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RESEARCH ARTICLE

Dimension of divergence sets for dispersive equation

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Abstract Consider the generalized dispersive equation defined by

$$\begin{cases} \mathrm{i}\,\partial_t u + \phi(\sqrt{-\Delta}\,)u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x), & f \in \mathscr{S}(\mathbb{R}^n), \end{cases} \tag{$*$}$$

where $\phi(\sqrt{-\Delta}\,)$ is a pseudo-differential operator with symbol $\phi(|\xi|)$. In the present paper, assuming that ϕ satisfies suitable growth conditions and the initial data in $H^s(\mathbb{R}^n)$, we bound the Hausdorff dimension of the sets on which the pointwise convergence of solutions to the dispersive equations $(*)$ fails. These upper bounds of Hausdorff dimension shall be obtained via the Kolmogorov-Seliverstov-Plessner method.

Keywords Dispersive equation, Hausdorff dimension, maximal operator
MSC 42B20, 42B25, 35S10

1 Introduction and main results

Let f be a Schwartz function in $\mathscr{S}(\mathbb{R}^n)$, and let

$$\mathrm{e}^{\mathrm{i}t\Delta} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi+\mathrm{i}t|\xi|^2} \hat{f}(\xi)\mathrm{d}\xi, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}.$$

Here, $\hat{f}$ denotes the Fourier transform of f defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\xi\cdot x} f(x)\mathrm{d}x.$$

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It is well known that

$$u(x,t) := \mathrm{e}^{\mathrm{i}t\Delta} f(x)$$

is the solution of the Schrödinger equation

$$\begin{cases} \mathrm{i}\,\partial_t u - \Delta u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x), & f \in \mathscr{S}(\mathbb{R}^n). \end{cases}$$

Carleson [5] proposed a problem: determining the optimal exponents s for which

$$\lim_{t\to 0} \mathrm{e}^{\mathrm{i}t\Delta} f(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^n, \tag{1}$$

holds whenever $f \in H^s(\mathbb{R}^n)$. Here, $H^s(\mathbb{R}^n)$ $(s \in \mathbb{R})$ is the non-homogeneous Sobolev space, defined by

$$H^s(\mathbb{R}^n) = \{G_s * f \colon f \in L^2(\mathbb{R}^n)\},$$

where G_s is the Bessel potential defined by

$$\hat{G}_s(\xi) = (1 + |\xi|^2)^{-s/2}.$$

Carleson [5] first considered this problem for one spatial dimension and showed that the pointwise convergence (1) holds for data in $H^s(\mathbb{R})$ with $s \geqslant 1/4$, and Dahlberg and Kenig [8] showed that $s \geqslant 1/4$ is sharp. Recently, in two spatial dimensions, Du et al. [12] showed that (1) holds for data in $H^s(\mathbb{R}^2)$ with $s > 1/3$, which improved the result of (1) that holds for $s > 3/8$ in [20]. Moreover, Bourgain [4] gave examples showing that such convergence (1) can fail for any $s < 1/3$ when $n = 2$, so the result in [12] is sharp up to the endpoint. In higher dimensions, Bourgain [3] showed that (1) holds for data in $H^s(\mathbb{R}^n)$ with

$$s > \frac{1}{2} - \frac{1}{4n}.$$

In particular, in the case f is a radial function, Prestini [22] proved that, if $f \in H^s(\mathbb{R}^n)$ $(n \geqslant 2)$, then the convergence (1) holds for $s = 1/4$. For more results on the pointwise convergence (1), see, e.g., [13,24,26–28,32].

Naturally, one may consider a refinement of this question which is the Hausdorff dimension of sets on which the pointwise convergence fails. In this direction, Sjögren and Sjölin [23] made the solution of Schrödinger equation precise by the ball or the disc method, and they had previously obtained some upper bound of the Hausdorff dimension of the divergence set (see [23, Theorem 2]). Carleson [5] and Sjögren and Sjölin [23] obtained some results of the pointwise convergence of the solution to the Schrödinger equation. Recently, Barceló et al. [1] and Bennett and Rogers [2] refined the above results of [5,23]. Barceló et al. [1] considered the pointwise convergence to the initial data of the solution to the fractional Schrödinger equation defined by

$$\begin{cases} \mathrm{i}\,\partial_t u + (-\Delta)^{m/2} u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases}$$

and bounded the Hausdorff dimension of the sets on which convergence fails. We denote by $\alpha_{m,n}(s)$ the supremum of

$$\dim_{\mathrm{H}}\{x \in \mathbb{R}^n \colon u(x,t_k) \nrightarrow u_0(x),\ k \to \infty\}$$

over all $u_0 \in H^s(\mathbb{R}^n)$ and all sequences $\{t_k\}$ which converge to zero. Here, $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension. More precisely, in one dimension, Barceló et al. [1] proved the following result for $f \in H^s(\mathbb{R})$.

Theorem A [1] *Assume $n = 1$ and $m > 1$. Then*

$$\alpha_{m,1}(s) = \begin{cases} 1, & s < \dfrac{1}{4}, \\ 1-2s, & \dfrac{1}{4} \leqslant s < \dfrac{1}{2}, \\ 0, & s \geqslant \dfrac{1}{2}. \end{cases}$$

And for the radial data, Bennett and Rogers [2] obtained the following result.

