damping term, one has shown the existence of local smooth solutions, and we refer to [3, 12, 13, 22, 23, 27, 34]. For the case Ω is a bounded domain in $\mathbb{R}^3$, Carbou and Fabrie studied a nonlinear dissipative LL equation (i.e., $\alpha > 0$) coupled with Maxwell equations in micromagnetism theory, and they proved the local existence and uniqueness of regular solutions for a so-called quasistatic model in [8]. Moreover, they showed global existence of regular solutions for small data in the 2D case for the LL equation (also see [16]). Recently, the local existence of very regular solution to LLG equation with electric current was addressed by applying the delicate Galerkin approximation method and adding compatibility initial-boundary condition in [9].

For the spin-magnetization system (1.1) that takes into account the diffusion process of the accumulation, García-Cervera and X. P. Wang [14] adopted Galerkin approximation and projection method to obtain the existence of weak solution. By using more refined harmonic analysis, X. K. Pu and W. D. Wang established the global existence and uniqueness of weak solutions to the simplified system (1.1) from $\mathbb{R}^2$ into $\mathbb{S}^2$ for large initial data in their paper [24], where the partial regularity was shown. Recently, Z. L. Jia and Y. D. Wang [19] employed a suitable auxiliary approximation equation and then took the Galerkin approximation method for the auxiliary equation as in [31] to get the global weak solution of (1.1)–(1.2). In particular, they also got the existence of weak solution to (1.1)–(1.2) without damping term (i.e., $\alpha = 0$, the coupling system of Schrödinger flow and diffusion equation). It seems that there are few results on its regular solutions to this coupling system in the literature.

1.3 Main Results and Strategy

Inspired by the method used in [9], we show the locally very regular solution of LLGSP when the underlying space Ω is a smooth bounded domain in $\mathbb{R}^3$. Our main result can be stated as following theorems.

Theorem 1.1. *Let $u_0 \in H^2(\Omega, \mathbb{S}^2)$ and $s_0 \in H^2(\Omega, \mathbb{R}^3)$ satisfy the compatibility condition:*

$$\begin{cases} \left.\dfrac{\partial u_0}{\partial \nu}\right|_{\partial\Omega} = 0, \\ \left.\dfrac{\partial s_0}{\partial \nu}\right|_{\partial\Omega} = 0. \end{cases}$$

Suppose that $D_0 \in C^2(\bar{\Omega})$ and $D_0(x) \geq c_0 > 0$ for some constant c_0, $0 < \theta < 1$ and $J_e \in L^\infty(\mathbb{R}^+, H^2(\Omega))$. Then there exists $T^* > 0$ depending only on the H^2-norm of (u_0, s_0) such that (1.2)–(1.3) admits a unique local solution (u, s) for any $T < T^*$, which satisfies

1. $|u|(x,t) = 1$ *in* $[0,T] \times \Omega$,
2. $(u, s) \in L^\infty([0,T], H^2(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^3(\Omega, \mathbb{R}^3))$.

Without additional compatibility condition, we can get a more regular solution to (1.3) when $(u_0, s_0) \in H^3(\Omega, \mathbb{R}^3)$.

Theorem 1.2. *Let $u_0 \in H^3(\Omega, \mathbb{S}^2)$ and $s_0 \in H^3(\Omega, \mathbb{R}^3)$ satisfy the same compatibility condition as in Theorem 1.1. Suppose that $D_0 \in C^3(\bar{\Omega})$ and $D_0 \geq c_0 > 0$, $0 < \theta < 1$, $J_e \in C^0(\mathbb{R}^+, H^2(\Omega))$ and $\partial_t J_e \in L^2(\mathbb{R}^+, H^1(\Omega))$. If (u, s) and T^* are respectively the solution and the existence time given in Theorem 1.1, then, for any $T < T^*$, there holds*

$$(u, s) \in L^\infty([0,T], H^3(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^4(\Omega, \mathbb{R}^3)).$$

In general, we can show the very regular solution to (1.3) under adding high order compatibility conditions.

Theorem 1.3. *Let* $k \geq 4$, $u_0 \in H^k(\Omega, \mathbb{S}^2)$ *and* $s_0 \in H^k(\Omega, \mathbb{R}^3)$ *satisfy the compatibility condition at* $[\frac{k}{2}] - 1$ *order, which is given in Definition* 2.1*. Let* (u, s) *and* $T^* > 0$ *be the same as in Theorem* 1.1*. In addition, we assume that* $D_0 \in C^k(\bar{\Omega})$*,* $D_0 \geq c_0 > 0$*, and for any* $i \leq k - [\frac{k}{2}] - 1$ *there holds*

$$\partial_t^i J_e \in C^0(\mathbb{R}^+, H^{2[\frac{k}{2}]-2i}(\Omega, \mathbb{R}^3)) \cap L^2(\mathbb{R}^+, H^{2[\frac{k}{2}]-2i+1}(\Omega, \mathbb{R}^3)).$$

Then, for any $T < T^*$*, we have*

$$(u, s) \in L^\infty([0, T], H^k(\Omega, \mathbb{R}^3)) \cap L^2([0, T], H^{k+1}(\Omega, \mathbb{R}^3)).$$

Differing from the LLG equation considered in [9], the system LLGSP is strictly parabolic, only provided the constraint $|u|^2 < \frac{1}{\theta}$. However, this condition cannot be maintained by applying Galerkin approximation to the system (1.3). Thus, we choose a suitable auxiliary equation of spin equation with respect to s (see the second equation of (3.1)) to overcome this difficulty. And hence, Theorem 1.1 is achieved by applying Galerkin approximation to the modified equation (3.1) of (1.3) and estimating some suitable energies directly. However, we cannot improve the regularity of strong solution

$$(u, s) \in L^\infty([0, T], H^2(\Omega, \mathbb{R}^3)) \cap L^2([0, T], H^3(\Omega, \mathbb{R}^3))$$

obtained in Theorem 1.1 by getting higher order energy estimates, since the right hand sides of system (1.3) do not satisfy the homogeneous Neumann boundary condition. To proceed, following Carbou's idea in [9], we consider the differential of Galerkin approximation to system (1.3) with respect to time and prove by the same way that $(\partial_t u, \partial_t s) \in L^\infty([0, T], H^1(\Omega, \mathbb{R}^3)) \cap L^2([0, T], H^2(\Omega, \mathbb{R}^3))$, where $0 < T < T^*$. Thus, we can show Theorem 1.2 by a bootstrap argument using equation (1.1).

However, we cannot enhance the regularity of solution (u, s) in Theorem 1.2 by applying directly the second order differential of Galerkin approximation to system (1.3), because of the lower regularity of Galerkin projection, although the Galerkin projection is the crucial analytic tool in our argument (see Lemma 2.7). Generally, to improve the regularity of solution (u, s), we need to impose so-called compatibility conditions of initial data. Let $k \geq 1$. Considering the equation of $(\partial_t^k u, \partial_t^k s)$ with compatibility condition (2.4), the very regular solution to system (1.3) can be shown. More precisely, we use the simplest case to explain the strategy of enhancing regularity $\mathscr{P}$:

(1) Assume $(u_0, s_0) \in H^3(\Omega, \mathbb{R}^3)$. By considering the differential of the Galerkin approximation of (1.3) with respect to time t, we can show

$$(u, s) \in L^\infty([0, T], H^3(\Omega, \mathbb{R}^3)) \cap L^2([0, T], H^4(\Omega, \mathbb{R}^3)).$$

(2) Assuming $(u_0, s_0) \in H^4(\Omega, \mathbb{R}^3)$, we can get a regular solution

$$(u_1, s_1) \in L^\infty([0,T], H^2(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^3(\Omega, \mathbb{R}^3))$$

to equation (4.6) of $(\partial_t u, \partial_t s)$. For this step, we need the compatibility condition at the boundary for $(\partial_t u, \partial_t s)$ when $t = 0$.

(3) The uniqueness guarantees $(\partial_t u, \partial_t s) = (u_1, s_1)$. It implies

$$(u, s) \in L^\infty([0,T], H^4(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^5(\Omega, \mathbb{R}^3))$$

by a bootstrap argument using the equation (1.1) again.

(4) Assuming $(u_0, s_0) \in H^5(\Omega, \mathbb{R}^3)$, we obtain

$$(u, s) \in L^\infty([0,T], H^5(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^6(\Omega, \mathbb{R}^3))$$

by repeating the argument in (1) to $(\partial_t u, \partial_t s)$.

Here $0 < T < T^*$. For higher order regularity, to add higher order compatibility conditions we can prove Theorem 1.3 by repeating the above process $\mathscr{P}$.

The rest of our paper is organized as follows. In Section 2, we introduce some basic notations on Sobolev space and some preliminary lemmas used later. Meanwhile the compatibility condition of initial data to system (1.3) will also be given. In Section 3, we prove Theorem 1.1 by employing Galerkin approximation method. Theorem 1.2 and Theorem 1.3 will be built up in Subsections 4.1 and 4.2 respectively.

2 Preliminary

2.1 Notations

In this subsection, we first recall some notations on Sobolev spaces which will be used in whole context. Let $u = (u_1, u_2, u_3) : \Omega \to \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ be a map. We set

$$H^k(\Omega, \mathbb{S}^2) = \{u \in W^{k,2}(\Omega, \mathbb{R}^3) : |u| = 1 \text{ a.e. in } \Omega\}$$

and

$$W_2^{k,l}(\Omega \times [0,T], \mathbb{S}^2) = \{u \in W_2^{k,l}(\Omega \times [0,T], \mathbb{R}^3) : |u| = 1 \text{ a.e. in } \Omega \times [0,T]\}$$

for k, $l \in \mathbb{N}$, and denote $H^0(\Omega, \mathbb{R}^3) = L^2(\Omega, \mathbb{R}^3)$.

Moreover, let $(B, \|\cdot\|_B)$ be a Banach space and $f : [0,T] \to B$ be a map. For any $p > 0$ and $T > 0$, we define

$$\|f\|_{L^p([0,T],B)} := \left(\int_0^T \|f\|_B^p dt \right)^{\frac{1}{p}},$$

and set

$$L^p([0,T], B) := \{f : [0,T] \to B : \|f\|_{L^p([0,T],B)} < \infty\}.$$

In particular, we denote

$$L^p([0,T],H^k(\Omega,\mathbb{S}^2))=\{u\in L^p([0,T],H^k(\Omega,\mathbb{R}^3)):|u|=1\text{ a.e. in }\Omega\times[0,T]\},$$

where $k\in\mathbb{N}$ and $p\geq 1$.

2.2 Estimates on h_d and Some Lemmas

For later application, we recall some regular results. Let $u:\Omega\to\mathbb{R}^3$ be a map. Then, in the sense of distributions, the induced vector field is defined by

$$h_d(u):=\nabla\int_\Omega \nabla N(x-y)u(y)\,dy,$$

where $N(x)=-\frac{1}{4\pi|x|}$ is the Newton potential on $\mathbb{R}^3$. Hence, the following estimates of h_d is a fundamental result in theory of singular integral operators. Its proof can be found in [8–10].

Lemma 2.1. *Let $p\in(1,\infty)$ and Ω be a bounded smooth domain in $\mathbb{R}^3$. Assume that $u\in W^{k,p}(\Omega,\mathbb{R}^3)$ for $k\in\mathbb{N}$. Then, the restriction of $h_d(u)$ to Ω belongs to $W^{k,p}(\Omega,\mathbb{R}^3)$. Moreover, there exists a constant $C_{k,p}$, which is independent of u, such that*

$$\|h_d(u)\|_{W^{k,p}(\Omega)}\leq C_{k,p}\|u\|_{W^{k,p}(\Omega)}.$$

In fact, $h_d:W^{k,p}(\Omega,\mathbb{R}^3)\to W^{k,p}(\Omega,\mathbb{R}^3)$ is a linear bounded operator.

The L^2 theory of Laplace operator with Neumann boundary condition implies the following lemma of equivalent norm, see [10,32].

Lemma 2.2. *Let Ω be a bounded smooth domain in $\mathbb{R}^m$ and $k\in\mathbb{N}$. There exists a constant $C_{k,m}$ such that, for all $u\in H^{k+2}(\Omega)$ with $\frac{\partial u}{\partial\nu}|_{\partial\Omega}=0$,*

$$\|u\|_{H^{2+k}(\Omega)}\leq C_{k,m}(\|u\|_{L^2(\Omega)}+\|\Delta u\|_{H^k(\Omega)}). \tag{2.1}$$

In particular, we define the H^{k+2}-norm of u as follows,

$$\|u\|_{H^{k+2}(\Omega)}:=\|u\|_{L^2(\Omega)}+\|\Delta u\|_{H^k(\Omega)}.$$

We also need some basic lemmas on the space $H^m(\Omega)$ with $m\geq 1$ below (cf. [9]).

Lemma 2.3. *Assume Ω be a bounded smooth domain in $\mathbb{R}^3$. Let $f\in H^1(\Omega)$ and $g\in H^m(\Omega)$ with $m\geq 2$. Then $f\cdot g\in H^1(\Omega)$.*

Applying the fact $H^2(\Omega)\subset W^{1,6}(\Omega)\subset L^\infty(\Omega)$, the following result can be obtained from the above lemma directly.

Lemma 2.4. *Assume Ω be a bounded smooth domain in $\mathbb{R}^3$. Let f and g in $H^m(\Omega)$ with $m\geq 2$, there holds*

$$f\cdot g\in H^m(\Omega).$$

In fact, $(H^m(\Omega),\cdot)$ is an algebra.

2.3 Comparison Theorem for ODE and Aubin–Simon's Compactness

In order to show the uniform estimates and the convergence of solutions to the approximated equation constructed in coming sections, we need to use the comparison theorem for ordinary differential equation (ODE) and the classical compactness result in [26].

Lemma 2.5. *Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a continuous function, which is locally Lipschitz on the second variable. Let $z : [0, T^*) \to \mathbb{R}$ be the maximal solution of the Cauchy problem:*

$$\begin{cases} z' = f(t, z), \\ z(0) = z_0. \end{cases}$$

Let $y : \mathbb{R}^+ \to \mathbb{R}$ be a C^1 function such that

$$\begin{cases} y' \leq f(t, y), \\ y(0) \leq z_0. \end{cases}$$

Then,

$$y(t) \leq z(t), \quad t \in [0, T^*).$$

Lemma 2.6 (Aubin–Simon Compactness Lemma)**.** *Let $X \subset B \subset Y$ be Banach spaces with compact embedding $X \hookrightarrow B$, F be a bounded set in $L^q([0,T], B)$ for $q > 1$. If F is bounded in $L^1([0,T], X)$ and $\frac{\partial F}{\partial t}$ is bounded in $L^1([0,T], Y)$, where $\frac{\partial F}{\partial t} = \{\frac{\partial f}{\partial t} : f \in F\}$, then F is relatively compact in $L^p([0,T], B)$, for any $1 \leq p < q$.*

2.4 Galerkin Basis and Galerkin Projection

Let Ω be a bounded smooth domain in $\mathbb{R}^m$, λ_i be the i^{th} eigenvalue of the operator $\Delta - I$ with Neumann boundary condition, whose corresponding eigenfunction is f_i. That is,

$$(\Delta - I) f_i = -\lambda_i f_i \quad \text{with } \left.\frac{\partial f_i}{\partial \nu}\right|_{\partial\Omega} = 0.$$

Without loss of generality, we assume $\{f_i\}_{i=1}^{\infty}$ are completely standard orthonormal basis of $L^2(\Omega, \mathbb{R}^1)$. Let $H_n = \text{span}\{f_1, \ldots, f_n\}$ be a finite subspace of L^2, $P_n : L^2(\Omega, \mathbb{R}^1) \to H_n$ be the canonical projection. In fact, for any $f \in L^2$,

$$f^n = P_n f = \sum_{i=1}^{n} \langle f, f_i \rangle_{L^2} f_i \quad \text{and} \quad \lim_{n \to \infty} \|f - f^n\|_{L^2} = 0.$$

For this canonical projection, we have the following uniform estimates, which is essential for our method to get very regularity of solution in Section 4. Its proof can be found in [9].

Lemma 2.7. *There exists a constant C such that for all n, the projection P_n satisfies the following properties:*

1. *For all $f \in H^1(\Omega, \mathbb{R}^1)$, $\|P_n(f)\|_{H^1(\Omega)} \le \|f\|_{H^1(\Omega)}$.*
2. *For all $f \in H^2(\Omega, \mathbb{R}^1)$ such that $\frac{\partial f}{\partial \nu}|_{\partial\Omega} = 0$, $\|P_n(f)\|_{H^2(\Omega)} \le C\|f\|_{H^2(\Omega)}$.*
3. *For all $f \in H^3(\Omega, \mathbb{R}^1)$ such that $\frac{\partial f}{\partial \nu}|_{\partial\Omega} = 0$, $\|P_n(f)\|_{H^3(\Omega)} \le C\|f\|_{H^3(\Omega)}$.*

Remark 2.1. Unfortunately, we cannot get such estimates for $f \in H^m(\Omega, \mathbb{R}^1)$ such that $\frac{\partial f}{\partial \nu}|_{\partial\Omega} = 0$, when $m \ge 4$.

2.5 Compatibility Conditions of the Initial Data

In order to get higher regularity of the solution to system (1.3) in Theorem 1.3, we now turn to define the compatibility conditions on the initial data. To clearly clarify the ideas of deriving compatibility conditions from equation (1.3), first of all we assume that (u, s) is a smooth solution.

