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RESEARCH ARTICLE

Weak and smooth solutions to incompressible Navier-Stokes-Landau-Lifshitz-Maxwell equations

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Abstract Considering the Navier-Stokes-Landau-Lifshitz-Maxwell equations, in dimensions two and three, we use Galerkin method to prove the existence of weak solution. Then combine the a priori estimates and induction technique, we obtain the existence of smooth solution.

Keywords Weak solution, smooth solution, Navier-Stokes-Landau-Lifshitz-Maxwell equations
MSC 35A01, 35D30, 35Q30

1 Introduction and main results

In this paper, we consider the following Navier-Stokes-Landau-Lifshitz-Maxwell system:

$$u_t + (u\cdot\nabla)u + \nabla P - \nu\Delta u = -\lambda\nabla\cdot(\nabla d\odot\nabla d) - (E + u\times H), \tag{1.1}$$

$$d_t + (u\cdot\nabla)d = \gamma(-d\times(d\times\Delta d) + d\times\Delta d), \tag{1.2}$$

$$\nabla\cdot u = 0, \tag{1.3}$$

$$\frac{\partial E}{\partial t} - \nabla\times H = -\sigma E + u, \tag{1.4}$$

$$\frac{\partial(H+\beta d)}{\partial t} + \nabla\times E = 0, \tag{1.5}$$

$$\nabla\cdot E = 0, \tag{1.6}$$

$$\nabla\cdot(H+\beta d) = 0, \tag{1.7}$$

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with initial and boundary conditions

$$u(x,0)=u_0(x),\quad \nabla\cdot u_0=0,\quad d(x,0)=d_0(x),\quad x\in\Omega, \tag{1.8}$$

$$u(x,t)=d_t(x,t)=0,\quad (x,t)\in\partial\Omega\times(0,T), \tag{1.9}$$

in $\Omega\times(0,T)$, for a bounded smooth domain Ω in $\mathbb{R}^3$ or in $\mathbb{R}^2$, $T\in(0,\infty)$. (1.1) and (1.3) are the well-known density independent Navier-Stokes equations, while (1.2) is the Landau-Lifshitz when $u\equiv 0$. Here, $u(x,t)\colon \Omega\times(0,T)\to\mathbb{R}^n$ represents the velocity field of the flow, $d(x,t)\colon\Omega\times(0,T)\to S^n$, the unit sphere in $\mathbb{R}^{n+1}$ ($n=1,2$), is a unit vector that represents the macroscopic molecular orientation of the liquid crystal material. $P(x,t)\colon\Omega\times(0,+\infty)\to\mathbb{R}$ represents the pressure function, and ν, λ, and γ are positive constants that represent viscosity, the competition between kinetic energy and potential energy, and the microscopic elastic relaxation time for the molecular orientation field, respectively. In this paper, we note

$$-(d|\nabla d|^2+d\times\Delta d)=f(d). \tag{1.10}$$

In system (1.1)–(1.5), the unusual term $\nabla\cdot(\nabla d\odot\nabla d)$ denotes the $n\times n$ matrix whose (i,j)-th entry is given by $d_{x_i}\cdot d_{x_j}$ for $1\leqslant i,j\leqslant n$, '$\times$' denotes the vector outer product, $\sigma\geqslant 0$ denotes the constant conductivity, and the constant β can be viewed as the magnetic permeability of free space.

For system (1.1)–(1.3), Kim [10] proved the following regularity criterion:

$$u\in L^s(0,t;L^{p,\infty}(\mathbb{R}^3)),\quad \frac{2}{s}+\frac{3}{p}=1,\ 3<p\leqslant\infty. \tag{1.11}$$

Fan et al. [7] extended (1.11) to the multiplier space

$$\dot{X}_r := M(\dot{H}^r,L^2)=\Big\{f \mid \|f\|_{\dot{X}_r}:=\sup\frac{\|fg\|_{L^2}}{\|g\|_{H^r}}<\infty\Big\}.$$

When the term $d\times\Delta d$ is omitted, (1.1) and (1.2) become

$$u_t+u\cdot\nabla u+\nabla P-\nu\Delta u=-\lambda\nabla\cdot(\nabla d\odot\nabla d), \tag{1.12}$$

$$d_t+u\cdot\nabla d=\gamma(\Delta d+d|\nabla d|^2), \tag{1.13}$$

respectively, and system (1.12)-(1.13)-(1.3) is a simplified version of the Ericksen-Leslie model, which reduces to the Ossen-Frank model in the static case, for the hydrodynamics of nematic liquid crystals developed during the period 1958–1968 [2,3,12].

It is a macroscopic continuum description of the time evolution of the materials under the influence of both the flow field $u(x,t)$ and the macroscopic description of the microscopic orientation configurations $d(x,t)$ of rod-like liquid crystals. Roughly speaking, system (1.12)-(1.13)-(1.3) is a coupling between the non-homogeneous Navier-Stokes equation and the transported flow of harmonic maps. It is probably the simplest mathematical model one can derive, without destroying the basic nonlinear structure, from the original equations in the continuum theory of nematic liquid crystals proposed by Ericksen [4,5] and Leslie [13]. Lin [14] and Lin and Liu [16,17] initiated the mathematical analysis of system (1.12)-(1.13)-(1.3).

More precisely, Lin and Liu [17] considered the Leslie system of variable length, i.e., the Dirichlet energy

$$\frac{1}{2}\int_\Omega |\nabla d|^2 \mathrm{d}x$$

for

$$d\colon \Omega \to S^{n-1}$$

is replaced by the Ginzburg-Landau energy

$$\int_\Omega \Big(\frac{1}{2}\,|\nabla d|^2 + \frac{(1-|d|^2)^2}{4\varepsilon^2}\Big)\mathrm{d}x$$

for

$$d\colon \Omega \to \mathbb{R}^n,$$

and proved the existence of global classical and weak solutions in dimensions two or three. In [16], they proved the partial regularity theorem for suitable weak solutions, similar to the classical theorem by Caffarelli-Kohn-Nirenberg [1] for the Navier-Stokes equation. However, as pointed out in [16,17], both their estimates and arguments depend on ε, and it is a challenging problem to study the convergence as ε tends to zero.

The Ericksen-Leslie theory has successfully predicted several effects and analyzed many others rather well; see, for example, the survey article [6] by Leslie. Most analytical work, however, has been carried out under rather special assumptions concerning either the type of flow or the form of the solutions. There are also other reasons that this system merits attention.

The coupling of this new Navier-Stokes-Landau-Lifshitz-Maxwell system can be derived from the full Maxwell system as follows:

$$\frac{\partial B}{\partial t} = -\Delta \times E, \quad \frac{\partial D}{\partial t} + \sigma E = \nabla \times H, \tag{1.14}$$

where E and H are the electric and magnetic fields, respectively, $\sigma > 0$ is the conductivity, D and B are the electric and magnetic displacements defined by

$$D = \varepsilon_0 E, \quad B = \mu_0(H + d),$$

respectively, where ε_0 is the permittivity of free space, μ_0 is the magnetic permeability of free space, u is the velocity field of the flow, and E is the electric polarization. Substituting these definitions into (1.14), we may couple u, E, H, and d by system (1.1)–(1.7), for the more information about (1.14), we refer to [9].

In the past few years, progress has been made on the analysis of model (1.1)–(1.7) by overcoming the supercritical nonlinearity $|\nabla d|^2 d$. The existence of weak solutions was established in [15]. To our best knowledge, however, there are no results available on weak solutions of the multi-dimensional problem (1.1)–(1.7) with supercritical nonlinearity. Since (1.1)–(1.7) is strongly coupled, it is not easy to obtain the weak solution by use of the theory of semigroups as Schein in [18]. We are going to use Galerkin method here. We are interested in global weak or smooth solutions to problem (1.1)–(1.9) in the domain $\Omega\times(0,T)=Q_T$.

This paper is organized as follows. In Section 2, we use the standard method to obtain the existence of weak solution for problem (1.1)–(1.9) in the domain $\Omega\times(0,T)=Q_T$, where $\Omega\subset\mathbb{R}^2$ or $\mathbb{R}^3$. In Section 3, we obtain the global smooth solution by establish a prior estimates and induction technique.

C is a generic constant and may assume different values in different formulates.

For the sake of simplicity, we denote

$$\|\cdot\|_{L^p(\Omega)}=\|\cdot\|_p,\quad p\geqslant 2.$$

Denote $H^m(\Omega)$, $m=1,2,\dots$, the Sobolev space of complex-valued functions with the norm

$$\|u\|_{H^m}=\Big(\int_\Omega\sum_{|\alpha|\leqslant m}|D^\alpha u|^2\mathrm{d}x\Big)^{1/2},$$

and denote

$H_0^1(\Omega)=$ closure of $C_0^\infty(\Omega,\mathbb{R}^n)$ in the norm $(\int_\Omega|\nabla u|^2\mathrm{d}x)^{1/2}$,

$H^{-1}(\Omega)=$ the dual of $H_0^1(\Omega)$, $V=C_0^\infty(\Omega,\mathbb{R}^n)\cap\{u\colon \nabla\cdot u=0\}$,

$J=$ closure of V in $L^2(\Omega,\mathbb{R}^n)$, $K=$ closure of V in $H^1(\Omega,\mathbb{R}^n)$.

Definition 1.1 A vector $(u(x,t),d(x,t),E(x,t),H(x,t))\in(L^\infty(0,T;L^2(\Omega)),$ $L^\infty(0,T;H^1(\Omega)),L^\infty(0,T;L^2(\Omega)),L^\infty(0,T;L^2(\Omega)))$ is called a weak solution to problem (1.1)–(1.9), if for any vector-valued test function $\Psi(x,t)\in C^1(Q_T)$ such that $\Psi(x,T)=0$, the following equalities hold:

$$\begin{aligned}
&-\iint_{Q_T}u\cdot\Psi_t\mathrm{d}x\mathrm{d}t+\iint_{Q_T}\nabla u\cdot\nabla\Psi\mathrm{d}x\mathrm{d}t\\
&\quad+\iint_{Q_T}u\cdot\nabla u\cdot\Psi\mathrm{d}x\mathrm{d}t+\iint_{Q_T}\nabla P\cdot\Psi\mathrm{d}x\mathrm{d}t\\
&=\int_\Omega u_0\Psi(x,0)\mathrm{d}x-\lambda\iint_{Q_T}(\nabla d\odot\nabla d)\cdot\nabla\Psi\mathrm{d}x\mathrm{d}t\\
&\quad-\iint_{Q_T}E\cdot\Psi\mathrm{d}x\mathrm{d}t-\iint_{Q_T}(u\times H)\cdot\Psi\mathrm{d}x\mathrm{d}t,
\end{aligned}\tag{1.15}$$

$$
\begin{aligned}
&-\iint_{Q_T} d\cdot\Psi_t \mathrm{d}x\mathrm{d}t+\iint_{Q_T}\nabla u\cdot\nabla\Psi\mathrm{d}x\mathrm{d}t+\iint_{Q_T} u\cdot\nabla d\cdot\Psi\mathrm{d}x\mathrm{d}t\\
&\quad=\gamma\iint_{Q_T}(-d\times(d\times\Delta d)+d\times\Delta d)\cdot\Psi\mathrm{d}x\mathrm{d}t, \qquad (1.16)
\end{aligned}
$$

$$
\begin{aligned}
&\iint_{Q_T} E\Psi_t \mathrm{e}^{\sigma t}\mathrm{d}x\mathrm{d}t-\sigma\iint_{Q_T}\mathrm{e}^{\sigma t}E\cdot\Psi\mathrm{d}x\mathrm{d}t\\
&+\iint_{Q_T}\mathrm{e}^{\sigma t}(\nabla\times\Psi)\cdot H\mathrm{d}x\mathrm{d}t+\int_\Omega E_0\cdot\Psi(x,0)\mathrm{d}x+\int_\Omega u\cdot\mathrm{e}^{\sigma t}\Psi\mathrm{d}x=0, \qquad (1.17)
\end{aligned}
$$

$$
\begin{aligned}
&\iint_{Q_T}(H+\beta d)\Psi_t\mathrm{d}x\mathrm{d}t-\iint_{Q_T}(\nabla\times\Psi)\cdot E\mathrm{d}x\mathrm{d}t+\int_\Omega(H_0+\beta d_0)\cdot\Psi(x,0)\mathrm{d}x\\
&\quad=0, \qquad (1.18)
\end{aligned}
$$

