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RESEARCH ARTICLE

Mean-field type forward-backward doubly stochastic differential equations and related stochastic differential games

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Abstract We study a kind of partial information non-zero sum differential games of mean-field backward doubly stochastic differential equations, in which the coefficient contains not only the state process but also its marginal distribution, and the cost functional is also of mean-field type. It is required that the control is adapted to a sub-filtration of the filtration generated by the underlying Brownian motions. We establish a necessary condition in the form of maximum principle and a verification theorem, which is a sufficient condition for Nash equilibrium point. We use the theoretical results to deal with a partial information linear-quadratic (LQ) game, and obtain the unique Nash equilibrium point for our LQ game problem by virtue of the unique solvability of mean-field forward-backward doubly stochastic differential equation.

Keywords Non-zero sum stochastic differential game, mean-field, backward doubly stochastic differential equation (BDSDE), Nash equilibrium point, maximum principle

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1 Introduction

Pardoux and Peng [14] first introduced the following backward doubly stochastic differential equations (BDSDEs):

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$$
\begin{aligned}
y(t) = \zeta + \int_t^T f(s, y(s), z(s))\mathrm{d}s + \int_t^T g(s, y(s), z(s)) \overleftarrow{\mathrm{d}} B(s) \\
- \int_t^T z(s) \overrightarrow{\mathrm{d}} W(s), \quad 0 \leqslant t \leqslant T.
\end{aligned} \tag{1}
$$

Pardoux and Peng [14] established the existence and uniqueness of solution for BDSDEs (1), and the probabilistic interpretation for the solutions of a class of semilinear stochastic partial differential equations (SPDEs). By virtue of Malliavin calculus, Wen and Shi [22] extended the nonlinear stochastic Feynman-Kac formula in [14] to non-Markovian situation. Peng and Shi [15] introduced a type of forward-backward doubly stochastic differential equations (FBDSDEs):

$$
\begin{cases}
p(t) = x + \int_0^t F(s, p(s), y(s), q(s), z(s))\mathrm{d}s - \int_0^t q(s) \overleftarrow{\mathrm{d}} B(s) \\
\qquad + \int_0^t G(s, p(s), y(s), q(s), z(s)) \overrightarrow{\mathrm{d}} W(s), \\
y(t) = \Phi(p(T)) + \int_t^T f(s, p(s), y(s), q(s), z(s))\mathrm{d}s - \int_t^T z(s) \overrightarrow{\mathrm{d}} W(s) \\
\qquad + \int_t^T g(s, p(s), y(s), q(s), z(s)) \overleftarrow{\mathrm{d}} B(s).
\end{cases} \tag{2}
$$

Peng and Shi [15] gave the existence and uniqueness of solution for FBDSDEs (2) by method of continuation. Based on FBDSDEs, the interest for doubly stochastic optimal control problems grew a lot (see [7,16,17,23,27,28,31,32]).

Game theory was first introduced by Von Neumann and Morgenstern [18]. Nash [12] gave the classical notion of Nash equilibrium point for non-cooperate games. In recent years, stochastic differential game problems driven by stochastic differential equations (SDEs) have appeared (see [5,6,19,24,30]). Recently, An and Øksendal [1,2] and Kieu et al. [9] researched partial information differential games of stochastic differential equations with jump (SDEJ). Then this kind of partial information game problems was widely discussed (see [3,8,10,13,20,21,26]).

In this paper, we discuss a kind of partial information non-zero sum differential games of mean-field backward doubly stochastic differential equations (MF-BDSDE). We establish a necessary maximum principle under partial information and a sufficient condition for Nash equilibrium point. We use the theoretical results to research a partial information linear-quadratic (LQ) game. In order to obtain the unique Nash equilibrium point, we study a new kind of fully coupled mean-field forward-backward doubly stochastic differential equations (MF-FBDSDE), and get the existence and uniqueness theorem for solutions to such kind of equations under some monotonicity conditions.

This paper is structured as follows. We state our partial information differential game problem of MF-BDSDE in Section 2. In Section 3, we study fully coupled MF-FBDSDE, and give the existence and uniqueness theorem for solutions to such kind of equations under some monotonicity conditions. Section 4 is devoted to the necessary optimality conditions under partial information. In Section 5, we obtain the sufficient maximum principle of differential game of MF-BDSDE under partial information. In Section 6, we give a partial information LQ game as an example to show the applications of our theoretical results, and obtain the unique Nash equilibrium point for our LQ game problem by virtue of the unique solvability of MF-FBDSDE.

2 Statement of problems

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space on which are defined two mutually independent Brownian motions $\{W_t\}_{t\geqslant 0}$ and $\{B_t\}_{t\geqslant 0}$, with values, respectively, in $\mathbb{R}^d$ and $\mathbb{R}^l$. We denote by

$$\mathscr{F}_t := \mathscr{F}_t^W \vee \mathscr{F}_{t,T}^B, \quad \forall\, t \in [0,T],$$

where

$$\mathscr{F}_t^W := \sigma\{W_r;\, 0 \leqslant r \leqslant t\} \vee \mathscr{N}, \quad \mathscr{F}_{t,T}^B := \sigma\{B_T - B_r;\, t \leqslant r \leqslant T\} \vee \mathscr{N},$$

with $\mathscr{N}$ is the class of P-null sets of $\mathscr{F}$. In this case, the collection $\{\mathscr{F}_t,\, t \in [0,T]\}$ is neither increasing nor decreasing, while $\{\mathscr{F}_t^W,\, t \in [0,T]\}$ is an increasing filtration and $\{\mathscr{F}_{t,T}^B,\, t \in [0,T]\}$ is a decreasing filtration. We use the usual inner product $\langle \cdot, \cdot \rangle$ and Euclidean norm $|\cdot|$ in $\mathbb{R}^n$, $\mathbb{R}^m, \mathbb{R}^{m\times l}$, and $\mathbb{R}^{n\times d}$. The notation '$\top$' appearing in the superscripts denotes the transpose of a matrix. All the equalities and inequalities mentioned in this paper are in the sense of $\mathrm{d}t \times \mathrm{d}P$ almost surely on $[0,T] \times \Omega$.

Let

$$(\Omega^2, \mathscr{F}^2, P^2) = (\Omega \times \Omega, \mathscr{F} \otimes \mathscr{F}, P \otimes P)$$

be the completion of the product probability space of the above $(\Omega, \mathscr{F}, P)$ with itself, where we define

$$\mathscr{F}_t^2 = \mathscr{F}_t \otimes \mathscr{F}_t, \quad t \in [0,T],$$

and $\mathscr{F}_t \otimes \mathscr{F}_t$ being the completion of $\mathscr{F}_t \times \mathscr{F}_t$. It is worthy of noting that any random variable $\xi = \xi(\omega)$ defined on Ω can be extended naturally to Ω^2 as $\xi'(\omega, \omega') = \xi(\omega)$ with $(\omega, \omega') \in \Omega^2$. For $H = \mathbb{R}^n$, etc., let $L^1(\Omega^2, \mathscr{F}^2, P^2; H)$ be the set of random variable $\xi\colon \Omega^2 \to H$, which is $\mathscr{F}^2$-measurable such that

$$\mathbb{E}^2|\xi| \equiv \int_{\Omega^2} |\xi(\omega', \omega)| P(\mathrm{d}\omega') P(\mathrm{d}\omega) < \infty.$$

For any $\eta \in L^1(\Omega^2, \mathscr{F}^2, P^2; H)$, we denote

$$\mathbb{E}'\eta(\omega, \cdot) := \int_\Omega \eta(\omega, \omega')P(\mathrm{d}\omega').$$

Particularly, for example, if $\eta_1(\omega, \omega') = \eta_1(\omega')$, then

$$\mathbb{E}'\eta_1 = \int_\Omega \eta_1(\omega')P(\mathrm{d}\omega') = \mathbb{E}\eta_1.$$

We denote some spaces:

- $M^2(0, T; \mathbb{R}^n)$ is the space of all $\mathscr{F}_t$-measurable $\mathbb{R}^n$-valued processes v such that

$$\mathbb{E}\int_0^T |v(t, \omega)|^2 \mathrm{d}t < \infty;$$

- $L^2(\Omega, \mathscr{F}_T, P; \mathbb{R}^n)$ is the space of all $\mathscr{F}_T$-measurable $\mathbb{R}^n$-valued random variable ξ such that $\mathbb{E}|\xi|^2 < \infty$.