Theorem B [2] *Assume that $n \geqslant 2$, $m > 1$, and f is radial. Then*

$$\alpha_{m,n}(s) = \begin{cases} n, & s < \dfrac{1}{4}, \\ n-2s, & \dfrac{1}{4} \leqslant s < \dfrac{1}{2}, \\ 0, & s > \dfrac{1}{2}. \end{cases}$$

A class of generalized dispersive equations is defined by

$$\begin{cases} \mathrm{i}\,\partial_t u + \phi(\sqrt{-\Delta}\,)u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = u_0(x) = f(x), & f \in \mathscr{S}(\mathbb{R}^n), \end{cases} \tag{2}$$

where $\phi(\sqrt{-\Delta}\,)$ is a pseudo-differential operator with symbol $\phi(|\xi|)$. For initial data u_0 belonging to the Schwartz class, the formal solution of equation (2) can be written as

$$u(x,t,\phi) = \mathrm{e}^{\mathrm{i}t\phi(\sqrt{-\Delta}\,)} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi + \mathrm{i}t\phi(|\xi|)} \hat{f}(\xi)\mathrm{d}\xi.$$

We would like to point out that some Strichartz estimates for the generalized dispersive equation (2) have been discussed recently. In fact, many dispersive equations can be reduced to this type. For instance, the half-wave equation ($\phi(r) = r$), the fractional Schrödinger equation ($\phi(r) = r^a\,(0 < a \neq 1)$), the Beam equation ($\phi(r) = \sqrt{1+r^4}$), Klein-Gordon or semirelativistic equation ($\phi(r) = \sqrt{1+r^2}\,$), iBq ($\phi(r) = r\sqrt{1+r^2}\,$), imBq ($\phi(r) = r/\sqrt{1+r^2}\,$), and the fourth-order Schrödinger equation ($\phi(r) = r^2 + r^4$) (see [6,7,14,16–19,29–31] and references therein).

Inspired by the work of Barceló et al. [1] and Bennett and Rogers [2], in the present paper, under $\phi\colon \mathbb{R}^+ \to \mathbb{R}$ satisfying suitable growth conditions and the initial data in Sobolev space $H^s(\mathbb{R}^n)$, we bound the Hausdorff dimension of the sets on which the pointwise convergence to the initial data of the solution to equation (2) fails. We denote by $\alpha_{\phi,n}(s)$ the supremum of

$$\dim_{\mathrm{H}}\{x \in \mathbb{R}^n\colon \mathrm{e}^{\mathrm{i}t_k\phi(\sqrt{-\Delta})}f(x) \nrightarrow u_0(x),\ k \to \infty\}$$

over all $u_0 \in H^s(\mathbb{R}^n)$ and all sequences (t_k) which converge to zero. Here, $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension.

For $u_0 \in H^s(\mathbb{R}^n)$, we may define u as in pointwise limit:

$$u(x,t) = \lim_{N\to\infty} S_{t,\phi}^N f(x)$$

whenever the limit exists, where the operator $S_{t,\phi}^N$ is defined by

$$S_{t,\phi}^N f(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} \psi\Big(\frac{|\xi|}{N}\Big)\mathrm{e}^{\mathrm{i}x\cdot\xi+\mathrm{i}t\phi(|\xi|)}\hat{f}(\xi)\mathrm{d}\xi.$$

Here, for convenience, we choose ψ to be the Gaussian $\psi(r) = \mathrm{e}^{-r^2}$. By standard argument, we can obtain

$$u(x,t) = S_{t,\phi}f(x) \quad \text{a.e. } x \in \mathbb{R}^n$$

with respect to the Lebesgue measure, where

$$S_{t,\phi}f(x) = \lim_{N\to\infty} S_{t,\phi}^N f(x)$$

in the L^2-sense. However, $u(\cdot,t)$ is also well defined regarding fractal measures when $s > 0$, see [1].

Let $0 \leqslant \alpha \leqslant n$. We say that a positive Borel measure μ is α-dimensional if

$$c_\alpha(\mu) := \sup_{x\in\mathbb{R}^n, r>0} \frac{\mu(B(x,r))}{r^\alpha} < \infty.$$

We denote by $\mathscr{M}^\alpha(\mathbb{A}^n)$ the α-dimensional probability measures which are supported in the unit annulus

$$\mathbb{A}^n = \Big\{x \in \mathbb{R}^n\colon \frac{1}{2} \leqslant |x| \leqslant 1\Big\}.$$

Assume that $\phi\colon \mathbb{R}^+ \to \mathbb{R}$ satisfies the following conditions:

(H1) there exists $m_1 > 1$ such that

$$|\phi'(r)| \sim r^{m_1-1}, \quad |\phi''(r)| \gtrsim r^{m_1-2}, \quad \forall\, r \in (0,1);$$

(H2) there exists $m_2 > 1$ such that

$$|\phi'(r)| \sim r^{m_2-1}, \quad |\phi''(r)| \gtrsim r^{m_2-2}, \quad \forall\, r \geqslant 1;$$

(H3) either $\phi''(r) > 0$ or $\phi''(r) < 0$ for all $r > 0$.

Our main result in this paper is as follows.

Theorem 1 *Assume that $n = 1$ and ϕ satisfies conditions* (H1)–(H3)*. Then*

(i) $\alpha_{\phi,1}(s) = 1 - 2s$ *for* $1/4 \leqslant s < 1/2$;

(ii) $\alpha_{\phi,1}(s) = 0$ *for* $s \geqslant 1/2$.

In higher dimensions, we obtain the following result for radial data.

Theorem 2 *Assume that $n \geqslant 2$ and ϕ satisfies conditions* (H1)–(H3)*. If f is radial, then*

(i) $\alpha_{\phi,n}(s) = n - 2s$ *for* $1/4 \leqslant s < 1/2$;

(ii) $\alpha_{\phi,n}(s) = 0$ *for* $s > 1/2$.