Theorem 1.1 *Assume* $(u_0,d_0,E_0,H_0)\in(L^2(\Omega),H^1(\Omega),L^2(\Omega),L^2(\Omega))$. *Then problem* (1.1)–(1.9) *admits at least one global initial-valued solution* $(u(x,t), d(x,t),E(x,t),H(x,t))$ *such that*

$$
\begin{aligned}
u&\in L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))\cap C(0,T;H^{-1}(\Omega)),\\
d&\in L^2(0,T;H^2(\Omega))\cap L^\infty(0,T;H^1(\Omega))\cap C^{(0,1/6)}(0,T;L^2(\Omega)),\\
E&\in L^\infty(0,T;L^2(\Omega))\cap C(0,T;H^{-1}(\Omega)),\\
H&\in L^\infty(0,T;L^2(\Omega))\cap C(0,T;H^{-1}(\Omega)).
\end{aligned}
$$

Theorem 1.2 *Let* $m\geqslant 2$, *and assume* $(u_0,d_0,E_0,H_0)\in(H^1(\Omega),H^2(\Omega), H^{m-1}(\Omega),H^{m-1}(\Omega))$. *Then there exists a unique smooth solution* $(u(x,t), d(x,t),E(x,t),H(x,t))$ *of problem* (1.1)–(1.9) *such that*

$$(u,d)\in(L^\infty(0,T;H^m(\Omega)))^2,\quad (E,H)\in(L^\infty(0,T;H^{m-1}(\Omega)))^2,$$

either

$$\dim\Omega=2$$

or

$$\dim\Omega=3,$$

and

$$\nu\geqslant\nu_0(u_0,d_0,E_0,H_0).$$

For various notations and definitions of function spaces in the statement of the above theorem and throughout the paper, we refer to [11,20].

2 Weak solution to (1.1)–(1.9)

The sizes of the viscosity constants ν, λ, and γ do not play important roles in our proof of Theorem 1.1, i.e., the global existence of weak solutions of the

Cauchy boundary-value problem (1.1)–(1.9). Since ν, λ, and γ are not crucial in this section, we assume that $\nu=\lambda=\gamma=1$ for simplicity.

Lemma 2.1 (Gagliardo-Nirenberg inequality) *Assume*

$$u\in L^q(\Omega),\quad D^m u\in L^r(\Omega),\quad \Omega\subseteq\mathbb{R}^n,\quad 1\leqslant q,r\leqslant\infty,\quad 0\leqslant j\leqslant m.$$

Let p and α satisfy

$$\frac{1}{p}=\frac{j}{n}+\alpha\Big(\frac{1}{r}-\frac{m}{n}\Big)+(1-\alpha)\frac{1}{q},\quad \frac{j}{m}\leqslant\alpha\leqslant 1.$$

Then

$$\|D^j u\|_p\leqslant C(p,m,j,q,r)\|D^m u\|_r^{\alpha}\,\|u\|_q^{1-\alpha},\tag{2.1}$$

where $C(p,m,j,q,r)$ is a positive constant.

Lemma 2.2 (Gronwall's inequality, [8]) *Let c be a constant, and let $b(t)$ and $u(t)$ be nonnegative continuous functions in the interval $[0,T]$, satisfying*

$$u(t)\leqslant c+\int_0^t b(\tau)u(\tau)\mathrm{d}\tau,\quad t\in[0,T].$$

Then $u(t)$ satisfies the estimate

$$u(t)\leqslant c\exp\bigg(\int_0^t b(\tau)\mathrm{d}\tau\bigg),\quad t\in[0,T].\tag{2.2}$$

Lemma 2.3 [19] *Assume that $X\subset E\subset Y$ are Banach spaces and $X\hookrightarrow\hookrightarrow E$. Then the following imbeddings are compact*:

$$L^q(0,T;X)\cap\Big\{\varphi\colon \frac{\partial\varphi}{\partial t}\in L^1(0,T;Y)\Big\}\hookrightarrow\hookrightarrow L^q(0,T;E),\quad 1\leqslant q\leqslant\infty,\tag{2.3}$$

$$L^\infty(0,T;X)\cap\Big\{\varphi\colon \frac{\partial\varphi}{\partial t}\in L^r(0,T;Y)\Big\}\hookrightarrow\hookrightarrow C(0,T;E),\quad 1<r\leqslant\infty.\tag{2.4}$$

Lemma 2.4 *If Ω is a smooth, bounded domain in $\mathbb{R}^n$ and $u|_{\partial\Omega}=0$, then we have*

$$\|u\|_4^4\leqslant 2\|\nabla u\|_2^2\,\|u\|_2^2,\quad \|u\|_4^8\leqslant C\|\nabla u\|_2^5\,\|\nabla^2 u\|_2^3,\quad n=2,\tag{2.5}$$

$$\|u\|_4^4\leqslant 4\|\nabla u\|_2^3\,\|u\|_2,\quad \|\nabla u\|_4^6\leqslant C\|u\|_\infty^5\,\|\nabla^3 u\|_2,\quad n=3.\tag{2.6}$$

2.1 Global weak solutions to (1.1)–(1.9)

First, we get the system of the ordinary differential equations (ODEs) (2.7)–(2.11) admits at least one continuously differentiable global solution. Second, we use the Galerkin method to obtain the existence of weak solution to problem (1.1)–(1.9).

Let $w_n(x)$, $n=1,2,\dots$, be the unit eigenfunctions satisfying the equation

$$\Delta w_n+\lambda_n w_n=0,\quad w_n(x)=0,\quad x\in\partial\Omega,$$

and let λ_n, $n = 1, 2, \ldots$, be the corresponding eigenvalues different from each other.

Denote the approximate solution of the problem with the following forms:

$$u_m(x,t) = \sum_{s=1}^{m} \alpha_{sm}(t) w_s(x), \quad d_m(x,t) = \sum_{s=1}^{m} \beta_{sm}(t) w_s(x),$$

$$P_m(x,t) = \sum_{s=1}^{m} \zeta_{sm}(t) w_s(x),$$

$$E_m(x,t) = \sum_{s=1}^{m} \gamma_{sm}(t) w_s(x), \quad H_m(x,t) = \sum_{s=1}^{m} \delta_{sm}(t) w_s(x),$$

where $\alpha_{sm}(t), \beta_{sm}(t), \gamma_{sm}(t), \delta_{sm}(t)$ ($t \in \mathbb{R}^+$, $s = 1, 2, \ldots, m$, $m = 1, 2, \ldots$) are n-dimensional vector-valued functions satisfying the following system of ODEs:

$$\begin{aligned} \int_\Omega u_{mt} w_s(x) \mathrm{d}x = \int_\Omega [(&-u_m \cdot \nabla u_m + \Delta u_m + \nabla P_m - \nabla \cdot (\nabla d_m \odot \nabla d_m) \\ &- E_m) w_s(x) - (u_m \times H_m) w_s(x)] \mathrm{d}x, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\int_\Omega (d_{mt} + u_m \cdot \nabla d_m) w_s(x) \mathrm{d}x \\ &\quad = \int_\Omega (-d_m \times (d_m \times \Delta d_m) + d_m \times \Delta d_m) w_s(x) \mathrm{d}x, \end{aligned} \tag{2.8}$$

$$\int_\Omega E_{mt} w_s(x) \mathrm{d}x = \int_\Omega (\nabla \times H_m + u_m - \sigma E_m) w_s(x) \mathrm{d}x, \tag{2.9}$$

$$\int_\Omega (H_{mt} + \beta d_{mt}) w_s(x) \mathrm{d}x = -\int_\Omega (\nabla \times E_m) w_s(x) \mathrm{d}x, \tag{2.10}$$

$$\int_\Omega \nabla P_m w_s(x) \mathrm{d}x = 0, \tag{2.11}$$

and the initial conditions

$$\int_\Omega u_m(x,0) w_s(x) \mathrm{d}x = \int_\Omega u_0(x) w_s(x) \mathrm{d}x, \tag{2.12}$$

$$\int_\Omega d_m(x,0) w_s(x) \mathrm{d}x = \int_\Omega d_0(x) w_s(x) \mathrm{d}x, \tag{2.13}$$

$$\int_\Omega E_m(x,0) w_s(x) \mathrm{d}x = \int_\Omega E_0(x) w_s(x) \mathrm{d}x, \tag{2.14}$$

$$\int_\Omega H_m(x,0) w_s(x) \mathrm{d}x = \int_\Omega H_0(x) w_s(x) \mathrm{d}x. \tag{2.15}$$

It follows from the standard theory on nonlinear ODEs that the problem (2.7)–(2.15) admits unique local solution. The following estimates can ensure the existence and uniqueness of the solution of (2.7)–(2.15) and also obtain the global solution to problem (1.1)–(1.9).

Lemma 2.5 *Assume* $(u_{m0}, d_{m0}, E_{m0}, H_{m0}) \in (L^2(\Omega), H^1(\Omega), L^2(\Omega), L^2(\Omega))$. *Then for the solutions of the initial-value problem* (2.7)–(2.15), *we have the following estimates*:

$$\sup_{0\leqslant t\leqslant T}(\|u_m\|_2^2+\|d_m\|_{H^1}^2+\|E_m\|_2^2+\|H_m\|_2^2)\leqslant C,$$
$$\|u_m\|_{L^2(0,T;H^1(\Omega))}+\|d_m\|_{L^2(0,T;H^2(\Omega))}\leqslant C,$$

where C is independent of m.

Proof Testing (2.9) by γ_{sm} and (2.10) by δ_{sm}, summing up the result for $s=1,2,\ldots,m$, and integrating by parts, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|E_m|^2+|H_m|^2)\mathrm{d}x+\sigma\int_\Omega|E_m|^2\mathrm{d}x+\beta\int_\Omega d_{mt}H_m\mathrm{d}x=\int_\Omega u_mE_m\mathrm{d}x. \tag{2.16}$$

Testing (2.9) by $\alpha_{sm}+\gamma_{sm}$, summing up the result for $s=1,2,\ldots,m$, and integrating by parts, we deduce that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|E_m|^2\mathrm{d}x+\int_\Omega u_mE_{mt}\mathrm{d}x&=\int_\Omega(E_m+u_m)(-\sigma E_m+u_m)\mathrm{d}x\\&\quad+\int_\Omega(\nabla\times H_m)(E_m+u_m)\mathrm{d}x.\end{aligned} \tag{2.17}$$

Adding (2.16) and (2.17), then testing the result by α_0, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(2\alpha_0|E_m|^2+2\alpha_0|H_m|^2)\mathrm{d}x+2\beta\alpha_0\int_\Omega d_{mt}H_m\mathrm{d}x\\&\quad+\alpha_0\int_\Omega u_mE_{mt}\mathrm{d}x+\sigma\alpha_0\int_\Omega|E_m|^2\mathrm{d}x\\&=\alpha_0\int_\Omega u_mE_m\mathrm{d}x+\alpha_0\int_\Omega(\nabla\times H_m)u_m\mathrm{d}x\\&\quad+\alpha_0\int_\Omega(E_m+u_m)(-\sigma E_m+u_m)\mathrm{d}x.\end{aligned} \tag{2.18}$$

Multiplying (2.7) by $\alpha_{sm}(t)$ and summing up the products for $s=1,2,\ldots,m$, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|u_m|^2\mathrm{d}x+\int_\Omega|\nabla u_m|^2\mathrm{d}x\\&\quad=-\int_\Omega(u_m\cdot\nabla)d_m\cdot\Delta d_m\mathrm{d}x-\int_\Omega E_mu_m\mathrm{d}x.\end{aligned} \tag{2.19}$$

Testing (2.8) by $-\lambda_m\beta_{sm}$, summing up the products for $s = 1, 2, \ldots, m$, integrating by parts, and adding the above equality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|u_m|^2+|\nabla d_m|^2)\mathrm{d}x+\int_\Omega(|\nabla u_m|^2+|\Delta d_m|^2\,|d_m|^2)\mathrm{d}x\\
&\quad=\int_\Omega(d_m\cdot\Delta d_m)^2\mathrm{d}x-\int_\Omega E_m u_m\mathrm{d}x\\
&\quad\leqslant\int_\Omega|\Delta d_m|^2\,|d_m|^2\mathrm{d}x-\int_\Omega E_m u_m\mathrm{d}x. \qquad (2.20)
\end{aligned}$$

