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RESEARCH ARTICLE

Higher moment of coefficients of Dedekind zeta function

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Abstract Let K_3 be a non-normal cubic extension over $\mathbb{Q}$. We study the higher moment of the coefficients $a_{K_3}(n)$ of Dedekind zeta function over sum of two squares $\sum_{n_1^2+n_2^2\leqslant x} a_{K_3}^l(n_1^2+n_2^2)$, where $2\leqslant l\leqslant 8$ and $n_1, n_2, l\in\mathbb{Z}$.

Keywords Non-normal cubic field, Dedekind zeta function
MSC 11F30, 11N45, 11R16, 11R42

1 Introduction

Let K be a number field of degree d over the rational field $\mathbb{Q}$, let $\mathscr{O}_K$ be the ring of integers of K, and let the Dedekind zeta function be defined by

$$\zeta_K(s)=\sum_{\mathfrak{a}}(N_{\mathfrak{a}})^{-s},\quad \operatorname{Re}s>1,$$

where the sum runs over all integral ideals in $\mathscr{O}_K$, and $N_{\mathfrak{a}}$ is the norm of the integral ideals $\mathfrak{a}$. We can rewrite the Dedekind zeta function as

$$\zeta_K(s)=\sum_{n=1}^{\infty}\frac{a_K(n)}{n^s},\quad \operatorname{Re}s>1,$$

where $a_K(n)$ denotes the number of integral ideals in K with norm n, which is the so-called coefficients of Dedekind zeta function. Since $a_K(n)$ is a multiplicative function, we get that for $\operatorname{Re}s>1$,

$$\zeta_K(s)=\sum_{n=1}^{\infty}\frac{a_K(n)}{n^s}=\prod_p\Big(1+\frac{a_K(p)}{p^s}+\frac{a_K(p^2)}{p^{2s}}+\cdots\Big).$$

It is known that for any $\varepsilon>0$,

$$a_K(n)\leqslant\tau(n)^d\ll n^{\varepsilon},\tag{1}$$

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where $\tau(n)$ is the divisor function and $d = [K : \mathbb{Q}]$.

It is a classical and important problem in number theory to study the arithmetic function $a_K(n)$. Landau [15] proved the asymptotic formula

$$\sum_{n \leqslant x} a_K(n) = cx + O(x^{\frac{d-1}{d+1}+\varepsilon})$$

for arbitrary algebraic number field of degree $d \geqslant 2$, where $c > 0$ is a positive constant depending on K. Chandrasekharan and Narasimhan [3] considered the second moment of $a_K(n)$ for a general extension $K/\mathbb{Q}$ of degree d. They proved that

$$\sum_{n \leqslant x} a_K^2(n) \ll x \log^{d-1} x.$$

Later, Chandrasekharan and Good [2] studied the l-th integral power sum of $a_K(n)$, and gave an asymptotic formula for the sum

$$\sum_{n \leqslant x} a_K^l(n), \quad l = 2, 3, \ldots.$$

Lü and Wang [18] improved the result of Chandrasekharan and Good.

Fomenko [4] considered a non-normal cubic extension $K_3/\mathbb{Q}$, which is given by an irreducible polynomial $h(x) = x^3 + ax^2 + bx + c$ of discriminant D. If $D < 0$, he proved that

$$\sum_{n \leqslant x} a_{K_3}^2(n) = c_1 x \log x + c_2 x + O(x^{\frac{9}{11}+\varepsilon})$$

and

$$\sum_{n \leqslant x} a_{K_3}^3(n) = xP(\log x) + O(x^{\frac{73}{79}+\varepsilon}),$$

where c_1 and c_2 are constants, and $P(t)$ is a polynomial in t of degree 4. Later, Lü [17] improved the results of Fomenko, and obtained the exponents $23/31$ and $235/259$ in place of $9/11$ and $73/79$, respectively. Recently, Yang [23] derived an asymptotic formula for the sum

$$\sum_{n_1^2+n_2^2 \leqslant x} a_{K_3}(n_1^2 + n_2^2).$$

In this paper, we will be interested in the estimation of the higher moment of the arithmetic function $a_{K_3}(n)$ over sum of two squares,

$$\sum_{n_1^2+n_2^2 \leqslant x} a_{K_3}^l(n_1^2 + n_2^2), \quad 2 \leqslant l \leqslant 8.$$

Our main result is the following theorem.