Consider the following MF-BDSDE:

$$\begin{cases} -\mathrm{d}y(t) = \mathbb{E}'f(t, y(t), z(t), y'(t), z'(t), v(t))\mathrm{d}t - z(t)\overrightarrow{\mathrm{d}}W(t) \\ \qquad\qquad + \mathbb{E}'g(t, y(t), z(t), y'(t), z'(t), v(t))\overleftarrow{\mathrm{d}}B(t), \\ y(T) = \xi, \end{cases} \tag{3}$$

where

$$\xi \in L^2(\Omega, \mathscr{F}_T, P; \mathbb{R}^n),$$
$$f\colon [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \to \mathbb{R}^n,$$
$$g\colon [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \to \mathbb{R}^{n\times l},$$

$v_1(\cdot)$ and $v_2(\cdot)$ are the control processes of Player 1 and Player 2, respectively, and $v(\cdot) = (v_1(\cdot), v_2(\cdot))$. We make use of the subscript i to mean the variables corresponding to Player i, $i = 1, 2$. The mean-field backward stochastic game system (3) means that, at the terminal time T, the two players have the same goal ξ.

We suppose that U_i is a nonempty convex subset of $\mathbb{R}^{k_i}$ $(i = 1, 2)$, and $\mathscr{E}_t^i \subseteq \mathscr{F}_t$ $(i = 1, 2)$ is a given sub-filtration which represents the information available to Player i at time $t \in [0, T]$, respectively. Our admissible control set is

$$\mathscr{U}_i = \left\{ v_i\colon [0, T] \times \Omega \to U_i \,\middle|\, v_i \text{ is } \mathscr{E}_t^i\text{-adapted, } \mathbb{E}\int_0^T |v_i(t)|^2 \mathrm{d}t < \infty \right\}, \quad i = 1, 2.$$

For Player i $(i = 1, 2)$, $v_i \in \mathscr{U}_i$ is called an open-loop admissible control.

We assume

(H1) f and g are continuously differentiable in (y, z, y', z', v_1, v_2);

(H2) the norm of $f_y, f_z, f_{y'}, f_{z'}, f_{v_1}, f_{v_2}, g_y, g_{y'}, g_{v_1}, g_{v_2}$ are bounded by $c > 0$, and the norm of $g_z, g_{z'}$ are bounded by $\alpha \in (0,1)$.

Now, if both $v_1(\cdot)$ and $v_2(\cdot)$ are admissible controls, and assumptions (H1) and (H2) hold, then MF-BDSDE (3) admits a unique solution

$$(y(\cdot), z(\cdot)) \in M^2(0,T;\mathbb{R}^n) \times M^2(0,T;\mathbb{R}^{n\times d})$$

(see [4,29,33]). Ensuring to achieve the goal ξ, the players have their own benefits, which are described by the following cost functionals:

$$J_i(v(\cdot)) = \mathbb{E}\left[\int_0^T \mathbb{E}' l_i(t, y(t), z(t), y'(t), z'(t), v(t))\mathrm{d}t + \Phi_i(y(0))\right],$$

where

$$v(\cdot) = (v_1(\cdot), v_2(\cdot)),$$
$$l_i \colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \to \mathbb{R},$$
$$\Phi_i \colon \mathbb{R}^n \to \mathbb{R}, \quad i = 1,2,$$

satisfying the condition

$$\mathbb{E}\left[\int_0^T |\mathbb{E}' l_i(t, y(t), z(t), y'(t), z'(t), v(t))|\mathrm{d}t + |\Phi_i(y(0))|\right] < \infty, \quad i = 1,2.$$

We also assume

(H3) l_i is continuously differentiable in (y, z, y', z', v_1, v_2), its partial derivatives are continuous in (y, z, y', z', v_1, v_2) and bounded by

$$c(1 + |y| + |z| + |y'| + |z'| + |v_1| + |v_2|);$$

(H4) Φ_i is continuously differentiable and Φ_{iy} is bounded by $c(1 + |y|)$.

Suppose that each player choose her/his appropriate admissible control $v_i(\cdot)$ $(i = 1,2)$ to maximize every cost functional $J_i(v_1(\cdot), v_2(\cdot))$. Then our game problem is to find a pair of admissible controls $(u_1(\cdot), u_2(\cdot)) \in \mathscr{U}_1 \times \mathscr{U}_2$

$$\begin{cases} J_1(u_1(\cdot), u_2(\cdot)) = \max\limits_{v_1(\cdot)\in\mathscr{U}_1} J_1(v_1(\cdot), u_2(\cdot)), \\ J_2(u_1(\cdot), u_2(\cdot)) = \max\limits_{v_2(\cdot)\in\mathscr{U}_2} J_2(u_1(\cdot), v_2(\cdot)). \end{cases} \tag{4}$$

We call the problem above a backward doubly stochastic differential game, and denote it by Problem (B). If an admissible controls $u(\cdot) = (u_1(\cdot), u_2(\cdot))$ satisfying (4) can be found, then we call it an equilibrium point of Problem (B) and denote the corresponding state trajectory by

$$(y(\cdot), z(\cdot)) = (y^u(\cdot), z^u(\cdot)).$$

3 Fully coupled MF-FBDSDE

To obtain the unique Nash equilibrium points for our above LQ game problem, we give an existence and uniqueness theorem of fully coupled MF-FBDSDE. Given an $n \times m$ full-rank matrix H. We denote some notations:

$$\zeta = \begin{pmatrix} p \\ y \\ q \\ z \end{pmatrix}, \quad \zeta' = \begin{pmatrix} p' \\ y' \\ q' \\ z' \end{pmatrix}, \quad A(t,\zeta,\zeta') = \begin{pmatrix} -H^\top f \\ HF \\ -H^\top g \\ HG \end{pmatrix}(t,\zeta,\zeta'),$$

where $H^\top g = (H^\top g_1, \dots, H^\top g_l)$, and $HG = (HG_1, \dots, HG_d)$.

Consider the following MF-FBDSDE:

$$\begin{cases} p(t) = \mathbb{E}'\Phi(y(0), y'(0)) + \int_0^t \mathbb{E}'F(s,\zeta(s),\zeta'(s))\mathrm{d}s - \int_0^t q(s)\overleftarrow{\mathrm{d}}B(s) \\ \qquad + \int_0^t \mathbb{E}'G(s,\zeta(s),\zeta'(s))\overrightarrow{\mathrm{d}}W(s), \\ y(t) = \xi - \int_t^T \mathbb{E}'f(s,\zeta(s),\zeta'(s))\mathrm{d}s - \int_t^T z(s)\overrightarrow{\mathrm{d}}W(s) \\ \qquad - \int_t^T \mathbb{E}'g(s,\zeta(s),\zeta'(s))\overleftarrow{\mathrm{d}}B(s), \end{cases} \tag{5}$$

where

$$f\colon \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times l} \times \mathbb{R}^{m\times d} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times l} \times \mathbb{R}^{m\times d} \to \mathbb{R}^m,$$
$$F\colon \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times l} \times \mathbb{R}^{m\times d} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times l} \times \mathbb{R}^{m\times d} \to \mathbb{R}^n,$$
$$g\colon \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times l} \times \mathbb{R}^{m\times d} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times l} \times \mathbb{R}^{m\times d} \to \mathbb{R}^{m\times l},$$
$$G\colon \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times l} \times \mathbb{R}^{m\times d} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times l} \times \mathbb{R}^{m\times d} \to \mathbb{R}^{n\times d},$$
$$\Phi\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n.$$

Definition 1 A quaternion $(p,y,q,z) \in M^2(0,T;\mathbb{R}^{n+m+n\times l+m\times d})$ is called an $\mathscr{F}_t$-measurable solution of MF-FBDSDEs (5), if (5) is satisfied.