Let $(u_k, s_k) = (\partial_t^k u, \partial_t^k s)$ with $k \ge 1$. Then a tedious but direct calculation shows (u_k, s_k) satisfies the following equation

$$\begin{cases} \partial_t u_k = \alpha\Delta u_k - u \times \Delta u_k + K_k(\nabla u_k, \nabla s_k) + L_k(u_k, s_k) + F_k(u, s), \\ \partial_t s_k = -\mathrm{div}(A(u)\nabla s_k) + Q_k(\nabla u_k, \nabla s_k) + T_k(u_k, s_k) + Z_k(u, s), \end{cases} \tag{2.2}$$

where

$$\begin{aligned}
&K_k(\nabla u_k, \nabla s_k) = 2\alpha(\nabla u_k \cdot \nabla u)u,\\
&Q_k(\nabla u_k, \nabla s_k) = -\mathrm{div}(u_k \otimes J_e),\\
&L_k(u_k, s_k) = \alpha|\nabla u|^2 u_k - \alpha u_k \times (u \times (\tilde{h}(u) + s)) - \alpha u \times (u_k \times (\tilde{h}(u) + s))\\
&\qquad\qquad - \alpha u \times \big(u \times (h_d(u_k) - \nabla^2\Phi(u) \cdot u_k + s_k)\big)\\
&\qquad\qquad - u \times \big(h_d(u_k) - \nabla^2\Phi(u) \cdot u_k + s_k\big) - u_k \times (h(u) + s),\\
&F_k(u, s) = \sum_{i+j+l=k,\, i,j,l<k} \nabla u_i \# \nabla u_j \# u_l + \sum_{i+j+l=k,\, i,j,l<k} u_i \# u_j \# (\bar{h}(u_l) + s_l)\\
&\qquad\qquad + \sum_{i+j=k,\, 0\le i,j<k} u_i \# (\bar{h}(u_j) + s_j) + \sum_{i+j=k,\, 0\le i,j<k} u_i \# \Delta u_j + R_k,\\
&T_k(u_k, s_k) = \theta\mathrm{div}(D_0 u_k \otimes (\nabla s \cdot u + D_0 u \otimes (\nabla s \cdot u_k)))\\
&\qquad\qquad - D_0 s_k - D_0 s_k \times u - D_0 s \times u_k,\\
&Z_k(u, s) = \mathrm{div}\Bigg(D_0\theta \sum_{i+j+l=k,\, i,j,l<k} u_i \# \nabla s_j \# u_l\Bigg)\\
&\qquad\qquad + \sum_{i+j=k,\, i,j<k} D_0 s_i \# u_j - \sum_{i+j=k,\, i<k} \mathrm{div}(u_i \# \partial_t^j J_e).
\end{aligned}$$

Here

$$\bar{h}(u_l) = h_d(u_l) - \nabla^2\Phi(u) \cdot u_l + \sum_{j_1+\cdots+j_i=l,\, i>1} \nabla_u^{i+1}\Phi(u) \# u_{j_1} \cdots \# u_{j_i},$$

$$R_k = u \times \Big(u \times \sum_{j_1+\cdots+j_i=k,\, i>1} \nabla_u^{i+1}\Phi(u)\#u_{j_1}\cdots\#u_{j_i}\Big) + u \times \sum_{j_1+\cdots+j_i=k,\, i>1} \nabla_u^{i+1}\Phi(u)\#u_{j_1}\cdots\#u_{j_i},$$

and # denotes the linear contraction.

Its initial data is

$$\begin{cases} V_k = u_k(x,0) = \alpha\Delta V_{k-1} - \alpha V_0 \times \Delta V_{k-1} + K_{k-1}(\nabla V_{k-1}, \nabla W_{k-1}) \\ \qquad\qquad\qquad + L_{k-1}(V_{k-1}, W_{k-1}) + F_{k-1}, \\ W_k = s_k(x,0) = -\mathrm{div}(A(V_0)\nabla W_{k-1}) + Q_{k-1}(\nabla V_{k-1}, \nabla W_{k-1}) \\ \qquad\qquad\qquad + T_{k-1}(V_{k-1}, W_{k-1}) + Z_{k-1}. \end{cases} \tag{2.3}$$

Here (u_l, s_l) has been replaced by (V_l, W_l) in the terms K_{k-1}, L_{k-1}, F_{k-1}, Q_{k-1}, T_{k-1} and Z_{k-1}, and $(V_0, W_0) = (u_0, s_0)$.

In particular, we have

$$\begin{cases} V_1 = \alpha\big(\Delta u_0 + |\nabla u_0|^2 u_0 - u_0 \times (u_0 \times (\tilde{h}(u_0) + s_0))\big) \\ \qquad - u_0 \times (\Delta u_0 + \tilde{h}(u_0) + s_0), \\ W_1 = -\mathrm{div} J_s(u_0, s_0) - D_0(x)\cdot s_0 - D_0(x)\cdot s_0 \times u_0. \end{cases}$$

Now, we are in the position to state the compatibility conditions on initial data (u_0, s_0), associated to equation (1.3), as follows.

Definition 2.1. Let $k \in \mathbb{N}$, $(u_0, s_0) \in H^{2k+1}(\Omega, \mathbb{R}^3)$ and $\partial_t^i J_e(x,0) \in H^1(\Omega, \mathbb{R}^3)$ for $0 \le i \le k$. We say (u_0, s_0) satisfies the compatibility condition at order k, if for any $j \in \{0, 1, \ldots, k\}$, there holds

$$\begin{cases} \dfrac{\partial V_j}{\partial \nu}\Big|_{\partial\Omega} = 0, \\ \dfrac{\partial W_j}{\partial \nu}\Big|_{\partial\Omega} = 0. \end{cases} \tag{2.4}$$

Intrinsically, we denote

$$\tau_{\Phi,s}(u) = \tau(u) - u \times (u \times (\tilde{h} + s)),$$

where $\tau(u) = \Delta u + |\nabla u|^2 u$ is the tension field. Then the first equation in (1.3) becomes

$$\partial_t u = \alpha\tau_{\Phi,s}(u) - u \times \tau_{\Phi,s}(u).$$

And hence, after taking k times derivatives at direction t for the above equation, u_k satisfies the following equation

$$\partial_t u_k = \partial_t^k \partial_t u = \alpha\partial_t^k \tau_{\Phi,s}(u) - \sum_{i+j=k} C_k^i \partial_t^i u \times \partial_t^j \tau_{\Phi,s}(u),$$

where $C_k^i = \frac{i!}{k!(k-i)!}$. When $k \geq 1$, letting $\tilde{V}_k = \partial_t^{k-1}\tau_{\Phi,s}(u)(x,0)$, there holds

$$V_k = \alpha\tilde{V}_k - u_0 \times \tilde{V}_k + \sum_{i+j=k, i\geq 1} C_k^i V_i \times \tilde{V}_j,$$

for the sake of convenience, where we denote $\tilde{V}_0 = V_0 = u_0$. In particular, there holds

$$\tilde{V}_1 = \tau(u_0) - u_0 \times (u_0 \times (\tilde{h}(u_0) + s_0)).$$

Therefore, it is not difficult to show that the k-order compatibility condition defined by Definition 2.1 has the below equivalent characterization.

Proposition 2.1. *Let $k \in \mathbb{N}$, $(u_0, s_0) \in H^{2k+1}(\Omega, \mathbb{R}^3)$ and $\partial_t^i J_e(x, 0) \in H^1(\Omega, \mathbb{R}^3)$ for $0 \leq i \leq k$. (u_0, s_0) satisfies the compatibility condition at order k if and only if for any $j \in \{0, 1, \ldots, k\}$, there holds*

$$\begin{cases} \left.\dfrac{\partial\tilde{V}_j}{\partial\nu}\right|_{\partial\Omega} = 0, \\ \left.\dfrac{\partial W_j}{\partial\nu}\right|_{\partial\Omega} = 0. \end{cases} \tag{2.5}$$

3 Regular Solution

In this section, we consider the existence of short-time regular solution to (1.3). To this end, we adopt the following equivalent equation

$$\begin{cases} \partial_t u = \alpha\big(\Delta u + |\nabla u|^2 u - u \times (u \times (\tilde{h} + s))\big) - u \times (h + s), & (x,t) \in \Omega \times \mathbb{R}^+, \\ \partial_t s = -\mathrm{div}(A(\mathfrak{J}(u))\nabla s + u \otimes J_e) - D_0(x)s - D_0(x)s \times u, & (x,t) \in \Omega \times \mathbb{R}^+, \end{cases} \tag{3.1}$$

with the initial-boundary condition (1.2) and $\alpha > 0$. Here,

$$-A(u) = D_0 \begin{pmatrix} 1-\theta u_1^2 & -\theta u_1 u_2 & -\theta u_1 u_3 \\ -\theta u_2 u_1 & 1-\theta u_2^2 & -\theta u_2 u_3 \\ -\theta u_3 u_1 & -\theta u_3 u_2 & 1-\theta u_3^2 \end{pmatrix}$$

is a positively definite matrix of functions with

$$0 < (1-\theta|u|^2)D_0|\xi|^2 \leq -\xi^T A(u)\xi \leq D_0|\xi|^2$$

for any vector ξ in $\mathbb{R}^3$ if $\theta|u|^2 < 1$,

$$\mathfrak{J}(u) = \frac{\sqrt{1+\delta}}{\sqrt{\delta + |u|^2}} u$$

with $\delta > 0$ to be determined later, and

$$\tilde{h}(u) = h_d(u) - \nabla_u \Phi(\mathfrak{J}(u)).$$

It should also be pointed out that the above $\Phi(u)$ has been extended to $\overline{B}_{\sqrt{1+\delta}}(0) \subset \mathbb{R}^3$. In fact, we extend $\Phi(z)$ by

$$\tilde{\Phi}(z) = \begin{cases} \zeta(|z|^2)\Phi\left(\dfrac{z}{|z|}\right), & |z|^2 > \delta_0, \\ 0, & |z|^2 \le \delta_0, \end{cases}$$

where $\zeta(t) : [0,2] \to [0,1]$ is a C^∞-smooth function with $\zeta(t) \equiv 0$ on $[0, 2\delta_0]$ $(2\delta_0 < 1)$ and $\zeta(t) = 1$ on $[1,2]$. It is easy to see that $\tilde{\Phi}$ is C^∞-smooth on $\overline{B}_{\sqrt{1+\delta}}(0)$. For the sake of simplicity, we still denote $\tilde{\Phi}$ by Φ.

3.1 Galerkin Approximation and A Priori Estimates

Let H_n be the n-dimensional subspace of $L^2(\Omega)$ defined in Subsection 2.4, P_n be the Galerkin projection. Next, we seek a solution (u^n, s^n) in H_n to the following Galerkin approximation equation associated to (3.1), i.e.,

$$\begin{cases} \partial_t u^n = P_n(\alpha(\Delta u^n + |\nabla u^n|^2 u^n - u^n \times (u^n \times (\tilde{h} + s^n))) - u^n \times (h + s^n)), \\ \partial_t s^n = P_n\left(-\mathrm{div}(A(\mathfrak{J}(u^n))\nabla s^n + u^n \times J_e) - D_0(x)s^n - D_0(x)s^n \times u^n\right) \end{cases} \tag{3.2}$$

with initial data $(u^n(\cdot,0), s^n(\cdot,0)) = (u_0^n, s_0^n)$.

Let $u^n = \sum_1^n g_i^n(t) f_i(x)$ and $s^n = \sum_1^n \gamma_i^n(t) f_i(x)$. For the sake of convenience, we denote $G^n(t) = \{g_1^n(t) \cdots g_n^n(t), \gamma_1^n(t)\gamma_n^n(t)\}$ as a vector-valued function. Then, a direct calculation shows that $G^n(t)$ satisfies the following ordinary differential equation

$$\begin{cases} \dfrac{\partial G^n}{\partial t} = F(t, G^n), \\ G^n(0) = (\langle u_0, f_1\rangle, \ldots, \langle u_0, f_n\rangle, \langle s_0, f_1\rangle, \ldots, \langle s_0, f_n\rangle), \end{cases} \tag{3.3}$$

where $F(G^n)$ is locally Lipschitz continuous with respect to G^n, since $\mathfrak{J}(f)$ is locally Lipschitz on f. Hence, there exists a solution (u^n, s^n) to (3.2) on $\Omega \times [0, T_0^n)$ for some $T_0^n > 0$.

If we choose (u^n, s^n) as a test function to multiply the two sides of equation (3.2), then it is easy to see that there hold

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |u^n|^2 dx + \alpha \int_\Omega |\nabla u^n|^2 dx = \alpha \int_\Omega |\nabla u^n|^2 |u^n|^2 dx \tag{3.4}$$

and

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |s^n|^2 dx + (1 - \theta(1+\delta))\int_\Omega D_0 |\nabla s^n|^2 dx \\ &\quad \le \int_\Omega \langle u^n \otimes J_e, \nabla s^n\rangle\, dx - \int_{\partial\Omega} u^n \cdot s^n \langle J_e, \nu\rangle\, d\mu_{\partial\Omega} \end{aligned}$$

$$
\begin{aligned}
&\leq C(\varepsilon, c_0)|u^n|^2_{L^\infty}\left(\int_\Omega |J_e|^2 dx + \int_{\partial\Omega} |J_e|^2 d\mu_{\partial\Omega}\right)\\
&\quad + \varepsilon\left(1+\frac{\tilde{C}}{c_0}\right)\int_\Omega D_0|\nabla s^n|^2 dx + \tilde{C}\varepsilon\int_\Omega |s^n|^2 dx. \qquad (3.5)
\end{aligned}
$$

Here, we have used the following fact

$$
\begin{aligned}
\int_\Omega \langle A(\mathfrak{J}(u^n))\nabla s^n, \nabla s^n\rangle\, dx &= \int_\Omega D_0|\nabla s^n|^2 dx - (1+\delta)\theta\int_\Omega D_0\frac{\langle\nabla s^n, u^n\rangle^2}{\delta+|u^n|^2}dx\\
&\geq (1-(1+\delta)\theta)\int_\Omega D_0|\nabla s^n|^2 dx,
\end{aligned}
$$

and the Trace Theorem to derive

$$
\begin{aligned}
\int_{\partial\Omega}|s^n|^2 dx &\leq \tilde{C}\left(\int_\Omega |\nabla s^n|^2 dx + \int_\Omega |s^n|^2 dx\right),\\
\int_{\partial\Omega}|J_e|^2 dx &\leq \tilde{C}\left(\int_\Omega |\nabla J_e|^2 dx + \int_\Omega |J_e|^2 dx\right).
\end{aligned}
$$

Therefore, by choosing $\delta < 1 - \frac{1}{\theta}$ and then suitable $\varepsilon > 0$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |u^n|^2 + |s^n|^2 dx + \alpha\int_\Omega |\nabla u^n|^2 dx + \frac{1}{2}(1-\theta(1+\delta))\int_\Omega D_0|\nabla s^n|^2 dx\\
&\quad\leq \alpha|u^n|^2_{L^\infty}\int_\Omega |\nabla u^n|^2 dx + C(\delta,\theta,c_0)\|J_e\|^2_{H^1}|u^n|^2_{L^\infty} + C(\delta,\theta,c_0)\int_\Omega |s^n|^2 dx\\
&\quad\leq C(\alpha,\delta,\theta,c_0)(1+\|J_e\|^2_{H^1})(U^2+U+S). \qquad (3.6)
\end{aligned}
$$

Here we denote $U = \|u^n\|^2_{H^2}$ and $S = \|s^n\|^2_{H^2}$.