Remark 1 From the results in [10,11], for ϕ satisfying conditions (H1)–(H3), $s \geqslant 1/4$, and initial data $f \in H^s(\mathbb{R})$ or f is radial and $f \in H^s(\mathbb{R}^n)$, the pointwise convergence

$$\lim_{t\to 0} \mathrm{e}^{\mathrm{i}t\phi(\sqrt{-\Delta})} f(x) = f(x) \quad \text{a.e. } x \in \mathbb{R}^n \tag{3}$$

holds with respect to Lebesgue measure. Hence, by the results of Theorems 1 and 2, we give a refinement of this question regarding the Hausdorff dimension of the set on convergence (3) fails.

Remark 2 There are many elements ϕ satisfying conditions (H1)–(H3), for instance, r^a $(a > 1)$, $\sqrt{1 + r^4}$, and $r^2 + r^4$. Therefore, Theorems 1 and 2 are apparently good extensions to the results of Theorems A and B, respectively.

2 Proof of Theorem 1

In this section, we will prove Theorem 1. To do this, we need two important lemmas (i.e., Lemmas 1 and 2 below), which play a key role in proving Theorems 1 and 2.

Lemma 1 *Assume that ϕ satisfies* (H1)–(H3) *and μ is a Schwartz function. If $1/2 \leqslant s < 1$, then*

$$\Big| \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x\xi + \mathrm{i}t\phi(|\xi|)} |\xi|^{-s} \mu\Big(\frac{\xi}{N}\Big) \mathrm{d}\xi \Big| \leqslant \frac{C}{|x|^{1-s}}, \quad x \in \mathbb{R},\, t \in \mathbb{R},\; N = 1, 2, \ldots.$$

Here, the constant C may depend on s, m_1, m_2, and μ, but not on x, t, and N.

Remark 3 The proof of Lemma 1 is similar to that of [10, Lemma 2.1], so we omit it here.

Lemma 2 *If*

$$\Big\|\sup_{k\geqslant 1}\sup_{N\geqslant 1}|S^N_{t_k,\phi}f|\Big\|_{L^1(\mathrm{d}\mu)}\leqslant C\sqrt{c_\alpha(\mu)}\,\|f\|_{H^s(\mathbb{R}^n)},\quad \forall\,\alpha>\alpha_0\geqslant n-2s,\tag{4}$$

whenever $\mu\in\mathscr{M}^\alpha(\mathbb{A}^n)$, $f\in H^s(\mathbb{R}^n)$, *and* $(t_k)\in\mathbb{R}^{\mathbb{N}}$, *then* $\alpha_{\phi,n}(s)\leqslant\alpha_0$.

Proof We choose a Schwartz function g such that

$$\|u_0-g\|_{H^s(\mathbb{R}^n)}<\varepsilon,$$

and notice that

$$|S^N_{t,\phi}u_0-u_0|\leqslant|S^N_{t,\phi}u_0-S^N_{t,\phi}g|+|S^N_{t,\phi}g-g|+|g-u_0|.$$

Thus, we have

$$\begin{aligned}&\mu\{x\colon \limsup_{k\to\infty}\limsup_{N\to\infty}|S^N_{t_k,\phi}u_0-u_0|>\lambda\}\\ &\leqslant\mu\Big\{x\colon \sup_{k\geqslant1}\sup_{N\geqslant1}|S^N_{t_k,\phi}(u_0-g)|>\frac{\lambda}{3}\Big\}\\ &\quad+\mu\Big\{x\colon \lim_{k\to\infty}\lim_{N\to\infty}|S^N_{t_k,\phi}g-g|>\frac{\lambda}{3}\Big\}+\mu\Big\{x\colon |g-u_0|>\frac{\lambda}{3}\Big\}\\ &=:\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3.\end{aligned}$$

Letting $t_k\to0$, we have

$$\mathrm{I}_2=0.\tag{5}$$

By the maximal inequality (4), we have

$$\mathrm{I}_1\leqslant C\lambda^{-1}\sqrt{c_\alpha(\mu)}\,\|u_0-g\|_{H^s(\mathbb{R}^n)}\leqslant C\lambda^{-1}\sqrt{c_\alpha(\mu)}\,\varepsilon.\tag{6}$$

Noting that

$$\|f\|_{L^1(\mathrm{d}\mu)}\leqslant C\sqrt{c_\alpha(\mu)}\,\|f\|_{H^s(\mathbb{R}^n)}$$

for $\mu\in\mathscr{M}^\alpha(\mathbb{A}^n)$ with $\alpha>n-2s$, see [2, p. 3], we have

$$\mathrm{I}_3\leqslant C\lambda^{-1}|g-u_0\|_{L^1(\mathrm{d}\mu)}\leqslant C\lambda^{-1}\sqrt{c_\alpha(\mu)}\,\|u_0-g\|_{H^s(\mathbb{R}^n)}\leqslant C\lambda^{-1}\sqrt{c_\alpha(\mu)}\,\varepsilon.\tag{7}$$

Hence, by (5)–(7), we obtain

$$\mu\{x\colon \lim_{k\to\infty}\lim_{N\to\infty}|S^N_{t_k,\phi}u_0-u_0|>\lambda\}\leqslant C\lambda^{-1}\sqrt{c_\alpha(\mu)}\,\varepsilon.$$