We assume

(A1) for each $\zeta, \zeta' \in \mathbb{R}^{n+m+n\times l+m\times d}$, $A(\cdot,\zeta,\zeta')$ is an $\mathscr{F}_t$-measurable process defined on $[0,T]$ with

$$A(\cdot,0,0) \in M^2(0,T;\mathbb{R}^{n+m+n\times l+m\times d+n+m+n\times l+m\times d});$$

(A2) $A(t,\zeta,\zeta')$ and $\Phi(y)$ satisfy the Lipschitz conditions: there exist constants $k>0$ and $\lambda \in (0,1/2)$ such that

$$|F(t,\zeta,\zeta') - F(t,\overline{\zeta},\overline{\zeta}')|^2 \leqslant k(|\hat{p}|^2 + |\hat{y}|^2 + |\hat{q}|^2 + |\hat{z}|^2 + |\hat{p}'|^2 + |\hat{y}'|^2 + |\hat{q}'|^2 + |\hat{z}'|^2),$$

$$|f(t,\zeta,\zeta')-f(t,\overline{\zeta},\overline{\zeta}')|^2 \leqslant k(|\hat p|^2+|\hat y|^2+|\hat q|^2+|\hat z|^2+|\hat p'|^2+|\hat y'|^2+|\hat q'|^2+|\hat z'|^2),$$
$$|G(t,\zeta,\zeta')-G(t,\overline{\zeta},\overline{\zeta}')|^2 \leqslant k(|\hat p|^2+|\hat y|^2+|\hat z|^2+|\hat p'|^2+|\hat y'|^2+|\hat z'|^2)+\lambda(|\hat q|^2+|\hat q'|^2),$$
$$|g(t,\zeta,\zeta')-g(t,\overline{\zeta},\overline{\zeta}')|^2 \leqslant k(|\hat p|^2+|\hat y|^2+|\hat q|^2+|\hat p'|^2+|\hat y'|^2+|\hat q'|^2)+\lambda(|\hat z|^2+|\hat z'|^2),$$
$$\forall\, \zeta,\zeta' \in \mathbb{R}^{n+m+n\times l+m\times d}, \quad \forall\, t\in[0,T],$$
$$\hat p = p-\overline{p}, \quad \hat y = y-\overline{y}, \quad \hat q = q-\overline{q}, \quad \hat z = z-\overline{z},$$
$$\hat p' = p'-\overline{p}', \quad \hat y' = y'-\overline{y}', \quad \hat q' = q'-\overline{q}', \quad \hat z' = z'-\overline{z}',$$
$$|\Phi(y,y')-\Phi(\overline{y},\overline{y}')| \leqslant k|y-\overline{y}|+k|y'-\overline{y}'|, \quad \forall\, y,y',\overline{y},\overline{y}'\in\mathbb{R}^n;$$

(A3) $A(t,\zeta,\zeta')$ and $\Phi(y)$ satisfy the monotonic conditions:

$$\begin{aligned}&\langle A(t,\zeta,\zeta')-A(t,\overline{\zeta},\overline{\zeta}'),\zeta-\overline{\zeta}\rangle\\ &\quad\leqslant -\mu_1(|H(p-\overline{p})|^2+|H(q-\overline{q})|^2)-\mu_2(|H^\top(y-\overline{y})|^2+|H^\top(z-\overline{z})|^2),\end{aligned}$$
$$\forall\,\zeta=(p,y,q,z)^\top, \zeta'=(p',y',q',z')^\top, \overline{\zeta}=(\overline{p},\overline{y},\overline{q},\overline{z})^\top,$$
$$\overline{\zeta}'=(\overline{p}',\overline{y}',\overline{q}',\overline{z}')^\top\in\mathbb{R}^{n+m+n\times l+m\times d}, \forall\, t\in[0,T],$$
$$\langle \Phi(y,y')-\Phi(\overline{y},\overline{y}'),y-\overline{y}\rangle \leqslant -\beta_2|H^\top(y-\overline{y})|^2, \quad \forall\, y,\overline{y}\in\mathbb{R}^n,$$

where μ_1, μ_2, and β_2 are given nonnegative constants with

$$\mu_1+\mu_2>0, \quad \mu_1+\beta_2>0.$$

Moreover, we have $\mu_1>0$ (resp., $\mu_2>0$, $\beta_2>0$) when $m<n$ (resp., $m>n$).

By the similar arguments of Yu and Ji [30], Wang and Yu [20], and Min et al. [11], we have the following existence and uniqueness theorem.

Theorem 1 *We assume that* (A1)–(A3) *hold. Then there exists a unique solution* $(p(t),y(t),q(t),z(t))\in M^2(0,T;\mathbb{R}^{n+m+n\times l+m\times d})$ *for MF-FBDSDE* (5).

Remark 1 The condition $\lambda\in(0,1/2)$ is necessary to construct the contractive mapping in the proof of Theorem 1, that is, when $1/2<\lambda<1$, the contractive mapping to prove Theorem 1 cannot be obtained.

4 A partial information necessary maximum principle

Let $u(\cdot)=(u_1(\cdot),u_2(\cdot))$ be an equilibrium point of Problem (B), and let $(y(\cdot),z(\cdot))$ be the corresponding optimal state trajectory of game system (3). Let $(v_1(\cdot),v_2(\cdot))$ satisfy

$$(u_1(\cdot)+v_1(\cdot),u_2(\cdot)+v_2(\cdot))\in\mathscr{U}_1\times\mathscr{U}_2.$$

Since $\mathscr{U}_1$ and $\mathscr{U}_2$ are convex, for any $\rho\in[0,1]$,

$$(u_1^\rho(\cdot),u_2^\rho(\cdot))=(u_1(\cdot)+\rho v_1(\cdot),u_1(\cdot)+\rho v_1(\cdot))$$

is also in $\mathscr{U}_1 \times \mathscr{U}_2$. For the controls $(u_1^\rho(\cdot), u_2(\cdot))$ and $(u_1(\cdot), u_2^\rho(\cdot))$, the corresponding state trajectories of system (3) are denoted as $(y^{u_1^\rho}(\cdot), z^{u_1^\rho}(\cdot))$ and $(y^{u_2^\rho}(\cdot), z^{u_2^\rho}(\cdot))$, respectively.

For $\varphi = f$, g, and l, respectively, we denote

$$\begin{aligned}
\varphi(t) &= \varphi(t, y(t), z(t), y'(t), z'(t), u_1(t), u_2(t)),\\
\varphi^v(t) &= \varphi(t, y(t), z(t), y'(t), z'(t), v_1(t), v_2(t)),\\
\varphi^{u_1^\rho}(t) &= \varphi(t, y(t), z(t), y'(t), z'(t), u_1^\rho(t), u_2(t)),\\
\varphi^{u_2^\rho}(t) &= \varphi(t, y(t), z(t), y'(t), z'(t), u_1(t), u_2^\rho(t)).
\end{aligned}$$

Our variational equations are the following: for $i = 1, 2$,

$$\begin{cases}
-\mathrm{d}y_i^1(t) = \mathbb{E}'[f_y(t)y_i^1(t) + f_z(t)z_i^1(t) + f_{y'}(t)(y_i^1(t))' \\
\qquad\qquad + f_{z'}(t)(z_i^1(t))' + f_{v_i}(t)v_i(t)]\mathrm{d}t - z_i^1(t)\overrightarrow{\mathrm{d}}W(t) \\
\qquad\qquad + \mathbb{E}'[g_y(t)y_i^1(t) + g_z(t)z_i^1(t) + g_{y'}(t)(y_i^1(t))' \\
\qquad\qquad + g_{z'}(t)(z_i^1(t))' + g_{v_i}(t)v_i(t)]\overleftarrow{\mathrm{d}}B(t), \\
y_i^1(T) = 0.
\end{cases} \tag{6}$$

By (H1)–(H4), we know that MF-BDSDE (6) admits a unique adapted solution $(y_i^1(t), z_i^1(t)) \in M^2(0,T;\mathbb{R}^n) \times M^2(0,T;\mathbb{R}^{n\times d})$, $i = 1, 2$.