Theorem 1.1 *Let K_3 be a non-normal cubic extension over $\mathbb{Q}$, which is given by an irreducible polynomial $h(x) = x^3 + ax^2 + bx + c$ of discriminant D. If $D < 0$, then, for arbitrarily small positive constant ε, we have*

$$\sum_{n_1^2+n_2^2\leqslant x} a_{K_3}^l(n_1^2+n_2^2) = xP_l(\log x) + O(x^{\vartheta_l+\varepsilon}),$$

where $P_l(t)$ are polynomials with degree η_l, and

$$\vartheta_2 = \frac{51}{59}, \quad \vartheta_3 = \frac{70}{73}, \quad \vartheta_4 = \frac{71}{72}, \quad \vartheta_5 = \frac{217}{218},$$
$$\vartheta_6 = \frac{1987}{1990}, \quad \vartheta_7 = \frac{6047}{6050}, \quad \vartheta_8 = \frac{18356}{18359}.$$

Here,

$$\eta_2 = 1, \quad \eta_3 = 4, \quad \eta_4 = 12, \quad \eta_5 = 33, \quad \eta_6 = 88, \quad \eta_7 = 232, \quad \eta_8 = 609.$$

2 Preliminaries

In this section, we begin with the representation of sum of two squares. Define

$$4r(n) = \sharp\{(n_1, n_2) \in \mathbb{Z}^2,\, n_1^2 + n_2^2 = n\}.$$

We know that (cf. [8, (1.51)])

$$r(n) = \sum_{d|n} \chi_4(d),$$

where $\chi_4(d)$ is the non-trivial Dirichlet character modulo 4. For the sake of simplicity, we denote

$$\chi := \chi_4.$$

By the completely multiplicative property, we get

$$r(p) = \sum_{d|p} \chi(d) = 1 + \chi(p).$$

Let K_3 be a non-normal cubic extension over $\mathbb{Q}$, which is given by an irreducible polynomial $h(x) = x^3 + ax^2 + bx + c$ of discriminant D. If $D < 0$, we learn from [4, (1)] that

$$\zeta_{K_3}(s) = \zeta(s)L(s, f), \tag{2}$$

where f is a holomorphic cusp form of weight 1 with respect to the congruence group $\Gamma_0(|D|)$, and

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n)e(nz).$$

Here, $\lambda_f(n)$ denotes the n-th Fourier coefficient of the holomorphic form f.

By using (2), the Euler product of Riemann zeta function $\zeta(s)$, and the Dirichlet L-function, one has

$$a_{K_3}(n) = \sum_{d|n} \lambda_f(d).$$

In particular, one has

$$a_{K_3}(p) = 1 + \lambda_f(p).$$

It is clear that

$$\sum_{n_1^2+n_2^2 \leqslant x} a_{K_3}^l(n_1^2+n_2^2) = \sum_{n\leqslant x} a_{K_3}^l(n) \sum_{m=n_1^2+n_2^2} 1 = 4\sum_{n\leqslant x} a_{K_3}^l(n) r(n).$$

The L-function defined for $\operatorname{Re} s > 1$ by

$$L_{K_3,l}(s) = \sum_{n=1}^{\infty} \frac{a_{K_3}^l(n) r(n)}{n^s} \tag{3}$$

has an analytic continuation to the whole complex plane.

In the following, we will give a decomposition of $L_{K_3,l}(s)$.

Lemma 2.1 *Let K_3 be a non-normal cubic extension over $\mathbb{Q}$. Suppose that $L_{K_3,l}(s)$ is defined as in* (3)*. Then we have*

$$L_{K_3,l}(s) = M_{K_3,l}(s) U_l(s), \quad l = 2, 3, \ldots, 8,$$

where

$$M_{K_3,2}(s) = \zeta^2(s) L^2(s,f) L(s,\mathrm{sym}^2 f) L^2(s,\chi) L^2(s, f\times\chi) L(s,\mathrm{sym}^2 f\times\chi),$$

$$\begin{aligned} M_{K_3,3}(s) = {} & \zeta^4(s) L^5(s,f) L^3(s,\mathrm{sym}^2 f) L(s,\mathrm{sym}^3 f) L^4(s,\chi) \\ & \times L^5(s,f\times\chi) L^3(s,\mathrm{sym}^2 f\times\chi) L(s,\mathrm{sym}^3 f\times\chi), \end{aligned}$$