Combining (2.20) with (2.18), we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|u_m|^2+|\nabla d_m|^2+2\alpha_0|E_m|^2+2\alpha_0|H_m|^2)\mathrm{d}x+\int_\Omega|\nabla u_m|^2\mathrm{d}x\\
&\qquad+2\beta\alpha_0\int_\Omega d_{mt}H_m\mathrm{d}x+\alpha_0\int_\Omega E_{mt}u_m\mathrm{d}x+\sigma\alpha_0\int_\Omega|E_m|^2\mathrm{d}x\\
&\quad\leqslant\alpha_0\int_\Omega(u_m-\sigma E_m)(E_m+u_m)\mathrm{d}x+\alpha_0\int_\Omega(\nabla\times H_m)u_m\mathrm{d}x\\
&\qquad+(\alpha_0-1)\int_\Omega E_m u_m\mathrm{d}x. \qquad (2.21)
\end{aligned}$$

In order to deal with the term $\int_\Omega d_{mt}H_m\mathrm{d}x$, we multiply (2.10) by $(2\beta\alpha_0-\beta)\beta_{sm}$ and sum up the product for $s = 1, 2, \ldots, m$ to obtain

$$\begin{aligned}
&(2\beta\alpha_0-\beta)\int_\Omega d_m H_{mt}\mathrm{d}x+\frac{\beta(2\beta\alpha_0-\beta)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|d_m|^2\mathrm{d}x\\
&\qquad\qquad+(2\beta\alpha_0-\beta)\int_\Omega(\nabla\times E_m)d_m\mathrm{d}x=0. \qquad (2.22)
\end{aligned}$$

Using (2.17), adding (2.21) and (2.22), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|u_m|^2+|\nabla d_m|^2+(2\alpha_0+1)|E_m|^2+(2\alpha_0+1)|H_m|^2)\mathrm{d}x\\
&\qquad+(2\beta\alpha_0-\beta)\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega d_m H_m\mathrm{d}x+\int_\Omega|\nabla u_m|^2\mathrm{d}x\\
&\qquad+\alpha_0\int_\Omega E_{mt}u_m\mathrm{d}x+(\sigma\alpha_0+\sigma)\int_\Omega|E_m|^2\mathrm{d}x\\
&\quad\leqslant\alpha_0\int_\Omega(u_m-\sigma E_m)(E_m+u_m)\mathrm{d}x+\alpha_0\int_\Omega(\nabla\times H_m)u_m\mathrm{d}x\\
&\qquad-(2\beta\alpha_0-\beta)\int_\Omega(\nabla\times E_m)d_m\mathrm{d}x\\
&\qquad-\frac{\beta(2\beta\alpha_0-\beta)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|d_m|^2\mathrm{d}x+(\alpha_0-2)\int_\Omega E_m u_m\mathrm{d}x. \qquad (2.23)
\end{aligned}$$

Similarly, to deal with the term $\int_\Omega u_m E_{mt}\mathrm{d}x$, we multiply (2.9) by $\alpha_0\alpha_{sm}$ and sum up the product for $s = 1, 2, \ldots, m$ to obtain

$$\alpha_0\int_\Omega E_{mt}u_m\mathrm{d}x - \alpha_0\int_\Omega |u_m|^2\mathrm{d}x + \alpha_0\sigma\int_\Omega E_m u_m\mathrm{d}x - \alpha_0\int_\Omega(\nabla\times H_m)\cdot u_m\mathrm{d}x = 0. \tag{2.24}$$

Take (2.24) into the inequality (2.23) and denote

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|u_m|^2 + |\nabla d_m|^2 + (2\alpha_0+1)|E_m|^2 + (2\alpha_0+1)|H_m|^2)\mathrm{d}x\\
&\quad + \alpha_0\int_\Omega |u_m|^2\mathrm{d}x + (2\beta\alpha_0-\beta)\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega d_m H_m\mathrm{d}x + \int_\Omega|\nabla u_m|^2\mathrm{d}x\\
&\quad + \frac{\beta(2\beta\alpha_0-\beta)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|d_m|^2\mathrm{d}x\\
&\leqslant \alpha_0\int_\Omega(u_m-\sigma E_m)(E_m+u_m)\mathrm{d}x + (2\beta\alpha_0-\beta)\int_\Omega(\nabla\times E_m)d_m\mathrm{d}x\\
&\quad - \sigma\alpha_0\int_\Omega|E_m|^2\mathrm{d}x + (\alpha_0\sigma+\alpha_0-2)\int_\Omega E_m u_m\mathrm{d}x\\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{2.25}$$

We estimate each term of the right-hand side of (2.25):

$$\begin{aligned}
I_1 &\leqslant \frac{\alpha_0}{2}\|u_m-\sigma E_m\|_2^2 + \frac{\alpha_0}{2}\|E_m+u_m\|_2^2,\\
I_2 &\leqslant \frac{2\beta\alpha_0-\beta}{2}(\|\nabla\times d_m\|_2^2 + \|E_m\|_2^2),\\
I_4 &\leqslant \frac{\alpha_0\sigma+\alpha_0-2}{2}(\|u_m\|_2^2+\|E_m\|_2^2).
\end{aligned}$$

So,

$$\begin{aligned}
I_1+I_2+I_4 &\leqslant \frac{\alpha_0}{2}\|u_m-\sigma E_m\|_2^2 + \frac{\alpha_0}{2}\|E_m+u_m\|_2^2\\
&\quad + \Big(\frac{2\beta\alpha_0-\beta}{2} + \frac{\alpha_0\sigma+\alpha_0-2}{2}\Big)\|E_m\|_2^2\\
&\quad + \frac{\alpha_0\sigma+\alpha_0-2}{2}\|u_m\|_2^2 + \frac{2\beta\alpha_0-\beta}{2}\|\nabla\times d_m\|_2^2\\
&\leqslant \frac{2\beta\alpha_0-\beta}{2}\|\nabla\times d_m\|_2^2 + \frac{\alpha_0\sigma+5\alpha_0-2}{2}\|u_m\|_2^2\\
&\quad + \frac{\alpha_0\sigma+3\alpha_0-2+2\alpha_0\sigma^2+2\beta\alpha_0-\beta}{2}\|E_m\|_2^2.
\end{aligned}$$

Taking the above inequalities into inequality (2.25), we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|u_m|^2+|\nabla d_m|^2+2\alpha_0|E_m|^2+2\alpha_0|H_m|^2)\mathrm{d}x\\&\quad+(2\beta\alpha_0-\beta)\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega d_mH_m\mathrm{d}x+\int_\Omega|\nabla u_m|^2\mathrm{d}x+\frac{\beta(2\beta\alpha_0-\beta)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|d_m|^2\mathrm{d}x\\&\leqslant\frac{\alpha_0\sigma+5\alpha_0-2}{2}\|u_m\|_2^2+\frac{\alpha_0\sigma+3\alpha_0-2+2\alpha_0\sigma^2+2\beta\alpha_0-\beta}{2}\|E_m\|_2^2\\&\quad+\frac{2\beta\alpha_0-\beta}{2}\|\nabla\times d_m\|_2^2.\end{aligned}\tag{2.26}$$

Integrating inequality (2.26) with the variant t, we deduce that

$$\begin{aligned}&\frac{1}{2}(\|u_m\|_2^2+\|\nabla d_m\|_2^2+2\alpha_0\|E_m\|_2^2+2\alpha_0\|H_m\|_2^2)\\&\quad-\frac{1}{2}(2\beta\alpha_0-\beta)(\|d_m\|_2^2+\|H_m\|_2^2)+\int_0^t\|\nabla u_m\|_2^2\mathrm{d}t+\frac{\beta(2\beta\alpha_0-\beta)}{2}\|d_m\|_2^2\\&\leqslant\frac{\alpha_0\sigma+5\alpha_0-2}{2}\int_0^t\|u_m\|_2^2\mathrm{d}t+\frac{2\beta\alpha_0-\beta}{2}\int_0^t\|\nabla\times d_m\|_2^2\mathrm{d}t\\&\quad+\frac{\alpha_0\sigma+3\alpha_0-2+2\alpha_0\sigma^2+2\beta\alpha_0-\beta}{2}\int_0^t\|E_m\|_2^2\mathrm{d}t.\end{aligned}$$

From the above inequality, we get

$$\beta>1,\quad \max\left\{\frac{2}{5+\sigma},\frac{2+\beta-2\beta\alpha_0}{\sigma+2\sigma^2+3},\frac{1}{2}\right\}<\alpha_0<\frac{1}{2(\beta-1)}.$$

Take

$$C_1=\frac{1}{2}\min\{2,4\alpha_0,\beta+2\alpha_0-2\beta\alpha_0,(\beta-1)(2\beta\alpha_0-\beta)\},$$

$$C_2=\frac{1}{2}\max\{\alpha_0\sigma+5\alpha_0-2,\alpha_0\sigma+3\alpha_0-2+2\alpha_0\sigma^2+2\beta\alpha_0-\beta,2\beta\alpha_0-\beta\},$$

so

$$\begin{aligned}&C_1(\|u_m\|_2^2+\|\nabla d_m\|_2^2+\|E_m\|_2^2+\|H_m\|_2^2)\\&\quad\leqslant C_2\int_0^t(\|u_m\|_2^2+\|E_m\|_2^2+\|\nabla\times d_m\|_2^2)\mathrm{d}t.\end{aligned}$$

By Gronwall's inequality, we have

$$\sup_{0\leqslant t\leqslant T}(\|u_m\|_2^2+\|d_m\|_{H^1}^2+\|E_m\|_2^2+\|H_m\|_2^2)\leqslant C,$$

where C is independent of m. □

Lemma 2.6 *Under the conditions of Lemma* 2.5, *for the solution* (u_m,d_m,E_m,H_m) *of system* (2.7)–(2.15), *there exists* $C>0$ *independent of* m *such that*

$$\int_0^t(\|u_{mt}\|_{H^{-2}}^2+\|d_{mt}\|_{H^{-2}}^2+\|E_{mt}\|_{H^{-2}}^2+\|H_{mt}\|_{H^{-2}}^2)\mathrm{d}t\leqslant C.$$

Proof For the function $\phi \in H_0^2(\Omega)$, ϕ can be represented by

$$\phi = \phi_m + \overline{\phi}_m, \quad \phi_m = \sum_{s=1}^{m} \eta_s w_s(x), \quad \overline{\phi}_m = \sum_{s=m+1}^{\infty} \eta_s w_s(x), \tag{2.27}$$

where, for $s \geqslant m+1$, we have

$$\int_\Omega d_{mt} w_s(x) \mathrm{d}x = 0.$$

First, we consider $\Omega \in \mathbb{R}^2$. By Lemma 2.5 and the Ladyzhenskaya inequality [11], there holds that

$$\begin{aligned}
&\Big| \int_\Omega d_{mt} \phi(x) \mathrm{d}x \Big| \\
&\quad = \Big| \int_\Omega d_{mt} \phi_m(x) \mathrm{d}x \Big| \\
&\quad = \Big| \int_\Omega (-u_m \cdot \nabla d_m + \Delta d_m + d_m |\nabla d_m|^2 + d_m \times \Delta d_m) \phi_m(x) \mathrm{d}x \Big| \\
&\quad \leqslant \|\phi_m\|_\infty \int_\Omega |u_m \cdot \nabla d_m| \mathrm{d}x + \int_\Omega |d_m \cdot \Delta \phi_m| \mathrm{d}x + \|\phi_m\|_\infty \int_\Omega d_m |\nabla d_m|^2 \mathrm{d}x \\
&\qquad + \int_\Omega |\nabla d_m \times \nabla d_m| \phi_m(x) \mathrm{d}x + \int_\Omega |d_m \times \nabla d_m| \nabla \phi_m(x) \mathrm{d}x \\
&\quad \leqslant (\|u_m\|_2 \, \|\nabla d_m\|_2 + \|d_m\|_2 + \|d_m\|_2 \, \|\nabla d_m\|_4^2)(\|\phi_m\|_\infty + \|\phi_m\|_2) \\
&\qquad + (\|\nabla d_m\|_2^2 + \|d_m\|_2 \, \|\nabla d_m\|_2)(\|\phi_m\|_\infty + \|\phi_m\|_2) \\
&\quad \leqslant (\|u_m\|_2 \, \|\nabla d_m\|_2 + \|d_m\|_2 + C \|d_m\|_2 \, \|\nabla d_m\|_2 \, \|\Delta d_m\|_2 \\
&\qquad + \|\Delta d_m\|_2 \, \|d_m\|_2) \|\phi\|_{H^2} \\
&\quad \leqslant C \|\Delta d_m\|_2 \, \|\phi\|_{H^2};
\end{aligned}$$