For $t \in [0, T]$ and $\rho > 0$, we set

$$\begin{aligned}
\widetilde{y}_i^\rho(t) &= \frac{y^{u_i^\rho}(t) - y_i(t)}{\rho} - y_i^1(t),\\
\widetilde{z}_i^\rho(t) &= \frac{z^{u_i^\rho}(t) - z_i(t)}{\rho} - z_i^1(t),
\end{aligned} \qquad i = 1, 2.$$

We derive the following result.

Lemma 1 *We assume that* (H1)–(H4) *hold. Then, for* $i = 1, 2$,

$$\lim_{\rho\to 0} \sup_{0\leqslant t\leqslant T} \mathbb{E}|\widetilde{y}_i^\rho(t)|^2 = 0, \tag{7}$$

$$\lim_{\rho\to 0} \mathbb{E}\int_0^T |\widetilde{z}_i^\rho(t)|^2\mathrm{d}t = 0. \tag{8}$$

Proof For $i = 1$, we have

$$\begin{cases}
-\mathrm{d}\widetilde{y}_1^\rho(t) = \Big[\dfrac{1}{\rho}\mathbb{E}'(f^{u_1^\rho}(t) - f(t)) - \mathbb{E}'(f_y(t)y_1^1(t) + f_z(t)z_1^1(t) \\
\qquad\qquad + f_{y'}(t)(y_1^1(t))' + f_{z'}(t)(z_1^1(t))' + f_{v_1}(t)v_1(t))\Big]\mathrm{d}t - \widetilde{z}_1^\rho(t)\overrightarrow{\mathrm{d}}W(t) \\
\qquad\qquad + \Big[\dfrac{1}{\rho}\mathbb{E}'(g^{u_1^\rho}(t) - g(t)) - \mathbb{E}'(g_y(t)y_1^1(t) + g_z(t)z_1^1(t) \\
\qquad\qquad + g_{y'}(t)(y_1^1(t))' + g_{z'}(t)(z_1^1(t))' + g_{v_1}(t)v_1(t))\Big]\overleftarrow{\mathrm{d}}B(t), \\
\widetilde{y}_1^\rho(T) = 0,
\end{cases}$$

or

$$
\begin{cases}
-\mathrm{d}\widetilde{y}_1^\rho(t) = \mathbb{E}'[A_1^\rho(t)\widetilde{y}_1^\rho(t) + B_1^\rho(t)\widetilde{z}_1^\rho(t) + \overline{A}_1^\rho(t)(\widetilde{y}_1^\rho(t)) + \overline{B}_1^\rho(t)(\widetilde{z}_1^\rho(t))' \\
\qquad\qquad + G_1^\rho(t)]\mathrm{d}t + \mathbb{E}'[C_1^\rho(t)\widetilde{y}_1^\rho(t) + D_1^\rho(t)\widetilde{z}_1^\rho(t) + \overline{C}_1^\rho(t)(\widetilde{y}_1^\rho(t)) \\
\qquad\qquad + \overline{D}_1^\rho(t)(\widetilde{z}_1^\rho(t))' + G_2^\rho(t)]\overleftarrow{\mathrm{d}}B(t) - \widetilde{z}_1^\rho(t)\overrightarrow{\mathrm{d}}W(t), \\
\widetilde{y}_1^\rho(T) = 0,
\end{cases}
$$

where we denote

$$
\begin{aligned}
(\Theta) = (&t, y_1(t) + \lambda\rho(y_1^1(t) + \widetilde{y}_1^\rho(t)), z_1(t) + \lambda\rho(z_1^1(t) + \widetilde{z}_1^\rho(t)), \\
&(y_1(t))' + \lambda\rho((y_1^1(t))' + (\widetilde{y}_1^\rho(t))'), (z_1(t))' + \lambda\rho((z_1^1(t))' + (\widetilde{z}_1^\rho(t))'), \\
&u_1(t) + \lambda\rho v_1(t), u_2(t)),
\end{aligned}
$$

and

$$
\begin{aligned}
A_1^\rho(t) &= \int_0^1 f_y(\Theta)\mathrm{d}\lambda, \quad B_1^\rho(t) = \int_0^1 f_z(\Theta)\mathrm{d}\lambda, \\
\overline{A}_1^\rho(t) &= \int_0^1 f_{y'}(\Theta)\mathrm{d}\lambda, \quad \overline{B}_1^\rho(t) = \int_0^1 f_{z'}(\Theta)\mathrm{d}\lambda, \\
C_1^\rho(t) &= \int_0^1 g_y(\Theta)\mathrm{d}\lambda, \quad D_1^\rho(t) = \int_0^1 g_z(\Theta)\mathrm{d}\lambda, \\
\overline{C}_1^\rho(t) &= \int_0^1 g_{y'}(\Theta)\mathrm{d}\lambda, \quad \overline{D}_1^\rho(t) = \int_0^1 g_{z'}(\Theta)\mathrm{d}\lambda,
\end{aligned}
$$

$$
\begin{aligned}
G_1^\rho(t) = &\int_0^1 (f_{v_1}(\Theta) - f_{v_1}(t))v_1(t)\mathrm{d}\lambda + [A_1^\rho(t) - f_y(t)]y^1(t) + [B_1^\rho(t) \\
&- f_z(t)]z^1(t) + [\overline{A}_1^\rho(t) - f_{y'}(t)](y^1(t))' + [\overline{B}_1^\rho(t) - f_{z'}(t)](z^1(t))', \\
G_2^\rho(t) = &\int_0^1 (g_{v_1}(\Theta) - g_{v_1}(t))v_1(t)\mathrm{d}\lambda + [C_1^\rho(t) - g_y(t)]y^1(t) + [D_1^\rho(t) \\
&- g_z(t)]z^1(t) + [\overline{C}_1^\rho(t) - g_{y'}(t)](y^1(t))' + [\overline{D}_1^\rho(t) - g_{z'}(t)](z^1(t))'.
\end{aligned}
$$

Using Itô's formula to $|\widetilde{y}_1^\rho(t)|^2$ on $[t,T]$, we get

$$
\begin{aligned}
&\mathbb{E}|\widetilde{y}_1^\rho(t)|^2 + \mathbb{E}\int_t^T |\widetilde{z}_1^\rho(s)|^2\mathrm{d}s \\
&= 2\mathbb{E}\mathbb{E}'\int_t^T |\langle \widetilde{y}_1^\rho(s), A_1^\rho(s,\cdot)\widetilde{y}_1^\rho(s) + B_1^\rho(s,\cdot)\widetilde{z}_1^\rho(s) + \overline{A}_1^\rho(s,\cdot)(\widetilde{y}_1^\rho(s))' \\
&\quad + \overline{B}_1^\rho(s,\cdot)(\widetilde{z}_1^\rho(s))' + G_1^\rho(s,\cdot)\rangle|\mathrm{d}s + \mathbb{E}\mathbb{E}'\int_t^T |C_1^\rho(s,\cdot)\widetilde{y}_1^\rho(s) \\
&\quad + D_1^\rho(s,\cdot)\widetilde{z}_1^\rho(s) + \overline{C}_1^\rho(s,\cdot)(\widetilde{y}_1^\rho(s))' + \overline{D}_1^\rho(s,\cdot)(\widetilde{z}_1^\rho(s))' + G_2^\rho(s,\cdot)|^2\mathrm{d}s \\
&\leqslant K_0\mathbb{E}\int_t^T |\widetilde{y}_1^\rho(s)|^2\mathrm{d}s + K_1\mathbb{E}\int_t^T |\widetilde{z}_1^\rho(s)|^2\mathrm{d}s \\
&\quad + K_2\alpha\bigg(\mathbb{E}\int_t^T |G_1^\rho(s)|^2 + \mathbb{E}\int_t^T |G_2^\rho(s)|^2\bigg)\mathrm{d}s.
\end{aligned}
$$