Letting $\varepsilon\to0$, and then $\lambda\to0$, we have

$$\mu\{x\colon u(x,t_k)\nrightarrow u_0(x),\ k\to\infty\}=0$$

whenever $\mu\in\mathscr{M}^\alpha(\mathbb{A}^n)$ with $\alpha>\alpha_0$. By Frostman' lemma [21, p. 112], we get

$$\mathscr{H}^\alpha\{x\in\mathbb{A}^n\colon u(x,t_k)\nrightarrow u(x,0),\ k\to\infty\}=0,$$

where $\mathscr{H}^\alpha$ denotes the α-Hausdorff measure. By translation invariance and the countable additivity of Hausdorff measure, we have

$$\mathscr{H}^\alpha\{x \in \mathbb{R}^n \colon u(x,t_k) \nrightarrow u(x,0),\ k \to \infty\} = 0,$$

which holds for every $\alpha > \alpha_0$. Thus,

$$\dim_{\mathrm{H}}\{x \in \mathbb{R}^n \colon u(x,t_k) \nrightarrow u_0(x),\ k \to \infty\} \leqslant \alpha_0$$

whenever $f \in H^s(\mathbb{R}^n)$ and $t_k \to 0$. We complete the proof. □

As a similar proof on [1, pp. 613, 614], when $1/4 \leqslant s < 1/2$, we can obtain the lower bound on the exponents $\alpha_{\phi,1}(s)$, that is, $\alpha_{\phi,1}(s) \geqslant 1-2s$. Hence, when $1/4 \leqslant s < 1/2$, to prove Theorem 1, it suffices to set up the following lemma.

Lemma 3 *Assume that* $n = 1$ *and* ϕ *satisfies* (H1)–(H3). *If* $1/4 \leqslant s \leqslant 1/2$ *and* $\alpha > 1-2s$, *then*

$$\Big\| \sup_{k\geqslant 1} \sup_{N\geqslant 1} |S^N_{t_k,\phi} f| \Big\|_{L^1(\mathrm{d}\mu)} \leqslant C\sqrt{c_\alpha(\mu)}\, \|f\|_{H^s(\mathbb{R})} \tag{8}$$

whenever $\mu \in \mathscr{M}^\alpha(\mathbb{A})$, $f \in H^s(\mathbb{R})$, *and* $(t_k) \in \mathbb{R}^{\mathbb{N}}$, *which yields* $\alpha_{\phi,1}(s) \leqslant 1-2s$.

Proof By the embedding property of the non-homogenous Sobolev spaces, we only prove inequality (8) holds when $1/4 \leqslant s < 1/2$. We recall the α-energy of μ defined by

$$I_\alpha(\mu) = \iint \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{|x-y|^\alpha}.$$

By the dyadic decomposition, for all $\mu \in \mathscr{M}^\alpha(\mathbb{A})$ with $\alpha > 1-2s$, we have

$$\begin{aligned} \iint \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{|x-y|^{1-2s}} &= \int \sum_{j=0}^\infty \int_{2^{-j}<|x-y|\leqslant 2^{-j+1}} \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{|x-y|^{1-2s}} \\ &\leqslant 2^\alpha c_\alpha(\mu) \int \sum_{j=0}^\infty 2^{-j\alpha} 2^{j(1-2s)} \mathrm{d}\mu(y) \\ &\leqslant C c_\alpha(\mu). \end{aligned}$$

Thus, the inequality

$$I_{1-2s}(\mu) \leqslant C c_\alpha(\mu)$$

holds for all $\mu \in \mathscr{M}^\alpha(\mathbb{A})$ with $\alpha > 1-2s$. Therefore, in order to prove (8), it suffices to prove

$$\int \sup_{k\geqslant 1} \sup_{N\geqslant 1} |S^N_{t_k,\phi} f(x)| \mathrm{d}\mu(x) \leqslant C\sqrt{I_{1-2s}(\mu)}\, \|f\|_{H^s(\mathbb{R})} \tag{9}$$

whenever $\mu \in \mathscr{M}^\alpha(\mathbb{A})$, $f \in H^s(\mathbb{R})$. Let $t(x)$, $N(x)$, and $\omega(x)$ be measurable functions with

$$t \colon (-1,1) \to \mathbb{R}, \quad N \colon (-1,1) \to [1,\infty), \quad \omega \colon (-1,1) \to \mathbb{S}^1.$$

By linearizing the maximal function, to prove (9), it suffices to prove

$$\Big|\int S_{t(x),\phi}^{N(x)}f(x)\omega(x)\mathrm{d}\mu(x)\Big|^2 \leqslant CI_{1-2s}(\mu)\|f\|_{H^s(\mathbb{R})}^2. \tag{10}$$

By Fubini's theorem, and invoking the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\Big|\int S_{t(x),\phi}^{N(x)}f(x)\omega(x)\mathrm{d}\mu(x)\Big|^2 \\
&\leqslant \int_{\mathbb{R}}|\xi|^{2s}|\hat{f}(\xi)|^2\mathrm{d}\xi\int\Big|\int\psi\Big(\frac{|\xi|}{N(x)}\Big)\mathrm{e}^{\mathrm{i}(x\xi+t(x)\phi(|\xi|))}\omega(x)\mathrm{d}\mu(x)\Big|^2\frac{\mathrm{d}\xi}{|\xi|^{2s}} \\
&\leqslant \|f\|_{H^s(\mathbb{R})}^2\int\Big|\int\psi\Big(\frac{|\xi|}{N(x)}\Big)\mathrm{e}^{\mathrm{i}(x\xi+t(x)\phi(|\xi|))}\omega(x)\mathrm{d}\mu(x)\Big|^2\frac{\mathrm{d}\xi}{|\xi|^{2s}}.
\end{aligned} \tag{11}$$