$$\begin{aligned}
\Big| \int_\Omega u_{mt} \phi(x) \mathrm{d}x \Big| &= \Big| \int_\Omega u_{mt} \phi_m(x) \mathrm{d}x \Big| \\
&\leqslant C \Big(\int_\Omega |u_m \cdot \nabla u_m \cdot \phi_m| \mathrm{d}x + \int_\Omega |\nabla u_m \cdot \nabla \phi_m| \mathrm{d}x \\
&\quad + \int_\Omega |\nabla \cdot (\nabla d_m \odot \nabla d_m) \cdot \phi_m| \mathrm{d}x + \int_\Omega |E_m \phi_m| \mathrm{d}x \\
&\quad + \int_\Omega |(u_m \times H_m) \phi_m| \mathrm{d}x \Big) \\
&\leqslant C (\|\nabla u_m\|_2 \, \|\nabla \phi_m\|_2 + \|u_m\|_2 \, \|\nabla u_m\|_2 \, \|\phi_m\|_\infty \\
&\quad + \|E_m\|_2 \, \|\phi_m\|_2 + \|u_m\|_2 \, \|H_m\|_2 \, \|\phi_m\|_\infty \\
&\quad + \|\nabla d_m\|_2 \, \|\Delta d_m\|_2 \, \|\phi_m\|_\infty),
\end{aligned}$$

so

$$\Big| \int_\Omega u_{mt} \phi(x) \mathrm{d}x \Big| \leqslant C (\|\nabla u_m\|_2^2 + |\Delta d_m\|_2^2) \|\phi_m\|_{H^2};$$

$$\begin{aligned}\Big|\int_\Omega E_{mt}\phi(x)\mathrm{d}x\Big| &= \Big|\int_\Omega E_{mt}\phi_m(x)\mathrm{d}x\Big|\\ &= \Big|-\int_\Omega [u_{mt}\phi_m + \sigma E_m\phi_m - (\nabla\times H_m)\phi_m]\mathrm{d}x\Big|\\ &\leqslant C(\|E_m\|_2\,\|\phi_m\|_2 + \|H_m\|_2\,\|\nabla\phi_m\|_2 + \|u_m\|_2\,\|\phi_m\|_2)\\ &\leqslant C\|\phi_m\|_{H^2}(\|\nabla u_m\|_2^2 + |\Delta d_m\|_2^2);\end{aligned}$$

$$\Big|\int_\Omega H_{mt}\phi(x)\mathrm{d}x\Big| = \Big|\int_\Omega H_{mt}\phi_m(x)\mathrm{d}x\Big| \leqslant C\|\phi_m\|_{H^2},$$

where the constant C is independent of m.

When $\Omega \in \mathbb{R}^3$, by Lemma 2.4, the above estimates also hold. Finally, integrate the above estimates with respect to t, the lemma is proved. □

Lemma 2.7 *Under the estimates of Lemma* 2.5 *for the solution* (u_m, d_m, E_m, H_m) *of system* (2.7)–(2.15), *there exists* $C > 0$ *independent of* m *such that*

$$\|d_m(\cdot,t_1) - d_m(\cdot,t_2)\|_2 \leqslant C|t_1 - t_2|^{1/6},$$
$$u_m, E_m, H_m \in C(0,T;H^{-1}(\Omega)).$$

Proof By the Sobolev interpolation of negative order, we have

$$\begin{aligned}\|d_m(\cdot,t_1) - d_m(\cdot,t_2)\|_2 &\leqslant C\|d_m(\cdot,t_1) - d_m(\cdot,t_2)\|_{H^{-2}}^{1/3}\,\|d_m(\cdot,t_1) - d_m(\cdot,t_2)\|_{H^1}^{2/3}\\ &\leqslant C\Big\|\int_{t_1}^{t_2} d_{mt}\mathrm{d}t\Big\|_{H^{-2}}^{1/3}\\ &\leqslant C\Big(\int_{t_1}^{t_2}\|d_{mt}\|_{H^{-2}}^2\mathrm{d}t\Big)^{1/6}\cdot|t_1 - t_2|^{1/6}\\ &\leqslant C|t_1 - t_2|^{1/6}.\end{aligned}$$

On the other hand, it follows from Lemma 2.3 and

$$L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \hookrightarrow H^{-2}(\Omega),$$

$$H_m \in L^\infty(0,T;L^2(\Omega)) \cap \Big\{\psi\colon \frac{\partial\psi}{\partial t} \in L^2(0,T;H^{-2}(\Omega))\Big\},$$

we have

$$H_m \in C([0,T];H^{-1}(\Omega)).$$

Similarly, we have

$$E_m \in C([0,T];H^{-1}(\Omega)), \quad u_m \in C([0,T];H^{-1}(\Omega)).$$ □

In fact, it follows from (2.21) and (2.22) that the solution (u_m, d_m, E_m, H_m) of system (2.7)–(2.15) does not blow up at finite from ODE problem, we have the following lemma.

Lemma 2.8 *Under the conditions of Lemma* 2.5, *the initial problem of the system of the ODEs* (2.7)–(2.15) *admits at least one continuously differentiable global solution* $(\overline{\alpha}_{sm}(t), \overline{\beta}_{sm}(t), \overline{\gamma}_{sm}(t), \overline{\delta}_{sm}(t))$.

2.2 Existence of weak solution for problem (1.1)–(1.9)

First of all, as in Definition 1.1, we may define the weak solution of problem (1.1)–(1.9). In the proof of Theorem 1.1, we must use the following lemma which is well known.

Lemma 2.9 *If $u_n \to u$ strongly in $L^2(Q_T)$ and $v_n \to v$ weakly in $L^2(Q_T)$, then $u_n v_n \to uv$ weakly in $L^1(Q_T)$ and in the sense of distribution.*

Now, we prove the existence of weak solution for problem (1.1)–(1.9) and finish the proof of Theorem 1.1.

Proof of Theorem 1.1 The uniform estimates for the approximate solution $(u_m(x,t), d_m(x,t), E_m(x,t), H_m(x,t))$ in the previous subsection yields that there is a subsequence of $(u_m(x,t), d_m(x,t), E_m(x,t), H_m(x,t))$ such that

$$u_m(x,t) \to u(x,t) \quad \text{weakly } * \text{ in } L^\infty(0,T;J), \tag{2.28}$$

$$u_m(x,t) \to u(x,t) \quad \text{weakly } * \text{ in } L^2(0,T;K), \tag{2.29}$$

$$d_m(x,t) \to d(x,t) \quad \text{weakly } * \text{ in } L^\infty(0,T;H^1(\Omega)), \tag{2.30}$$

$$d_m(x,t) \to d(x,t) \quad \text{strongly in } L^2(0,T;H^1(\Omega)), \tag{2.31}$$

$$d_m(x,t) \to d(x,t) \quad \text{weakly } * \text{ in } L^2(0,T;H^2(\Omega)), \tag{2.32}$$

$$H_m(x,t) \to H(x,t) \quad \text{weakly } * \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{2.33}$$

$$E_m(x,t) \to E(x,t) \quad \text{weakly } * \text{ in } L^\infty(0,T;L^2(\Omega)). \tag{2.34}$$

For any vector-value test function $\Psi(x,t) \in C^1(\overline{Q}_T)$, $\Psi(x,T) = 0$. We define an approximate sequence

$$\Psi_m(x,t) = \sum_{s=1}^{m} \eta_s(t) w_s(x),$$

where

$$\eta_s(t) = \int_\Omega \Psi(x,t) w_s(x) \mathrm{d}x.$$

Then

$$\Psi_m(x,t) \to \Psi(x,t) \ \text{ in } C^1(Q_T) \text{ and in } L^p(Q_T),\ \forall p > 1. \tag{2.35}$$

Making the scalar product of $\eta_s(t)$ in (2.7), (2.8), (2.10) and $\mathrm{e}^{\sigma t}\eta_s(t)$ in (2.9), summing up the result for $s = 1, 2, \ldots, m$ and integrating by parts we have

$$\begin{aligned}
&\iint_{Q_T} u_m\Psi_{mt}\mathrm{d}x\mathrm{d}t+\int_\Omega u_m(\cdot,0)\Psi_m(\cdot,0)\mathrm{d}x\\
&\quad=\iint_{Q_T}(u_m\cdot\nabla u_m-\Delta u_m+\nabla P+\nabla\cdot(\nabla d_m\odot\nabla d_m)+E_m)\Psi_m\mathrm{d}x\mathrm{d}t\\
&\qquad+(u_m\times H_m)\Psi_m\mathrm{d}x\mathrm{d}t\\
&\quad=\iint_{Q_T}[(u_m\cdot\nabla u_m)\Psi_m-\Delta u_m\Psi_m+\nabla\cdot(\nabla d_m\odot\nabla d_m)\Psi_m+\nabla P\Psi_m\\
&\qquad+(E_m+(u_m\times H_m))\Psi_m]\mathrm{d}x\mathrm{d}t, \qquad (2.36)
\end{aligned}$$

$$\begin{aligned}
&\iint_{Q_T} d_m\Psi_{mt}\mathrm{d}x\mathrm{d}t+\int_\Omega d_m(\cdot,0)\Psi_m(\cdot,0)\mathrm{d}x\\
&=\iint_{Q_T}[(u_m\cdot\nabla)d_m\Psi_m-(d_m\times\Delta d_m)\cdot(d_m\times\Psi_m)-(d_m\times\Delta d_m)\Psi_m]\mathrm{d}x\mathrm{d}t, \qquad (2.37)
\end{aligned}$$

$$\begin{aligned}
&\iint_{Q_T} E_m\Psi_{mt}\mathrm{e}^{\sigma t}\mathrm{d}x\mathrm{d}t+\iint_{Q_T} u_m\Psi_m\mathrm{e}^{\sigma t}\mathrm{d}x\mathrm{d}t\\
&+\iint_{Q_T}(\nabla\times\Psi_m)\cdot H_m\mathrm{e}^{\sigma t}\mathrm{d}x\mathrm{d}t+\int_\Omega E_m(\cdot,0)\Psi_m(\cdot,0)\mathrm{d}x=0, \qquad (2.38)
\end{aligned}$$

$$\begin{aligned}
&\iint_{Q_T}(H_m+\beta d_m)\Psi_{mt}\mathrm{d}x\mathrm{d}t+\int_\Omega(H_m(\cdot,0)+\beta d_m(\cdot,0))\Psi_m(\cdot,0)\mathrm{d}x\\
&\qquad-\iint_{Q_T}(\nabla\times\Psi_m)\cdot E_m\mathrm{d}x\mathrm{d}t=0. \qquad (2.39)
\end{aligned}$$