$$\begin{aligned} M_{K_3,4}(s) = {} & \zeta^9(s) L^{12}(s,f) L^9(s,\mathrm{sym}^2 f) L^4(s,\mathrm{sym}^3 f) L(s,\mathrm{sym}^4 f) \\ & \times L^9(s,\chi) L^{12}(s,f\times\chi) L^9(s,\mathrm{sym}^2 f\times\chi) \\ & \times L^4(s,\mathrm{sym}^3 f\times\chi) L(s,\mathrm{sym}^4 f\times\chi), \end{aligned}$$

$$\begin{aligned} M_{K_3,5}(s) = {} & \zeta^{21}(s) L^{30}(s,f) L^{25}(s,\mathrm{sym}^2 f) L^{13}(s,\mathrm{sym}^3 f) L^5(s,\mathrm{sym}^4 f) \\ & \times L(s,\mathrm{sym}^4 f\times f) L^{21}(s,\chi) L^{30}(s,f\times\chi) L^{25}(s,\mathrm{sym}^2 f\times\chi) \\ & \times L^{13}(s,\mathrm{sym}^3 f\times\chi) L^5(s,\mathrm{sym}^4 f\times\chi) L(s,\mathrm{sym}^4 f\times f\times\chi), \end{aligned}$$

$$\begin{aligned} M_{K_3,6}(s) = {} & \zeta^{51}(s) L^{76}(s,f) L^{68}(s,\mathrm{sym}^2 f) L^{38}(s,\mathrm{sym}^3 f) L^{19}(s,\mathrm{sym}^4 f) \\ & \times L^6(s,\mathrm{sym}^4 f\times f) L(s,\mathrm{sym}^4 f\times\mathrm{sym}^2 f) L^{51}(s,\chi) L^{76}(s,f\times\chi) \\ & \times L^{68}(s,\mathrm{sym}^2 f\times\chi) L^{38}(s,\mathrm{sym}^3 f\times\chi) L^{19}(s,\mathrm{sym}^4 f\times\chi) \\ & \times L^6(s,\mathrm{sym}^4 f\times f\times\chi) L(s,\mathrm{sym}^4 f\times\mathrm{sym}^2 f\times\chi), \end{aligned}$$

$$
\begin{aligned}
M_{K_3,7}(s) = {} & \zeta^{127}(s)L^{195}(s,f)L^{182}(s,\mathrm{sym}^2 f)L^{106}(s,\mathrm{sym}^3 f)L^{63}(s,\mathrm{sym}^4 f) \\
& \times L^{26}(s,\mathrm{sym}^4 f\times f)L^{7}(s,\mathrm{sym}^4 f\times \mathrm{sym}^2 f)L(s,\mathrm{sym}^4 f\times \mathrm{sym}^3 f) \\
& \times L^{127}(s,\chi)L^{195}(s,f\times\chi)L^{182}(s,\mathrm{sym}^2 f\times\chi)L^{106}(s,\mathrm{sym}^3 f\times\chi) \\
& \times L^{63}(s,\mathrm{sym}^4 f\times\chi)L^{26}(s,\mathrm{sym}^4 f\times f\times\chi) \\
& \times L^{7}(s,\mathrm{sym}^4 f\times \mathrm{sym}^2 f\times\chi)L(s,\mathrm{sym}^4 f\times \mathrm{sym}^3 f\times\chi),
\end{aligned}
$$

$$
\begin{aligned}
M_{K_3,8}(s) = {} & \zeta^{322}(s)L^{504}(s,f)L^{483}(s,\mathrm{sym}^2 f)L^{288}(s,\mathrm{sym}^3 f)L^{195}(s,\mathrm{sym}^4 f) \\
& \times L^{96}(s,\mathrm{sym}^4 f\times f)L^{34}(s,\mathrm{sym}^4 f\times \mathrm{sym}^2 f)L^{8}(s,\mathrm{sym}^4 f\times \mathrm{sym}^3 f) \\
& \times L(s,\mathrm{sym}^4 f\times \mathrm{sym}^4 f)L^{322}(s,\chi)L^{504}(s,f\times\chi)L^{483}(s,\mathrm{sym}^2 f\times\chi) \\
& \times L^{288}(s,\mathrm{sym}^3 f\times\chi)L^{195}(s,\mathrm{sym}^4 f\times\chi)L^{96}(s,\mathrm{sym}^4 f\times f\times\chi) \\
& \times L^{34}(s,\mathrm{sym}^4 f\times \mathrm{sym}^2 f\times\chi)L^{8}(s,\mathrm{sym}^4 f\times \mathrm{sym}^3 f\times\chi) \\
& \times L(s,\mathrm{sym}^4 f\times \mathrm{sym}^4 f\times\chi).
\end{aligned}
$$

Here, χ is the non-trivial Dirichlet character modulo 4, *$U_l(s)$ is a Dirichlet series which converges absolutely in the half plane* $\mathrm{Re}\, s > 1/2$, *and* $U_l(1+\mathrm{i}t) \neq 0$.