By Grownwall's inequality, we easily derive the desired result (7). Similarly, we can show that conclusion (8) holds for $i = 2$. □

Since $(u_1(\cdot), u_2(\cdot))$ is an equilibrium point of Problem (B), we have

$$\rho^{-1}[J_1(u_1^\rho(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)] \leqslant 0, \tag{9}$$

$$\rho^{-1}[J_2(u_1(\cdot), u_2^\rho(\cdot)) - J_2(u_1(\cdot), u_2(\cdot)] \leqslant 0. \tag{10}$$

Combining Lemma 1, (9), and (10), we derive the following variational inequality.

Lemma 2 *We assume that* (H1)–(H4) *hold. Then*

$$\begin{aligned}&\mathbb{E}\int_0^T \mathbb{E}'[l_{iy}(t)y_i^1(t) + l_{iz}(t)z_i^1(t) + l_{iy'}(t)(y_i^1(t))' \\ &+ l_{iz'}(t)(z_i^1(t))' + l_{iv_i}(t)v_i(t)]\mathrm{d}t + \mathbb{E}[\Phi_{iy}(y(0))y_i^1(0)] \leqslant 0, \quad i = 1, 2. \end{aligned}\tag{11}$$

Proof For $i = 1$, from (7), we derive

$$\begin{aligned}&\rho^{-1}[\Phi_1(y^{u_1^\rho}(0)) - \Phi_1(y(0))] \\ &= \rho^{-1}\mathbb{E}\int_0^1 \Phi_{1y}(y(0) + \lambda(y^{u_1^\rho}(0) - y(0)))(y^{u_1^\rho}(0) - y(0))\mathrm{d}\lambda \\ &\to \mathbb{E}[\Phi_{1y}(y(0))y_1^1(0)], \quad \rho \to 0.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\rho^{-1}\Big\{\mathbb{E}\int_0^T \mathbb{E}'[l_1^{u_1^\rho}(t) - l_1(t)]\mathrm{d}t\Big\} \\ &\to \mathbb{E}\int_0^T \mathbb{E}'[l_{1y}(t)y_1^1(t) + l_{1z}(t)z_1^1(t) + l_{1y'}(t)(y_1^1(t))' \\ &\quad + l_{1z'}(t)(z_1^1(t))' + l_{1v_1}(t)v_1(t)]\mathrm{d}t, \quad \rho \to 0.\end{aligned}$$

Let $\rho \to 0$ in (9). Then we get that (11) holds for $i = 1$. Similarly, from (10), we can show that the conclusion holds for $i = 2$. □

We define the Hamiltonian function

$$H_i \colon [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \mathbb{R}^n \to \mathbb{R}, \quad i = 1, 2,$$

as follows:

$$\begin{aligned}&H_i(t, y, z, y', z', v_1, v_2, p_i) \\ &= -\langle f(t, y, z, y', z', v_1, v_2), p_i\rangle - \langle g(t, y, z, y', z', v_1, v_2), q_i\rangle \\ &\quad + l_i(t, y, z, y', z', v_1, v_2), \quad i = 1, 2.\end{aligned}$$

Let

$$\left.\begin{aligned}H_i(t) &= H_i(t, y, z, y', z', u_1, u_2, p_i), \\ H_i^{v_1,v_2}(t) &= H_i(t, y, z, y', z', v_1, v_2, p_i),\end{aligned}\right. \quad i = 1, 2.$$

We introduce the following adjoint equation:

$$\begin{cases} \mathrm{d}p_i(t) = -\mathbb{E}'[H_{iy}^{v_1,v_2}(t) + H_{iy'}^{v_1,v_2}(t)]\mathrm{d}t \\ \qquad\qquad -\mathbb{E}'[H_{iz}^{v_1,v_2}(t) + H_{iz'}^{v_1,v_2}(t)]\overrightarrow{\mathrm{d}}W(t) - q_i(t)\overleftarrow{\mathrm{d}}B(t), \quad i=1,2. \\ p_i(0) = -\Phi_{iy}(y(0)), \end{cases} \tag{12}$$

Starting from the variational inequality (11), we can now state the necessary optimality conditions.

Theorem 2 (Partial information necessary maximum principle) *We assume that* (H1) *and* (H2) *hold. Suppose that* $(u_1(\cdot),u_2(\cdot))$ *is an equilibrium point of Problem* (B) *and* $(y(\cdot),z(\cdot))$ *is the corresponding state trajectory. Then we have*

$$\mathbb{E}[\langle H_{1v_1}(t), v_1 - u_1(t)\rangle \mid \mathscr{E}_t^1] \leqslant 0,$$
$$\mathbb{E}[\langle H_{2v_2}(t), v_2 - u_2(t)\rangle \mid \mathscr{E}_t^2] \leqslant 0,$$

hold for any $(v_1,v_2)\in U_1\times U_2$*, a.e., a.s., where* $p_i(\cdot)$ $(i=1,2)$ *is the solution of the adjoint equation* (12).

Proof For $i=1$, using Itô's formula to $\langle y_1^1(t), p_1(t)\rangle$, we obtain

$$\begin{aligned}\mathbb{E}\int_0^T &\mathbb{E}'[l_{1y}(t)y_1^1(t) + l_{1z}(t)z_1^1(t) + l_{1y'}(t)(y_1^1(t))' \\ &+ l_{1z'}(t)(z_1^1(t))' + l_{1v_1}(t)v_1(t)]\mathrm{d}t + \mathbb{E}[\Phi_{1y}(y(0))y_1^1(0)] \\ &= \mathbb{E}\langle -f_{v_1}^\top(t)p_1(t) + l_{1v_1}(t), v_1(t)\rangle \mathrm{d}t.\end{aligned}$$

From Lemma 2, we have

$$\mathbb{E}\int_0^T \langle H_{1v_1}(t), v_1(t)\rangle \mathrm{d}t \leqslant 0.$$

Because $v_1(t)$ satisfies $u_1(t)+v_1(t)\in\mathscr{U}_1$, we have

$$\mathbb{E}\int_0^T \langle H_{1v_1}(t), v_1 - u_1(t)\rangle \mathrm{d}t \leqslant 0, \quad \forall\, v_1 \in U_1,$$

which implies that

$$\mathbb{E}\langle H_{1v_1}(t), v_1 - u_1(t)\rangle \leqslant 0, \quad \forall\, v_1\in U_1.$$

Now, let $v_1(t)\in U_1$ be a deterministic element and F be an arbitrary element of the σ-algebra $\mathscr{E}_t^1$. And set

$$w_1(t) = v_1(t)\mathbf{1}_F + u_1(t)\mathbf{1}_{\Omega-F}.$$

It is obvious that w_1 is an admissible control.