Now, we are in the position to prove that $(u(x,t),d(x,t),E(x,t),H(x,t))$ is a weak solution of (1.1)–(1.9). To this aim, one should set $m\to\infty$ in (2.36)–(2.39). From (2.28)–(2.35) and Lemma 2.9, it suffices to deal with the nonlinear terms in (2.36)–(2.39). By (2.28) and (2.29), we have

$$\begin{aligned}
&\iint_{Q_T}(\nabla u_m\cdot\nabla\Psi_m-\nabla u\cdot\nabla\Psi)\mathrm{d}x\mathrm{d}t\\
&\quad=\int_0^T\int_\Omega(\nabla u_m-\nabla u)\cdot\nabla\Psi_m\mathrm{d}x\mathrm{d}t+\nabla u\cdot(\nabla\Psi_m-\nabla\Psi)\mathrm{d}x\mathrm{d}t\\
&\quad\leqslant\int_0^T\|\nabla\Psi_m\|_2\,\|\nabla u_m-\nabla u\|_2\mathrm{d}t+\int_0^T\|\nabla u\|_2\,\|\nabla\Psi_m-\nabla\Psi\|_2\mathrm{d}t\\
&\quad\to 0,\quad m\to\infty,
\end{aligned}$$

$$\begin{aligned}
&\iint_{Q_T}(u_m\cdot\nabla u_m-u\cdot\nabla u)\mathrm{d}x\mathrm{d}t\\
&\quad=\iint_{Q_T}[(u_m-u)\cdot\nabla u_m+u\cdot(\nabla u_m-\nabla u)]\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$\to 0, \quad m \to \infty,$$

$$\begin{aligned}&\iint_{Q_T} (u_m \times H_m - u \times H)\Phi \mathrm{d}x\mathrm{d}t \\ &\quad \leqslant \iint_{Q_T} [(H_m \times (u_m - u))\Phi + (u_m \times (H_m - H))\Phi]\mathrm{d}x\mathrm{d}t \\ &\quad \to 0, \quad m \to \infty.\end{aligned}$$

By Lemma 2.9, we only need to prove

$$\nabla \cdot (\nabla d_m \odot \nabla d_m) \rightharpoonup \nabla \cdot (\nabla d \odot \nabla d) \quad \text{weakly in } L^2(Q_T).$$

In fact, by (2.31) and (2.32), for any $\Psi \in C^1(Q_T)$,

$$\begin{aligned}&\Big| \iint_{Q_T} (\nabla \cdot (\nabla d_m \odot \nabla d_m) - \nabla \cdot (\nabla d \odot \nabla d)) \cdot \Psi \mathrm{d}x\mathrm{d}t \Big| \\ &\leqslant \iint_{Q_T} |(\nabla |\nabla d_m|^2 - \nabla |\nabla d|^2) \cdot \Psi| \mathrm{d}x\mathrm{d}t \\ &\quad + \iint_{Q_T} |(\Delta d_m \cdot \nabla d_m - \Delta d \cdot \nabla d) \cdot \Psi| \mathrm{d}x\mathrm{d}t \\ &\leqslant \int_0^T \int_\Omega |(|\nabla d_m|^2 - |\nabla d|^2) \cdot \nabla \Psi| \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega |\nabla d_m \cdot (\Delta d_m - \Delta d)\Psi| \mathrm{d}x\mathrm{d}t \\ &\quad + \int_0^T \int_\Omega |\Delta d \cdot (\nabla d_m - \nabla d)\Psi| \mathrm{d}x\mathrm{d}t \\ &\leqslant \|\nabla \Psi\|_{L^\infty(Q_T)} \|\nabla d_m - \nabla d\|_{L^2(Q_T)} + \int_0^T \|\nabla d_m\|_2 \|\Delta d_m - \Delta d\|_2 \|\Psi\|_\infty \mathrm{d}t \\ &\quad + \int_0^T \|\Delta d\|_2 \|\nabla d_m - \nabla d\|_2 \|\Psi\|_\infty \mathrm{d}t \\ &=: I_1^m + I_2^m + I_3^m.\end{aligned}$$

(2.31) implies that $I_1^m, I_3^m \to 0$ as $m \to \infty$, and the fact that ∇d_m is uniformly bounded in $L^2(Q_T)$ and (2.32) yield that $I_2^m \to 0$ as $m \to \infty$. We will prove

$$d_m \times \Delta d_m \to d \times \Delta d \quad \text{weakly in } L^2(Q_T).$$

By virtue of (2.31) and (2.32), we have

$$\begin{aligned}&\iint_{Q_T} |(d_m \times \Delta d_m - d \times \Delta d)\Psi| \mathrm{d}x\mathrm{d}t \\ &= \iint_{Q_T} |[(d_m - d) \times \Delta d_m + d \times (\Delta d_m - \Delta d)]\Psi| \mathrm{d}x\mathrm{d}t \\ &\leqslant \|\Psi\|_\infty \|d_m - d\|_{L^2(Q_T)} \|\Delta d_m\|_{L^2(Q_T)} + \int_0^T \|d\|_2 \|\Delta d_m - \Delta d\|_2 \|\Psi\|_\infty \mathrm{d}t \\ &\to 0, \quad m \to \infty. \qquad (2.40)\end{aligned}$$

To prove the existence of the generalized solution, it remains to prove

$$\iint_{Q_T}(d_m\times\Delta d_m)\cdot(d_m\times\Psi_m)\mathrm{d}x\mathrm{d}t\to\iint_{Q_T}(d\times\Delta d)\cdot(d\times\Psi)\mathrm{d}x\mathrm{d}t.$$

In fact,

$$\begin{aligned}&\iint_{Q_T}(d_m\times\Delta d_m)\cdot(d_m\times\Psi_m)\mathrm{d}x\mathrm{d}t-\iint_{Q_T}(d\times\Delta d)\cdot(d\times\Psi)\mathrm{d}x\mathrm{d}t\\&=\iint_{Q_T}(d_m\times\Delta d_m-d\times\Delta d)\cdot(d\times\Psi)\mathrm{d}x\mathrm{d}t\\&\quad+\iint_{Q_T}d_m\times\Delta d_m\cdot(d_m\times\Psi_m-d\times\Psi)\mathrm{d}x\mathrm{d}t\\&=:J_1^m+J_2^m.\end{aligned}$$

It follows from (2.40) that $J_1^m\to 0$, and moreover,

$$\begin{aligned}J_2^m&\leqslant\|d_m\times\Delta d_m\|_{L^2(Q_T)}\Big(\iint_{Q_T}|d_m\times\Psi_m-d\times\Psi|^2\mathrm{d}x\mathrm{d}t\Big)^{1/2}\\&\leqslant C\Big(\iint_{Q_T}|d_m\times(\Psi_m-\Psi)+(d_m-d)\times\Psi|^2\mathrm{d}x\mathrm{d}t\Big)^{1/2}\\&\to 0.\end{aligned}$$

Finally, from the above arguments, one may take $m\to\infty$ in (2.36)–(2.39) to obtain that (u,d,E,H) is a global weak solution of (1.1)–(1.9), which completes the proof of Theorem 1.1. □

3 Global smooth solution for problem (1.1)–(1.9)

By applying the Banach compression mapping theorem and induction technique, we can obtain the smooth solution for problem (1.1)–(1.9), then we get Theorem 1.2. In order to prove that there exists a global smooth solution for problem (1.1)–(1.9), one needs to establish a priori estimate.

In this section, we first consider the case $n=2$.

Remark 3.1 We consider a classical solution (u,d,E,H) of problem (1.1)–(1.9). In fact,

$$|d|\equiv 1\quad\text{if}\quad |d_0|=1.\tag{3.1}$$

Multiplying (1.2) by d, we obtain

$$\partial_t|d|^2+u\cdot\nabla|d|^2=0,$$

i.e.,

$$\partial_t(|d|^2-1)+u\cdot\nabla(|d|^2-1)=0.\tag{3.2}$$

Multiplying (3.2) by $|d|^2-1$ and then integrating by parts over Ω to deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|d|^2-1)^2\mathrm{d}x=0,$$

we immediately verify (3.1).

Due to Remark 3.1, equation (1.2) equals to

$$d_t+(u\cdot\nabla)d=\gamma(|\nabla d|^2d+\Delta d+d\times\Delta d). \tag{3.3}$$

In the following, we will consider problem (1.1), (3.3), and (1.3)–(1.7).

Lemma 3.1 *Assume* $(u_0,d_0,E_0,H_0)\in(H^1(\Omega),H^2(\Omega),H^1(\Omega),H^1(\Omega))$. *Then there exists a smooth solution* $(u(x,t),d(x,t),E(x,t),H(x,t))$ *of problem* (1.1)–(1.9) *satisfying*

$$\sup_{0\leqslant t\leqslant T}(\|\Delta u\|_2^2+\|\Delta d\|_2^2+\|\nabla E\|_2^2+\|\nabla H\|_2^2)\leqslant C \tag{3.4}$$

when $\dim\Omega=2$.

Proof Testing (1.2) by $\Delta(\Delta d+H)$, by Young and Sobolev inequality we have

$$\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\Delta d|^2\mathrm{d}x+\int_\Omega d_t\cdot\Delta H\mathrm{d}x+\int_\Omega|\nabla\Delta d|^2\mathrm{d}x\\
&=\int_\Omega\nabla u\cdot\nabla d\cdot\nabla\Delta d\mathrm{d}x+\int_\Omega u\cdot\nabla^2d\cdot\nabla\Delta d\mathrm{d}x+\int_\Omega\nabla u\cdot\nabla d\cdot\nabla H\mathrm{d}x\\
&\quad-\int_\Omega\nabla d\cdot|\nabla d|^2\cdot\nabla\Delta d\mathrm{d}x-\int_\Omega d\cdot\nabla|\nabla d|^2\cdot\nabla\Delta d\mathrm{d}x-\int_\Omega\nabla d\cdot|\nabla d|^2\cdot\nabla H\mathrm{d}x\\
&\quad-\int_\Omega d\cdot\nabla|\nabla d|^2\cdot\nabla H\mathrm{d}x+\int_\Omega\Delta d\cdot\Delta H\mathrm{d}x-\int_\Omega(\nabla d\times\Delta d)\cdot\nabla\Delta d\mathrm{d}x\\
&\quad-\int_\Omega(\nabla d\times\Delta d)\cdot\nabla H\mathrm{d}x-\int_\Omega(d\times\nabla\Delta d)\cdot\nabla H\mathrm{d}x+\int_\Omega u\cdot\nabla^2d\cdot\nabla H\mathrm{d}x\\
&\leqslant C(\Omega)(\|\nabla u\|_2\,\|\nabla d\|_\infty\,\|\nabla\Delta d\|_2+\|u\|_4\,\|\Delta d\|_4\,\|\nabla\Delta d\|_2+\|\nabla u\|_4\,\|\nabla d\|_4\,\|\nabla H\|_2\\
&\quad+\|u\|_4\,\|\Delta d\|_4\,\|\nabla H\|_2+\|\nabla d\|_\infty\,\||\nabla d|^2\|_2\,\|\nabla\Delta d\|_2+\|d\|_4\,\|\nabla|\nabla d|^2\|_4\,\|\nabla\Delta d\|_2\\
&\quad+\|\nabla d\|_\infty\,\||\nabla d|^2\|_2\,\|\nabla H\|_2+\|d\|_4\,\|\nabla|\nabla d|^2\|_4\,\|\nabla H\|_2+\|\nabla d\|_4\,\|\Delta d\|_4\,\|\nabla\Delta d\|_2\\
&\quad+\|d\|_\infty\,\|\nabla\Delta d\|_2\,\|\nabla H\|_2+\|\nabla d\|_4\,\|\Delta d\|_4\,\|\nabla H\|_2)\\
&\leqslant\varepsilon\|\nabla\Delta d\|_2^2+C(\|\nabla H\|_2^2+\|\Delta d\|_2^2+\|\Delta d\|_4^4+\|\Delta u\|_4^4+\|\Delta u\|_2^2).
\end{aligned}\tag{3.5}$$

Here, we have used

$$\begin{aligned}
\|\nabla d\|_\infty\,\||\nabla d|^2\|_2\,\|\nabla\Delta d\|_2&\leqslant C(\Omega)\|\nabla d\|_2^{1/2}\,\|\nabla\Delta d\|_2^{1/2}\,\|\nabla d\|_4^2\,\|\nabla\Delta d\|_2\\
&\leqslant C(\Omega)\|\nabla\Delta d\|_2^{3/2}\,\|\nabla d\|_4^2\\
&\leqslant C(\Omega)(\varepsilon\|\nabla\Delta d\|_2^2+\|\nabla d\|_4^8),
\end{aligned}$$

$$\|\nabla d\|_4^8\leqslant C(\Omega)\|\nabla d\|_2^4\,\|\Delta d\|_2^4\leqslant C(\Omega)\|\Delta d\|_4^4,$$

$$\|d\|_\infty \|\nabla d\|_4 \|\Delta d\|_4 \|\nabla\Delta d\|_2 \leqslant C(\Omega)(\|\nabla d\|_4^2 \|\Delta d\|_4^2 + \varepsilon\|\nabla\Delta d\|_2^2)$$
$$\leqslant C(\Omega)(\|\Delta d\|_4^4 + \varepsilon\|\nabla\Delta d\|_2^2).$$

Taking the same procedure to (1.1), we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Delta u|^2\mathrm{d}x + \int_\Omega |\nabla\Delta u|^2\mathrm{d}x + \int_\Omega u_t \cdot \Delta E\mathrm{d}x \\ \leqslant \varepsilon(\|\nabla\Delta d\|_2^2 + \|\nabla\Delta u\|_2^2) \\ + C(\|\nabla H\|_2^2 + \|\nabla E\|_2^2 + \|\Delta d\|_2^2 + \|\Delta u\|_2^2 + \|\Delta u\|_4^4),\end{aligned} \tag{3.6}$$

here, we can choose $\varepsilon = 1/8$.