In order to get the H^3-energy estimates, we choose $(v, w) = (\Delta^2 u^n, \Delta^2 s^n)$ as the test function. By using integration by parts, we take a simple computation to derive

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\Delta u^n|^2 dx + \alpha\int_\Omega |\nabla\Delta u^n|^2 dx\\
&= -\alpha\int_\Omega \langle\nabla(|\nabla u^n|^2 u^n), \nabla\Delta u^n\rangle dx \qquad (3.7)\\
&\quad + \alpha\int_\Omega \langle\nabla(u^n\times(u^n\times(\tilde{h}+s^n))), \nabla\Delta u^n\rangle dx\\
&\quad + \int_\Omega \langle\nabla(u^n\times(h+s^n)), \nabla\Delta u^n\rangle\, dx\\
&=: I + II + III \qquad (3.8)
\end{aligned}
$$

and

$$
\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\Delta s^n|^2 dx = -\int_\Omega \left\langle \operatorname{div}\left(A(\mathfrak{J}(u^n))\nabla s^n + u^n\otimes J_e\right), \Delta^2 s^n\right\rangle dx
$$

$$
\begin{aligned}
&+\int_\Omega \langle \nabla(D_0 s^n + D_0 s^n \times u^n), \nabla\Delta s^n\rangle\,dx \\
&=: IV + V.
\end{aligned} \tag{3.9}
$$

By direct calculations, we have

$$
\begin{aligned}
|I| &= \alpha\left|\int_\Omega \big\langle \nabla(|\nabla u^n|^2 u^n), \nabla\Delta u^n\big\rangle\,dx\right| \\
&\le 2\alpha\int_\Omega |\nabla^2 u^n||\nabla u^n||u^n||\nabla\Delta u^n|dx + 2\alpha\int_\Omega |\nabla u^n|^3|\nabla\Delta u^n|dx \\
&\le 2\alpha\|u^n\|_{L^\infty}\|\nabla u^n\|_{L^6}\|\nabla^2 u^n\|_{L^3}\|\nabla\Delta u^n\|_{L^2} + 2\alpha\|\nabla u^n\|_{L^6}^3\|\nabla\Delta u^n\|_{L^2} \\
&= 2\alpha(I_1 + I_2).
\end{aligned} \tag{3.10}
$$

For the above two terms I_1 and I_2, there hold

$$
\begin{aligned}
I_1 &= \|u^n\|_{L^\infty}\|\nabla u^n\|_{L^6}\|\nabla^2 u^n\|_{L^3}\|\nabla\Delta u^n\|_{L^2} \\
&\le C\|u^n\|_{H^2}^2\|\nabla^2 u\|_{L^2}^{\frac12}\|\nabla^2 u\|_{L^6}^{\frac12}\|\nabla\Delta u^n\|_{L^2} \\
&\le C\|u^n\|_{H^2}^{2+\frac12}\|u^n\|_{H^3}^{\frac12}\|\nabla\Delta u^n\|_{L^2} \\
&\le C\|u^n\|_{H^2}^3\|\nabla\Delta u^n\|_{L^2} + C\|u^n\|_{H^2}^{2+\frac12}\|\nabla\Delta u^n\|_{L^2}^{\frac32} \\
&\le \varepsilon\|\nabla\Delta u^n\|_{L^2}^2 + C(\varepsilon)(\|u^n\|_{H^2}^6 + \|u^n\|_{H^2}^{10}) \\
&\le \varepsilon\|\nabla\Delta u^n\|_{L^2}^2 + C(\varepsilon)(U^3 + U^5)
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \|\nabla u^n\|_{L^6}^3\|\nabla\Delta u^n\|_{L^2} \\
&\le C(\varepsilon)\|u^n\|_{H^2}^6 + \varepsilon\|\nabla\Delta u^n\|_{L^2}^2 \\
&\le C(\varepsilon)U^3 + \varepsilon\|\nabla\Delta u^n\|_{L^2}^2.
\end{aligned}
$$

Here we have used the following facts

$$
H^2(\Omega) \hookrightarrow W^{1,6}(\Omega) \hookrightarrow L^\infty(\Omega),
$$

and

$$
\|f\|_{L^3(\Omega)} \le \|f\|_{L^2(\Omega)}^{1/2}\|f\|_{L^6(\Omega)}^{1/2}.
$$

For the term II, there holds

$$
\begin{aligned}
|II| &\le \int_\Omega |\nabla u^n||u^n|(|h_d(u^n)| + |\nabla_u\Phi(\mathfrak{J}(u^n))| + |s^n|)|\nabla\Delta u^n|dx \\
&\quad + \int_\Omega |u^n|^2(|\nabla h_d(u^n)| + |\nabla_u^2\Phi(\mathfrak{J}(u^n))||\nabla\mathfrak{J}(u^n)| + |\nabla s^n|)|\nabla\Delta u^n|dx \\
&= II_1 + II_2.
\end{aligned} \tag{3.11}
$$

By direct calculations, we have

$$\begin{aligned}II_1 &\le \|u^n\|_{L^\infty}\|\nabla u^n\|_{L^6}(\|h_d(u^n)\|_{L^3}+\|\nabla_u\Phi(\mathfrak{J}(u^n))\|_{L^3}+\|s^n\|_{L^3})\|\nabla\Delta u^n\|_{L^2}\\ &\le C(\varepsilon,\Phi)\|u^n\|_{H^2}^4(\|u^n\|_{H^2}^2+\|s^n\|_{H^1}^2+1)+\varepsilon\|\nabla\Delta u^n\|_{L^2}^2\\ &\le C(\varepsilon,\Phi)U^2(U+S+1)+\varepsilon\|\nabla\Delta u^n\|_{L^2}^2\end{aligned}$$

and

$$\begin{aligned}II_2 &\le \|u^n\|_{L^\infty}^2(\|\nabla h_d(u^n)\|_{L^2}+\|\nabla_u^2\Phi(\mathfrak{J}(u^n))\|_{L^\infty}\sqrt{1+1/\delta}\|\nabla u^n\|_{L^2}\\ &\quad+\|\nabla s^n\|_{L^2})\|\nabla\Delta u^n\|_{L^2}\\ &\le C(\varepsilon,\delta,\Phi)\|u^n\|_{H^2}^4(\|u^n\|_{H^1}^2+\|s^n\|_{H^1}^2)+\varepsilon\|\nabla\Delta u^n\|_{L^2}^2\\ &\le C(\varepsilon,\delta,\Phi)U^2(U+S)+\varepsilon\|\nabla\Delta u^n\|_{L^2}^2.\end{aligned}$$

For the term III, we can show

$$\begin{aligned}|III| &\le \int_\Omega|\nabla u^n|(|\Delta u^n|+|h_d(u^n)|+|\nabla_u\Phi(\mathfrak{J}(u^n))|+|s^n|)|\nabla\Delta u^n|dx\\ &\quad+\int_\Omega|u^n|(|\nabla h_d(u^n)|+|\nabla^2\Phi(\mathfrak{J}(u^n))||\nabla u^n|+|\nabla s^n|)|\nabla\Delta u^n|dx\\ &\le C(\varepsilon,\delta,\Phi)(\|u^n\|_{H^2}^6+\|u^n\|_{H^2}^4+\|u^n\|_{H^2}^2\|s^n\|_{H^2}^2+1)+2\varepsilon\|\nabla\Delta u^n\|_{L^2}^2\\ &\le C(\varepsilon,\delta,\Phi)(U^3+U^2+US+1)+2\varepsilon\|\nabla\Delta u^n\|_{L^2}^2.\end{aligned}\tag{3.12}$$

Here, we have cancelled the term $\langle u\times\nabla\Delta u,\nabla\Delta u\rangle=0$.

By combining inequalities (3.10)–(3.12), and choosing suitable ε, we have

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega|\Delta u^n|^2dx+\frac{\alpha}{2}\int_\Omega|\nabla\Delta u^n|^2dx\le C(\delta,\alpha,\Phi)(U+S+1)^5.\tag{3.13}$$

On the other hand, for equation (3.9), we have

$$\begin{aligned}|IV| &= \int_\Omega\big\langle D_0\Delta s^n,\Delta^2s^n\big\rangle dx-\theta\int_\Omega\big\langle\nabla D_0\cdot(\mathfrak{J}(u^n)\otimes(\nabla s^n\cdot\mathfrak{J}(u^n))),\Delta^2s^n\big\rangle dx\\ &\quad+\int_\Omega\big\langle\nabla D_0\cdot\nabla s^n,\Delta^2s^n\big\rangle dx-\theta\int_\Omega\big\langle D_0\nabla\mathfrak{J}(u^n)\otimes(\nabla s^n\cdot\mathfrak{J}(u^n)),\Delta^2s^n\big\rangle dx\\ &\quad-\theta\int_\Omega\big\langle D_0\mathfrak{J}(u^n)\otimes(\Delta s^n\cdot\mathfrak{J}(u^n)),\Delta^2s^n\big\rangle dx-\int_\Omega\big\langle\nabla u^n\cdot J_e,\Delta^2s^n\big\rangle dx\\ &\quad-\theta\int_\Omega\big\langle D_0\mathfrak{J}(u^n)\otimes(\nabla s^n\cdot\nabla\mathfrak{J}(u^n)),\Delta^2s^n\big\rangle dx-\int_\Omega\big\langle u^n\mathrm{div}J_e,\Delta^2s^n\big\rangle dx\\ &=: IV_1+IV_2+IV_3+IV_4+IV_5+IV_6+IV_7+IV_8.\end{aligned}\tag{3.14}$$

By direct calculations, there hold

$$IV_1\le-\int_\Omega D_0|\nabla\Delta s^n|^2dx+C(\varepsilon,\theta,c_0)|\nabla D_0|_{L^\infty}^2S+\varepsilon\theta\int_\Omega D_0|\nabla\Delta s^n|^2dx,$$

$$
\begin{aligned}
IV_2 &\le C(\varepsilon,\delta,c_0)\theta(|\nabla^2 D_0|_{L^\infty}^2+|\nabla D_0|_{L^\infty}^2)(U+S+SU)+\varepsilon\theta\int_\Omega D_0|\nabla\Delta s^n|^2dx,\\
IV_3 &\le C(\varepsilon,\theta,c_0)(|\nabla^2 D_0|_{L^\infty}^2+|\nabla D_0|_{L^\infty}^2)S+\varepsilon\theta\int_\Omega D_0|\nabla\Delta s^n|^2dx,\\
IV_4 &\le \theta\int_\Omega |\nabla D_0||\nabla\mathfrak{J}(u^n)||\nabla s^n||\mathfrak{J}(u^n)||\nabla\Delta s^n|dx\\
&\quad+\theta\int_\Omega |D_0||\nabla^2\mathfrak{J}(u^n)||\nabla s^n||\mathfrak{J}(u^n)||\nabla\Delta s^n|dx\\
&\quad+\theta\int_\Omega D_0|\nabla\mathfrak{J}(u^n)||\Delta s^n||\mathfrak{J}(u^n)||\nabla\Delta s^n|dx\\
&\quad+\theta\int_\Omega D_0|\nabla\mathfrak{J}(u^n)|^2|\nabla s^n||\nabla\Delta s^n|dx\\
&\le C(\delta,\varepsilon,c_0)(|\nabla D_0|_{L^\infty}^2+|D_0|_{L^\infty})(US^2+US+U^2S)\\
&\quad+\varepsilon\theta\|\nabla\Delta u^n\|_{L^2}^2+\varepsilon\theta\int_\Omega D_0|\nabla\Delta s^n|^2dx,\\
IV_5 &\le C(\delta)\theta|\nabla D_0|_{L^\infty}\int_\Omega|\Delta s^n||\nabla\Delta s^n|dx+\theta\int_\Omega|\mathfrak{J}(u^n)\cdot\nabla\Delta s^n|^2dx\\
&\quad+\theta C(\delta)\int_\Omega D_0|\nabla u^n||\Delta s^n||\nabla\Delta s^n|dx\\
&\le(\varepsilon+1+\delta)\theta\int_\Omega D_0|\nabla\Delta s^n|^2dx+C(\delta,\varepsilon,c_0)\|D_0\|_{C^1}\theta(S+US+U^2S),\\
IV_6+IV_8 &\le C(\varepsilon,c_0)\theta\|J_e\|_{H^2}^2U+\varepsilon\theta\int_\Omega D_0|\nabla\Delta s^n|^2dx.
\end{aligned}\tag{3.15}
$$

In addition, IV_7 has the same estimates as IV_4.

Finally, we show the estimate of term V as follows,

$$
|V|\le C(\varepsilon,\theta,c_0)\|D_0\|_{C^1}(S+US)+\varepsilon\theta\int_\Omega D_0|\nabla\Delta s^n|^2dx. \tag{3.16}
$$

Substituting inequalities (3.15) and (3.16) into the formula (3.9), by choosing suitable ε, there holds

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega|\Delta s^n|^2dx+\frac{1}{2}(1-(1+\delta)\theta)\int_\Omega D_0|\nabla\Delta s^n|^2dx\\
&\quad\le C(\delta,\theta,c_0,\|D_0\|_{C^2},\|J_e\|_{H^2})(U+S+1)^2.
\end{aligned}\tag{3.17}
$$

Therefore, by combining (3.6), (3.13) and (3.17), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}(U+S)+\frac{\alpha}{2}\int_\Omega|\nabla\Delta u^n|^2dx+\frac{1}{2}(1-(1+\delta)\theta)\int_\Omega D_0|\nabla\Delta s^n|^2dx\\
&\quad\le C(\alpha,\delta,\theta,c_0,\Phi,\|D_0\|_{C^2},\|J_e\|_{H^2})(U+S+1)^5.
\end{aligned}\tag{3.18}
$$

Here we should point out that δ is a fixed positive number such that $\delta<\frac{1}{\theta}-1$.

By using the comparison theorem of ODE in Lemma 2.5, the desired estimates of approximated solution (u^n, s^n) are obtained from (3.18). Hence, we conclude

Lemma 3.1. *Let* $(u_0, s_0) \in H^2(\Omega, \mathbb{R}^3)$. *Suppose* $J_e \in L^\infty(\mathbb{R}^+, H^2(\Omega, \mathbb{R}^3))$ *and* $D_0 \in C^2(\bar{\Omega})$ *with* $D_0 \geq c_0 > 0$ *for some constant* c_0. *Then there exists a solution* $(u^n, s^n) \in L^\infty([0, T^*), H^2(\Omega, \mathbb{R}^3)) \cap W_2^{3,1}(\Omega \times [0, T^*), \mathbb{R}^3)$ *to system* (3.2), *where* T^* *is dependent on* $\|u_0\|_{H^2(\Omega)} + \|s_0\|_{H^2(\Omega)}$. *Moreover, for any* $T < T^*$, *there exists a constant* $C(T)$ *independent on* (u^n, s^n), *such that the following a priori estimate holds*

$$\sup_{0<t\leq T}(\|u^n\|_{H^2} + \|s^n\|_{H^2}) + \alpha \int_0^T \|\nabla\Delta u^n\|_{L^2}^2 dt + (1 - (1+\delta)\theta)c_0 \int_0^T \|\nabla\Delta s^n\|_{L^2}^2 dt \leq C(T). \tag{3.19}$$

Moreover, the above estimate and equation (3.2) *follow*

$$\sup_{0<t\leq T}(\|\partial_t u^n\|_{L^2}^2 + \|\partial_t s^n\|_{L^2}^2) + \int_0^T (\|\nabla\partial_t u^n\|_{L^2}^2 + \|\nabla\partial_t s^n\|_{L^2}^2) dt \leq C(T).$$

Proof. Let $y(t) = U(t) + S(t)$. Then the estimate (3.18) implies y satisfies the below ODE inequality:

$$\begin{cases} y' \leq C(y+1)^5, \\ y(0) = \|u_0^n\|_{H^2}^2 + \|s_0^n\|_{H^2}^2 \leq C(\|u_0\|_{H^2}^2 + \|s_0\|_{H^2}^2). \end{cases}$$

Here the constant C is dependent on $\|D_0\|_{C^2}$ and $\|J_e\|_{L^\infty(\mathbb{R}^+, H^2(\Omega))}$. If we let $z : [0, T^*) \to \mathbb{R}$ be the maximal solution to

$$\begin{cases} z' = C(z+1)^5, \\ z(0) = C(\|u_0\|_{H^2}^2 + \|s_0\|_{H^2}^2), \end{cases}$$

where T^* is dependent only on $\|u_0\|_{H^2}^2 + \|s_0\|_{H^2}^2$, then Lemma 2.5 shows

$$\sup_{0<t\leq T}(U(t) + S(t)) \leq C(T).$$

Moreover, by considering estimate (3.18), it implies (3.19). On the other hand, by using equation (3.2), it is not difficult to show the following estimate

$$\sup_{0<t\leq T}(\|\partial_t u^n\|_{L^2}^2 + \|\partial_t s^n\|_{L^2}^2) + \int_0^T (\|\nabla\partial_t u^n\|_{L^2}^2 + \|\nabla\partial_t s^n\|_{L^2}^2) dt \leq C(T).$$

□

3.2 Regular Solutions to LLG System with Spin-polarized Transport

In this subsection, we consider the compactness of the approximation solution (u^n, s^n) to (3.2) constructed in the above. The main tool to achieve the compactness is the well-known Alaoglu's theorem and the Aubin–Simons' compactness (see Lemma 2.6 in Section 2). Thus, Lemma 3.1 implies that there exists a subsequence of $\{(u^n, s^n)\}$ (we still denote it by $\{(u^n, s^n)\}$) and a $(u, s) \in L^\infty([0,T], H^2(\Omega, \mathbb{R}^3)) \cap W_2^{3,1}(\Omega \times [0,T], \mathbb{R}^3)$ such that

$$(u^n, s^n) \rightharpoonup (u, s) \quad \text{weakly* in } L^\infty([0,T], H^2(\Omega, \mathbb{R}^3)),$$
$$(u^n, s^n) \rightharpoonup (u, s) \quad \text{weakly in } W_2^{3,1}(\Omega \times [0,T], \mathbb{R}^3),$$

where $0 < T < T^*$. Next, let $X = H^3(\Omega, \mathbb{R}^3)$, $B = H^2(\Omega, \mathbb{R}^3)$ and $Y = L^2(\Omega, \mathbb{R}^3)$. Then, Lemma 2.6 tells us

$$(u^n, s^n) \to (u, s) \quad \text{strongly in } L^p([0,T], H^2(\Omega, \mathbb{R}^3))$$

for any $p < \infty$. It follows that u is a strong solution to equation (3.1).

Theorem 3.1. *The limiting map* $(u, s) \in L^\infty([0,T], H^2(\Omega, \mathbb{R}^3)) \cap W_2^{3,1}(\Omega \times [0,T], \mathbb{R}^3)$ *is a locally strong solution to* (3.1) *for any* $0 < T < T^*$. *Moreover, there exists a constant* $C(T)$ *such that the following estimate holds*

$$\begin{aligned}\sup_{0<t\le T} \left(\|(u,s)\|_{H^2}^2 + \|(\partial_t u, \partial_t s)\|_{L^2}^2\right) + \int_0^T \|(u,s)\|_{H^3}^2 dt \\ + \int_0^T \|(\nabla\partial_t u, \nabla\partial_t s)\|_{L^2}^2 \le C(T).\end{aligned} \tag{3.20}$$

Proof. For any $\varphi \in C^\infty(\bar{\Omega}\times[0,T])$ and $k \in \mathbb{N}$, we set $\varphi_k = P_k(\varphi) = \sum_{i=1}^k g_i(t) f_i$, where $g_i(t) = \langle \varphi, f_i\rangle_{L^2(\Omega)}$. Thus, $\|\varphi - \varphi_k\|_{L^\infty([0,T],L^2(\Omega))} \to 0$ as $k \to \infty$.