Proof We will take $l = 8$ for an example, and give a detailed proof. Other cases can be obtained by the similar approaches.

For $\mathrm{Re}\, s > 1$, the Rankin-Selberg L-function attached to $\mathrm{sym}^M f$ and $\mathrm{sym}^N f$ are defined by

$$
\begin{aligned}
L(s,\mathrm{sym}^M f\times \mathrm{sym}^N f) &= \prod_p \prod_{0\leqslant i\leqslant M} \prod_{0\leqslant j\leqslant N} (1-\alpha_p^{M-i}\beta_p^{i}\alpha_p^{N-j}\beta_p^{j}p^{-s})^{-1} \\
&=: \sum_{n\geqslant 1} \frac{\lambda_{\mathrm{sym}^M f\times \mathrm{sym}^N f}(n)}{n^s},
\end{aligned}
$$

where

$$
\alpha_p + \beta_p = \lambda_f(p), \quad \alpha_p\beta_p = 1.
$$

For $l = 8$, we obtain

$$
\begin{aligned}
a_{K_3}^8(p)r(p) &= (1+\alpha_p+\beta_p)^8(1+\chi(p)) \\
&= \{(\alpha_p+\beta_p)^8 + 8(\alpha_p+\beta_p)^7 + 28(\alpha_p+\beta_p)^6 \\
&\quad + 56(\alpha_p+\beta_p)^5 + 70(\alpha_p+\beta_p)^4 + 56(\alpha_p+\beta_p)^3 \\
&\quad + 28(\alpha_p+\beta_p)^2 + 8(\alpha_p+\beta_p) + 1\}(1+\chi(p)).
\end{aligned}
$$

It is clear that

$$
\begin{aligned}
(\alpha_p+\beta_p)^2 &= \lambda_{\mathrm{sym}^2 f}(p) + 1, \\
(\alpha_p+\beta_p)^3 &= \lambda_{\mathrm{sym}^3 f}(p) + 2\lambda_f(p), \\
(\alpha_p+\beta_p)^4 &= \lambda_{\mathrm{sym}^4 f}(p) + 3\lambda_{\mathrm{sym}^2 f}(p) + 2, \\
(\alpha_p+\beta_p)^5 &= \lambda_{\mathrm{sym}^4 f\times f}(p) + 3\lambda_{\mathrm{sym}^3 f}(p) + 5\lambda_f(p),
\end{aligned}
$$

$$
\begin{aligned}
(\alpha_p+\beta_p)^6 &= \lambda_{\mathrm{sym}^4 f\times \mathrm{sym}^2 f}(p)+4\lambda_{\mathrm{sym}^4 f}(p)+8\lambda_{\mathrm{sym}^2 f}(p)+5,\\
(\alpha_p+\beta_p)^7 &= \lambda_{\mathrm{sym}^4 f\times \mathrm{sym}^3 f}(p)+5\lambda_{\mathrm{sym}^4 f\times f}(p)+8\lambda_{\mathrm{sym}^3 f}(p)+13\lambda_f(p),\\
(\alpha_p+\beta_p)^8 &= \lambda_{\mathrm{sym}^4 f\times \mathrm{sym}^4 f}(p)+6\lambda_{\mathrm{sym}^4 f\times \mathrm{sym}^2 f}(p)\\
&\quad +13\lambda_{\mathrm{sym}^4 f}(p)+21\lambda_{\mathrm{sym}^2 f}(p)+13.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
a_{K_3}^8(p)r(p) = {}&(\lambda_{\mathrm{sym}^4 f\times \mathrm{sym}^4 f}(p)+8\lambda_{\mathrm{sym}^4 f\times \mathrm{sym}^3 f}(p)+34\lambda_{\mathrm{sym}^4 f\times \mathrm{sym}^2 f}(p)\\
&+96\lambda_{\mathrm{sym}^4 f\times f}(p)+195\lambda_{\mathrm{sym}^4 f}(p)+288\lambda_{\mathrm{sym}^3 f}(p)\\
&+483\lambda_{\mathrm{sym}^2 f}(p)+504\lambda_f(p)+322)(1+\chi(p)).
\end{aligned}
$$