Making the scalar product of ΔE with (1.4) and ΔH with (1.5), respectively, and then integrating the resulting equation with respective to $x \in \Omega$, we obtain

$$\begin{aligned}\int_\Omega \Big[\frac{\partial E}{\partial t}\Delta E - (\nabla \times H)\cdot\Delta E + \sigma E\Delta E \\ + \frac{\partial(H+\beta d)}{\partial t}\Delta H - u\Delta E + (\nabla\times E)\Delta H\Big]\mathrm{d}x = 0,\end{aligned}$$

i.e.,

$$\begin{aligned}-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\nabla E|^2 + |\nabla H|^2)\mathrm{d}x + \int_\Omega u\Delta E\mathrm{d}x + \beta\int_\Omega d_t\Delta H\mathrm{d}x - \sigma\int_\Omega |\nabla E|^2\mathrm{d}x \\ = 0.\end{aligned} \tag{3.7}$$

Differentiating (1.1) with respect to t, and then making the scalar product with respect to u_t, we obtain

$$\begin{aligned}u_{tt}\cdot u_t + [(u_t\cdot\nabla)u + (u\cdot\nabla)u_t]u_t + \nabla P_t\cdot u_t - \Delta u_t\cdot u_t \\ = -[\nabla\cdot(\nabla d\odot\nabla d)]_t\cdot u_t - E_t u_t - (u\times H_t)u_t.\end{aligned} \tag{3.8}$$

Integrating (3.8) over Ω, and combining the Sobolev imbedding theorem, we deduce

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_t|^2\mathrm{d}x + \int_\Omega |\nabla u_t|^2\mathrm{d}x \\ &\leqslant \int_\Omega |\nabla d|\,|\nabla d_t|\,|\nabla u_t|\mathrm{d}x + \int_\Omega |u\cdot\nabla u_t\cdot u_t|\mathrm{d}x \\ &\quad + \int_\Omega |(u_t\cdot\nabla)u\cdot u_t|\mathrm{d}x + \int_\Omega (|E_t u_t| + |(u\times H_t)u_t|)\mathrm{d}x \\ &\leqslant \|\nabla d\|_4\,\|\nabla d_t\|_4\,\|\nabla u_t\|_2 + \|u\|_4\,\|u_t\|_4\,\|\nabla u_t\|_2 \\ &\quad + \|u_t\|_4\,\|\nabla u\|_2\,\|u_t\|_4 + \|E_t\|_2\,\|u_t\|_2 + \|u\|_\infty\,\|u_t\|_2\,\|H_t\|_2 \\ &\leqslant \varepsilon(\|\nabla u_t\|_2^2 + \|\nabla\Delta d\|_2^2) + C(\Omega)(\|u_t\|_2^2 + \|\nabla d_t\|_2^2 + \|\Delta u\|_2^2 \\ &\quad + \|\nabla H\|_2^2 + \|\nabla E\|_2^2 + \|\nabla d_t\|_4^4 + \|u_t\|_4^4 + \|\Delta d\|_2^2).\end{aligned}$$

Take $\varepsilon = 1/8$, whence

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_t|^2 \mathrm{d}x + \int_\Omega |\nabla u_t|^2 \mathrm{d}x \\
&\quad \leqslant \frac{1}{8}(\|\nabla u_t\|_2^2 + \|\nabla \Delta d\|_2^2) + C(\Omega)(\|u_t\|_2^2 + \|\nabla d_t\|_2^2 + \|\Delta u\|_2^2 \\
&\qquad + \|\nabla H\|_2^2 + \|\nabla E\|_2^2 + \|\nabla d_t\|_4^4 + \|u_t\|_4^4 + \|\Delta d\|_2^2). \qquad (3.9)
\end{aligned}$$

Applying ∂_t to (1.2), multiplying by Δd_t, and integrating by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla d_t|^2 \mathrm{d}x + \int_\Omega |\Delta d_t|^2 \mathrm{d}x \\
&\leqslant \int_\Omega |(u_t\cdot\nabla)d\cdot\Delta d_t + (u\cdot\nabla)d_t\cdot\Delta d_t|\mathrm{d}x \\
&\quad + \int_\Omega |[d_t|\nabla d|^2 + d(|\nabla d|^2)_t]\cdot\Delta d_t|\mathrm{d}x + \int_\Omega |(d_t\times\Delta d)\cdot\Delta d_t|\mathrm{d}x \\
&\leqslant \|u_t\|_4\,\|\nabla d\|_4\,\|\Delta d_t\|_2 + \|u\|_4\,\|\nabla d_t\|_4\,\|\Delta d_t\|_2 + \|d_t\|_\infty\,\|\nabla d\|_4^2\,\|\Delta d_t\|_2 \\
&\quad + \|d\|_\infty\,\|\nabla d\|_4\,\|\nabla d_t\|_4\,\|\Delta d_t\|_2 + \|d_t\|_4\,\|\Delta d\|_4\,\|\Delta d_t\|_2 \\
&\leqslant C(\|\nabla d_t\|_2^2 + \|\Delta d\|_4^4 + \|\nabla d_t\|_4^4 + \|\Delta u\|_2^2) \\
&\quad + \frac{1}{8}(\|\nabla u_t\|_2^2 + \|\Delta d_t\|_2^2 + \|\nabla\Delta d\|_2^2). \qquad (3.10)
\end{aligned}$$

Whence (3.5)–(3.7), (3.9), and (3.10) imply (3.4) and

$$\begin{aligned}
&u_t \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),\\
&\nabla d_t \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).
\end{aligned} \qquad (3.11)$$

□

Lemma 3.2

$$u_t \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)).$$

Proof Applying ∂_t to (1.1), multiplying by Δu_t, and integrating by parts, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u_t|^2 \mathrm{d}x + \int_\Omega |\Delta u_t|^2 \mathrm{d}x \\
&\leqslant \int_\Omega [|u_t\cdot\nabla u\cdot\Delta u_t| + |u\cdot\nabla u_t\cdot\Delta u_t| + |\nabla d_t\cdot\Delta d\cdot\Delta u_t| \\
&\quad + |E_t\cdot\Delta u_t| + |\nabla d\cdot\Delta d_t\cdot\Delta u_t| + ||\nabla d|_t^2\cdot d\cdot\Delta u_t| \\
&\quad + ||\nabla d|^2\cdot d_t\cdot\Delta u_t| + |(u\times H_t)\cdot\Delta u_t|]\mathrm{d}x \\
&\leqslant \|\nabla u\|_4\,\|u_t\|_4\,\|\Delta u_t\|_2 + \|u\|_\infty\,\|\nabla u_t\|_2\,\|\Delta u_t\|_2 + \|\nabla d_t\|_\infty\,\|\Delta d\|_2\,\|\Delta u_t\|_2 \\
&\quad + \|\nabla d\|_\infty\,\|\Delta d_t\|_2\,\|\Delta u_t\|_2 + \|\nabla d_t\|_4\,\|\nabla d\|_4\,\|d\|_\infty\,\|\Delta u_t\|_2 \\
&\quad + \|\nabla d\|_4^2\,\|d_t\|_\infty\,\|\Delta u_t\|_2 + \|E_t\|_2\,\|\Delta u_t\|_2 + \|u\|_\infty\,\|H_t\|_2\,\|\Delta u_t\|_2 \\
&\leqslant C(\|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \|\nabla d_t\|_2^2 + \|\Delta d_t\|_2^2 + \|E_t\|_2^2 + \|H_t\|_2^2) + \frac{1}{8}\|\Delta u_t\|_2^2,
\end{aligned}$$

and then, together with Lemma 3.1, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u_t|^2\mathrm{d}x + \int_\Omega |\Delta u_t|^2\mathrm{d}x\\ &\quad\leqslant C(\|u_t\|_2^2 + \|\nabla u_t\|_2^2 + \|\nabla d_t\|_2^2 + \|\Delta d_t\|_2^2) + \frac{1}{8}\|\Delta u_t\|_2^2. \end{aligned}\tag{3.12}$$

By Gronwall's inequality, (3.11), and (3.12), we deduce

$$\nabla u_t \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \qquad \square$$

Lemma 3.3 *Assume* $(u_0, d_0, E_0, H_0) \in (H^1(\Omega), H^2(\Omega), H^{m-1}(\Omega), H^{m-1}(\Omega))$ $(m \geqslant 3)$. *Then there exists a unique smooth solution* $(u(x,t), d(x,t), E(x,t), H(x,t))$ *of problem* (1.1)–(1.9) *satisfying*

$$\sup_{0\leqslant t\leqslant T}(\|u\|_{H^m}^2 + \|d\|_{H^m}^2 + \|E\|_{H^{m-1}}^2 + \|H\|_{H^{m-1}}^2) \leqslant C. \tag{3.13}$$

Proof Existence part. We will prove the result by the induction for m. From Theorem 1.1 and Lemma 3.1, the estimate holds for $m = 1, 2$.

Now, we assume that the estimate holds for $m = M \geqslant 3$, i.e.,

$$\sup_{0\leqslant t\leqslant T}(\|u\|_{H^M}^2 + \|d\|_{H^M}^2 + \|E\|_{H^{M-1}}^2 + \|H\|_{H^{M-1}}^2) \leqslant C. \tag{3.14}$$

We will prove that (3.14) holds for $m = M + 1$.

Making the scalar product of $\Delta^M(\Delta d + H)$ with (1.2) and integrating the resulting equation with respect to $x \in \Omega$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^{M+1}d\|_2^2 + \int_\Omega D^M d_t\cdot D^M H\mathrm{d}x - \int_\Omega |D^M\Delta d|^2\mathrm{d}x\\ &\quad + \int_\Omega D^M u\cdot\nabla d\cdot D^M H\mathrm{d}x + (-1)^M\int_\Omega u\cdot D^{M+1}d\cdot D^M H\mathrm{d}x\\ &\quad + \int_\Omega D^M u\cdot\nabla d\cdot D^M\Delta d\mathrm{d}x + \int_\Omega u\cdot D^{M+1}d\cdot D^M\Delta d\mathrm{d}x\\ &\quad - \int_\Omega D^M\Delta d\cdot D^M H\mathrm{d}x\\ &= \int_\Omega D^M d\cdot|\nabla d|^2\cdot D^M\Delta d\mathrm{d}x + \int_\Omega d\cdot D^M(|\nabla d|^2)\cdot D^M\Delta d\mathrm{d}x\\ &\quad + \int_\Omega D^M d\cdot|\nabla d|^2\cdot D^M H\mathrm{d}x + \int_\Omega d\cdot D^M|\nabla d|^2\cdot D^M H\mathrm{d}x\\ &\quad + \int_\Omega (D^M d\times\Delta d)\cdot D^M(\Delta d)\mathrm{d}x + \int_\Omega (D^M d\times\Delta d)\cdot D^M H\mathrm{d}x\\ &\quad + \int_\Omega (D^{M+2}d\times d)\cdot D^M H\mathrm{d}x. \end{aligned}\tag{3.15}$$

By Sobolev's inequality and Young's inequality, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^{M+1}d\|_2^2+\|D^{M+2}d\|_2^2+\int_\Omega D^M d_t\cdot D^M H\mathrm{d}x\\&\quad\leqslant\frac{1}{8}\|D^{M+2}d\|_2^2+C(\|D^{M+1}d\|_2^2+\|D^{M+1}u\|_2^2+\|D^M H\|_2^2).\end{aligned}\tag{3.16}$$