We firstly fix k and let $n \ge k$. Since (u^n, s^n) is a locally strong solution of (3.2), it follows

$$\begin{aligned}&\int_0^T \int_\Omega \langle \partial_t u^n, \varphi_k\rangle\, dxdt - \alpha \int_0^T \int_\Omega \langle \Delta u^n + |\nabla u^n| u^n, \varphi_k\rangle\, dxdt \\ &\quad = -\alpha \int_0^T \int_\Omega \langle u^n \times (u^n \times (\tilde{h} + s^n)), \varphi_k\rangle dxdt - \int_0^T \int_\Omega \langle u^n \times (h + s^n), \varphi_k\rangle\, dxdt\end{aligned}$$

and

$$\begin{aligned}&\int_0^T \int_\Omega \langle \partial_t s^n, \varphi_k\rangle\, dxdt + \int_0^T \int_\Omega \langle \mathrm{div}(A(\mathfrak{J}(u^n)\nabla s^n) + u^n \times J_e), \varphi_k\rangle\, dxdt \\ &\quad = -\int_0^T \int_\Omega \langle D_0(x) s^n + D_0(x) s^n \times u^n, \varphi_k\rangle\, dxdt.\end{aligned}$$

The above conclusions on compactness tell us that, for any $1 \le p < \infty$,

$$(\partial_t u^n, \partial_t s^n) \rightharpoonup (\partial_t u, \partial_t s) \quad \text{weakly in } L^2([0,T], L^2(\Omega, \mathbb{R}^3)),$$

and

$$(u^n, s^n) \to (u, s) \quad \text{strongly in } L^p([0,T], H^2(\Omega, \mathbb{R}^3)).$$

It follows

- $h_d(u^n) \to h_d(u) \quad \text{strongly in } L^p([0,T], H^2(\Omega, \mathbb{R}^3))$,
- $(u^n, s^n) \to (u, s) \quad \text{strongly in } L^p([0,T], L^\infty(\Omega, \mathbb{R}^3))$,
- $u^n \times h(u^n) \to u \times h(u) \quad \text{strongly in } L^2([0,T], L^2(\Omega, \mathbb{R}^3))$,
- $u^n \times (u^n \times \tilde{h}(u^n)) \to u \times (u \times \tilde{h}(u)) \quad \text{strongly in } L^2([0,T], L^2(\Omega, \mathbb{R}^3))$,
- $\mathrm{div}(A(u^n)\nabla s^n) \to \mathrm{div}(A(u)\nabla s) \quad \text{strongly in } L^2([0,T], L^2(\Omega, \mathbb{R}^3))$.

Here Lemma 2.1 has been used.

Therefore, according to the dominated convergence theorem and the definition of weak convergence, we infer the desired conclusions as $n \to \infty$ and then $k \to \infty$. It remains that we need to check the Neumann boundary condition.

We only check the condition $\frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0$. The other can be gotten by a similar argument. Since for any $\xi \in C^\infty(\bar{\Omega} \times [0,T])$, there holds

$$\int_0^T \int_\Omega \langle \Delta u^n, \xi \rangle \, dxdt = -\int_0^T \int_\Omega \langle \nabla u^n, \nabla \xi \rangle \, dxdt,$$

letting $n \to \infty$, we have

$$\int_0^T \int_\Omega \langle \Delta u, \xi \rangle \, dxdt = -\int_0^T \int_\Omega \langle \nabla u, \nabla \xi \rangle \, dxdt,$$

which means

$$\frac{\partial u}{\partial \nu}|_{\partial\Omega \times [0,T]} = 0.$$

Eventually, estimate (3.20) is obtained from (3.19) by taking $n \to \infty$ by the lower semi-continuity. □

Next, we show $|u| = 1$ for any $0 < t < T^*$, which follows that (u, s) is a strong solution to (1.3).

Proposition 3.1. *The solution $u \in L^\infty([0,T], H^2(\Omega, \mathbb{R}^3)) \cap W_2^{3,1}(\Omega \times [0,T], \mathbb{R}^3)$ obtained in the above satisfies $|u| = 1$.*

Proof. Let $\omega = |u|^2 - 1$, which satisfies the following equation

$$\begin{cases} \partial_t \omega - \alpha \Delta w = 2\alpha\omega|\nabla u|^2, & (x,t) \in \Omega \times (0,T], \\ \omega(x,0) = 0, & x \in \Omega, \\ \dfrac{\partial \omega}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times [0,T]. \end{cases}$$

If we choose ω as a test function, there holds

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\omega|^2 dx \le 2\alpha|\nabla u|^2_{L^\infty}\int_\Omega |\omega|^2 dx \le C\|u\|^2_{H^3(\Omega)}\int_\Omega |\omega|^2 dx.$$

Since $\gamma(t) = \|u\|^2_{H^3}(t)$ is in $L^1[0,T]$, the Gronwall inequality implies

$$\int_\Omega |\omega|^2 dx(t) \le C(T)\int_\Omega |\omega(x,0)|^2 dx = 0, \quad 0 < t \le T$$

for some $C(T)$. It follows $|u| = 1$ for almost every $(x,t) \in \Omega \times [0,T]$. □

Now, we turn to showing the uniqueness of the solution (u,s) to (1.3) obtained in the above.

Proposition 3.2. *There is a unique solution to* (1.3) *in* $L^\infty([0,T], H^2(\Omega,\mathbb{R}^3)) \cap W_2^{3,1}(\Omega \times [0,T], \mathbb{R}^3)$.

Proof. Let (u,s) and $(\tilde{u},\tilde{s})$ be two solutions to (1.3) in $L^\infty([0,T], H^2(\Omega,\mathbb{R}^3)) \cap W_2^{3,1}(\Omega \times [0,T], \mathbb{R}^3)$ and set $(\bar{u},\bar{s}) = (u-\tilde{u}, s-\tilde{s})$. Then, $(\bar{u},\bar{s})$ satisfies the following equation

$$\begin{cases} \partial_t \bar{u} = \Delta \bar{u} + R_1, \\ \partial_t \bar{s} = -\mathrm{div}(A(u)\nabla \bar{s}) + R_2 \end{cases}$$

with initial-boundary condition

$$\begin{cases} u(\cdot,0) = 0, & \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0, \\ s(\cdot,0) = 0, & \left.\dfrac{\partial s}{\partial \nu}\right|_{\partial\Omega} = 0. \end{cases}$$

Here

$$\begin{aligned} R_1 &= |\nabla u|^2\bar{u} - (|\nabla u|^2 - |\nabla\tilde{u}|^2)\tilde{u} + \bar{u} \times (u \times (\tilde{h}(u) + s)) \\ &\quad + \tilde{u} \times (\bar{u} \times (\tilde{h} + s)) + \tilde{u} \times (\tilde{u} \times (\tilde{h}(u) - \tilde{h}(\tilde{u}) + \bar{s})) \\ &\quad + \bar{u} \times (h(u) + s) + \tilde{u} \times (h(u) - h(\tilde{u}) + \bar{s}), \\ R_2 &= \mathrm{div}(\bar{u} \otimes J_e) - \theta\left(\mathrm{div}(D_0 \cdot (u \otimes (\nabla s \cdot u))) - D_0 \cdot (\tilde{u} \otimes (\nabla\tilde{s} \cdot \tilde{u}))\right) \\ &\quad - D_0\bar{s} - D_0\bar{s} \times u - D_0\tilde{s} \times \bar{u}. \end{aligned}$$

By choosing test functions $(\bar{u},\bar{s})$ and $(\Delta\bar{u},\Delta\bar{s})$, we take a direct calculation to show

$$\frac{\partial}{\partial t}(\|\bar{u}\|^2_{H^1}+\|\bar{s}\|^2_{H^1})+\alpha\int_\Omega|\Delta\bar{u}|^2dx+(1-\theta)\int_\Omega D_0|\Delta\bar{s}|^2dx \\ \le C(\alpha,\theta,\|D_0\|_{C^1},c_0,\|J_e\|_{H^2})F(t)(\|\bar{u}\|^2_{H^1}+\|\bar{s}\|^2_{H^1})$$

for a.e. $t\in[0,T]$. Here $F(t)=\|u\|^2_{H^3}+\|\tilde{u}\|^2_{H^3}+\|s\|^2_{H^3}+\|\tilde{s}\|^2_{H^3}+1$, which is in $L^1[0,T]$. Thus, the Gronwall inequality implies

$$\|\bar{u}\|^2_{H^1}+\|\bar{s}\|^2_{H^1}=0,\quad \text{for any } t\in[0,T].$$

Therefore, the proof is complete. □

Proof of Theorem 1.1. Immediately, Theorem 1.1 follows Theorem 3.1, Propositions 3.1 and 3.2. Thus, we finish the proof. □

4 Very Regular Solution

However, when $k\ge 3$, the H^{k+1}-estimates of solution cannot be derived by multiplying the two sides of the Galerkin approximation of (3.1) in the above section by

$$(-1)^k(\Delta^k u^n,\Delta^k s^n),$$

since the right hand side of equation (3.2) doesn't satisfy the homogeneous Neumann boundary condition. Thus, to improve the regularity of the solution (u,s) to (1.3) constructed in the previous section, we apply the method used in [9].

Roughly speaking, under suitable compatibility initial-boundary conditions, which are defined in Subsection 2.5, we consider equation (2.2) for $k=1$. By using Galerkin approximation method, we can get a solution $(u_1,s_1)\in L^\infty([0,T], H^2(\Omega,\mathbb{R}^3))\cap W_2^{3,1}(\Omega\times[0,T],\mathbb{R}^3)$, where $0<T<T^*$ and T^* is determined in Theorem 1.1. Fortunately, a uniqueness argument guarantees that $(u_1,s_1)\equiv(\partial_t u,\partial_t s)$. Then, we can get $(u,s)\in L^\infty([0,T],H^4(\Omega,\mathbb{R}^3))\cap W_2^{5,1}(\Omega\times[0,T],\mathbb{R}^3)$ by a bootstrap argument using equation (1.3). Therefore, we can get very higher regularity of (u,s) step by step, by considering the compatible equation (2.2) for $k>1$.

Next, we divide this section into two parts. In the first part, we give a detailed process to enhance the regularity of (u,s) to $L^\infty([0,T],H^5(\Omega,\mathbb{R}^3))\cap W_2^{6,1}(\Omega\times[0,T],\mathbb{R}^3)$. And then in the other part, the results on very regular solution can be obtained by using method of induction with a similar argument as that in the first part.

4.1 $\boldsymbol{H^6}$-regularity of Solution

Let (u^n, s^n) be the solution of (3.2) given in the previous section, and let $(u_t^n, s_t^n) = (\partial_t u^n, \partial_t s^n)$. Then, (u_t^n, s_t^n) satisfies the following equation

$$\begin{cases}\partial_t u_t^n = \alpha\Delta u_t^n + T_1 + T_2 + T_3 + T_4 + T_5, & (x,t) \in \Omega \times [0, T^*),\\ \partial_t u_t^n = -P_n\left(\operatorname{div}(A(\mathfrak{J}(u^n)))\nabla s_t^n + u_t^n \otimes J_e\right) + I_1 + I_2, & (x,t) \in \Omega \times [0, T^*)\end{cases} \tag{4.1}$$

with initial condition

$$\begin{cases}u_t^n(x,0) = \sum_{i=1}^n \partial_t g_i^n(0) f_i, \quad x \in \Omega,\\ s_t^n(x,0) = \sum_{i=1}^n \partial_t \gamma_i^n(0) f_i, \quad x \in \Omega.\end{cases}$$

Here

$$\begin{cases}T_1 = \alpha P_n(u^n \times (u^n \times (h_d(u_t^n) - \partial_t \nabla_u \Phi(\mathfrak{J}(u^n) + s_t^n)))),\\ T_2 = \alpha P_n(u_t^n \times (u^n \times (h_d(u^n) - \nabla_u \Phi(\mathfrak{J}(u^n) + s^n))))\\ \qquad + \alpha P^n(u^n \times (u_t^n \times (h_d(u^n) - \nabla_u \Phi(\mathfrak{J}(u^n) + s^n)))),\\ T_3 = -P_n(u^n \times (\Delta u_t^n + h_d(u_t^n) - \partial_t \nabla_u \Phi(\mathfrak{J}(u^n)) + s_t^n)),\\ T_4 = -P_n(u_t^n \times (\Delta u^n + h_d(u^n) - \nabla_u \Phi(\mathfrak{J}(u^n)) + s^n)),\\ T_5 = \alpha P_n(|\nabla u^n|^2 u_t^n + 2\langle \nabla u_t^n, \nabla u^n\rangle u^n),\\ I_1 = -P_n(\operatorname{div}(\partial_t A(\mathfrak{J}(u^n))\nabla s^n + u^n \otimes \partial_t J_e)),\\ I_2 = P_n(D_0 s_t^n + D_0 s_t^n \times u^n + D_0 s^n \times u_t^n).\end{cases}$$

For the sake of convenience, we also denote

$$I_0 = -P_n(\operatorname{div}(A(\mathfrak{J}(u^n))\nabla s_t^n + u_t^n \otimes J_e)).$$

In Lemma 3.1, we have shown that

$$\|u_t^n\|_{L^2}^2 + \|s_t^n\|_{L^2}^2 = g(t) \le C(T), \quad 0 \le t \le T < T^*.$$

In order to show the H^2-estimate of (u_t^n, s_t^n), we choose $(-\Delta u_t^n, -\Delta s_t^n)$ as a test function for the above equation (4.1). Then, there hold true

$$\frac{1}{2}\frac{\partial}{\partial_t}\int_\Omega |\nabla u_t^n|^2 dx + \alpha \int_\Omega |\Delta u_t^n|^2 dx = -\sum_{i=1}^5 \int_\Omega \langle T_i, \Delta u_t^n\rangle\, dx$$

and

$$\frac{1}{2}\frac{\partial}{\partial_t}\int_\Omega |\nabla s_t^n|^2 dx = -\sum_{i=0}^3 \langle I_i, \Delta s_t^n\rangle\, dx.$$

Next, we estimate the terms on the right hand sides of the above equations step by step.

$$\left|\int_\Omega \langle T_1, \Delta u_t^n\rangle\, dx\right| \le C(\varepsilon,\alpha,\delta)U^2(\|u_t^n\|_{L^2}^2 + \|s_t^n\|_{L^2}^2) + \varepsilon\alpha\|\Delta u_t^n\|_{L^2}^2,$$

$$\left|\int_\Omega \langle T_2, \Delta u_t^n\rangle\, dx\right| \le C(\varepsilon,\alpha,\delta)U(U+S+1)\|u_t^n\|_{L^2}^2 + \varepsilon\alpha\|\Delta u_t^n\|_{L^2}^2,$$

$$\left|\int_\Omega \langle T_3, \Delta u_t^n\rangle\, dx\right| \le C(\varepsilon,\alpha,\delta)U(\|u_t^n\|_{L^2}^2 + \|s_t^n\|_{L^2}^2) + \varepsilon\alpha\|\Delta u_t^n\|_{L^2}^2,$$

$$\left|\int_\Omega \langle T_4, \Delta u_t^n\rangle\, dx\right| \le C(\varepsilon,\alpha,\delta)(\|u^n\|_{H^3}^2 + S + 1)\|u_t^n\|_{H^1}^2 + \varepsilon\alpha\|\Delta u_t^n\|_{L^2}^2,$$

$$\left|\int_\Omega \langle T_5, \Delta u_t^n\rangle\, dx\right| \le C(\varepsilon,\alpha,\delta)U(U + \|u^n\|_{H^3}^2)\|u_t^n\|_{H^1}^2 + \varepsilon\alpha\|\Delta u_t^n\|_{L^2}^2,$$

$$\begin{aligned} -\int_\Omega \langle I_0, \Delta s_t^n\rangle\, dx \le{}& C(\varepsilon,\alpha,\delta,\|D_0\|_{C^1})(U + \|J_e\|_{H^2}^2 + 1)(\|u_t^n\|_{H^1}^2 + \|\nabla s_t^n\|_{L^2}^2) \\ &- (1-(1+\delta+\varepsilon)\theta)\int_\Omega D_0|\Delta s_t^n|^2 dx, \end{aligned}$$

$$\begin{aligned} \left|\int_\Omega \langle I_1, \Delta s_t^n\rangle\, dx\right| \le{}& C(\varepsilon,\theta,\delta,\|D_0\|_{C^1})(\|s^n\|_{H^3}^2 + S\|u^n\|_{H^3}^2)\|u_t^n\|_{H^1}^2 \\ &+ CU\|\partial_t J_e\|_{H^1}^2 + \varepsilon\theta\int_\Omega D_0|\Delta s_t^n|^2 dx, \end{aligned}$$

$$\begin{aligned} \left|\int_\Omega \langle I_2, \Delta s_1^n\rangle\, dx\right| \le{}& C(\varepsilon,\theta,c_0)(U+S+1)(\|u_t^n\|_{L^2}^2 + \|s_t^n\|_{L^2}^2) \\ &+ \varepsilon\theta\int_\Omega D_0|\Delta s_t^n|^2 dx. \end{aligned}$$

By combining the above inequalities, we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\int_\Omega |\nabla u_t^n|^2 + |\nabla s_t^n|^2 dx\right) + \alpha\int_\Omega |\Delta u_t^n|^2 dx + (1-(1+\delta)\theta)\int_\Omega D_0|\Delta s_t^n|^2 dx \\ &\quad\le f(t)(\|u_t^n\|_{L^2}^2 + \|s_t^n\|_{L^2}^2 + 1) + f(t)\left(\int_\Omega |\nabla u_t^n|^2 dx + \int_\Omega |\nabla s_t^n|^2 dx\right). \qquad (4.2) \end{aligned}$$

Here

$$f(t) = C(\alpha,\delta,\theta,\|D_0\|_{C^1})(U^2 + S^2 + (S+1)\|u^n\|_{H^3}^2 + \|J_e\|_{H^2}^2 + U\|\partial_t J_e\|_{H^1}^2 + 1),$$

which is in $L^1([0,T])$ for any $0 < T < T^*$. Here we have used that $J_e \in L^2([0,T], H^2(\Omega))$ and $\partial_t J_e \in L^2([0,T], H^1(\Omega))$.

