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RESEARCH ARTICLE

Lump wave and hybrid solutions of a generalized (3 + 1)-dimensional nonlinear wave equation in liquid with gas bubbles

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Abstract We investigate a generalized (3 + 1)-dimensional nonlinear wave equation, which can be used to depict many nonlinear phenomena in liquid containing gas bubbles. By employing the Hirota bilinear method, we derive its bilinear formalism and soliton solutions succinctly. Meanwhile, the first-order lump wave solution and second-order lump wave solution are well presented based on the corresponding two-soliton solution and four-soliton solution. Furthermore, two types of hybrid solutions are systematically established by using the long wave limit method. Finally, the graphical analyses of the obtained solutions are represented in order to better understand their dynamical behaviors.

Keywords Generalized (3 + 1)-dimensional nonlinear wave equation, bilinear formalism, soliton solutions, lump solutions, hybrid solutions
MSC 35Q51, 35Q53, 37K40

1 Introduction

It is well known that the research of studying integrable properties and constructing exact solutions for the nonlinear evolution equations (NLEEs) is one of the most meaningful work in mathematical physics field [20]. In the past few decades, finding exact solutions of NLEEs is also hot topic for research workers all the time. Moreover, all kinds of methods, including the inverse scattering transformation method [1], Lie group method [4], Darboux transformation [34], Bäcklund transformation [43], Hirota bilinear method [23], have been successively proposed. Recently, lump solutions attract particular attention of many mathematicians and physicists. It was originally found in

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1977 by Manakov et al. [33]. The lump wave, as a special localized wave, is a rational solution in all space directions. In view of the Hirota bilinear method and symbolic computation [3,8,12,13,16–18,21,41,45–49,53,57,63], a sea of studies about lump solutions have been done [5–7,10,11,15,22,24,28–32, 35–38,55,56,58,60–62,64–68]. Inspired by the above studies, we further consider the lump wave solutions and semi-rational solutions.

Very recently, many researchers have considered the work of propagation of waves in bubbly liquids. Moreover, the bubble-liquid mixture equations have been constructed in the field of the liquid with gas bubbles to depict the propagation of weakly nonlinear waves [25,26]. In this paper, we are going to investigate a generalized $(3+1)$-dimensional nonlinear wave equation as follows:

$$(u_t + h_1 u u_x + h_2 u_{xxx} + h_3 u_x)_x + h_4 u_{yy} + h_5 u_{zz} = 0, \tag{1}$$

where $u = u(x, y, z, t)$ is a differentiable function with space coordinates x, y, z and time coordinate t, and h_i $(i = 1, 2, 3, 4, 5)$ are all arbitrary constants. This equation can be used to describe some nonlinear physical phenomena in liquid containing gas bubble. Its Bäcklund transformation, infinite conservation laws, N-soliton solutions, and periodic wave solutions were reported in detail in [54]. In addition, Eq. (1) can be reduced to the $(3 + 1)$-dimensional generalized Kadomtsev-Petviashvili equation when $h_3 = 0$. It is worth pointing that its rogue wave, bright-dark soliton, and traveling wave solutions were also derived in [39].

To the best of our knowledge, some results have been reported for the generalized $(3+1)$-dimensional nonlinear wave equation [9], but there has been no discussion with regard to lump solutions and semi-rational solutions. The main purpose of this paper is to study high order lump solutions and semi-rational solutions by using the long wave limit method [2,42,44] with the aid of the corresponding soliton solutions.

The structure of this paper is given as follows. In Section 2, we construct the bilinear formalism of Eq. (1) under the appropriate transformation by virtue of Bell polynomial. In Section 3, based on soliton solutions of Eq. (1), its first-order lump wave solution and second-order lump wave solution are presented in detail. In Section 4, we further derive two types of hybrid solutions, which are the hybrid between first-order lump solution and single-soliton and the hybrid between second-order lump solution and single-soliton. Finally, short conclusions and discussion of this paper are represented in the last section.

2 Bilinear formalism

In this section, our goal is to construct the bilinear representation of equation (1). We first introduce the following hypothesis:

$$u = c(t) q_{xx}, \tag{2}$$

where q is a real function with respect to variables x, y, z, and t, and $c(t)$ is a function to be determined later. Then, substituting the above expression (2) into Eq. (1) and integrating the obtained equation with respect to x twice, one can get

$$E(q) = q_{xt} + h_2(q_{4x} + 3q_{xx}^2) + h_3 q_{xx} + h_4 q_{yy} + h_5 q_{zz} = \sigma,$$

where σ is a constant of integration with the help of $c(t) = 6h_2/h_1$. By means of the previous results in [14,19,27,40,50–52,59], taking $\sigma = 0$, one has

$$E(q) = P_{xt} + h_2 P_{4x} + h_3 P_{xx} + h_4 P_{yy} + h_5 P_{zz} = 0.$$

Then the bilinear formalism of Eq. (1) is given by

$$F(D_x, D_y, D_z, D_t) := (D_x D_t + h_2 D_x^4 + h_3 D_x^2 + h_4 D_y^2 + h_5 D_z^2) f \cdot f = 0, \quad (3)$$

under the variable transformation

$$q = 2\log f \Longleftrightarrow u = c(t) q_{xx} = \frac{12h_2}{h_1} (\log f)_{xx}.$$

It is noted that the aforementioned expression (3) is equivalent to the form

$$\begin{aligned}(f_{xt}f - f_x f_t) + h_2(f_{4x}f - 4f_x f_{3x} + 3f_{xx}^2) + h_3(f_{xx}f - f_x^2)\\ + h_4(f_{yy}f - f_y^2) + h_5(f_{zz}f - f_z^2) = 0.\end{aligned}$$