Applying the above inequality with w_1, we get

$$\mathbb{E}[\mathbf{1}_F\langle H_{1v_1}(t), v_1 - u_1(t)\rangle] \leqslant 0, \quad \forall\, F \in \mathscr{E}_t^1,$$

which implies that

$$\mathbb{E}[\langle H_{1v_1}(t), v_1 - u_1(t)\rangle | \mathscr{E}_t^1] \leqslant 0, \quad \forall\, v_1 \in U_1, \text{ a.e., a.s.}$$

Proceeding in the same way as the above arguments, we can show that the other inequality holds for any $v_2 \in U_2$. Then the proof is completed. □

5 A partial information sufficient maximum principle

In this section, we investigate a sufficient maximum principle for Problem (B). Let $(y(t), z(t), u_1(t), u_2(t))$ be a quintuple satisfying (3) and suppose that there exists a solution $p_i(t)$ of the corresponding adjoint forward SDE (12). We assume

(H5) for $i = 1, 2$, and for all $t \in [0, T]$, $H_i(t, y, z, y', z', v_1, v_2, p_i)$ is convex in (y, z, y', z', v_1, v_2), and $\Phi_i(y)$ is convex in y.

Let

$$\begin{aligned} H_i(t) &= H_i(t, y(t), z(t), y'(t), z'(t), u_1(t), u_2(t), p_i(t)), \\ H_i^{v_1}(t) &= H_i(t, y(t), z(t), y'(t), z'(t), v_1(t), u_2(t), p_i(t)), \qquad i = 1, 2, \\ H_i^{v_2}(t) &= H_i(t, y(t), z(t), y'(t), z'(t), u_1(t), v_2(t), p_i(t)), \end{aligned}$$

and

$$\begin{aligned} \varphi(t) &= \varphi(t, y(t), z(t), y'(t), z'(t), u_1(t), u_2(t)), \\ \varphi^{v_1}(t) &= \varphi(t, y(t), z(t), y'(t), z'(t), v_1(t), u_2(t)), \\ \varphi^{v_2}(t) &= \varphi(t, y(t), z(t), y'(t), z'(t), u_1(t), v_2(t)), \end{aligned}$$

where $\varphi = f$, g, and l, respectively.

Theorem 3 (Partial information sufficient maximum principle) *We assume that* (H1)–(H5) *hold. Moreover, the following partial information maximum conditions hold*:

$$\mathbb{E}[H_1(t) \mid \mathscr{E}_t^1] = \max_{v_1 \in \mathscr{U}_1} \mathbb{E}[H_1^{v_1}(t) \mid \mathscr{E}_t^1], \tag{13}$$

$$\mathbb{E}[H_2(t) \mid \mathscr{E}_t^2] = \max_{v_2 \in \mathscr{U}_2} \mathbb{E}[H_2^{v_2}(t) \mid \mathscr{E}_t^2]. \tag{14}$$

Then $(u_1(\cdot), u_2(\cdot))$ *is an equilibrium point of Problem* (B).

Proof For any $v_1(\cdot) \in \mathscr{U}_1$, we consider

$$J_1(u_1(\cdot), u_2(\cdot)) - J_1(v_1(\cdot), u_2(\cdot)) = \mathbf{I}_1 + \mathbf{I}_2,$$

where

$$\mathbf{I}_1 := \mathbb{E}\int_0^T \mathbb{E}'[l_1(t) - l_1^{v_1}(t)]\mathrm{d}t,$$

$$\mathbf{I}_2 := \mathbb{E}[\Phi_1(y(0)) - \Phi_1(y^{v_1}(0))].$$

Now, using Itô's formula to $\langle p_1(t), y(t) - y^{v_1}(t)\rangle$ on $[0,T]$, we get

$$\begin{aligned}
&\mathbb{E}\langle \Phi_{1y}(y(0)), y(0) - y^{v_1}(0)\rangle \\
&\quad = \mathbb{E}\int_0^T \mathbb{E}'[\langle y(t) - y^{v_1}(t), -H_{1y}(t)\rangle + \langle y'(t) - (y^{v_1}(t))', -H_{1y'}(t)\rangle]\mathrm{d}t \\
&\qquad + \mathbb{E}\int_0^T \mathbb{E}'[\langle z(t) - z^{v_1}(t), -H_{1z}(t)\rangle + \langle z'(t) - (z^{v_1}(t))', -H_{1z'}(t)\rangle]\mathrm{d}t \\
&\qquad - \mathbb{E}\int_0^T \mathbb{E}'[\langle p_1(t), f(t) - f^{v_1}(t)\rangle]\mathrm{d}t.
\end{aligned} \tag{15}$$

Moreover, by virtue of (15) and the convexity of Φ_1, it instantly follows that

$$\mathbf{I}_2 \geqslant \mathbb{E}\langle \Phi_{1y}(y(0)), y(0) - y^{v_1}(0)\rangle = -\Xi_1 + \Xi_2, \tag{16}$$

where

$$\begin{aligned}
\Xi_1 := {}& \mathbb{E}\int_0^T \mathbb{E}'[\langle y(t) - y^{v_1}(t), H_{1y}(t)\rangle + \langle y'(t) - (y^{v_1}(t))', H_{1y'}(t)\rangle]\mathrm{d}t \\
& + \mathbb{E}\int_0^T \mathbb{E}'[\langle z(t) - z^{v_1}(t), H_{1z}(t)\rangle + \langle z'(t) - (z^{v_1}(t))', H_{1z'}(t)\rangle]\mathrm{d}t,
\end{aligned}$$

$$\Xi_2 := -\mathbb{E}\int_0^T \mathbb{E}'[\langle p_1(t), f(t) - f^{v_1}(t)\rangle]\mathrm{d}t.$$

Noting the definition of H_1 and $\mathbf{I}_1$, we have

$$\begin{aligned}
\mathbf{I}_1 &= \mathbb{E}\int_0^T \mathbb{E}'[H_1(t) - H_1^{v_1}(t)]\mathrm{d}t + \mathbb{E}\int_0^T \mathbb{E}'[\langle p_1(t), f(t) - f^{v_1}(t)\rangle]\mathrm{d}t \\
&= \Xi_3 - \Xi_2,
\end{aligned}$$

where

$$\Xi_3 := \mathbb{E}\int_0^T \mathbb{E}'[H_1(t) - H_1^{v_1}(t)]\mathrm{d}t. \tag{17}$$

Using the convexity of $H_1(t, y, z, y', z', v_1, v_2, p_1)$ with respect to (y, z, y', z', v_1, v_2), we obtain

$$\begin{aligned}
H_1(t) - H_1^{v_1}(t) \geqslant{}& H_{1y}(t)(y(t) - y^{v_1}(t)) + H_{1z}(t)(z(t) - z^{v_1}(t)) \\
& + H_{1y'}(t)((y'(t) - y^{v_1}(t))') \\
& + H_{1z'}(t)((z'(t) - z^{v_1}(t))') + H_{1u_1}(t)(u_1(t) - v_1(t)).
\end{aligned} \tag{18}$$

Since $v_1 \to \mathbb{E}[H_1^{v_1}(t)|\mathscr{E}_t^1]$, $v_1 \in \mathscr{U}_1$, is maximal at $u_1(t)$, and $v_1(t)$ and $u_1(t)$ are $\mathscr{E}_t^1$-measurable, we get

$$\mathbb{E}[H_{1u_1}(t) \mid \mathscr{E}_t^1](u_1(t) - v_1(t)) = \mathbb{E}[H_{1u_1}(t)(u_1(t) - v_1(t)) \mid \mathscr{E}_t^1] \geqslant 0. \tag{19}$$

Hence, combining (17)–(19), we obtain

$$\begin{aligned}\Xi_3 \geqslant\ & \mathbb{E}\int_0^T \mathbb{E}'[\langle y(t) - y^{v_1}(t), -H_{1y}(t)\rangle + \langle y'(t) - (y^{v_1}(t))', -H_{1y'}(t)\rangle]\mathrm{d}t \\ & + \mathbb{E}\int_0^T \mathbb{E}'[\langle z(t) - z^{v_1}(t), -H_{1z}(t)\rangle + \langle z'(t) - (z^{v_1}(t))', -H_{1z'}(t)\rangle]\mathrm{d}t \\ =\ & \Xi_1.\end{aligned} \tag{20}$$