Applying Fubini's theorem and by Lemma 1, we get

$$\begin{aligned}
&\int\Big|\int\psi\Big(\frac{|\xi|}{N(x)}\Big)\mathrm{e}^{\mathrm{i}(x\xi+t(x)\phi(|\xi|))}\omega(x)\mathrm{d}\mu(x)\Big|^2\frac{\mathrm{d}\xi}{|\xi|^{2s}} \\
&=\iiint\psi\Big(\frac{|\xi|}{N(x)}\Big)\psi\Big(\frac{|\xi|}{N(y)}\Big)\mathrm{e}^{\mathrm{i}[(y-x)\xi+(t(y)-t(x))\phi(|\xi|)]}\frac{\mathrm{d}\xi}{|\xi|^{2s}}\,\omega(x)\omega(y)\mathrm{d}\mu(x)\mathrm{d}\mu(y) \\
&\leqslant C\iint\frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{|x-y|^{1-2s}} \\
&=CI_{1-2s}(\mu)
\end{aligned} \tag{12}$$

uniformly in the functions t, N, and ω. Hence, inequality (10) holds from (11) and (12). Hence, it follows that (8) holds, and applying Lemma 2, we get $\alpha_{\phi,1}(s)\leqslant 1-2s$. Thus, we complete the proof. $\square$

When $s>1/2$, to prove Theorem 1, it suffices to set up the following lemma.

Lemma 4 *Assume that ϕ satisfies* (H1)–(H3). *If $n=1$ and $s>1/2$, then*

$$\Big\|\sup_{k\geqslant 1}\sup_{N\geqslant 1}|S_{t_k,\phi}^{N}f|\Big\|_{L^1(\mathrm{d}\mu)}\leqslant C\|f\|_{H^s(\mathbb{R})} \tag{13}$$

whenever $\mu\in\mathscr{M}^{\alpha}(\mathbb{A})$, $f\in H^s(\mathbb{R})$, and $(t_k)\in\mathbb{R}^{\mathbb{N}}$, which yields $\alpha_{\phi,1}(s)=0$.

Proof Let $f=G_s*g$. To prove (13), it suffices to prove

$$\int_{\mathbb{A}}\sup_{k\geqslant 1}\sup_{N\geqslant 1}\Big|\int_{\mathbb{R}}\frac{\psi(|\xi|/N)\hat{g}(\xi)\mathrm{e}^{\mathrm{i}(x\cdot\xi+t_k\phi(|\xi|))}}{(1+|\xi|^2)^{s/2}}\,\mathrm{d}\xi\Big|\mathrm{d}\mu(x)\leqslant C\|g\|_{L^2(\mathbb{R})} \tag{14}$$

for all $\mu\in\mathscr{M}^{\alpha}(\mathbb{A})$, $g\in L^2(\mathbb{R})$, and $(t_k)\in\mathbb{R}^{\mathbb{N}}$. Note that μ is a probability measure. Thus, in order to prove (14), it suffices to prove

$$\Big|\int_{\mathbb{R}}\frac{\psi(|\xi|/N)\hat{g}(\xi)\mathrm{e}^{\mathrm{i}(x\cdot\xi+t\phi(|\xi|))}}{(1+|\xi|^2)^{s/2}}\,\mathrm{d}\xi\Big|\leqslant C\|g\|_{L^2(\mathbb{R})} \tag{15}$$

uniformly in $N \geqslant 1$, $t \in \mathbb{R}$, $x \in \mathbb{A}$, and $g \in L^2(\mathbb{R})$. Note that for $s > 1/2$,

$$\Big(\int_0^\infty \frac{1}{(1+|r|^2)^s}\,\mathrm{d}r\Big)^{1/2} \leqslant C.$$

Then, applying the Cauchy-Schwarz inequality and Plancherel's theorem, we obtain

$$\begin{aligned}\Big|\int_{\mathbb{R}} \frac{\psi(|\xi|/N)\hat{g}(\xi)\mathrm{e}^{\mathrm{i}(x\cdot\xi+t\phi(|\xi|))}}{(1+|\xi|^2)^{s/2}}\,\mathrm{d}\xi\Big| &\leqslant C\int_{\mathbb{R}} \frac{|\hat{g}(\xi)|}{(1+|\xi|^2)^{s/2}}\,\mathrm{d}\xi \\ &\leqslant C\Big(\int_{\mathbb{R}} |\hat{g}(\xi)|^2\mathrm{d}\xi\Big)^{1/2}\Big(\int_{\mathbb{R}} \frac{\mathrm{d}\xi}{(1+|\xi|^2)^s}\Big)^{1/2} \\ &\leqslant C\|g\|_{L^2(\mathbb{R})}\end{aligned}$$

uniformly in $N \geqslant 1$, $t \in \mathbb{R}$, $x \in \mathbb{A}$, and $g \in L^2(\mathbb{R})$. Thus, we verify estimate (15). It follows that (13) holds. Thus, applying Lemma 2, when $s > 1/2$, we get $\alpha_{\phi,1}(s) = 0$. We complete the proof. $\square$

3 Proof of Theorem 2

Let $H^s_{\mathrm{rad}}(\mathbb{R}^n)$ be the set of all radial elements of $H^s(\mathbb{R}^n)$, and let $L^2_{\mathrm{rad}}(\mathbb{R}^n)$ be the set of all radial elements of $L^2(\mathbb{R}^n)$. When $s > 1/2$, to prove Theorem 2, it suffices to set up the following lemma.