Thus, the Gronwall inequality and (4.2) imply

$$\sup_{0<\le t\le T}\left(\int_\Omega |\nabla u_t^n|^2 dx + \int_\Omega |\nabla s_t^n|^2 dx\right) + \alpha\int_0^T\int_\Omega |\Delta u_t^n|^2 dx + (1-(1+\delta)\theta)$$

$$\times \int_0^T \int_\Omega D_0|\Delta s_1^n|^2 dx \le C(T, \|\nabla u_t^n\|_{L^2}^2(0) + \|\nabla s_t^n\|_{L^2}^2(0), \|f\|_{L^1([0,T])}). \tag{4.3}$$

On the other hand, equation (3.2) implies

$$\begin{cases} u_t^n(x,0) = P_n(\alpha(\Delta u_0^n + |\nabla u_0^n|^2 u_0^n)) \\ \qquad\qquad -P_n(u_0^n \times (u_0^n \times (\tilde{h} + s_0^n)) - u_0^n \times (h + s_0^n)), \\ u_t^n(x,0) = -P_n(\mathrm{div}(A(\mathfrak{J}(u_0^n))\nabla s_0^n + u_0^n \times J_e)) \\ \qquad\qquad -P_N(D_0(x)\cdot s_0^n - D_0(x)\cdot s_0^n \times u_0^n). \end{cases} \tag{4.4}$$

A direct computation shows

$$\begin{aligned} \|\nabla u_t^n\|_{L^2}^2(0) + \|\nabla s_t^n\|_{L^2}^2(0) &\le C(\|u_0^n\|_{H^3(\Omega)} + \|s_0^n\|_{H^3(\Omega)} + 1) \\ &\le C(\|u_0\|_{H^3(\Omega)} + \|s_0\|_{H^3(\Omega)} + 1), \end{aligned}$$

since we have the following estimates

$$\begin{aligned} \|u_0^n\|_{H^3(\Omega)} &\le C\|u_0\|_{H^3(\Omega)}, \\ \|s_0^n\|_{H^3(\Omega)} &\le C\|s_0\|_{H^3(\Omega)}, \end{aligned}$$

and $J_e \in C^0(\mathbb{R}^+, H^2(\Omega))$.

Now, we can give the proof of Theorem 1.2 by using the above estimate of (u_t^n, s_t^n) and a bootstrap argument.

Proof of Theorem 1.2. The estimates in (4.3) indicates

$$(u_t^n, s_t^n) \rightharpoonup (\partial_t u, \partial_t s) \quad \text{weakly in } L^2([0,T], H^2(\Omega,\mathbb{R}^3))$$

and

$$(u_t^n, s_t^n) \rightharpoonup (\partial_t u, \partial_t s) \quad \text{weakly* in } L^\infty([0,T], H^1(\Omega,\mathbb{R}^3)).$$

Thus,

$$(\partial_t u, \partial_t s) \in L^\infty([0,T], H^1(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^2(\Omega,\mathbb{R}^3)).$$

Next, by using the property of cross-product, the equation (1.1) can be rewritten as the form

$$\begin{cases} \Delta u = -|\nabla u|^2 u + u \times (u \times (\tilde{h} + s)) - \dfrac{1}{1+\alpha^2}\left(\alpha \partial_t u + u \times \partial_t u\right), \\ Ls = \mathrm{div}(A(u)\nabla s) = -\partial_t s + \mathrm{div}(u \otimes J_e) + D_0(x)s + D_0(x)s \times u. \end{cases} \tag{4.5}$$

We claim: Δu, $\Delta s \in L^\infty([0,T], H^1(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^2(\Omega,\mathbb{R}^3))$.

Here, we shall only prove $\Delta u \in L^\infty([0,T], H^1(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^2(\Omega,\mathbb{R}^3))$, since we can take a similar argument to prove the fact Δs belongs to the same space as Δu.

From the first equation of (4.5) and the fact $u \in L^\infty([0,T], H^2(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^3(\Omega,\mathbb{R}^3))$, we can easily get that

$$\sup_{0<t\le T} \|\Delta u\|_{L^3} \le C(T).$$

Since

$$\begin{aligned}\nabla\Delta u = &- (|\nabla u|^2\nabla u + 2\nabla^2 u\cdot\nabla u u) + \nabla\big(u\times(u\times(\tilde h + s))\big)\\ &+ \frac{1}{1+\alpha^2}(\alpha\nabla\partial_t u + \nabla u\times\partial_t u + u\times\nabla\partial_t u),\end{aligned}$$

by a simple calculation we obtain

$$\|\nabla\Delta u\|_{L^2}^2 \le C(U+S)^3 + U\|\nabla^2 u\|_{L^3}^2 + (U+1)\|\partial_t u\|_{H^1}^2 \le C(T),$$

where we have used the global L^p estimate (see [32, Theorem 2.3])

$$\|\nabla^2 u\|_{L^3} \le C(\|\Delta u\|_{L^3} + \|u\|_{L^3}).$$

Thus, we have $u \in L^\infty([0,T], H^3(\Omega,\mathbb{R}^3))$ by classical L^p-estimate.

In the sense of distribution, we have

$$\begin{aligned}\nabla^2\Delta u = F = &- \nabla(|\nabla u|^2\nabla u + 2\nabla^2 u\cdot\nabla u u) + \nabla^2\big(u\times(u\times(\tilde h + s))\big)\\ &+ \frac{1}{1+\alpha^2}\nabla(\alpha\nabla\partial_t u + \nabla u\times\partial_t u + u\times\nabla\partial_t u).\end{aligned}$$

By a simple calculation, we get

$$\|F\|_{L^2}^2 \le C(U+S+1)^2 + (U+1)\|u\|_{H^3}^4 + \|\partial_t u\|_{H^1}^2\|u\|_{H^3}^2 + \|\nabla^2\partial_t u\|_{L^2}^2,$$

which implies that

$$\Delta u \in L^2([0,T], H^2(\Omega,\mathbb{R}^3)).$$

Thus, we have

$$u \in L^2([0,T], H^4(\Omega,\mathbb{R}^3)).$$

Therefore, we have finished the proof of the claim on Δu.

Now we turn to considering the regularity of Δs. Since we have shown that

$$u \in L^\infty([0,T], H^3(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^4(\Omega,\mathbb{R}^3))$$

and $Ls = \operatorname{div}(A(u)\nabla s)$ is a uniformly elliptic operator, a similar argument to that in the above implies

$$\Delta s \in L^\infty([0,T], H^1(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^2(\Omega,\mathbb{R}^3)).$$

Here we need $D_0 \in C^3(\bar\Omega)$. Thus, it follows $s \in L^\infty([0,T], H^3(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^4(\Omega,\mathbb{R}^3))$. $\square$

Remark 4.1. We can also show $(u, s) \in C^0([0, T], H^3(\Omega, \mathbb{R}^3))$ by the embedding Theorem II.5.14 in [7].

Remark 4.2. We cannot get H^3-estimate of (u_t, s_t) as that for (u, s) in the previous section by applying test function $(\Delta^2 u_t^n, \Delta^2 s_t^n)$, since the following inequality does not hold true

$$\|u_0^n\|_{H^4} \leq C\|u_0\|_{H^4},$$

which is beyond of our knowledge. Meanwhile, the same reason implies that we cannot get higher regularity of (u, s) by considering the equation of $(\partial_t^2 u^n, \partial_t^2 s^n)$.

The above remark indicates that one can not proceed to enhance the regularity except for one adds some new restrictive conditions on the initial data. Motivated by [9], we intend to add some suitable compatibility boundary conditions to enhance the regularity (see [9]). Hence, we consider the following equation

$$\begin{cases} \partial_t u_1 = \alpha \Delta u_1 - u \times \Delta u_1 + K_1(\nabla u_1, \nabla s_1) + L_1(s_1, u_1), & (x, t) \in \Omega \times \mathbb{R}^+, \\ \partial_t s_1 = -\mathrm{div}\,(A(u)\nabla s_1) + \hat{Q}_1(\nabla u_1, \nabla s_1) + T_1(u_1, s_1), & (x, t) \in \Omega \times \mathbb{R}^+, \\ \dfrac{\partial u_1}{\partial \nu} = 0, & (x, t) \in \partial\Omega \times \mathbb{R}^+, \\ \dfrac{\partial s_1}{\partial \nu} = 0, & (x, t) \in \partial\Omega \times \mathbb{R}^+ \end{cases} \tag{4.6}$$

with initial-boundary condition

$$\begin{cases} u_1(x, \cdot) = V_1(u_0, s_0), & \dfrac{\partial V_1}{\partial \nu}\Big|_{\partial\Omega} = 0, \\ s_1(x, \cdot) = W_1(u_0, s_0), & \dfrac{\partial W_1}{\partial \nu}\Big|_{\partial\Omega} = 0. \end{cases} \tag{4.7}$$

Here,

$$\begin{aligned} K_1(\nabla u_1, \nabla s_1) &= 2\alpha(\nabla u_1 \cdot \nabla u)u, \\ \hat{Q}_1(\nabla u_1, \nabla s_1) &= -\mathrm{div}(u_1 \otimes J_e) - \mathrm{div}(u \otimes \partial_t J_e), \\ L_1(u_1, s_1) &= \alpha|\nabla u|^2 u_1 - \alpha u_1 \times (u \times (\tilde{h}(u) + s)) - \alpha u \times (u_1 \times (\tilde{h}(u) + s)) \\ &\quad - \alpha u_1 \times (u \times (\bar{h}(u_1) + s_1)) - u \times (\bar{h}(u_1) + s_1) - u_1 \times (h(u) + s), \\ T_1(u_1, s_1) &= \theta \mathrm{div}(D_0 u_1 \otimes (\nabla s \cdot u + D_0 u \otimes (\nabla s \cdot u_1))) - D_0 s_1 \\ &\quad - D_0 s_1 \times u - D_0 s \times u_1, \\ V_1(u_0, s_0) &= \alpha\big(\Delta u_0 + |\nabla u_0|^2 u_0 - u_0 \times (u_0 \times (\tilde{h}(u_0) + s_0))\big) \\ &\quad - u_0 \times (\Delta u_0 + \tilde{h}(u_0) + s_0), \\ W_1(u_0, s_0) &= -\mathrm{div}(A(u_0)\nabla s_0 + u_0 \otimes J_e(x, 0)) - D_0(x) \cdot s_0 - D_0(x) \cdot s_0 \times u_0, \end{aligned}$$

where $\bar{h}(u_1) = h_d(u_1) - \nabla^2\Phi(u)\cdot u_1$.

The above equation (4.6) is a linear system with respect to u_1 and s_1, since h_d is a linear operator. Next, as before, we also need to consider the following Galerkin approximation equation associated to (4.6)

$$\begin{cases}\partial_t u_1^n = P_n(\alpha\Delta u_1^n - u\times\Delta u_1^n + K_1(\nabla u_1^n,\nabla s_1^n)+L_1(s_1^n,u_1^n)), & (x,t)\in\Omega\times\mathbb{R}^+,\\ \partial_t s_1^n = P_n(-\mathrm{div}(A(u)\nabla s_1^n) + \hat{Q}_1(\nabla u_1^n,\nabla s_1^n) + T_1(u_1^n,s_1^n)), & (x,t)\in\Omega\times\mathbb{R}^+,\end{cases} \tag{4.8}$$

with initial data

$$\begin{cases}u_1^n(x,\cdot) = P_n(V_1(u_0,s_0)),\\ s_1^n(x,\cdot) = P_n(W_1(u_0,s_0)).\end{cases} \tag{4.9}$$

By almost the same argument as that on equation (3.2) in Section 3, it is not difficult to prove that there exists a solution (u_1^n, s_1^n) in $L^\infty([0,T^*),H^2(\Omega,\mathbb{R}^3))\cap L^2([0,T^*),H^3(\Omega,\mathbb{R}^3))$ with energy estimates as follows,

$$\begin{aligned}&\sup_{0<t\le T}\big(\|u_1^n\|_{H^2(\Omega)} + \|s_1^n\|_{H^2(\Omega)}\big) + \alpha\int_0^T \|\nabla\Delta u_1^n\|^2_{L^2(\Omega)}dt\\ &\quad + (1-\theta)c_0\int_0^T \|\nabla\Delta s_1^n\|^2_{L^2(\Omega)}dt\\ &\le C(T,\|(u_0,s_0)\|_{H^2(\Omega)},\|(V_1,W_1)\|_{H^2(\Omega)})\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}&\sup_{0<t\le T}(\|\partial_t u_1^n\|^2_{L^2(\Omega)} + \|\partial_t s_1^n\|^2_{L^2(\Omega)}) + \int_0^T(\|\nabla\partial_t u_1^n\|^2_{L^2(\Omega)} + \|\nabla\partial_t s_1^n\|^2_{L^2(\Omega)})dt\\ &\le C(T,\|(u_0,s_0)\|_{H^2(\Omega)},\|(V_1,W_1)\|_{H^2(\Omega)})\end{aligned} \tag{4.11}$$

for any $0<T<T^*$, since the system (4.6) is linear. Here we need to assume

$$J_e\in C^0([0,T],H^3(\Omega,\mathbb{R}^3)) \quad \text{and} \quad \partial_t J_e\in L^\infty([0,T],H^2(\Omega,\mathbb{R}^3)),$$

and use the estimate:

$$\|(P_n(V_1),P_n(W_1))\|_{H^2(\Omega)} \le C\|(V_1,W_1)\|_{H^2(\Omega)} \le C(\|(u_0,s_0)\|_{H^4(\Omega)}).$$

By the Alaoglu's theorem and Aubin–Simon Compactness Lemma 2.6, we know that there exists a subsequence of $\{(u_1^n,s_1^n)\}$ (we still denote it by $\{(u_1^n, s_1^n)\}$) and a map

$$(u_1,s_1)\in L^\infty([0,T],H^2(\Omega,\mathbb{R}^3))\cap W_2^{3,1}(\Omega\times[0,T],\mathbb{R}^3)$$

such that, for any $1<p<\infty$,

- $(u_1^n,s_1^n)\rightharpoonup(u_1,s_1)$ weakly* in $L^\infty([0,T],H^2(\Omega,\mathbb{R}^3))$,

- $(u_1^n, s_1^n) \rightharpoonup (u_1, s_1)$ weakly in $W_2^{3,1}(\Omega \times [0,T], \mathbb{R}^3)$,
- $(u_1^n, s_1^n) \to (u_1, s_1)$ strong in $L^p([0,T], H^2(\Omega, \mathbb{R}^3))$.

It follows that (u_1, s_1) is a strong solution to (4.6). On the other hand, it is not difficult to show (u_t, s_t) is a solution to (4.6) in $L^\infty([0,T], H^1(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^2(\Omega, \mathbb{R}^3))$, since $(u, s) \in C^0([0,T], H^3(\Omega, \mathbb{R}^3))$.

If the uniqueness of solution to (4.6) in the space $L^\infty([0,T], H^1(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^2(\Omega, \mathbb{R}^3))$ can be established well, then one can see easily that

$$(u_t, s_t) \equiv (u_1, s_1),$$

which implies higher regularity. So, we need to prove the following

Proposition 4.1. *There is a unique solution to* (4.6) *in* $L^\infty([0,T], H^1(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^2(\Omega, \mathbb{R}^3))$.

Proof. Assume that (u_1, s_1) and $(\tilde{u}_1, \tilde{s}_1)$ in $L^\infty([0,T^*), H^1(\Omega, \mathbb{R}^3)) \cap W_2^{2,1}(\Omega \times [0,T^*), \mathbb{R}^3)$ are two solutions to (4.6) in $L^\infty([0,T^*), H^1(\Omega, \mathbb{R}^3)) \cap W_2^{2,1}(\Omega \times [0,T^*), \mathbb{R}^3)$. Let $(\bar{u}_1, \bar{s}_1) = (u_1 - \tilde{u}_1, s_1 - \tilde{s}_1)$. It satisfies the following equation

$$\begin{cases} \partial_t \bar{u}_1 = \alpha \Delta \bar{u}_1 - u \times \Delta \bar{u}_1 + K_1(\nabla \bar{u}_1, \nabla \bar{s}_1) + L_1(\bar{u}_1, \bar{s}_1), \\ \partial_t \bar{s}_1 = -\mathrm{div}(A(u)\nabla \bar{s}_1) + Q_1(\nabla \bar{u}_1, \nabla \bar{s}_1) + T_1(\bar{u}_1, \bar{s}_1), \end{cases}$$

with initial-boundary condition

$$\begin{cases} \bar{u}_1(\cdot, 0) = 0, & \left.\dfrac{\partial \bar{u}_1}{\partial \nu}\right|_{\partial\Omega} = 0, \\ \bar{s}_1(\cdot, 0) = 0, & \left.\dfrac{\partial \bar{s}_1}{\partial \nu}\right|_{\partial\Omega} = 0. \end{cases}$$

The above equation indicates $(\partial_t \bar{u}_1, \partial_t \bar{s}_1) \in L^2([0,T] \times \Omega)$. Then by taking $(\bar{u}_1, \bar{s}_1)$ as a test function of the above equation, there holds

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega (|\bar{u}_1|^2 + |\bar{s}_1|^2)dx \\ &\quad \le C(\alpha, \theta, D_0, \|\partial_t J_e\|_{H^2} + \|J_e\|_{H^2})(\|(u,s)\|_{H^3}^2 + U + S + 1)\int_\Omega (|\bar{u}_1|^2 + |\bar{s}_1|^2)dx \\ &\quad \le F(t)\int_\Omega (|\bar{u}_1|^2 + |\bar{s}_1|^2)dx, \end{aligned}$$

where $F(t) \in L^1([0,T])$ for any $0 < T < T^*$. Hence, the Gronwall inequality implies $(\bar{u}_1, \bar{s}_1) = (0,0)$ and the proof is finished. □

Proposition 4.1 implies $(u_t, s_t) \in L^\infty([0,T], H^2(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^3(\Omega, \mathbb{R}^3))$. If $(u_0, s_0) \in H^5(\Omega, \mathbb{R}^3)$ and $D_0 \in C^5(\bar{\Omega})$, then a similar argument with that in the proof of Theorem 1.2 shows

$$(u, s) \in L^\infty([0,T], H^5(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^6(\Omega, \mathbb{R}^3)).$$

4.2 Very Regular Solutions

In this subsection, we will apply the method of induction to show the existence of very regular solution (u,s) to (1.3) by considering the initial-Neumann boundary value problem of the equation of ($u_k=\partial_t^k u$, $s_k=\partial_t^k s$) with matching initial-boundary data, that is to prove Theorem 1.3. In fact, in the above Subsection 4.1, we have enhanced regularity of (u,s) by the strategy $\mathscr{P}$ in Section 1.