Therefore, it follows from (13), (16), and (20) that

$$\begin{aligned}J_1(u_1(\cdot), u_2(\cdot)) - J_1(v_1(\cdot), u_2(\cdot)) &\geqslant \Xi_3 - \Xi_2 - \Xi_1 + \Xi_2 \\ &\geqslant \Xi_1 - \Xi_2 - \Xi_1 + \Xi_2 \\ &= 0.\end{aligned}$$

Then it implies that

$$J_1(u_1(\cdot), u_2(\cdot)) = \max_{v_1(\cdot)\in\mathscr{U}_1} J_1(v_1(\cdot), u_2(\cdot)).$$

In the same way, from (14),

$$J_2(u_1(\cdot), u_2(\cdot)) = \max_{v_2(\cdot)\in\mathscr{U}_2} J_2(u_1(\cdot), v_2(\cdot)).$$

Hence, we draw the desired conclusion. The proof is completed. □

6 A partial information LQ case

In this section, we apply the above results to study the partial information LQ differential games of MF-BDSDE. For notational simplification, we assume

$$n = d = l = k_1 = k_2 = 1, \quad U_1 = U_2 = \mathbb{R}, \quad \mathscr{E}_t^1 = \mathscr{E}_t^2 = \mathscr{E}_t \subseteq \mathscr{F}_t.$$

Consider

$$\begin{cases} -\mathrm{d}y(t) = \mathbb{E}'[A(t)y(t) + B(t)z(t) + \overline{A}(t)y'(t) + \overline{B}(t)z'(t) \\ \qquad\qquad + E_1(t)v_1(t) + E_2(t)v_2(t)]\mathrm{d}t - z(t)\overrightarrow{\mathrm{d}}W(t) \\ \qquad\qquad + \mathbb{E}'[C(t)y(t) + D(t)z(t) + \overline{C}(t)y'(t) + \overline{D}(t)z'(t)]\overleftarrow{\mathrm{d}}B(t), \\ y(T) = \xi. \end{cases}$$

The cost functional is

$$J_i(v_1(\cdot),v_2(\cdot)) = \frac{1}{2}\mathbb{E}\bigg[\int_0^T \mathbb{E}'(M_i(t)v_i^2(t) + N_i(t)(y(t))^2 + \overline{N}_i(t)(y'(t))^2)\mathrm{d}t + L_i(y(0))^2\bigg], \quad i=1,2,$$

where constants $L_i \geqslant 0$, $i=1,2$. Functions $A(\cdot)$, $\overline{A}(\cdot)$, $B(\cdot)$, $\overline{B}(\cdot)$, $C(\cdot)$, $\overline{C}(\cdot)$, $D(\cdot)$, $\overline{D}(\cdot)$, $E_1(\cdot)$, $E_2(\cdot)$ are bounded and deterministic; $N_i(\cdot)$, $\overline{N}_i(\cdot)$, $i=1,2$, are nonnegative, deterministic, and bounded; $M_i(\cdot)$, $i=1,2$, are positive, deterministic, and bounded, and $M_i^{-1}(\cdot)$, $i=1,2$, are also bounded. Our task is to find $(u_1(\cdot),u_2(\cdot)) \in \mathscr{U}_1 \times \mathscr{U}_2$ satisfying (4).

Theorem 4 *The mapping*

$$\begin{cases} u_1(t) = M_1^{-1}(t)E_1(t)\mathbb{E}[p_1(t) \mid \mathscr{E}_t], \\ u_2(t) = M_2^{-1}(t)E_2(t)\mathbb{E}[p_2(t) \mid \mathscr{E}_t], \end{cases}$$

is a Nash equilibrium point for the above LQ game problem, where $(p_1(t), p_2(t), q_1(t), q_2(t), y(t), z(t))$ *is the solution of the following MF-FBDSDE: for* $i=1,2$,

$$\begin{cases} \mathrm{d}p_i(t) = \mathbb{E}'[A(t)p_i(t) + \overline{A}(t)p_i'(t) + C(t)q_i(t) + \overline{C}(t)q_i'(t) \\ \qquad\qquad - N_i(t)y(t) - \overline{N}_i(t)y'(t)]\mathrm{d}t + q_i(t)\overleftarrow{\mathrm{d}}B(t) \\ \qquad\qquad + \mathbb{E}'[B(t)p_i(t) + \overline{B}(t)p_i'(t) + D(t)q_i(t) + \overline{D}(t)q_i'(t)]\overrightarrow{\mathrm{d}}W(t), \\ -\mathrm{d}y(t) = \mathbb{E}'\{A(t)y(t) + B(t)z(t) + \overline{A}(t)y'(t) + \overline{B}(t)z'(t) \\ \qquad\qquad + E_1^2(t)M_1^{-1}(t)\mathbb{E}[p_1(t)|\mathscr{E}_t] + E_2^2(t)M_2^{-1}(t)\mathbb{E}[p_2(t)|\mathscr{E}_t]\}\mathrm{d}t \\ \qquad\qquad + \mathbb{E}'\{C(t)y(t) + D(t)z(t) + \overline{C}(t)y'(t) + \overline{D}(t)z'(t)\}\overleftarrow{\mathrm{d}}B(t) \\ \qquad\qquad - z(t)\overrightarrow{\mathrm{d}}W(t), \\ p_i(0) = -L_iy(0), \quad y(T) = \xi. \end{cases} \tag{21}$$

Proof We first prove the existence of the solution of equation (21). We set

$$\hat{\theta}(t) = \mathbb{E}[\theta(t) \mid \mathscr{E}_t], \quad \theta = y, z, y', z', p_1, p_2, q_1, q_2.$$

Similar to [25, Lemma 5.4], the optimal filter

$$(\hat{y}(t), \hat{z}(t), \hat{y}'(t), \hat{z}'(t), \hat{p}_1(t), \hat{q}_1(t), \hat{p}_2(t), \hat{q}_2(t))$$

of

$$(y(t), z(t), y'(t), z'(t), p_1(t), q_1(t), p_2(t), q_2(t))$$

satisfies

$$\begin{cases} \mathrm{d}\hat{p}_i(t) = \mathbb{E}'[A(t)\hat{p}_i(t) + \overline{A}(t)\hat{p}_i'(t) + C(t)\hat{q}_i(t) + \overline{C}(t)\hat{q}_i'(t) \\ \qquad\qquad - N_i(t)\hat{y}(t) - \overline{N}_i(t)\hat{y}'(t)]\mathrm{d}t + \hat{q}_i(t)\overleftarrow{\mathrm{d}}B(t) \\ \qquad\qquad + \mathbb{E}'[B(t)\hat{p}_i(t) + \overline{B}(t)\hat{p}_i'(t) + D(t)\hat{q}_i(t) + \overline{D}(t)\hat{q}_i'(t)]\overrightarrow{\mathrm{d}}W(t), \\ -\mathrm{d}\hat{y}(t) = \mathbb{E}'\{A(t)\hat{y}(t) + B(t)\hat{z}(t) + \overline{A}(t)\hat{y}'(t) + \overline{B}(t)\hat{z}'(t) \\ \qquad\qquad + E_1^2(t)M_1^{-1}(t)\hat{p}_1(t) + E_2^2(t)M_2^{-1}(t)\hat{p}_2(t)\}\mathrm{d}t - \hat{z}(t)\overrightarrow{\mathrm{d}}W(t) \\ \qquad\qquad + \mathbb{E}'\{C(t)\hat{y}(t) + D(t)\hat{z}(t) + \overline{C}(t)\hat{y}'(t) + \overline{D}(t)\hat{z}'(t)\}\overleftarrow{\mathrm{d}}B(t), \\ \hat{p}_i(0) = -L_i\hat{y}(0), \quad \hat{y}(T) = \mathbb{E}[\xi|\mathscr{E}_T], \end{cases} \tag{22}$$

for $i = 1, 2$. Due to the above analysis, the candidate equilibrium point $(u_1(\cdot), u_2(\cdot))$ can be rewritten as