Lemma 5 *Assume that ϕ satisfies* (H1)–(H3). *If $n \geqslant 2$ and $s > 1/2$, then*

$$\big\|\sup_{k\geqslant 1}\sup_{N\geqslant 1}|S^N_{t_k,\phi}f|\big\|_{L^1(\mathrm{d}\mu)} \leqslant C\|f\|_{H^s(\mathbb{R}^n)} \tag{16}$$

whenever $\mu \in \mathscr{M}^\alpha(\mathbb{A}^n)$, $f \in H^s_{\mathrm{rad}}(\mathbb{R}^n)$, and $(t_k) \in \mathbb{R}^{\mathbb{N}}$, which yields $\alpha_{\phi,n}(s) = 0$.

Proof Writing $f = G_s * g$, to prove (16), it suffices to prove

$$\int_{\mathbb{A}^n}\sup_{k\geqslant 1}\sup_{N\geqslant 1}\Big|\int_{\mathbb{R}^n} \frac{\psi(|\xi|/N)\hat{g}(\xi)\mathrm{e}^{\mathrm{i}(x\cdot\xi+t_k\phi(|\xi|))}}{(1+|\xi|^2)^{s/2}}\,\mathrm{d}\xi\Big|\mathrm{d}\mu(x) \leqslant C\|g\|_{L^2(\mathbb{R}^n)} \tag{17}$$

for all $\mu \in \mathscr{M}^\alpha(\mathbb{A}^n)$, $g \in L^2_{\mathrm{rad}}(\mathbb{R}^n)$, and $(t_k) \in \mathbb{R}^{\mathbb{N}}$. Noting that μ is a probability measure, in order to prove (17), it suffices to prove

$$\Big|\int_{\mathbb{R}^n} \frac{\psi(|\xi|/N)\hat{g}(\xi)\mathrm{e}^{\mathrm{i}(x\cdot\xi+t\phi(|\xi|))}}{(1+|\xi|^2)^{s/2}}\,\mathrm{d}\xi\Big| \leqslant C\|g\|_{L^2(\mathbb{R}^n)} \tag{18}$$

uniformly in $N \geqslant 1$, $t \in \mathbb{R}$, $x \in \mathbb{A}^n$, and $g \in L^2_{\mathrm{rad}}(\mathbb{R}^n)$. Using the polar coordinates, we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n}\frac{\psi(\frac{|\xi|}{N})\hat{g}(\xi)\mathrm{e}^{\mathrm{i}(x\cdot\xi+t\phi(|\xi|))}}{(1+|\xi|^2)^{s/2}}\,\mathrm{d}\xi\Big|\\
&\quad=\Big|\int_0^\infty\frac{\psi(r/N)r^{n-1}\hat{g}(r)\mathrm{e}^{\mathrm{i}t\phi(r)}}{(1+|r|^2)^{s/2}}\int_{\mathbb{S}^{n-1}}\mathrm{e}^{\mathrm{i}rx\cdot\xi'}\mathrm{d}\sigma(\xi')\mathrm{d}r\Big|\\
&\quad=C_n\Big|\int_0^\infty\frac{\psi(r/N)r^{n-1}\hat{g}(r)\mathrm{e}^{\mathrm{i}t\phi(r)}}{(1+|r|^2)^{s/2}}\,\frac{J_{(n-2)/2}(r|x|)}{(r|x|)^{(n-2)/2}}\,\mathrm{d}r\Big|,
\end{aligned}$$

where C_n is a constant which dependent on n, $J_{(n-2)/2}$ denotes the Bessel function of order $(n-2)/2$, and the Bessel function $J_m(r)$ is defined by

$$J_m(r)=\frac{(r/2)^m}{\Gamma(m+\frac{1}{2})\pi^{1/2}}\int_{-1}^1\mathrm{e}^{\mathrm{i}rt}(1-t^2)^{m-\frac{1}{2}}\mathrm{d}t,\quad m>-\frac{1}{2}.$$

By the result on [15, pp. 430, 431], we have

$$(r|x|)^{1/2}J_{(n-2)/2}(r|x|)\leqslant c_n.$$

Thus, applying the Cauchy-Schwarz inequality and Plancherel's theorem, we obtain

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n}\frac{\psi(|\xi|/N)\hat{g}(\xi)\mathrm{e}^{\mathrm{i}(x\cdot\xi+t\phi(|\xi|))}}{(1+|\xi|^2)^{s/2}}\,\mathrm{d}\xi\Big|\\
&\quad\leqslant C\frac{1}{|x|^{(n-1)/2}}\int_0^\infty\frac{r^{(n-1)/2}|\hat{g}(r)|}{(1+|r|^2)^{s/2}}\,\mathrm{d}r\\
&\quad\leqslant C\frac{1}{|x|^{(n-1)/2}}\Big(\int_0^\infty r^{n-1}|\hat{g}(r)|^2\mathrm{d}r\Big)^{1/2}\Big(\int_0^\infty\frac{1}{(1+|r|^2)^s}\,\mathrm{d}r\Big)^{1/2}\\
&\quad\leqslant C\frac{1}{|x|^{(n-1)/2}}\|g\|_{L^2(\mathbb{R}^n)}\\
&\quad\leqslant C\|g\|_{L^2(\mathbb{R}^n)}
\end{aligned}$$

uniformly in $N\geqslant 1$, $t\in\mathbb{R}$, $x\in\mathbb{A}^n$, and $g\in L^2_{\mathrm{rad}}(\mathbb{R}^n)$. Thus, we verify estimate (18). It follows that (16) holds. Then, applying Lemma 2, when $s>1/2$, we get $\alpha_{\phi,n}(s)=0$. Thus, we complete the proof. $\square$

On the other hand, similar to the proof on [2, p. 8], when $1/4\leqslant s<1/2$, we can get the lower bound on the exponents $\alpha_{\phi,n}(s)$, that is, $\alpha_{\phi,n}(s)\geqslant n-2s$. Hence, when $1/4\leqslant s<1/2$, to prove Theorem 2, it suffices to set up the following lemma.