Similarly, Making the scalar product of $\Delta^M(\Delta u+E)$ with (1.1) and integrating the resulting equation with respect to $x\in\Omega$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^{M+1}u\|_2^2+\|D^{M+2}u\|_2^2+\int_\Omega D^M u_t\cdot D^M E\mathrm{d}x\\&\quad\leqslant C(\|D^{M+1}d\|_2^2+\|D^{M+1}u\|_2^2+\|D^M E\|_2^2+\|D^M H\|_2^2)\\&\qquad+\frac{1}{8}(\|D^{M+2}u\|_2^2+\|D^{M+2}d\|_2^2).\end{aligned}\tag{3.17}$$

Making the scalar product of $\Delta^M E$ with (1.4) and $\Delta^M H$ with (1.5), respectively, and then integrating the resulting equation with respective to $x\in\Omega$, we obtain

$$\begin{aligned}&\int_\Omega\Big[\frac{\partial E}{\partial t}\Delta^M E-(\nabla\times H)\cdot\Delta^M E+\sigma E\cdot\Delta^M E\\&+\frac{\partial(H+\beta d)}{\partial t}\cdot\Delta^M H+(\nabla\times E)\cdot\Delta^M H-u\Delta^M E\Big]\mathrm{d}x=0,\end{aligned}$$

i.e.,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|D^M E\|_2^2+\|D^M H\|_2^2)-\int_\Omega D^M u\cdot D^M E\mathrm{d}x\\&\quad+\beta\int_\Omega D^M d_t\cdot D^M H\mathrm{d}x+\sigma\int_\Omega|D^M E|^2\mathrm{d}x=0.\end{aligned}\tag{3.18}$$

Next, we will estimate the terms $\int_\Omega D^M u_t\mathrm{d}x$ and $\int_\Omega D^M d_t\mathrm{d}x$.

Taking ∂_t of (1.1), multiplying by $\Delta^M u_t$, and integrating the resulting over Ω, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|D^M u_t|^2\mathrm{d}x+\int_\Omega|D^{M+1}u_t|^2\mathrm{d}x\\&\quad\leqslant\|D^M u_t\|_2\,\|\nabla u\|_4\,\|D^M u_t\|_4+\|u_t\|_4\,\|D^{M+1}u\|_2\,\|D^M u_t\|_4\\&\qquad+\|\nabla u_t\|_2\,\|D^M u\|_4\,\|D^M u_t\|_4+\|D^{M+1}u_t\|_2\,\|u\|_\infty\,\|D^M u_t\|_2\\&\qquad+\|D^{M+1}d_t\|_2\,\|\Delta d\|_4\,\|D^M u_t\|_4+\|\nabla d_t\|_4\,\|D^{M+2}d\|_2\,\|D^M u_t\|_4\\&\qquad+\|D^{M+1}d\|_4\,\|\Delta d_t\|_2\,\|D^M u_t\|_4+\|\nabla d\|_\infty\,\|D^{M+1}d_t\|_2\,\|D^{M+1}u_t\|_2\\&\qquad+\|\Delta d\|_4\,\|D^{M+1}d_t\|_2\,\|D^M u_t\|_4+\|D^{M+1}d\|_4\,\|\nabla d_t\|_2\,\|d\|_\infty\,\|D^M u_t\|_4\\&\qquad+\|\nabla d\|_\infty\,\|D^{M+1}d_t\|_2\,\|D^M u_t\|_2+\|D^M|\nabla d|^2\|_2\,\|d_t\|_\infty\,\|D^M u_t\|_2\end{aligned}$$

$$
\begin{aligned}
&+\|\nabla d_t\|_2\,\|D^M u_t\|_2\,\|D^M d\|_\infty\,\|\nabla d\|_\infty+\||\nabla d|^2\|_\infty\,\|D^M u_t\|_2\,\|D^M d_t\|_2\\
&+\|D^M E_t\|_2\,\|D^M u_t\|_2+\|D^M(u\times H)_t\|_2\,\|D^M u_t\|_2\\
\leqslant\ & C(\|D^M u_t\|_2^2+\|D^M d_t\|_2^2+\|D^M E\|_2^2+\|D^M H\|_2^2+\|D^M u\|_2^2)\\
&+\frac{1}{8}\,(\|D^{M+1}u_t\|_2^2+\|D^{M+1}d_t\|_2^2).
\end{aligned}
$$

So,

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |D^M u_t|^2\mathrm{d}x+\int_\Omega |D^{M+1}u_t|^2\mathrm{d}x\\
&\leqslant C(\|D^M u_t\|_2^2+\|\Delta d_t\|_2^4+\|\nabla d_t\|_2^4)+\frac{1}{8}\,\|D^{M+1}u_t\|_2^2+\|D^{M+1}d_t\|_2^2).
\end{aligned}
$$

Taking the similar procedure to (1.2), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |D^M d_t|^2\mathrm{d}x+\int_\Omega |D^{M+1}d_t|^2\mathrm{d}x\\
&\leqslant C(\|D^M d_t\|_2^2+\|D^M u_t\|_2^2)+\frac{1}{8}\,\|D^{M+1}d_t\|_2^2.
\end{aligned}
$$

Whence, together with Gronwall's inequality and (3.16)–(3.18), we deduce

$$
\begin{gathered}
\|D^M u_t\|_2+\|D^M d_t\|_2\leqslant C,\quad M\geqslant 2,\\
\sup\nolimits_{0\leqslant t\leqslant T}(\|u\|_{H^{M+1}}^2+\|d\|_{H^{M+1}}^2+\|E\|_{H^M}^2+\|H\|_{H^M}^2)\leqslant C.
\end{gathered}
$$

Uniqueness part. Next, we will give the proof of the uniqueness of the solution. Suppose that $(\overline{u},\overline{d},\overline{E},\overline{H})$ and $(\overline{\overline{u}},\overline{\overline{d}},\overline{\overline{E}},\overline{\overline{H}})$ are two solutions of (1.1)–(1.9) as obtained in Lemma 3.1, and let

$$
u^*=\overline{u}-\overline{\overline{u}},\quad d^*=\overline{d}-\overline{\overline{d}},\quad E^*=\overline{E}-\overline{\overline{E}},\quad H^*=\overline{H}-\overline{\overline{H}}.
$$

Then we have the following energy estimates of (u^*,d^*):

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega(|u^*|^2+|\nabla d^*|^2)\mathrm{d}x(T)\\
&\leqslant\frac{1}{2}\int_\Omega(|u^*|^2+|\nabla d^*|^2)\mathrm{d}x(0)-\int_0^T\int_\Omega(|\nabla u^*|^2+|\Delta d^*|^2)\mathrm{d}x\mathrm{d}t\\
&\quad-\int_0^T\int_\Omega(u^*\nabla\overline{u}u^*+\Delta\overline{d}\nabla d^*u^*+\overline{u}\nabla d^*\Delta d^*-(f(\overline{d})-f(\overline{\overline{d}}))\Delta d^*)\mathrm{d}x\mathrm{d}t\\
&\quad-\int_0^T\int_\Omega(E^*u^*+(\overline{\overline{u}}\times H^*)u^*+(u^*\times\overline{H})u^*)\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{3.19}
$$

On the other hand, we deduce

$$
E_t^*-\nabla\times H^*=-\sigma E^*+u^*,\tag{3.20}
$$

$$H_t^* + \beta d_t^* + \nabla \times E^* = 0. \tag{3.21}$$

Testing (3.20) and (3.21) with E^* and H^*, respectively, we have

$$\begin{aligned}\int_\Omega (|E^*|^2 + |H^*|^2)\mathrm{d}x(T) = &\int_\Omega (|E^*|^2 + |H^*|^2)\mathrm{d}x(0) - \sigma \int_0^T \int_\Omega |E^*|^2 \mathrm{d}x\mathrm{d}t \\ &- \beta \int_0^T \int_\Omega d_t^* H^* \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega u^* E^* \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.22}$$

By using Lemma 2.4 and the zero boundary condition, we have

$$\begin{aligned}\int_\Omega u^* \nabla \overline{u} u^* \mathrm{d}x &= \int_\Omega u^* \nabla \overline{\overline{u}} \overline{u} \mathrm{d}x \\ &= \int_\Omega u^* \nabla u^* \overline{u} \mathrm{d}x \\ &\leqslant \|\nabla u^*\|_2 \, \|u^*\|_4 \, \|\overline{u}\|_4 \\ &\leqslant \varepsilon \|\nabla u^*\|_2^2 + C\|u^*\|_2^2,\end{aligned}$$

where ε is an arbitrary small number, and C is a constant. We also note that $\overline{u}$ is the good solution as in Lemma 3.1, so that $\|\overline{u}\|_4$ is bounded for all t.

The same argument also works for the other terms:

$$\begin{aligned}\int_\Omega \Delta \overline{d} \nabla d^* u^* \mathrm{d}x &\leqslant \|\nabla d^*\|_4 \, \|u^*\|_4 \, \|\Delta \overline{d}\|_2 \\ &\leqslant C\|\nabla d^*\|_2 \, \|u^*\|_2 + \varepsilon \|\Delta d^*\|_2 \, \|\nabla u^*\|_2 \\ &\leqslant C(\|\nabla d^*\|_2^2 + \|u^*\|_2^2) + \varepsilon(\|\Delta d^*\|_2^2 + \|\nabla u^*\|_2^2),\end{aligned}$$

$$\int_\Omega \overline{u} \nabla d^* \Delta d^* \mathrm{d}x \leqslant \|\Delta d^*\|_2 \, \|\overline{u}\|_4 \, \|\nabla d^*\|_4 \leqslant \varepsilon \|\Delta d^*\|_2^2 + C\|\nabla d^*\|_2^2,$$

$$\int_\Omega (f(\overline{d}) - f(\overline{\overline{d}}))\Delta d^* \mathrm{d}x \leqslant \|\Delta d^*\|_2 \, \|f(\overline{d}) - f(\overline{\overline{d}})\|_2 \leqslant \varepsilon \|\Delta d^*\|_2^2 + C\|\nabla d^*\|_2^2,$$

$$\begin{aligned}\int_\Omega d_t^* H^* \mathrm{d}x &\leqslant \|d_t^*\|_2 \, \|H^*\|_2 \\ &\leqslant (\|u^* \nabla \overline{d}\|_2 + \|\overline{\overline{u}} \nabla d^*\|_2 + \|\Delta d^*\|_2 + \|f(\overline{d}) - f(\overline{\overline{d}})\|_2)\|H^*\|_2 \\ &\leqslant C(\|u^*\|_2^2 + \|H^*\|_2^2 + \|\nabla d^*\|_2^2) + \varepsilon \|\Delta d^*\|_2^2,\end{aligned} \tag{3.23}$$

$$\int_\Omega (\overline{\overline{u}} \times H^*) u^* \mathrm{d}x \leqslant \|\overline{\overline{u}}\|_\infty \, \|H^*\|_2 \, \|u^*\|_2 \leqslant C(\|H^*\|_2^2 + \|u^*\|_2^2), \tag{3.24}$$

where $f(d)$ is defined as in (1.10). Whence (3.19) and (3.22)–(3.24) yield that

$$\begin{aligned}&\frac{1}{2} \int_\Omega (|u^*|^2 + |\nabla d^*|^2 + |E^*|^2 + |H^*|^2)\mathrm{d}x(T) \\ &\quad \leqslant \frac{1}{2} \int_\Omega (|u^*|^2 + |\nabla d^*|^2 + |E^*|^2 + |H^*|^2)\mathrm{d}x(0) \\ &\qquad + C \int_0^T \int_\Omega (|u^*|^2 + |\nabla d^*|^2 + |E^*|^2 + |H^*|^2)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Since

$$u^*(0) = d^*(0) = E^*(0) = H^*(0) = 0,$$

Gronwall's inequlity yields

$$u^* = d^* = E^* = H^* = 0. \qquad \square$$

When the dimension of Ω is 3, we see that the size of ν plays a rather crucial role while the other viscosity constants λ, γ do not as long as λ, γ are positive constants. This fact can be seen from the following calculations. Thus, we shall, for simplicity, assume that $\lambda = \gamma = 1$.