For the case of $k>1$, to prove Theorem 1.3 we need to repeat the process of the strategy $\mathscr{P}$ by showing the following property $\mathscr{P}(k)$:

1. If $(u_0,s_0)\in H^{2k}(\Omega,\mathbb{R}^3)$ with compatibility condition at order $k-1$, then the solution (u,s) with initial data (u_0,s_0) satisfies

$$(\partial_t^i u,\partial_t^i s)\in L^\infty([0,T],H^{2k-2i}(\Omega,\mathbb{R}^3))\cap L^2([0,T],H^{2k-2i+1}(\Omega,\mathbb{R}^3))$$

for any $0<T<T^*$, where $i\in\{0,\dots,k\}$;

2. Moreover, if $(u_0,s_0)\in H^{2k+1}(\Omega,\mathbb{R}^3)$, then there holds

$$(\partial_t^i u,\partial_t^i s)\in L^\infty([0,T],H^{2k-2i+1}(\Omega,\mathbb{R}^3))\cap L^2([0,T],H^{2k-2i+2}(\Omega,\mathbb{R}^3))$$

for any $0<T<T^*$, where $i\in\{0,\dots,k\}$.

In fact, it is not difficult to see that, in the previous subsection, we have shown this property holds for $k=1$ and $k=2$.

Next, suppose that $\mathscr{P}(k)$ is already established for $k\geq 2$, then we want to prove that $\mathscr{P}(k+1)$ is true. Therefore, we assume $(u_0,s_0)\in H^{2(k+1)}(\Omega,\mathbb{R}^3)$ satisfies the compatibility condition (2.4) at order k. By the property $\mathscr{P}(k)$, we have

$$(\partial_t^i u,\partial_t^i s)\in L^\infty([0,T],H^{2k-2i+1}(\Omega,\mathbb{R}^3))\cap L^2([0,T],H^{2k-2i+2}(\Omega,\mathbb{R}^3)),$$

where $0<T<T^*$ and $i\in\{0,\dots,k\}$.

Furthermore, we know that

$$(u_k,s_k)\in L^\infty([0,T],H^1(\Omega,\mathbb{R}^3))\cap L^2([0,T],H^2(\Omega,\mathbb{R}^3))$$

satisfies the following equation

$$\begin{cases}\partial_t v=\alpha\Delta v-u\times\Delta v+K_k(\nabla v,\nabla w)+L_k(v,w)+F_k(u,s), & (x,t)\in\Omega\times\mathbb{R}^+,\\ \partial_t w=-\mathrm{div}(A(u)\nabla w)+Q_k(\nabla v,\nabla w)+T_k(v,w)+Z_k(u,s), & (x,t)\in\Omega\times\mathbb{R}^+,\\ \dfrac{\partial v}{\partial\nu}=0, & (x,t)\in\partial\Omega\times\mathbb{R}^+,\\ \dfrac{\partial w}{\partial\nu}=0, & (x,t)\in\partial\Omega\times\mathbb{R}^+,\\ (v(x,0),w(x,0))=(V_k(u_0,s_0),W_k(u_0,s_0)), & x\in\Omega.\end{cases} \tag{4.12}$$

Next, we will adopt the same procedure as in the strategy $\mathscr{P}$ for $(\partial_t^k u, \partial_t^k s)$ to get the regular property $\mathscr{P}(k+1)$. Hence, we need to show the following three claims.

1. If $(u_0, s_0) \in H^{2k+2}(\Omega, \mathbb{R}^3)$, then we can get a regular solution to (4.12)

$$(v, w) \in L^\infty([0,T], H^2(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^3(\Omega, \mathbb{R}^3))$$

with the k-order compatibility condition of initial data at boundary.

2. There holds true $(\partial_t^k u, \partial_t^k s) \equiv (v, w)$ as long as one can show the uniqueness of solution to (4.12) in $L^\infty([0,T], H^1(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^2(\Omega, \mathbb{R}^3))$. Moreover, it implies

$$(u, s) \in L^\infty([0,T], H^{2k+2}(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^{2k+3}(\Omega, \mathbb{R}^3))$$

by using equation (2.2) again.

3. If $(u_0, s_0) \in H^{2k+3}(\Omega, \mathbb{R}^3)$, then one can infer that

$$(u, s) \in L^\infty([0,T], H^{2k+3}(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^{2k+4}(\Omega, \mathbb{R}^3))$$

by repeating the same arguments as one proves item (1) of the property $\mathscr{P}$ on $(\partial_t^k u, \partial_t^k s)$.

In the below context, we will show the above three claims step by step.

4.2.1 Regular Solution to (4.12)

Now, we repeat the process of Galerkin approximation in Section 3 to seek a regular solution $(v, w) \in L^\infty([0,T], H^2(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^3(\Omega, \mathbb{R}^3))$ to equation (4.12) when $(V_k, W_k) \in H^2(\Omega, \mathbb{R}^3)$. By a similar argument of enhancing regularity to that in the previous Subsection 4.1, we can obtain that $(v, w) \in L^\infty([0,T], H^3(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^4(\Omega, \mathbb{R}^3))$ under an improved initial value assumption that $(V_k, W_k) \in H^3(\Omega, \mathbb{R}^3)$. To this end, first we need to show the estimates of the nonhomogeneous terms F_k and Z_k in (4.12) satisfy the following

Proposition 4.2. *Assume that the property $\mathscr{P}(k)$ has been established. Suppose that for any $0 \le i \le k$,*

$$\partial_t^i J_e \in L^\infty([0,T], H^{2k-2i+2}(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^{2k-2i+3}(\Omega, \mathbb{R}^3)).$$

Then there holds

$$(F_i, Z_i) \in L^\infty([0,T], H^{2k-2i+1}(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^{2k-2i+2}(\Omega, \mathbb{R}^3)),$$

where $0 < T < T^*$.

Proof. Firstly, we show

$$F_i \in L^\infty([0,T], H^{2k-2i+1}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2k-2i+2}(\Omega,\mathbb{R}^3)).$$

According to the definition of F_i in (2.2), we only need to consider the following terms.

(a) $I = \nabla u_s \# \nabla u_j \# u_l$, where $s+j+l=i$ and $0 \le s,j,l \le i-1$. For $l \in \{1,\dots,i-1\}$,

$$u_l \in L^\infty([0,T], H^{2k-2i+3}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2k-2i+4}(\Omega,\mathbb{R}^3)).$$

Thus ∇u_s, ∇u_j and u_l are all in $L^\infty([0,T], H^{2k-2i+2}(\Omega,\mathbb{R}^3))$. Since $2k-2i+2 \ge 2$, by Lemma 2.4 we have

$$I \in L^\infty([0,T], H^{2k-2i+2}(\Omega,\mathbb{R}^3)).$$

(b) $II = \nabla u_s \# \nabla u_j \# (\bar{h}(u_l) + s_l)$, where $s+j+l=i$ and $0 \le s,j,l \le i-1$. Almost the same argument as that in (a) shows

$$II \in L^\infty([0,T], H^{2k-2i+2}(\Omega,\mathbb{R}^3)).$$

(c) $III = \nabla u_j \# (\bar{h}(u_l) + s_l)$, where $j+l=i$ and $0 \le j,l \le i-1$. Almost the same argument as that in (a) shows

$$III \in L^\infty([0,T], H^{2k-2i+2}(\Omega,\mathbb{R}^3)).$$

(d) $IV = u_l \# \Delta u_j$, where $j+l=i$ and $0 \le j,l \le i-1$. Since

$$u_l \in L^\infty([0,T], H^{2k-2i+3}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2k-2i+4}(\Omega,\mathbb{R}^3))$$

and

$$\Delta u_j \in L^\infty([0,T], H^{2k-2i+1}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2k-2i+2}(\Omega,\mathbb{R}^3)),$$

Lemma 2.4 tells us

$$IV \in L^\infty([0,T], H^{2k-2i+1}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2k-2i+2}(\Omega,\mathbb{R}^3)).$$

(e) $V = R_i$. By almost the same argument as that in (a) we also have

$$V \in L^\infty([0,T], H^{2k-2i+2}(\Omega,\mathbb{R}^3)).$$

Next, we turn to the estimates of Z_i and take a similar discussion to that we derive the estimate of F_i. It is not difficult to find that one needs only to consider the following two terms.

(f) $I' = u_s \# \nabla^2 s_j \# u_l$, where $s+j+l=i$ and $0 \le s,j,l \le i-1$. For $l \in \{1,\dots,i-1\}$, it is easy to see u_s and u_l are in $L^\infty([0,T], H^{2k-2i+3}(\Omega,\mathbb{R}^3))$. Then, by Lemma 2.4 we have

$$u_s * u_l \in L^\infty([0,T], H^{2k-2i+3}(\Omega,\mathbb{R}^3)).$$

Thus, combining the fact and the following

$$\nabla^2 s_j \in L^\infty([0,T], H^{2k-2i+1}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2k-2i+2}(\Omega,\mathbb{R}^3)),$$

we have

$$I' \in L^\infty([0,T], H^{2k-2i+1}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2k-2i+2}(\Omega,\mathbb{R}^3)),$$

where we have used Lemma 2.3.

(g) $II' = \nabla u_s \# \partial_t^j J_e + u_s \# \nabla \partial_t^j J_e$, where $s+j=i$ and $s<i$. Since there hold

$$u_s \in L^\infty([0,T], H^{2k-2i+3}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2k-2i+4}(\Omega,\mathbb{R}^3))$$

and

$$\partial_t^j J_e \in L^\infty([0,T], H^{2k-2i+2}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2k-2i+3}(\Omega,\mathbb{R}^3)),$$

it is easy to conclude that there holds

$$II' \in L^\infty([0,T], H^{2k-2i+1}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2k-2i+2}(\Omega,\mathbb{R}^3)).$$

□

Remark 4.3. Moreover, if

$$\partial_t^{k+1} J_e \in L^\infty([0,T], L^2(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^1(\Omega,\mathbb{R}^3)),$$

by almost the same argument as that in above Proposition 4.2, we obtain

$$(\partial_t F_i, \partial_t Z_i) \in L^2([0,T], H^{2k-2i}(\Omega,\mathbb{R}^3)),$$

where $i \in \{0,\dots,k\}$.

Now, we turn to considering the Galerkin approximation associated to (4.12) as following

$$\begin{cases} \partial_t v^n = P_n(\alpha\Delta v^n - u\times\Delta v^n + K_k(\nabla v^n,\nabla w^n) + L_k(v^n,w^n) + F_k), \\ \partial_t w^n = P_n(-\mathrm{div}(A(u)\nabla w^n) + Q_k(\nabla v^n,\nabla w^n) + T_k(v^n,w^n) + Z_k), \\ (v^n(x,0), w^n(x,0)) = (P_n(V_k), P_n(W_k)). \end{cases} \tag{4.13}$$

Obviously, the equation admits a solution (v^n,w^n) in H_n, defined on $\Omega\times[0,T^n)$, where T^n is the maximal existence time. In fact, it is easy to see that $T^n = T^*$, since equation (4.12) is a linear system and its coefficient is well controlled on $[0,T]$ for any $T<T^*$.

To show the H^3-estimates of v^n, we firstly choose v^n and $-\Delta^2 v^n$ as test functions. A simple computation shows

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial_t}\int_\Omega |v^n|^2dx+\alpha\int_\Omega|\nabla v^n|^2dx\\&\quad\le C(\alpha,\Phi)(\|u\|^2_{H^3}+\|u\|^4_{H^2}+\|s\|^4_{H^2}+1)\int_\Omega|v^n|^2dx+C\int_\Omega|w^n|^2dx+\|F_k\|^2_{L^2}\end{aligned}\tag{4.14}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial_t}\int_\Omega |\Delta v^n|^2dx+\alpha\int_\Omega|\nabla\Delta v^n|^2dx\\&\quad=-\int_\Omega\langle\nabla u\times\Delta v^n+\nabla K_k+\nabla L_k+\nabla F_k,\nabla\Delta v^n\rangle\,dx.\end{aligned}\tag{4.15}$$

Here

$$\begin{aligned}&\left|\int_\Omega\langle\nabla u\times\Delta v^n,\nabla\Delta v^n\rangle\,dx\right|\le C(\varepsilon,\alpha)\|u\|^2_{H^3(\Omega)}\int_\Omega|\Delta v^n|^2dx+\varepsilon\alpha\int_\Omega|\nabla\Delta v^n|^2dx,\\&\left|\int_\Omega\langle\nabla K_k,\nabla\Delta v^n\rangle\,dx\right|\le C(\varepsilon,\alpha,\Phi)(\|u\|^2_{H^3(\Omega)}+\|u\|^4_{H^2(\Omega)}+1)\|v^n\|^2_{H^2(\Omega)}\\&\qquad\qquad+\varepsilon\alpha\int_\Omega|\nabla\Delta v^n|^2dx,\\&\left|\int_\Omega\langle\nabla L_k,\nabla\Delta v^n\rangle\,dx\right|\\&\quad\le C(\varepsilon,\alpha,\Phi)(\|s\|^2_{H^3(\Omega)}+\|u\|^2_{H^2(\Omega)}+1)(\|s\|^2_{H^2(\Omega)}+\|u\|^2_{H^2(\Omega)}+1)\\&\qquad\times(\|v^n\|^2_{H^1(\Omega)}+\|w^n\|^2_{H^1(\Omega)})+\varepsilon\alpha\int_\Omega|\nabla\Delta v^n|^2dx.\end{aligned}$$

By combining the above inequalities, (4.15) can be rewritten as the following

$$\begin{aligned}&\frac{\partial}{\partial_t}\int_\Omega |\Delta v^n|^2dx+\alpha\int_\Omega|\nabla\Delta v^n|^2dx\\&\quad\le C(\alpha,\Phi)(\|u\|^2_{H^3(\Omega)}+\|s\|^2_{H^3(\Omega)}+\|u\|^4_{H^2(\Omega)}+1)(\|s\|^2_{H^2(\Omega)}+\|u\|^2_{H^2(\Omega)}+1)\\&\qquad\times(\|v^n\|^2_{H^2(\Omega)}+\|w^n\|^2_{H^2(\Omega)})+\|F_k\|^2_{H^1(\Omega)}.\end{aligned}\tag{4.16}$$

Moreover, to show the H^2-estimates of w^n, we choose w^n and $-\Delta^2 w^n$ as test functions. A simple computation shows

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial_t}\int_\Omega |w^n|^2dx+(1-\theta)\int_\Omega|D_0\nabla w^n|^2dx\\&\quad\le C(\theta,\|D_0\|_{C^1},\|J_e\|_{H^2(\Omega)})\|v^n\|_{H^1(\Omega)}+C\int_\Omega|w^n|^2dx+\|Z_k\|^2_{L^2(\Omega)}\end{aligned}\tag{4.17}$$

and

$$\frac{1}{2}\frac{\partial}{\partial_t}\int_\Omega |\Delta w^n|^2 dx$$
$$= -\int_\Omega \langle -\nabla(\mathrm{div}(A(u)\nabla w^n)) + \nabla Q_k + \nabla T_k + \nabla Z_k, \nabla\Delta w^n\rangle\, dx. \qquad (4.18)$$

We can see easily that

$$\int_\Omega \langle \nabla(\mathrm{div}(A(u)\nabla w^n)), \nabla\Delta w^n\rangle\, dx$$
$$\leq C(\theta, \|D_0\|_{C^2}, c_0)(\|u\|^2_{H^3(\Omega)} + \|u\|^4_{H^3(\Omega)} + 1)\|w^n\|^2_{H^2(\Omega)}$$
$$- \frac{3}{4}(1-\theta)\int_\Omega D_0|\nabla\Delta w^n|^2 dx,$$
$$\left|\int_\Omega \langle \nabla Q_k, \nabla\Delta w^n\rangle\, dx\right|$$
$$\leq C(\varepsilon, \theta, \|D_0\|_{C^2}, c_0, \|J_e\|_{H^2(\Omega)})\|v^n\|^2_{H^2(\Omega)}$$
$$+ \varepsilon\theta\int_\Omega |\nabla\Delta w^n|^2 dx$$

and

$$\left|\int_\Omega \langle \nabla T_k, \nabla\Delta w^n\rangle\, dx\right|$$
$$\leq C(\varepsilon, \theta, \|D_0\|_{C^2}, c_0, \|J_e\|_{H^2(\Omega)})(\|s\|^2_{H^3(\Omega)} + \|u\|^2_{H^3(\Omega)}\|u\|^2_{H^3(\Omega)}$$
$$+ \|u\|^2_{H^3(\Omega)} + 1)(\|v^n\|^2_{H^2(\Omega)} + \|w^n\|^2_{H^2(\Omega)}) + \varepsilon\theta\int_\Omega |\nabla\Delta w^n|^2 dx.$$