Thus, for $\operatorname{Re} s > 1$, we can write

$$
\begin{aligned}
&\zeta^{322}(s)L^{504}(s,f)L^{483}(s,\mathrm{sym}^2 f)L^{288}(s,\mathrm{sym}^3 f)L^{195}(s,\mathrm{sym}^4 f)\\
&\quad\times L^{96}(s,\mathrm{sym}^4 f\times f)L^{34}(s,\mathrm{sym}^4 f\times \mathrm{sym}^2 f)L^{8}(s,\mathrm{sym}^4 f\times \mathrm{sym}^3 f)\\
&\quad\times L(s,\mathrm{sym}^4 f\times \mathrm{sym}^4 f)L^{322}(s,\chi)L^{504}(s,f\times\chi)L^{483}(s,\mathrm{sym}^2 f\times\chi)\\
&\quad\times L^{288}(s,\mathrm{sym}^3 f\times\chi)L^{195}(s,\mathrm{sym}^4 f\times\chi)L^{96}(s,\mathrm{sym}^4 f\times f\times\chi)\\
&\quad\times L^{34}(s,\mathrm{sym}^4 f\times \mathrm{sym}^2 f\times\chi)L^{8}(s,\mathrm{sym}^4 f\times \mathrm{sym}^3 f\times\chi)\\
&\quad\times L(s,\mathrm{sym}^4 f\times \mathrm{sym}^4 f\times\chi)
\end{aligned}
$$

as an Euler product of the form

$$\prod_p \Big(1+\frac{A(p)}{p^s}+\frac{A(p^2)}{p^{2s}}+\cdots\Big),$$

where

$$A(p)=a_{K_3}^8(p)r(p).$$

Then, we derive that

$$
\begin{aligned}
L_{K_3,8}(s) = {}&\zeta^{322}(s)L^{504}(s,f)L^{483}(s,\mathrm{sym}^2 f)L^{288}(s,\mathrm{sym}^3 f)L^{195}(s,\mathrm{sym}^4 f)\\
&\times L^{96}(s,\mathrm{sym}^4 f\times f)L^{34}(s,\mathrm{sym}^4 f\times \mathrm{sym}^2 f)L^{8}(s,\mathrm{sym}^4 f\times \mathrm{sym}^3 f)\\
&\times L(s,\mathrm{sym}^4 f\times \mathrm{sym}^4 f)L^{322}(s,\chi)L^{504}(s,f\times\chi)L^{483}(s,\mathrm{sym}^2 f\times\chi)\\
&\times L^{288}(s,\mathrm{sym}^3 f\times\chi)L^{195}(s,\mathrm{sym}^4 f\times\chi)L^{96}(s,\mathrm{sym}^4 f\times f\times\chi)\\
&\times L^{34}(s,\mathrm{sym}^4 f\times \mathrm{sym}^2 f\times\chi)L^{8}(s,\mathrm{sym}^4 f\times \mathrm{sym}^3 f\times\chi)\\
&\times L(s,\mathrm{sym}^4 f\times \mathrm{sym}^4 f\times\chi)\\
&\times \prod_p\Big(1+\frac{a_{K_3}^8(p^2)r(p^2)-A(p^2)}{p^{2s}}+\cdots\Big)\\
=:{}& M_{K_3,8}(s)U_8(s),
\end{aligned}
$$

where $U_8(s)$ is a Dirichlet series which converges absolutely in the half plane $\operatorname{Re} s > 1/2$, and $U_8(1+\mathrm{i}t)\neq 0$. □

Remark 2.2 The famous work of Gelbart and Jacquet [5], Kim [12], and Kim and Shahidi [13,14] showed that $L(s, \mathrm{sym}^M f)$ $(1 \leqslant M \leqslant 4)$ is a general L-function, which has an analytic continuation as an entire function in the whole complex plane $\mathbb{C}$ and satisfies a certain functional equation of Riemann zeta-type of degree $M+1$. Due to the work of Jacquet and Shalika [9,10], Shahidi [20,21], and Rudnick and Sarnak [19], the Rankin-Selberg L-function $L(s, \mathrm{sym}^M f \times \mathrm{sym}^N f)$ has an analytic continuation as an entire function in the whole complex plane $\mathbb{C}$ $(1 \leqslant M, N \leqslant 4$, $M \neq N$ and except possibly for simple poles at $s = 0, 1$ for $M = N)$ and satisfies a certain functional equation of Riemann zeta-type of degree $(M+1)(N+1)$.