Lemma 6 *Assume that $n\geqslant 2$ and ϕ satisfies* (H1)–(H3). *If $1/4\leqslant s<1/2$, then, for $\alpha>n-2s$, we have*

$$\Big\|\sup_{k\geqslant 1}\sup_{N\geqslant 1}|S^N_{t_k,\phi}f|\Big\|_{L^1(\mathrm{d}\mu)}\leqslant C\sqrt{c_\alpha(\mu)}\,\|f\|_{H^s(\mathbb{R}^n)} \tag{19}$$

whenever $\mu \in \mathscr{M}^\alpha(\mathbb{A}^n)$, $f \in H^s_{\mathrm{rad}}(\mathbb{R}^n)$, *and* $(t_k) \in \mathbb{R}^{\mathbb{N}}$, *which yields* $\alpha_{\phi,n}(s) \leqslant n - 2s$.

Proof Denote $f = G_s * g$. To prove (19), it suffices to prove

$$\int \sup_{k\geqslant 1}\sup_{N\geqslant 1} |S^N_{t_k,\phi} G_s * g(x)| \mathrm{d}\mu(x) \leqslant C\sqrt{c_\alpha(\mu)}\, \|g\|_{L^2(\mathbb{R}^n)} \tag{20}$$

whenever $\mu \in \mathscr{M}^\alpha(\mathbb{A}^n)$ and $g \in L^2_{\mathrm{rad}}(\mathbb{R}^n)$. Notice that the operator $S^N_{t,\phi}$ maps radial functions to radial functions. Thus, it suffices to consider the radial measures. We write $\mathrm{d}\mu_0(v) = \mathrm{d}\mu(x)$ when $|x| = v$, and recall the energy I_{1-2s} defined by

$$I_{1-2s}(\mu_0) = \int_0^1 \int_0^1 \frac{\mathrm{d}\mu_0(v)\mathrm{d}\mu_0(\omega)}{|\omega - v|^{1-2s}}.$$

And noting that the inequality

$$I_{1-2s}(\mu_0) \leqslant C c_\alpha(\mu)$$

holds for all $\mu \in \mathscr{M}^\alpha(\mathbb{A}^n)$ with $\alpha > n - 2s$. Using the polar coordinates, to prove (20), it suffices to prove

$$\int \sup_{k\geqslant 1}\sup_{N\geqslant 1} |S^N_{t_k,\phi} G_s * g(v)| v^{(n-1)/2} \mathrm{d}\mu_0(v) \leqslant C\sqrt{I_{1-2s}(\mu_0)}\, \|g\|_{L^2(\mathbb{R}^n)}. \tag{21}$$

As in the proof of Lemma 5, we have

$$S^N_{t_k,\phi} G_s * g(v) = \frac{C_n}{v^{(n-1)/2}} \int_0^\infty \frac{\psi(r/N)\hat{g}(r)(rv)^{1/2} J_{(n-1)/2}(rv)\mathrm{e}^{\mathrm{i}t_k\phi(r)} r^{(n-2)/2}\mathrm{d}r}{(1+r^2)^{s/2}}.$$

Let $t(v)$ be a measurable function with $[0,1] \to \mathbb{R}$, and let $N(v)$ be a measurable function with $[0,1] \to [0,\infty)$. By linearizing the maximal function, in order to prove (21), it suffices to prove

$$\begin{aligned} &\Big| \int_0^1 S^{N(v)}_{t(v),\phi} G_s * g(v) v^{(n-1)/2} h(v) \mathrm{d}\mu_0(v) \Big| \\ &\qquad \leqslant C\sqrt{I_{1-2s}(\mu_0)}\, \|g\|_{L^2(\mathbb{R}^n)}\, \|h\|_{L^\infty(\mathrm{d}\mu_0)}, \end{aligned} \tag{22}$$

where $h \in L^\infty(\mathrm{d}\mu_0)$. By the duality, to show (22), it suffices to show that

$$\|Th\|_{L^2(0,\infty)} \leqslant C\sqrt{I_{1-2s}(\mu_0)}\, \|h\|_{L^\infty(\mathrm{d}\mu_0)}, \tag{23}$$

where

$$Th(r) = (1+r^2)^{-s/2} \int_0^1 \psi\Big(\frac{r}{N(v)}\Big)(rv)^{1/2} J_{(n-2)/2}(rv)\mathrm{e}^{\mathrm{i}t(v)\phi(r)} h(v)\mathrm{d}\mu_0(v).$$

To prove inequality (23), we need the following lemma.

Lemma 7 [25, p. 158] *As $r \to \infty$,*

$$J_m(r) = \sqrt{\frac{2}{\pi r}} \cos\Big(r - \frac{\pi m}{2} - \frac{\pi}{4}\Big) + O(r^{-3/2}).$$

In particular, $J_m(r) = O(r^{-1/2})$ as $r \to \infty$.