In view of the above three inequalities, from (4.18) we deduce the following inequality

$$\frac{\partial}{\partial t}\int_\Omega |\Delta w^n|^2 dx + (1-\theta)\int_\Omega D_0|\nabla\Delta w^n|^2 dx$$
$$\leq C(\theta, c_0, \|D_0\|_{C^2}, \|J_e\|_{H^2(\Omega)})(\|s\|^4_{H^3(\Omega)} + \|u\|^2_{H^3(\Omega)} + 1)$$
$$\times (\|v^n\|^2_{H^2(\Omega)} + \|w^n\|^2_{H^2(\Omega)}) + \|Z_k\|^2_{H^1(\Omega)}. \qquad (4.19)$$

Hence, by combining inequalities (4.14), (4.16), (4.17) and (4.19) we get the following

$$\frac{\partial}{\partial t}(\|v^n\|^2_{H^2(\Omega)} + \|w^n\|^2_{H^2(\Omega)}) + \alpha\int_\Omega |\nabla\Delta v^n|^2 dx + (1-\theta)\int_\Omega D_0|\nabla\Delta w^n|^2 dx$$
$$\leq C(\theta, \alpha, \Phi, c_0, \|D_0\|_{C^2}, \|J_e\|_{H^2(\Omega)})(\|v^n\|^2_{H^2(\Omega)} + \|w^n\|^2_{H^2(\Omega)})p(t) + q(t). \qquad (4.20)$$

Here

$$q(t) = \|F_k\|^2_{H^1(\Omega)} + \|Z_k\|^2_{H^1(\Omega)}$$

and

$$p(t) = (\|s\|^4_{H^3(\Omega)} + \|u\|^4_{H^3(\Omega)} + 1)(\|s\|^2_{H^3(\Omega)} + \|u\|^2_{H^3(\Omega)} + 1).$$

Moreover, Proposition 4.2 tells us $q(t) \le C(T)$ and

$$p(t) = (\|s\|^4_{H^3(\Omega)} + \|u\|^4_{H^3(\Omega)} + 1)(\|s\|^2_{H^3(\Omega)} + \|u\|^2_{H^3(\Omega)} + 1) \le C(T).$$

Then, by the Gronwall inequality we infer from (4.20)

$$\begin{aligned} &\sup_{0<t\le T}(\|v^n\|^2_{H^2(\Omega)} + \|w^n\|^2_{H^2(\Omega)}) + \alpha\int_0^T\int_\Omega |\nabla\Delta v^n|^2 dxdt \\ &\quad + (1-\theta)\int_0^T\int_\Omega |\nabla\Delta w^n|^2 dxdt \le C(T, \|V_k\|_{H^2(\Omega)} + \|W_k\|_{H^2(\Omega)}), \end{aligned} \tag{4.21}$$

where $0 < T < T^*$. Here we have used the fact

$$\|V_k\|_{H^2(\Omega)} + \|W_k\|_{H^2(\Omega)} \le C(\alpha, \theta, \|D_0\|_{C^{2k+1}(\Omega)}, \|u_0\|_{H^{2k+2}(\Omega)} + \|s_0\|_{H^{2k+2}(\Omega)}).$$

Therefore, the Compactness Lemma (Lemma 2.6) claims that the limiting map

$$(v, w) \in L^\infty([0,T], H^2(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^3(\Omega, \mathbb{R}^3))$$

of sequence $\{(v^n, w^n)\}$ as $n \to \infty$ is just a solution to (4.12), which satisfies the same estimates as (4.20) and (4.21) with replacing (v^n, w^n) by (v, w).

4.2.2 Uniqueness of Solution to (4.12)

In this part, we will show the uniqueness of the solutions to (4.12) in the space $L^\infty([0,T], H^1(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^2(\Omega, \mathbb{R}^3))$.

Proposition 4.3. *In the space* $L^\infty([0,T], H^1(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^2(\Omega, \mathbb{R}^3))$, *there exists a unique solution to* (4.12).

Proof. Let (v, w) and $(\tilde{v}, \tilde{w})$ be two solutions. We denote $(\bar{v}, \bar{w}) = (v - \tilde{v}, w - \tilde{w})$, which satisfies the following equation

$$\begin{cases} \partial_t \bar{v} = \alpha\Delta\bar{v} - u \times \Delta\bar{v} + K_k(\nabla\bar{v}, \nabla\bar{w}) + L_k(\bar{v}, \bar{w}), & (x,t) \in \Omega \times (0, T^*), \\ \partial_t \bar{w} = -\mathrm{div}(A(u)\nabla\bar{w}) + Q_k(\nabla\bar{v}, \nabla\bar{w}) + T_k(\bar{v}, \bar{w}), & (x,t) \in \Omega \times (0, T^*), \\ \dfrac{\partial\bar{v}}{\partial\nu} = 0, & (x,t) \in \partial\Omega \times [0, T^*), \\ \dfrac{\partial\bar{w}}{\partial\nu} = 0, & (x,t) \in \partial\Omega \times [0, T^*), \\ (\bar{v}(x,0), \bar{w}(x,0)) = (0,0), & x \in \Omega. \end{cases}$$

Thus, for any fixed $0 < T < T^*$, by choosing $(\bar{v}, \bar{w})$ as test function to the above equation, there hold true

$$\begin{aligned}&\frac{\partial}{\partial t}\int_\Omega |\bar{v}|^2 dx + \alpha \int_\Omega |\nabla \bar{v}|^2 dx \\ &\qquad \le C(\alpha, \Phi)(\|u\|^2_{H^3(\Omega)} + \|s\|^2_{H^2(\Omega)} + 1)\int_\Omega |\bar{v}|^2 dx + C\int_\Omega |\bar{w}|^2 dx\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial}{\partial t}\int_\Omega |\bar{w}|^2 dx + (1-\theta)\int_\Omega |\nabla \bar{w}|^2 dx \\ &\qquad \le C(\theta, \|D_0\|_{C^1(\Omega)}\|J_e\|^2_{H^2(\Omega)})\|s\|^2_{H^3(\Omega)}\left(\int_\Omega (|\bar{v}|^2 + |\bar{w}|^2)dx\right).\end{aligned}$$

It follows

$$\begin{aligned}\frac{\partial}{\partial t}\int_\Omega (|\bar{v}|^2 + |\bar{w}|^2)dx &\le C(\alpha, \theta, \Phi, \|D_0\|_{C^1(\Omega)}, \|J_e\|_{H^2(\Omega)}) \\ &\qquad \times (\|u\|^2_{H^3(\Omega)} + \|s\|^2_{H^3(\Omega)} + 1)\left(\int_\Omega (|\bar{v}|^2 + |\bar{w}|^2)dx\right).\end{aligned}$$

Thus, the Gronwall inequality tells us that, for any $0 < t < T^*$,

$$(\bar{v}(x,t), \bar{w}(x,t)) = (\bar{v}(x,0), \bar{w}(x,0)) = (0,0).$$

□

Immediately, $(u_k, s_k) \equiv (v, w)$ follows the uniqueness result, since (u_k, s_k) is also a solution to (4.12) in the space $L^\infty([0,T], H^1(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^2(\Omega,\mathbb{R}^3))$.

4.2.3 The Proof of Property $\mathscr{P}(k+1)$

Next, we are in the position to show the item (1) of property $\mathscr{P}(k+1)$ holds true if $(u_0, s_0) \in H^{2k+2}(\Omega,\mathbb{R}^3)$ and the solution to (3.1) satisfies the property $\mathscr{P}(k)$.

Proposition 4.4. *If $(u_0, s_0) \in H^{2k+2}(\Omega,\mathbb{R}^3)$ and the property $\mathscr{P}(k)$ holds, then we have*

$$(\partial_t^i u, \partial_t^i s) \in L^\infty([0,T], H^{2(k+1)-2i}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2(k+1)-2i+1}(\Omega,\mathbb{R}^3)), \tag{4.22}$$

where $0 < T < T^*$ and $i \in \{0, \dots, k+1\}$.

Proof. We use mathematical induction on $k+1-l$. When $l = 0$, we have

$$\begin{cases} u_{k+1} = \alpha\Delta u_k - u \times \Delta u_k + K_k(\nabla u_k, \nabla s_k) + L_k(s_k, u_k) + F_k(u,s), \\ s_{k+1} = -\operatorname{div}(A(u)\nabla s_k) + Q_k(\nabla u_k, \nabla s_k) + T_k(u_k, s_k) + Z_k(u,s). \end{cases}$$

A direct computation shows

$$(u_{k+1}, s_{k+1}) \in L^\infty([0,T], L^2(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^1(\Omega,\mathbb{R}^3)),$$

where we have used Proposition 4.2 and estimate (4.21).

Thus, we have shown the result holds when $l = 0$ and 1. Now, we assume when $l = i \geq 1$, the result has been proved. Then we need to establish it for $l = i+1$, where $i \leq k-1$. Since $u_{k+1-i} = \partial_t u_{k-i}$, it follows

$$\begin{cases} \alpha\Delta u_{k-i} - u \times \Delta u_{k-i} = u_{k+1-i} - K_{k-i}(\nabla u_{k-i}, \nabla s_{k-i}) \\ \qquad\qquad\qquad\qquad -L_{k-i}(s_{k-i}, u_{k-i}) + F_{k-i}(u,s), \\ \operatorname{div}(A(u)\nabla s_{k-i}) = -s_{k+1-i} + Q_{k-i}(\nabla u_{k-i}, \nabla s_{k-i}) \\ \qquad\qquad\qquad\qquad +T_{k-i}(u_{k-i}, s_{k-i}) + Z_{k-i}(u,s). \end{cases}$$

Next, we consider the first equation in order to obtain the estimate of u_{k-i}. By utilizing the properties $\mathscr{P}(k)$ and Proposition 4.2, we have

- $u_{k-i} \in L^\infty([0,T], H^{2i+1}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2i+2}(\Omega,\mathbb{R}^3))$,
- $u \in L^\infty([0,T], H^{2k+1}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2k+2})$,
- $F_{k-i} \in L^\infty([0,T], H^{2i+1}) \cap L^2([0,T], H^{2i+2}(\Omega,\mathbb{R}^3))$,
- and by the assumption of induction,

$$u_{k+1-i} \in L^\infty([0,T], H^{2i}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2i+1}(\Omega,\mathbb{R}^3)).$$

For the term K_{k-i}, since ∇u_{k-i} and ∇s_{k-i} are in $L^\infty([0,T], H^{2i}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2i+1}(\Omega,\mathbb{R}^3))$, u is in $L^\infty([0,T], H^{2k+1}(\Omega,\mathbb{R}^3))$, then

$$K_{k-i} \in L^\infty([0,T], H^{2i}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2i+1}(\Omega,\mathbb{R}^3)),$$

where Lemma 2.4 has been used.

Since $\Delta u \in L^\infty([0,T], H^{2i+1}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2i+2}(\Omega,\mathbb{R}^3))$ by $2k-1 \geq 2i+1$, it follows

$$u_{k-i} \times \Delta u \in L^\infty([0,T], H^{2i}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2i+1}(\Omega,\mathbb{R}^3)).$$

By almost the same argument as that for K_{k-i}, we know

$$L_{k-i} \in L^\infty([0,T], H^{2i}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2i+1}(\Omega,\mathbb{R}^3)),$$

since $h_d(u_{k-i})$ and s_{k-i} are in $L^\infty([0,T], H^{2i+1}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2i+2}(\Omega,\mathbb{R}^3))$.

Now, it is not difficult to see that the above estimates imply

$$\alpha\Delta u_{k-i} - u \times \Delta u_{k-i} \in L^\infty([0,T], H^{2i}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2i+1}(\Omega,\mathbb{R}^3)).$$

Furthermore, the L^p-estimate of elliptic equation shows

$$u_{k-i} \in L^\infty([0,T], H^{2i+2}(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^{2i+3}(\Omega, \mathbb{R}^3)).$$

On the other hand, in order to show the same estimate of s_{k-i} we need to take almost the same argument as that for u_{k-i} except for that we need to control the following term $u_{k-i} * \nabla^2 s * u$. Since u_{k-i}, $\nabla^2 s$ and u are all in $L^\infty([0,T], H^{2i+1}(\Omega, \mathbb{R}^3))$, it follows

$$u_{k-i} * \nabla^2 s * u \in L^\infty([0,T], H^{2i}(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^{2i+1}(\Omega, \mathbb{R}^3)).$$

Note that, to control the term Q_k, here we need $J_e \in L^\infty([0,T], H^{2i}(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^{2i+1}(\Omega, \mathbb{R}^3))$.

Thus, we have

$$\operatorname{div}(A(u)\nabla s_{k-i}) \in L^\infty([0,T], H^{2i}(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^{2i+1}(\Omega, \mathbb{R}^3)).$$

It follows

$$s_{k-i} \in L^\infty([0,T], H^{2i+2}(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^{2i+3}(\Omega, \mathbb{R}^3)).$$

Therefore, we finish the induction argument. In particular, when $l = k$, we have

$$(\partial_t u, \partial_t s) \in L^\infty([0,T], H^{2k}(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^{2k+1}(\Omega, \mathbb{R}^3)).$$

Finally, we need to show the result when $l = k+1$. Since $(\partial_t u, \partial_t s)$ satisfies the following equations

$$\begin{cases} \Delta u = -|\nabla u|^2 u + u \times (u \times (\tilde{h} + s)) - \dfrac{1}{1+\alpha^2}(\alpha \partial_t u + u \times \partial_t u), \\ \operatorname{div}(A(u) \cdot \nabla s) = -\partial_t s + \operatorname{div}(u \otimes J_e) + D_0(x)s + D_0(x)s \times u, \end{cases}$$

in view of the fact

$$(u, s) \in L^\infty([0,T], H^{2k+1}(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^{2k+2}(\Omega, \mathbb{R}^3)),$$

we take a bootstrap argument to show

$$(\Delta u, \Delta s) \in L^\infty([0,T], H^{2k}(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^{2k+1}(\Omega, \mathbb{R}^3)).$$

Hence, it implies

$$(u, s) \in L^\infty([0,T], H^{2k+2}(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^{2k+3}(\Omega, \mathbb{R}^3))$$

by L^p-estimates. □

Now, assume that $(u_0, s_0) \in H^{2(k+1)+1}(\Omega, \mathbb{R}^3)$. We want to show the item (2) in the property $\mathscr{P}(k+1)$. Here, we will only give the sketch of proof to this property, since the proof goes almost the same as that in Subsection 4.1 and the proof of Proposition 4.4. First of all, we prove the following result, which is analogous to Theorem 1.2.

Proposition 4.5. *If* $(u_0, s_0) \in H^{2(k+1)+1}(\Omega, \mathbb{R}^3)$ *and the property* $\mathscr{P}(k)$ *holds, then*

$$(u_{k+1}, s_{k+1}) \in L^\infty([0,T], H^1(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^2(\Omega, \mathbb{R}^3)) \tag{4.23}$$

for any $0 < T < T^*$.

Proof. By the Galerkin approximation equation (4.13) associated to (4.12), $(v_t^n, w_t^n) = (\partial_t v^n, \partial_t w^n)$ satisfies the following equation

$$\begin{cases} \partial_t v_t^n = \alpha \Delta v_t^n + P_n \partial_t \left(-u \times \Delta v^n + K_k(\nabla v^n, \nabla w^n) + L_k(v^n, w^n) + F_k \right), \\ \partial_t w_t^n = P_n \partial_t \left(-\mathrm{div}(A(u) \nabla w^n) + Q_k(\nabla v^n, \nabla w^n) + T_k(v^n, w^n) + Z_k \right). \end{cases} \tag{4.24}$$

By the assumption of $\mathscr{P}(k+1)$ and the previous induction arguments, we can combine the estimates in Proposition 4.2, Proposition 4.4 and estimate (4.21) to obtain

- $(v^n, w^n) \in L^\infty([0,T], H^2(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^3(\Omega, \mathbb{R}^3))$,
- $(\partial_t^i u, \partial_t^i s) \in L^\infty([0,T], H^{2k+2-2i}(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^{2k+3-2i}(\Omega, \mathbb{R}^3))$, where $i \in \{0, \dots, k+1\}$,
- $(F_k, Z_k) \in L^\infty([0,T], H^1(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^2(\Omega, \mathbb{R}^3))$,
- $(\partial_t F_k, \partial_t Z_k) \in L^2([0,T], L^2(\Omega, \mathbb{R}^3))$.