Next, we shall recall some lemmas, which play an important role in the proof of our results.

Lemma 2.3 *For any $\varepsilon > 0$, we have*

$$\zeta(\sigma + \mathrm{i}t) \ll_\varepsilon (1 + |t|)^{\max\{(1-\sigma)/3, 0\} + \varepsilon}$$

uniformly for $1/2 \leqslant \sigma \leqslant 1 + \varepsilon$ and $|t| \geqslant 1$.

Proof See [22, Theorem II 3.6]. □

The current best result is due to [1, Theorem 5], which states that

$$\zeta(\sigma + \mathrm{i}t) \ll_\varepsilon (1 + |t|)^{\max\{13(1-\sigma)/42, 0\} + \varepsilon} \tag{4}$$

uniformly for $1/2 \leqslant \sigma \leqslant 1 + \varepsilon$ and $|t| \geqslant 1$. For the average bounds, we have the well-known estimates

$$\int_1^T \left|\zeta\Big(\frac{1}{2} + \mathrm{i}t\Big)\right|^A \mathrm{d}t \ll T^{1+\varepsilon}, \quad A = 2, 4. \tag{5}$$

Combining the Phragmen-Lindelöf principle for a strip [8, Theorem 5.53] with the estimate given by Heath-Brown [7, (1.1)], we can derive a similar sub-convexity bound for Dirichlet L-function:

$$L(\sigma + \mathrm{i}t, \chi) \ll_\varepsilon (1 + |t|)^{\frac{1-\sigma}{3} + \varepsilon} \tag{6}$$

uniformly for $1/2 \leqslant \sigma \leqslant 1$ and $|t| \geqslant 1$, where χ is a Dirichlet character χ modulo q, and q is an integer.

We learn from [6, Corollary] that

$$L\Big(\frac{1}{2} + \mathrm{i}t, f\Big) \ll (1 + |t|)^{\frac{1}{3} + \varepsilon}, \quad t \in \mathbb{R}.$$

Similarly, we have the following result.

Lemma 2.4 *Let f be a primitive holomorphic cusp form with respect to the congruence group $\Gamma_0(|D|)$. Then, for any $\varepsilon > 0$, we have*

$$L(\sigma + \mathrm{i}t, f) \ll_\varepsilon (1 + |t|)^{\frac{2(1-\sigma)}{3} + \varepsilon}$$

uniformly for $1/2 \leqslant \sigma \leqslant 1$ *and* $|t| \geqslant 1$.

For the symmetric square L-function $L(s, \mathrm{sym}^2 f)$, we have the following sub-convexity bound.

Lemma 2.5 *For any* $\varepsilon > 0$, *we have*

$$L(\sigma + \mathrm{i}t, \mathrm{sym}^2 f) \ll (1 + |t|)^{\max\{11(1-\sigma)/8, 0\} + \varepsilon}$$

uniformly for $1/2 \leqslant \sigma \leqslant 1 + \varepsilon$ *and* $|t| \geqslant 1$.

Proof See [16, Corollary 1.2]. □

For the general L-function, we have the following convexity bound.

Lemma 2.6 *Let* $L(s, g)$ *be a Dirichlet series with the Euler product of degree* $m \geqslant 2$, *which means that*

$$L(s, g) = \sum_{n=1}^{\infty} \frac{L_g(n)}{n^s} = \prod_p \prod_{j=1}^{m} \Big(1 - \frac{\alpha_g(p, j)}{p^s}\Big)^{-1},$$

where $\alpha_g(p, j)$, $j = 1, 2, \ldots, m$, *are the local parameters of* $L(s, g)$ *at prime* p *and* $L_g(n) \ll n^{\varepsilon}$. *Assume that this series and its Euler product are absolutely convergent for* $\mathrm{Re}\, s > 1$. *Assume also that it admits a meromorphic continuation to the whole complex plane* $\mathbb{C}$ *and satisfies a functional equation of Riemann type. Then for* $0 \leqslant \sigma \leqslant 1$, *we have*

$$L(\sigma + \mathrm{i}t, g) \ll_{\varepsilon} (1 + |t|)^{\frac{m(1-\sigma)}{2} + \varepsilon}.$$