Applying Lemma 7 and by [22] or [9, p. 15], we may get the following estimates:

$$|t^{1/2} J_{(n-2)/2}(t) - (b_1 \mathrm{e}^{\mathrm{i}t} + b_2 \mathrm{e}^{-\mathrm{i}t})| \leqslant \frac{C}{t}, \quad t > 1, \tag{24}$$

and

$$|t^{1/2} J_{(n-2)/2}(t) - (b_1 \mathrm{e}^{\mathrm{i}t} + b_2 \mathrm{e}^{-\mathrm{i}t})| \leqslant C, \quad 0 < t \leqslant 1, \tag{25}$$

where b_1 and b_2 are the constants depending on n. Invoking (24) and (25), we have

$$Th(r) =: b_1 B_1(r) + b_2 B_2(r) + C(r), \tag{26}$$

where

$$B_1(r) = (1 + r^2)^{-s/2} \int_0^1 \psi\Big(\frac{r}{N(v)}\Big) \mathrm{e}^{\mathrm{i}(rv + t(v)\phi(r))} h(v) \mathrm{d}\mu_0(v),$$

$$B_2(r) = (1 + r^2)^{-s/2} \int_0^1 \psi\Big(\frac{r}{N(v)}\Big) \mathrm{e}^{\mathrm{i}(-rv + t(v)\phi(r))} h(v) \mathrm{d}\mu_0(v),$$

and

$$|C(r)| \leqslant C(1 + r^2)^{-s/2} \int_0^1 \min\Big\{1, \frac{1}{rv}\Big\} |h(v)| \mathrm{d}\mu_0(v).$$

From the proof of [2, Theorem 3.3], we have

$$\Big(\int_0^\infty |C(r)|^2 \mathrm{d}r\Big)^{1/2} \leqslant C\sqrt{I_{1-2s}(\mu_0)}\, \|h\|_{L^\infty(\mathrm{d}\mu_0)}. \tag{27}$$

Set

$$B(\xi) = (1 + \xi^2)^{-s/2} \int_0^1 \psi\Big(|\xi|/N(v)\Big) \mathrm{e}^{\mathrm{i}(\xi v + t(v)\phi(|\xi|))} h(v) \mathrm{d}\mu_0(v), \quad \xi \in \mathbb{R}.$$

We first prove

$$\Big(\int_{\mathbb{R}} |B(\xi)|^2 \mathrm{d}\xi\Big)^{1/2} \leqslant C\sqrt{I_{1-2s}(\mu_0)}\, \|h\|_{L^\infty(\mathrm{d}\mu_0)}. \tag{28}$$

Notice that

$$\Big(\int_{\mathbb{R}} |B(\xi)|^2 \mathrm{d}\xi\Big)^{1/2} \leqslant \Big(\int_{\mathbb{R}} |B'(\xi)|^2 \mathrm{d}\xi\Big)^{1/2}, \tag{29}$$

where

$$B'(\xi)=|\xi|^{-s}\int_{\mathbb{R}}\psi\Big(\frac{|\xi|}{N(v)}\Big)\mathrm{e}^{\mathrm{i}(\xi v+t(v)\phi(|\xi|))}h(v)\mathrm{d}\mu_0(v),\quad \xi\in\mathbb{R}.$$

By Fubini's theorem, we have

$$\int_{\mathbb{R}}|B'(\xi)|^2\mathrm{d}\xi=\iint I(v,\omega)h(v)\overline{h(\omega)}\,\mathrm{d}\mu_0(v)\mathrm{d}\mu_0(\omega),$$

where

$$I(v,\omega)=\int_{\mathbb{R}}\psi\Big(\frac{|\xi|}{N(v)}\Big)\psi\Big(\frac{|\xi|}{N(\omega)}\Big)\mathrm{e}^{\mathrm{i}[(v-\omega)\xi-(t(\nu)-t(\omega))\phi(|\xi|)]}|\xi|^{-2s}\mathrm{d}\xi.$$

By Lemma 1, we have

$$I(v,\omega)\leqslant C\frac{1}{|v-\omega|^{1-2s}}. \tag{30}$$

By (29) and (30), we have

$$\begin{aligned}\Big(\int_{\mathbb{R}}|B(\xi)|^2\mathrm{d}\xi\Big)^{1/2}&\leqslant\int_0^1\int_0^1\frac{|h(v)|\,|h(\omega)|}{|v-\omega|^{1-2s}}\,\mathrm{d}\mu_0(v)\mathrm{d}\mu_0(\omega)\\&\leqslant C\sqrt{I_{1-2s}(\mu_0)}\,\|h\|_{L^\infty(\mathrm{d}\mu_0)},\end{aligned}$$

which is estimate (28). Note that

$$\Big(\int_0^\infty|B_i(\xi)|^2\mathrm{d}\xi\Big)^{1/2}\leqslant C\Big(\int_{\mathbb{R}}|B(\xi)|^2\mathrm{d}\xi\Big)^{1/2},\quad i=1,2. \tag{31}$$

Thus, by (28) and (31), we have

$$\Big(\int_0^\infty|B_i(\xi)|^2\mathrm{d}\xi\Big)^{1/2}\leqslant C\sqrt{I_{1-2s}(\mu_0)}\,\|h\|_{L^\infty(\mathrm{d}\mu_0)},\quad i=1,2. \tag{32}$$

Hence, (22) holds from (26), (27), and (32). Thus, estimate (19) holds, and then applying Lemma 2, when $1/4\leqslant s<1/2$, we get $\alpha_{\phi,n}(s)\leqslant n-2s$. Thus, we complete the proof of Lemma 6. □

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