In the following context, we aim at proving

$$(v_t^n, w_t^n) \in L^\infty([0,T], H^1(\Omega, \mathbb{R}^3)) \cap L^2([0,T], H^2(\Omega, \mathbb{R}^3)).$$

From equation (4.13) and the estimate of (v^n, w^n), we can get easily

$$\|(v_t^n, w_t^n)\|_{L^\infty([0,T], L^2(\Omega))} + \|(v_t^n, w_t^n)\|_{L^2([0,T], H^1(\Omega))} \le C(T).$$

By choosing $(-\Delta v_t^n, -\Delta w_t^n)$ as a test function, we can show the H^2-estimate as follows,

$$\begin{aligned} &\frac{\partial}{\partial t} \int_\Omega |\nabla v_t^n|^2 dx + \alpha \int_\Omega |\Delta v_t^n|^2 dx \\ &\quad \le C(\alpha) \left(\int_\Omega |K_k(\nabla v_t^n, \nabla w_t^n)|^2 dx + \int_\Omega |L_k(v_t^n, w_t^n)|^2 dx + \int_\Omega |\tilde{F}_k|^2 dx \right) \end{aligned}$$

$$=: C(\alpha)(I_1 + II_1 + III_1), \tag{4.25}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_\Omega |\nabla w_t^n|^2 dx + (1-\theta)\int_\Omega D_0|\Delta w_t^n|^2 dx \\
&\le C(c_0,\theta,\|D_0\|_{C^1})\left(\int_\Omega |Q_k(\nabla v_t^n, \nabla w_t^n)|^2 dx + \int_\Omega |T_k(v_t^n, w_t^n)|^2 dx\right) \\
&\quad + C(c_0,\theta,\|D_0\|_{C^1})\left(\int_\Omega |\tilde{Z}_k|^2 dx + \int_\Omega |\nabla w_t^n|^2 dx + \int_\Omega |\nabla w_t^n|^2|\nabla u|^2 dx\right) \\
&=: C(c_0,\theta,\|D_0\|_{C^1})(I_2 + II_2 + III_2 + IV_2 + V_2),
\end{aligned} \tag{4.26}$$

where

$$\begin{aligned}
\tilde{F}_k &= -u_t \times \Delta v^n + \partial_t K_k(\nabla v^n, \nabla w^n) + \partial_t L_k(v^n, w^n) + \partial_t F_k, \\
\tilde{Z}_k &= -\mathrm{div}\,(\partial_t A(u)\nabla w^n) + \partial_t Q_k(\nabla v^n, \nabla w^n) + \partial_t T_k(v^n, w^n) + \partial_t Z_k.
\end{aligned}$$

By a direct computation, we get the below estimates.

$$\begin{aligned}
I_1 &\le C\int_\Omega |\nabla v_t^n|^2|\nabla u|^2\,dx \le C\|u\|_{H^3}^2\int_\Omega |\nabla v_t^n|^2 dx, \\
II_1 &\le 2\int_\Omega |v_t^n|^2|\nabla u|^4\,dx + \int_\Omega |v_t^n|^2(|\tilde{h}|^2 + |s|^2)dx \\
&\quad + C\int_\Omega (|h_d(v_t^n)|^2 + |v_t|^2 + |w_t^n|^2)dx + \int_\Omega |v^n|^2(|h(u)|^2 + |s|^2)dx \\
&\le C(\|u\|_{H^3(\Omega)}^4 + \|s\|_{H^2(\Omega)}^2 + 1)\int_\Omega |v_t^n|^2 + C\int_\Omega |w_t^n|^2 dx + C\|u\|_{H^3}^2\|v_t^n\|_{H^1} \\
&\le C(T) + C(T)\int_\Omega |\nabla v_t^n|^2\,dx, \\
III_1 &\le \int_\Omega |\partial_t K_k(\nabla v^n, \nabla w^n)|^2\,dx + \int_\Omega |\partial_t L_k(v^n, w^n)|^2\,dx \\
&\quad + \int_\Omega |\partial_t F_k|^2 dx + \int_\Omega |u_t|^2|\Delta v^n|^2\,dx \\
&\le \|v^n\|_{H^2(\Omega)}^2\|u_t\|_{H^2(\Omega)}^2 + \|u\|_{H^3(\Omega)}^2\|\partial_t u\|_{H^2(\Omega)}^2\int_\Omega |\nabla v^n|^2\,dx \\
&\quad + \int_\Omega |\partial_t L_k(v^n, w^n)|^2\,dx + \int_\Omega |\partial_t F_k|^2 dx + \|u_t\|_{H^2(\Omega)}^2\int_\Omega |\Delta v^n|^2\,dx \\
&\le C(T) + \int_\Omega |\partial_t F_k|^2 dx.
\end{aligned}$$

Here,

$$\begin{aligned}
&\int_\Omega |\partial_t L_k(v^n, w^n)|^2\,dx \\
&\le C\|u\|_{H^2(\Omega)}^2\|v^n\|_{H^2(\Omega)}^2\|u_t\|_{H^2(\Omega)} + C\|v^n\|_{H^2(\Omega)}\|u_t\|_{H^2(\Omega)}^2(1 + \|s\|_{L^2(\Omega)}^2)
\end{aligned}$$

$$
\begin{aligned}
&+ C\|v^n\|^2_{H^2(\Omega)}(\|u_t\|^2_{L^2(\Omega)} + \|s_t\|^2_{L^2(\Omega)}) \\
&+ C\|u_t\|^2_{L^2(\Omega)}(\|v^n\|^2_{H^2(\Omega)} + \|w^n\|^2_{H^2(\Omega)}) + \|v^n\|^2_{H^2(\Omega)} \int_\Omega |\Delta u_t|^2 \, dx \\
\leq{}& C(T).
\end{aligned}
$$

Thus, there holds

$$
\begin{aligned}
&\frac{\partial}{\partial_t} \int_\Omega |\nabla v^n_t|^2 dx + \alpha \int_\Omega |\Delta v^n_t|^2 dx \\
&\qquad \leq C(T) + C(T) \int_\Omega |\partial_t F_k|^2 dx + C(T) \int_\Omega |\nabla v^n_t|^2 dx.
\end{aligned}
$$

By Gronwall inequality, it follows

$$
\sup_{0<t\leq T} \|v^n_t\|^2_{H^1(\Omega)} + \int_0^T \|v^n_t\|^2_{H^2(\Omega)} \, dt \leq C(T, \|v^n_t|_{t=0}\|^2_{H^1(\Omega)}).
$$

Here we have used the fact

$$
\int_\Omega |\partial_t F_k|^2 dx + \int_\Omega |\nabla v^n_t|^2 dx \in L^1([0,T]).
$$

Now, we turn to showing the H^2-estimate of w^n_t. We need to control the terms in the right hand side of inequality (4.26) as follows,

$$
\begin{aligned}
I_2 \leq{}& C\|J_e\|^2_{H^2(\Omega)} \|v^n_t\|^2_{H^1(\Omega)} \leq C(T, \|v^n_t|_{t=0}\|^2_{H^1}), \\
II_2 \leq{}& C(\|D_0\|_{C^1}, c_0, \theta) \|s\|^2_{H^3(\Omega)} \|v^n_t\|^2_{H^1(\Omega)} \\
&+ C(\|D_0\|_{C^1}, c_0, \theta) \left(\|s\|^2_{H^3(\Omega)} \|u\|^2_{H^3(\Omega)} \|v^n_t\|^2_{L^2(\Omega)} + \int_\Omega |w^n_t|^2 dx \right) \\
&+ C(\|D_0\|_{C^1}, c_0, \theta) \left(\|s\|^2_{H^2} \int_\Omega |v^n_t|^2 dx \right) \\
\leq{}& C(T), \\
III_2 \leq{}& \int_\Omega |\partial_t Q_k(\nabla v^n, \nabla w^n)|^2 \, dx + \int_\Omega |\partial_t T_k(v^n, w^n)|^2 \, dx + \int_\Omega |\partial_t Z_k|^2 \, dx \\
\leq{}& C\|\partial_t J_e\|^2_{H^1(\Omega)} \|v^n\|^2_{H^2(\Omega)} + \int_\Omega |\partial_t Z_k|^2 \, dx + C(T).
\end{aligned}
$$

Here, we have used the below estimates

$$
\begin{aligned}
&\int_\Omega |\partial_t T_k(v^n, w^n)|^2 \, dx \\
&\quad \leq C(\|D_0\|_{C^1}) \left(\|v^n\|^2_{H^2(\Omega)} \int_\Omega |\nabla s_t|^2 \, dx + \|u_t\|^2_{H^2(\Omega)} \|v^n\|^2_{H^2} \int_\Omega |\nabla s|^2 \, dx \right)
\end{aligned}
$$

$$
\begin{aligned}
&+ C(\|D_0\|_{C^1})\left(\|s\|^2_{H^3(\Omega)}\int_\Omega |\partial_t v^n|^2\,dx + \|v^n\|^2_{H^2(\Omega)}\int_\Omega |\nabla s|^2\,dx\right)\\
&+ C(\|D_0\|_{C^1})\left(\|u_t\|^2_{H^2(\Omega)}\int_\Omega |w^n|^2\,dx + \|s_t\|^2_{H^2(\Omega)}\int_\Omega |v^n|^2\,dx\right)\\
&\le C(T)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\Omega |\mathrm{div}(D_0\theta u_t\otimes u\cdot\nabla w^n)|^2\,dx\\
&\quad\le C(\|D_0\|_{C^1})\left(\|u_t\|^2_{H^2(\Omega)}\|w^n\|^2_{H^2(\Omega)} + \|u_t\|^2_{H^2(\Omega)}\|u\|^2_{H^3(\Omega)}\int_\Omega |\nabla w^n|^2\,dx\right)\\
&\quad\le C(T).
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
&\frac{\partial}{\partial_t}\int_\Omega |\nabla w_t^n|^2 dx + (1-\theta)\int_\Omega D_0|\Delta w_t^n|^2 dx\\
&\quad\le C(T) + C(T)\int_\Omega |\partial_t Z_k|^2\,dx + C(T)\int_\Omega |\nabla w_t^n|^2 dx.
\end{aligned}
$$

Hence, the Gronwall inequality implies

$$
\sup_{0<t\le T}\|w_t^n\|^2_{H^1(\Omega)} + \int_0^T \|w_t^n\|^2_{H^2(\Omega)}\,dt \le C(T, \|v_t^n|_{t=0}\|^2_{H^1(\Omega)}, \|w_t^n|_{t=0}\|^2_{H^1(\Omega)}).
$$

Finally, we need to show the assumption of initial data (u_0, s_0) can guarantee the bound of $\|\nabla v_t^n\|_{L^2} + \|\nabla w_t^n\|_{L^2}$, hence the proof is complete. Since the initial data of (4.24) satisfies

$$
\begin{cases}
v_t^n(x,0) = \alpha\Delta P_n(V_k) + P_n(-u_0\times P_n(V_k) + K_k|_{t=0}(\nabla P_n(V_k), \nabla P_n(W_k))\\
\qquad\qquad + L_k|_{t=0}(P_n(V_k), P_n(W_k)) + F_k|_{t=0}),\\
w_t^n(x,0) = P_n(-\mathrm{div}(A(u_0)\nabla P_n(W_k)) + Q_k|_{t=0}(\nabla P_n(V_k), \nabla P_n(W_k))\\
\qquad\qquad + T_k|_{t=0}(P_n(V_k), P_n(W_k)) + Z_k|_{t=0}),
\end{cases}
$$

a simple calculation shows

$$
\int_\Omega |\nabla v_t^n|_{t=0}|^2\,dx \le C(\|P_n(V_k)\|^2_{H^3(\Omega)} + \|P_n(W_k)\|^2_{H^3(\Omega)} + 1)
$$

and

$$
\int_\Omega |\nabla w_t^n|_{t=0}|^2\,dx \le C(\|P_n(V_k)\|^2_{H^3(\Omega)} + \|P_n(W_k)\|^2_{H^3(\Omega)} + 1).
$$

Thus, there holds

$$\int_\Omega |\nabla v_t^n|_{t=0}|^2\,dx + \int_\Omega |\nabla w_t^n|_{t=0}|^2\,dx \le C(\|V_k\|^2_{H^3(\Omega)} + \|W_k\|^2_{H^3(\Omega)} + 1)$$
$$\le C(\|u_0\|_{H^{2k+3}(\Omega)}, \|s_0\|_{H^{2k+3}(\Omega)}).$$

Here, we have applied the formula of V_k and Lemma 2.7 about estimate of P_n. □

By summarizing the estimates in Proposition 4.4 and Proposition 4.5, and taking almost the same argument as in the proof of Proposition 4.4, we can prove

$$(\partial_t^i u, \partial_t^i s) \in L^\infty([0,T], H^{2(k+1)-2i+1}(\Omega,\mathbb{R}^3)) \cap L^2([0,T], H^{2(k+1)-2i+2}(\Omega,\mathbb{R}^3)), \tag{4.27}$$

where $0 < T < T^*$ and $i \in \{0,\ldots,k+1\}$. Hence, the property $\mathscr{P}(k+1)$ is established.

Acknowledgements The authors are supported partially by NSFC (No. 11731001). The author Y. D. Wang is supported partially by NSFC (No. 11971400) and Guangdong Basic and Applied Basic Research Foundation (No. 2020A1515011019).

References

1. Alouges F., Soyeur A., On global weak solutions for Landau–Lifshitz equations: existence and nonuniqueness. Nonlinear Anal., 1992, 18(11): 1071–1084
2. Bejenaru I., Global results for Schrödinger maps in dimensions $n \ge 3$. Comm. Partial Differential Equations, 2008, 33(1–3): 451–477
3. Bejenaru I., On Schrödinger maps. Amer. J. Math., 2008, 130(4): 1033–1065
4. Bejenaru I., Ionescu A.D., Kenig C.E., Global existence and uniqueness of Schrödinger maps in dimensions $n \ge 4$. Adv. Math., 2007, 215(1): 263–291
5. Bejenaru I., Ionescu A.D., Kenig C.E., Tataru D., Global Schrödinger maps in dimensions $n \ge 2$: small data in the critical Sobolev spaces. Ann. of Math. (2), 2011, 173(3): 1443–1506
6. Bonithon G., Landau–Lifschitz–Gilbert equation with applied electric current. Discrete Contin. Dyn. Syst., 2007, (suppl.): 138–144
7. Boyer F., Fabrie P., Mathematical Tools for the Study of the Incompressible Navier–Stokes Equations and Related Models. Applied Mathematical Sciences, Vol. 183. New York: Springer, 2013
8. Carbou G., Fabrie P., Regular solutions for Landau–Lifschitz equation in a bounded domain. Differential Integral Equations, 2001, 14(2): 213–229
9. Carbou G., Jizzini R., Very regular solutions for the Landau–Lifschitz equation with electric current. Chinese Ann. Math. Ser. B, 2018, 39(5): 889–916
10. Chen B., Wang Y.D., Global weak solutions for Landau–Lifshitz flows and heat flows associated to micromagnetic energy functional. Commun. Pure Appl. Anal., 2021, 20(1): 319–338

11. Ding S.J., Wang C.Y., Finite time singularity of the Landau–Lifshitz–Gilbert equation. Int. Math. Res. Not. IMRN, 2007, 2007(4): Art. ID rnm012, 25 pp.
12. Ding W.Y., Wang Y.D., Schrödinger flow of maps into symplectic manifolds. Sci. China Ser. A, 1998, 41(7): 746–755
13. Ding W.Y., Wang Y.D., Local Schrödinger flow into Kähler manifold. Sci. China Ser. A, 2001, 44(11): 1446–1464
14. García-Cervera C.J., Wang X.-P., Spin-polarized transport: existence of weak solutions. Discre. Contin. Dyn. Syst. Ser. B, 2007, 7(1): 87–100
15. Gilbert T.L., A Lagrangian formulation of the gyromagnetic equation of the magnetization field. Phys. Rev., 1955, 100(4): 1243
16. Guo B.L., Hong M.C., The Landau–Lifshitz equation of the ferromagnetic spin chain and harmonic maps. Calc. Var. Partial Differential Equations, 1993, 1(3): 311–334
17. Ionescu A.D., Kenig C.E., Low-regularity Schrödinger maps. II. Global well-posedness in dimensions $d \geq 3$. Comm. Math. Phys., 2007, 271(2): 523–559
18. Jia Z.L., Wang Y.D., Global weak solutions to Landau–Lifshitz equations into compact Lie algebras. Front. Math. China, 2019, 14(6): 1163–1196
19. Jia Z.L., Wang Y.D., Global weak solutions to Landau–Lifshitz systems with spin-polarized transport. Discrete Contin. Dyn. Syst., 2020, 40(3): 1903–1935
20. Landau L.D., Lifshitz E.M., On the theory of the dispersion of magnetic permeability in ferromagnetic bodies. Phys. Zeitsch. der Sow., 1935, 8: 153–169
21. Melcher C., Global solvability of the Cauchy problem for the Landau–Lifshitz–Gilbert equation in higher dimensions. Indiana Univ. Math. J., 2012, 61(3): 1175–1200
22. Pang P.Y.H., Wang H.-Y., Wang Y.D., Schrödinger flow for maps into Kähler manifolds. Asian J. Math., 2001, 5(3): 509–534
23. Pang P.Y.H., Wang H.-Y., Wang Y.D., Schrödinger flow on Hermitian locally symmetric spaces. Comm. Anal. Geom., 2002, 10(4): 653–681
24. Pu X.K., Wang W.D., Partial regularity to the Landau–Lifshitz equation with spin accumulation. J. Differential Equations, 2020, 268(2): 707–737
25. Shpiro A., Levy P.M., Zhang S.F., Self-consistent treatment of nonequilibrium spin torques in magnetic multilayers. Phys. Rev. B, 2003, 67: 104430, 17 pp.
26. Simon J., Compact sets in the space $L^p(0, T; B)$. Ann. Mat. Pure Appl. (4), 1987, 146: 65–96
27. Sulem P.-L., Sulem C., Bardos C., On the continuous limit for a system of classical spins. Comm. Math. Phys., 1986, 107(3): 431–454
28. Tilioua M., Current-induced magnetization dynamics. Global existence of weak solutions. J. Math. Anal. Appl., 2011, 373(2): 635–642
29. Visintin A., On Landau–Lifshitz' equations for ferromagnetism. Japan J. Appl. Math., 1985, 2(1): 69–84
30. Wang C.Y., On Landau–Lifshitz equation in dimensions at most four. Indiana Univ. Math. J., 2006, 55(5): 1615–1644
31. Wang Y.D., Heisenberg chain systems from compact manifolds into S^2. J. Math. Phys., 1998, 39(1): 363–371
32. Wehrheim K., Uhlenbeck Compactness. EMS Series of Lectures in Mathematics, Zürich: Eur. Math. Soc., 2004.
33. Zhang S., Levy P.M., Fert A., Mechanisms of spin-polarized current-driven magnetication switching. Phys. Rev. Lett., 2002, 88: 236601, 4 pp.
34. Zhou L.Y., Guo B.L., Tan S.B., Existence and uniqueness of smooth solution for system of ferro-magnetic chain. Sci. China Ser. A, 1991, 34(3): 257–266