Proof See [8, Theorem 5.41]. □

It should be remarked that in Lemma 2.6, we consider only the t-aspect in the analytic conductor introduced by Iwaniec and Kowalski [8, Theorem 5.41]. By means of Remark 2.2 and Lemma 2.6, for the Rankin-Selberg L-function $L(s, \mathrm{sym}^M f \times \mathrm{sym}^N f)$ and any $\varepsilon > 0$, we have

$$L(s, \mathrm{sym}^M f \times \mathrm{sym}^N f) \ll_{\varepsilon} (1 + |t|)^{\frac{(M+1)(N+1)(1-\sigma)}{2} + \varepsilon},$$

where $1 \leqslant M, N \leqslant 4$.

For the general L-function, we have the following average sub-convexity bounds.

Lemma 2.7 *For any* $\varepsilon > 0$, *we have*

$$\int_1^T \Big|L\Big(\frac{1}{2} + \mathrm{i}t, f\Big)\Big|^2 \mathrm{d}t \sim CT \log T, \quad \int_1^T \Big|L\Big(\frac{1}{2} + \mathrm{i}t, f\Big)\Big|^6 \mathrm{d}t \ll T^{2+\varepsilon},$$

uniformly for $T \geqslant 1$.

Proof See [6, Theorem] and [11, (0.9)]. □

3 Proof of Theorem 1.1

In this section, we will take $l = 2$ for an example, and give a detailed proof. The cases of $3 \leqslant l \leqslant 8$ can be obtained by the similar approaches. By using sharper bounds and mean values of $\zeta(s)$ and $L(s,f)$, we can obtain better results. However, it will be improved a little. Thus, for the sake of simplicity, we would rather use the bound of Riemann zeta function in Lemma 2.3 than refer to (4).

Assume that K_3 is a cubic non-normal extension over $\mathbb{Q}$. By the Perron formula and (1), we get

$$\sum_{n\leqslant x} a_{K_3}^2(n)r(n) = \frac{1}{2\pi\mathrm{i}}\int_{1+\varepsilon-\mathrm{i}T}^{1+\varepsilon+\mathrm{i}T} L_{K_3,2}(s)\frac{x^s}{s}\,\mathrm{d}s + O\Big(\frac{x^{1+\varepsilon}}{T}\Big).$$

Then we move the integration to the parallel segment with $\operatorname{Re} s = \frac{1}{2}+\varepsilon =: b$. By the Cauchy residue theorem, we have

$$\begin{aligned}\sum_{n\leqslant x} a_{K_3}^2(n)r(n) &= \frac{1}{2\pi\mathrm{i}}\int_{1+\varepsilon-\mathrm{i}T}^{1+\varepsilon+\mathrm{i}T} L_{K_3,2}(s)\frac{x^s}{s}\,\mathrm{d}s + O\Big(\frac{x^{1+\varepsilon}}{T}\Big)\\ &= \frac{1}{2\pi\mathrm{i}}\bigg\{\int_{b-\mathrm{i}T}^{b+\mathrm{i}T} + \int_{1+\varepsilon-\mathrm{i}T}^{b-\mathrm{i}T} + \int_{b+\mathrm{i}T}^{1+\varepsilon+\mathrm{i}T}\bigg\} L_{K_3,2}(s)\frac{x^s}{s}\,\mathrm{d}s\\ &\quad + \operatorname*{Res}_{s=1}\Big\{L_{K_3,2}(s)\frac{x^s}{s}\Big\} + O\Big(\frac{x^{1+\varepsilon}}{T}\Big)\\ &=: I_{2,1} + I_{2,2} + xP_2(\log x) + O\Big(\frac{x^{1+\varepsilon}}{T}\Big),\end{aligned}$$

where

$$I_{2,1} = \frac{1}{2\pi\mathrm{i}}\int_{b-\mathrm{i}T}^{b+\mathrm{i}T} L_{K_3,2}(s)\frac{x^s}{s}\,\mathrm{d}s,$$
$$I_{2,2} = \frac{1}{2\pi\mathrm{i}}\bigg\{\int_{1+\varepsilon-\mathrm{i}T}^{b-\mathrm{i}T} + \int_{b+\mathrm{i}T}^{1+\varepsilon+\mathrm{i}T}\bigg\} L_{K_3,2}(s)\frac{x^s}{s}\,\mathrm{d}s,$$

and P_2 is a polynomial of degree η_2. We see from complex analysis that η_2+1 equals to the order of the pole $s=1$ of $L_{K_3,2}(s)$. By Lemma 2.1 and Remark 2.2, we know that only $\zeta(s)$ has a pole at $s=1$ in the factorization of $L_{K_3,2}(s)$. So we have $\eta_2 = 1$.