$$\begin{cases} u_1(t) = M_1^{-1}(t)E_1(t)\hat{p}_1(t), \\ u_2(t) = M_2^{-1}(t)E_2(t)\hat{p}_2(t), \end{cases}$$

where $\hat{p}_i(t)$, $i = 1, 2$, admits MF-FBDSDE (22). We introduce a new MF-FBDSDE:

$$\begin{cases} \mathrm{d}P(t) = \mathbb{E}'[A(t)P(t) + \overline{A}(t)P'(t) + C(t)Q(t) + \overline{C}(t)Q'(t) \\ \qquad\qquad - (E_1^2(t)M_1^{-1}(t)N_1(t) + E_2^2(t)M_2^{-1}(t)N_2(t))Y(t) \\ \qquad\qquad - (E_1^2(t)M_1^{-1}(t)\overline{N}_1(t) + E_2^2(t)M_2^{-1}(t)\overline{N}_2(t))Y'(t)]\mathrm{d}t \\ \qquad\qquad + \mathbb{E}'[B(t)P(t) + \overline{B}(t)P'(t) + D(t)Q(t) + \overline{D}(t)Q'(t)]\overrightarrow{\mathrm{d}}W(t) \\ \qquad\qquad + Q(t)\overleftarrow{\mathrm{d}}B(t), \\ -\mathrm{d}Y(t) = \mathbb{E}'[A(t)Y(t) + B(t)Z(t) + \overline{A}(t)Y'(t) + \overline{B}(t)Z'(t) + P(t)]\mathrm{d}t \\ \qquad\qquad + \mathbb{E}'[C(t)Y(t) + D(t)Z(t) + \overline{C}(t)Y'(t) + \overline{D}(t)Z'(t)]\overleftarrow{\mathrm{d}}B(t) \\ \qquad\qquad - Z(t)\overrightarrow{\mathrm{d}}W(t), \\ P(0) = -[E_1^2(0)M_1^{-1}(0)L_1 + E_2^2(0)M_2^{-1}(0)L_2]Y(0), \quad Y(T) = \xi. \end{cases} \tag{23}$$

Based on the analysis above, we can say that the existence and uniqueness of MF-FBDSDE (22) are equivalent to that of MF-FBDSDE (23). It is easy to check that MF-FBDSDE (23) satisfies assumptions (A1)–(A3) with

$$H = 1, \quad \mu_1 = 1, \quad \mu_2 = \beta_2 = 0.$$

According to Theorem 2, there exists a unique solution $(P(t), Q(t), Y(t), Z(t))$ of MF-FBDSDE (23), where

$$\begin{aligned} P(t) &= E_1^2(t)M_1^{-1}(t)\hat{p}_1(t) + E_2^2(t)M_2^{-1}(t)\hat{p}_2(t), \\ Q(t) &= E_1^2(t)M_1^{-1}(t)\hat{q}_1(t) + E_2^2(t)M_2^{-1}(t)\hat{q}_2(t), \\ Y(t) &= \hat{y}(t), \quad Z(t) = \hat{z}(t). \end{aligned}$$

Then there exists a unique solution $(\hat{p}_1(t), \hat{p}_2(t), \hat{y}(t), \hat{z}(t))$ of MF-FBDSDE (22). Furthermore, there exists at most one equilibrium point for the underlying game.

Now, we try to prove that $(u_1(\cdot), u_2(\cdot))$ is a Nash equilibrium point for our backward LQ game problem. We only prove

$$J_1(u_1(\cdot), u_2(\cdot)) = \max_{v_1(\cdot)\in\mathscr{U}_1} J_1(v_1(\cdot), u_2(\cdot)).$$

It is similar to get another inequality of (4). Denote $(y^{v_1}(t), z^{v_1}(t))$ the solution of the system

$$\begin{cases} -\mathrm{d}y^{v_1}(t) = \mathbb{E}'[A(t)y^{v_1}(t) + B(t)z^{v_1}(t) + \overline{A}(t)(y^{v_1}(t))' \\ \qquad\qquad + \overline{B}(t)(z^{v_1}(t))' + E_1(t)v_1(t) + E_2(t)u_2(t)]\mathrm{d}t \\ \qquad\qquad + \mathbb{E}'[C(t)y^{v_1}(t) + D(t)z^{v_1}(t) + \overline{C}(t)(y^{v_1}(t))' \\ \qquad\qquad + \overline{D}(t)(z^{v_1}(t))']\overleftarrow{\mathrm{d}}B(t) - z^{v_1}(t)\overrightarrow{\mathrm{d}}W(t), \\ y^{v_1}(T) = \xi. \end{cases}$$

Then

$$\begin{aligned} &J_1(u_1(\cdot), u_2(\cdot)) - J_1(v_1(\cdot), u_2(\cdot)) \\ &\quad = \frac{1}{2}\mathbb{E}\bigg[\int_0^T \mathbb{E}'(M_1(t)(u_1(t) - v_1(t))^2 + 2M_1(t)v_1(t)(u_1(t) - v_1(t)) \\ &\qquad + N_1(t)(y(t) - y^{v_1}(t))^2 + 2N_1(t)y(t)(y(t) - y^{v_1}(t)) \\ &\qquad + \overline{N}_1(t)(y'(t) - (y^{v_1}(t))')^2 + 2\overline{N}_1(t)(y^{v_1}(t))'(y'(t) - (y^{v_1}(t))')\mathrm{d}t \\ &\qquad + L_1(y(0) - y^{v_1}(0))^2 + 2L_1y^{v_1}(0)(y(0) - y^{v_1}(0))\bigg]. \end{aligned}$$

Applying Itô's formula to $(y(t) - y^{v_1}(t))\hat{p}_1(t)$, we have

$$\begin{aligned} &\mathbb{E}\{L_1y(0)(y(0) - y^{v_1}(0))\} \\ &\quad = -\mathbb{E}\int_0^T \mathbb{E}'[E_1(t)\hat{p}_1(t)(u_1(t) - v_1(t)) + N_1(t)y(t)(y(t) - y^{v_1}(t)) \\ &\qquad + \overline{N}_1(t)y'(t)(y'(t) - (y^{v_1}(t))')]\mathrm{d}t. \end{aligned}$$

Since

$$M_1(t) > 0, \quad N_1(t) \geqslant 0, \quad \overline{N}_1(t) \geqslant 0, \quad \forall\, t \in [0, T], \quad L_1 \geqslant 0,$$

noting that

$$u_1(t) = M_1^{-1}(t)B_1(t)\hat{p}_1(t),$$

we have

$$\begin{aligned} &J_1(u_1(\cdot), u_2(\cdot)) - J_1(v_1(\cdot), u_2(\cdot)) \\ &\quad \geqslant \mathbb{E}\int_0^T \mathbb{E}'[(M_1(t)u_1(t) - E_1(t)\hat{p}_1(t))(u_1(t) - v_1(t))]\mathrm{d}t \\ &\quad = 0. \end{aligned}$$

So

$$(u_1(t), u_2(t)) = (M_1^{-1}(t)E_1(t)\hat{p}_1(t), M_2^{-1}(t)E_2(t)\hat{p}_2(t))$$

is a Nash equilibrium point for our backward LQ nonzero-sum differential game problem. □

7 Conclusion

In this paper, we research a kind of non-zero sum differential games of mean-field backward doubly stochastic differential equations (MF-BDSDE) under partial information. First, we establish a partial information necessary maximum principle and a sufficient condition for the Nash equilibrium point. Then we use the theoretical results to research a partial information linear-quadratic (LQ) game. In order to obtain the unique Nash equilibrium point for our LQ game problem, we study a new kind of fully coupled mean-field forward-backward doubly stochastic differential equations (MF-FBDSDE), and get the existence and uniqueness theorem for solutions to such kind of equations under some monotonicity conditions. To our knowledge, under the full information case, the result in our paper is also new.

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