Denote

$$A^2(t) = \int_\Omega (|\Delta u|^2 + |\Delta d|^2 + |\nabla E|^2 + |\nabla H|^2)\mathrm{d}x,$$

$$B^2(t) = \int_\Omega (|\nabla u|^2 + |\Delta d|^2)\mathrm{d}x, \quad D^2(t) = \int_\Omega (|\Delta u|^2 + |\Delta d|^2)\mathrm{d}x.$$

Lemma 3.4 *Assume* $(u_0, d_0, E_0, H_0) \in (H^1(\Omega), H^2(\Omega), H^1(\Omega), H^1(\Omega))$ ($\Omega \subseteq \mathbb{R}^3$). *Then there exists a unique smooth solution* $(u(x,t), d(x,t), E(x,t), H(x,t))$ *of problem* (1.1)–(1.9) *such that*

$$(u, d) \in (L^\infty(0,T; H^2(\Omega)))^2, \quad (E, H) \in (L^\infty(0,T; H^1(\Omega)))^2, \tag{3.25}$$

when

$$\nu \geqslant \nu_0(\lambda, \gamma, u_0, d_0, E_0, H_0).$$

Proof First, we prove that

$$A^2(t) \in L^2(0,T).$$

Multiplying (1.1) and (1.2) with u and $\Delta d + |\nabla d|^2 d$, respectively,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|u|^2 + |\nabla d|^2)\mathrm{d}x = -\int_\Omega (|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2 + E \cdot u)\mathrm{d}x,$$

i.e.,

$$\sup_{0\leqslant t\leqslant T}\int_\Omega (|u|^2 + |\nabla d|^2)\mathrm{d}x + 2\int_0^T (\|\nabla u\|_2^2 + |\Delta d + |\nabla d|^2 d|^2)\mathrm{d}t + \int_0^T\int_\Omega Eu\mathrm{d}x\mathrm{d}t$$
$$\leqslant \int_\Omega (|u_0|^2 + |\nabla d_0|^2)\mathrm{d}x. \tag{3.26}$$

By using Theorem 1.1, (3.26), and the elliptic estimates, for $\forall\, t \in [0, T-1]$, there is a $t_1 \in [t, t+1]$ such that

$$B^2(t_1) \leqslant 2M, \tag{3.27}$$

where

$$M=\|u_0\|_2^2+\|\nabla d_0\|_2^2+TC_0,$$

with

$$C_0\geqslant\|E\|_2^2+\|u\|_2^2.$$

Then by the similar procedure, we deduce

$$D^2(t_1)\leqslant CM+\|\nabla E\|_2^2(t_1)+\|\nabla H\|_2^2(t_1). \tag{3.28}$$

By (3.7) and (3.27), for the above t_1 we chosen, there holds that

$$\int_\Omega(|\nabla E|^2+|\nabla H|^2)\mathrm{d}x(t_1)\leqslant C(B^2(0)+\|\nabla E_0\|_2^2+\|\nabla H_0\|_2^2). \tag{3.29}$$

Therefore,

$$A^2(t_1)\leqslant 2M', \tag{3.30}$$

where

$$M'=\|u_0\|_2^2+\|\nabla d_0\|_2^2+C_0T+\|\nabla E_0\|_2^2+\|\nabla H_0\|_2^2.$$

We calculate

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}A^2}{\mathrm{d}t}=&\int_\Omega[-(\Delta(\Delta u),u_t)+(\Delta d,\Delta d_t)]\mathrm{d}x\\&+\int_\Omega u_t\Delta E\mathrm{d}x-\beta\int_\Omega\nabla d_t\cdot\nabla H\mathrm{d}x-\sigma\int_\Omega|\nabla E|^2\mathrm{d}x\\=&-(\|\nabla\Delta u\|_2^2+\|\nabla\Delta d\|_2^2)+\int_\Omega[u\nabla u\Delta(\Delta u)+\Delta(\Delta u)\nabla d\Delta d\\&+\Delta d(\Delta d-u\nabla d)+\Delta d(-\Delta u\nabla d-u\nabla\Delta d-2\nabla u\cdot\nabla^2 d)]\mathrm{d}x\\&-\int_\Omega E\cdot\Delta(\Delta u)\mathrm{d}x-\int_\Omega(u\times H)\cdot\Delta(\Delta u)\mathrm{d}x\\&+\int_\Omega u_t\Delta E\mathrm{d}x-\beta\int_\Omega\nabla d_t\cdot\nabla H\mathrm{d}x-\sigma\int_\Omega|\nabla E|^2\mathrm{d}x.\end{aligned} \tag{3.31}$$

Now, we can work with the right-hand side of (3.31) term by term, for simplicity, we just give some terms that may be difficulties for the calculation:

$$\begin{aligned}\int_\Omega u\nabla u\Delta(\Delta u)\mathrm{d}x&\leqslant\|\nabla u\|_4^2\,\|\nabla\Delta u\|_2+\|u\|_4\,\|\Delta u\|_4\,\|\nabla\Delta u\|_2\\&\leqslant C(\|\nabla u\|_2^{1/2}\,\|\Delta u\|_2^{3/2}\,\|\nabla\Delta u\|_2+\|\Delta u\|_2^{1/4}\,\|\nabla\Delta u\|_2^{7/4})\\&\leqslant C(\|\nabla u\|_2^2\,\|\nabla\Delta u\|_2^2+\|\Delta u\|_2^2\,\|\nabla\Delta u\|_2^2+\|\nabla\Delta u\|_2^2),\end{aligned}$$

$$\int_\Omega u\nabla u\Delta E\mathrm{d}x\leqslant\|\Delta u\|_2^2\,\|\nabla E\|_2^2+\|\Delta u\|_2^2+\|\nabla E\|_2^2\,\|\nabla\Delta u\|_2^2,$$

$$\int_\Omega\Delta u\Delta E\mathrm{d}x\leqslant C(\|\nabla\Delta u\|_2^2+\|\nabla E\|_2^2),$$

$$-\int_\Omega (u\times H)\cdot\Delta(\Delta u)\mathrm{d}x \leqslant C(\|\nabla H\|_2^2+\|\Delta u\|_2^2\,\|\nabla\Delta u\|_2^2+\|\nabla\Delta u\|_2^2),$$

$$\int_\Omega (u\times H)\Delta E\mathrm{d}x \leqslant C(\|\nabla H\|_2^2+\|\Delta u\|_2^2\,\|\nabla E\|_2^2+\|\nabla E\|_2^2),$$

$$\begin{aligned}\int_\Omega & u\cdot\nabla d\cdot\Delta\Delta d\mathrm{d}x\\
&= -\int_\Omega \nabla u\cdot\nabla d\cdot\nabla\Delta d\mathrm{d}x-\int_\Omega u\cdot\nabla^2 d\cdot\nabla\Delta d\mathrm{d}x\\
&\leqslant C(\|\nabla u\|_\infty\,\|\nabla d\|_2\,\|\nabla\Delta d\|_2+\|u\|_\infty\,\|\Delta d\|_2\,\|\nabla\Delta d\|_2)\\
&\leqslant C\Big(\frac{1}{\nu}\,\|\nabla\Delta d\|_2^2+\frac{1}{\nu}\,\|\nabla\Delta u\|_2^2+\nu\|\Delta u\|_2^2+\frac{1}{\nu}\,A^2(t)\|\nabla\Delta d\|_2^2\Big)\\
&\leqslant C\Big(\frac{1}{\nu}\,\|\nabla\Delta u\|_2^2+\frac{A^2(t)+C}{\nu}\,\|\nabla\Delta d\|_2^2+\nu A^2(t)\Big),\end{aligned}$$

$$\int_\Omega |\nabla d|^2 d\Delta\Delta d\mathrm{d}x=-\int_\Omega \nabla|\nabla d|^2 d\nabla\Delta d\mathrm{d}x-\int_\Omega |\nabla d|^2\nabla d\nabla\Delta d\mathrm{d}x.$$

Using Lemma 2.4, we deduce

$$\begin{aligned}\int_\Omega (\nabla|\nabla d|^2)d\nabla\Delta d\mathrm{d}x &\leqslant \|d\|_\infty\,\|\nabla d\|_4\,\|\Delta d\|_4\,\|\nabla\Delta d\|_2\\
&\leqslant C(\Omega)\Big(\nu\|\nabla d\|_4^2\,\|\Delta d\|_4^2+\frac{1}{\nu}\,\|\nabla\Delta d\|_2^2\Big)\\
&\leqslant C(\Omega)\Big(\nu^3\|\Delta d\|_4^4+\frac{1}{\nu}\,\|\nabla\Delta d\|_2^2+\frac{1}{\nu}\,\|d\|_\infty^{10/3}\,\|\nabla^3 d\|_2^{2/3}\Big)\\
&\leqslant C(\Omega)\Big(\nu^3\|\Delta d\|_4^4+\frac{1}{\nu}\,\|\nabla\Delta d\|_2^2\Big).\end{aligned}$$

Taking the above inequality into (3.31), we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}A^2}{\mathrm{d}t} &\leqslant -\Big(\nu-CA^2-\frac{1}{\nu}\Big)\|\nabla\Delta u\|_2^2\\
&\quad-\Big(C-\frac{A^2(t)+C}{\nu}\Big)\|\nabla\Delta d\|_2^2+(C+CA^2)A^2.\end{aligned}\tag{3.32}$$

By setting $\widetilde{A}^2=A^2+1$, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}\widetilde{A}^2}{\mathrm{d}t} &\leqslant -\Big(\frac{\nu^2-\widetilde{A}^2C}{\nu}\Big)\|\nabla\Delta u\|_2^2\\
&\quad-\Big(C-\frac{\widetilde{A}^2(t)C}{\nu}\Big)\|\nabla\Delta d\|_2^2+(\widetilde{A}^4+C\widetilde{A}^2).\end{aligned}\tag{3.33}$$

Next, we shall assume that ν is so large that

$$\nu\geqslant 2C[\widetilde{A}^4(0)+1+4M'].\tag{3.34}$$

Then, initially, there is some $T_0 > 0$ such that

$$\frac{\nu^2 - \widetilde{A}^2 C}{\nu} \geqslant 0, \quad C - \frac{\widetilde{A}^2(t) C}{\nu} \geqslant 0, \quad \forall\, t \in [0, T_0]. \tag{3.35}$$

We assume that T_* is the largest such T_0. Then we claim $T_* = T$.

To see this, we first show

$$T_* \geqslant \min\{1, T\}.$$

Indeed, by (3.32), (3.34), and (3.35), we have

$$\widetilde{A}^2(t) \leqslant \widetilde{A}^2(0) + 2C \int_0^t (\widetilde{A}^2(t) + \widetilde{A}^4(t)) \mathrm{d}t \leqslant \widetilde{A}^2(0) + 2C(M' + M'^2), \quad \forall\, t \in [0, T_*].$$

For $T > 1$, to see $T = T_*$, we use (3.30). In fact, there is a $t_* \in [T_* - \frac{1}{2}, T_*]$ such that

$$\widetilde{A}^2(t_*) \leqslant 4M',$$

and then inequality (3.35) is valid at t_* in the strict sense by our choice of ν. We repeat the above reasoning with t replaced by $t - t_*$ to conclude a contradiction if $T < T_*$.

Then, from the above computation, we have (3.25) under the condition of (3.34).

The uniqueness can be proved exactly as in the 2D case. □

Lemma 3.5 *Assume* $(u_0, d_0, E_0, H_0) \in (H^1(\Omega), H^2(\Omega), H^{m-1}(\Omega), H^{m-1}(\Omega))$ $(m \geqslant 3)$. *Then there exists a unique smooth solution* $(u(x,t), d(x,t), E(x,t), H(x,t))$ *of problem* (1.1)–(1.9) *satisfying*

$$\sup_{0 \leqslant t \leqslant T} (\|u\|_{H^m}^2 + \|d\|_{H^m}^2 + \|E\|_{H^{m-1}}^2 + \|H\|_{H^{m-1}}^2) \leqslant C, \tag{3.36}$$

when

$$\nu \geqslant \nu_0(u_0, d_0, E_0, H_0).$$

Proof This lemma can be proved by the induction for m, the procedure is similar to Lemmas 3.3 and 3.4, we omit it for simplicity. □

So Theorem 1.2 is proved.

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