For $I_{2,1}$, from Lemma 2.1, we have

$$I_{2,1} \ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon}\int_1^T |M_{K_3,2}(b+\mathrm{i}t)|t^{-1}\mathrm{d}t.$$

By Cauchy's inequality, we get

$$I_{2,1} \ll x^{\frac{1}{2}+\varepsilon} \max_{T_1 \leqslant T} \Big\{ \frac{1}{T_1} |(L^2(s,\chi)L^2(s,f\times\chi) \times L(s,\mathrm{sym}^2 f)L(s,\mathrm{sym}^2 f\times\chi))_{s=b+\mathrm{i}T_1}| \times \Big(\int_{T_1/2}^{T_1} |\zeta(b+\mathrm{i}t)|^4 \mathrm{d}t\Big)^{1/2} \Big(\int_{T_1/2}^{T_1} |L(b+\mathrm{i}t,f)|^4 \mathrm{d}t\Big)^{1/2} \Big\} + x^{\frac{1}{2}+\varepsilon}.$$

By Lemma 2.7 and Cauchy's inequality, we obtain

$$\int_{T_1/2}^{T_1} |L(b+\mathrm{i}t,f)|^4 \mathrm{d}t \ll \Big(\int_{T_1/2}^{T_1} |L(b+\mathrm{i}t,f)|^2 \mathrm{d}t\Big)^{1/2} \Big(\int_{T_1/2}^{T_1} |L(b+\mathrm{i}t,f)|^6 \mathrm{d}t\Big)^{1/2} \ll T^{\frac{3}{2}+\varepsilon}.$$

Then, from Lemmas 2.4–2.6, (5), and (6), we get

$$I_{2,1} \ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} T^{\frac{1}{3}+\frac{2}{3}+\frac{3}{4}+\frac{11}{16}+\frac{1}{2}+\frac{3}{4}-1} \ll x^{\frac{1}{2}+\varepsilon} T^{\frac{43}{16}}.$$

For $I_{2,2}$, we derive that

$$I_{2,2} \ll \int_b^{1+\varepsilon} x^\sigma |M_{K_3,2}(\sigma+\mathrm{i}T)| T^{-1} \mathrm{d}\sigma \ll \max\{x^\sigma T^{\frac{55}{8}(1-\sigma)-1+\varepsilon}\} \ll x^{\frac{1}{2}+\varepsilon} T^{39/16} + \frac{x^{1+\varepsilon}}{T}.$$

Thus, we obtain

$$\sum_{n\leqslant x} a_{K_3}^2(n) r(n) = x P_2(\log x) + O\Big(x^{\frac{1}{2}+\varepsilon} T^{43/16} + \frac{x^{1+\varepsilon}}{T}\Big).$$

On taking $T = x^{8/59}$, we obtain

$$\sum_{n\leqslant x} a_{K_3}^2(n) r(n) = x P_2(\log x) + O(x^{\frac{51}{59}+\varepsilon}).$$

Remark 3.1 In the case $l=2$, only $\zeta(s)$ contributes orders of $s=1$ in the factorization of $L_{K_3,l}(s)$. However, for larger l, we know from [4] that in this case $L(s,\mathrm{sym}^3 f)$ also has a simple pole at $s=1$ though in most cases it is entire. So for $l \geqslant 3$, $\eta_l + 1$ equals to the sum of the degree of $\zeta(s)$ and $L(s,\mathrm{sym}^3 f)$ instead of the degree of $\zeta(s)$ itself. This explains the values of η_l in Theorem 1.1.

Remark 3.2 For the sake of simplicity, we only use the individual convexity or sub-convexity bounds of Riemann zeta function and the general L-function to derive nontrivial bounds in the cases of $3 \leqslant l \leqslant 8$.

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