3 Lump wave solutions

In what follows, we will systematically construct the lump wave solutions of Eq. (1) by taking a long wave limit for the corresponding soliton solutions. By employing the Hirota bilinear method, the soliton solutions of Eq. (1) are ascertained. The solution of Eq. (3) can be written as

$$f = f_N = \sum_{\mu=0,1} \exp\left(\sum_{1\leqslant i<j\leqslant N} \mu_i \mu_j A_{ij} + \sum_{i=1}^{N} \mu_i \eta_i \right), \quad (4)$$

with

$$\begin{aligned}\eta_i &= k_i[x + p_i y + q_i z - (h_2 k_i^2 + h_3 + h_4 p_i^2 + h_5 q_i^2)t] + \eta_i^0,\\ \exp(A_{ij}) &= \frac{3h_2(k_i - k_j)^2 - h_4(p_i - p_j)^2 - h_5(q_i - q_j)^2}{3h_2(k_i + k_j)^2 - h_4(p_i - p_j)^2 - h_5(q_i - q_j)^2}, \quad (5)\end{aligned}$$

where k_i, p_i, q_i, and η_i^0 are all free parameters. The notation $\sum_{\mu=0,1}$ denotes a summation that takes over all possible combinations $\mu_i = 0, 1$ $(i = 1, 2, \dots, N)$. For instance, we give the first three solutions in (4) as below:

$$f_1 = 1 + \exp(\eta_1),$$

$$f_2 = 1 + \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_1 + \eta_2 + A_{12}),$$

$$\begin{aligned} f_3 = 1 &+ \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_3) \\ &+ \exp(\eta_1 + \eta_2 + A_{12}) + \exp(\eta_1 + \eta_3 + A_{13}) + \exp(\eta_2 + \eta_3 + A_{23}) \\ &+ \exp(\eta_1 + \eta_2 + \eta_3 + A_{12} + A_{13} + A_{23}). \end{aligned}$$

Remark 1 Equation (1) admits the N-soliton solution (4) for the case $N > 2$ if and only if the following condition holds:

$$\sum_{\sigma_1,\sigma_2,\sigma_3=\pm 1} F(\sigma_1 e_1+\sigma_2 e_2+\sigma_3 e_3)F(\sigma_1 e_1-\sigma_2 e_2)F(\sigma_2 e_2-\sigma_3 e_3)F(\sigma_1 e_1-\sigma_3 e_3) = 0,$$

i.e.,

$$h_2^2 h_4 h_5 k_1^4 k_2^4 k_3^4 (-p_1 q_3 + p_2 q_3 + p_1 q_2 - p_3 q_2 + p_3 q_1 - p_2 q_1)^2 = 0. \tag{6}$$

In order to seek lump wave solutions of Eq. (1), we take

$$\exp(\eta_i^0) = -1, \quad 1 \leqslant i \leqslant N,$$

and let a limit $k_i \to 0$ in (4). We have the following theorem.

Theorem 3.1 *Equation* (1) *has the high order lump wave solutions in the following form*:

$$u = \frac{12 h_2}{h_1} (\log f_N)_{xx},$$

with

$$\begin{aligned} f_N = \prod_{i=1}^{N} \theta_i + \frac{1}{2}\sum_{i,j}^{N} B_{ij} \prod_{1\leqslant l\leqslant N,\, l\neq i,j} \theta_l + \frac{1}{2!2^2}\sum_{i,j,s,r}^{N} B_{ij}B_{sr} \prod_{1\leqslant l\leqslant N,\, l\neq i,j,s,r} \theta_l \\ + \cdots + \frac{1}{M!2^M}\sum_{i,j,\ldots,m,n}^{N} \overbrace{B_{ij}B_{vl}\cdots B_{mn}}^{M} \prod_{1\leqslant p\leqslant N,\, p\neq i,j,v,l,\ldots,m,n} \theta_p + \cdots, \end{aligned} \tag{7}$$

and

$$\theta_i = x + p_i y + q_i z - (h_3 + h_4 p_i^2 + h_5 q_i^2)t, \tag{8}$$

$$B_{ij} = \frac{12}{h_4(p_i - p_j)^2 + h_5(q_i - q_j)^2}, \tag{9}$$

where $\sum_{i,j,\ldots,m,n}^{N}$ *represents the summation roundly feasible combinations of* $i, j, \ldots, m, n$, *which are chosen from* $1, 2, \ldots, N$ *and they are all distinct. A class of nonsingular lump solutions can be derived, if we choose the parameters*

$$p_{n+i} = p_i^*, \quad q_{n+i} = q_i^*, \quad i = 1, 2, \ldots, n,$$

for $N = 2n$ *under the condition* $B_{ij} > 0$.

Remark 2 Equation (1) admits the N-lump solution (7) for the case $N > 1$ if and only if condition (6) holds.

3.1 First-order lump-wave solutions

In this part, we will calculate the first-order lump wave solutions of Eq. (1) from the corresponding two-soliton solutions, that is, $n = 1$, $N = 2$. Here, equation (7) can be rewritten as

$$f_2 = \theta_1\theta_2 + B_{12},$$

with

$$\theta_i = x + p_i y + q_i z - (h_3 + h_4 p_i^2 + h_5 q_i^2)t, \quad i = 1, 2,$$

$$B_{12} = \frac{12}{h_4(p_1 - p_2)^2 + h_5(q_1 - q_2)^2}.$$

By taking

$$p_2 = p_1^*, \quad q_2 = q_1^*,$$

we have a nonsingular solution

$$f_2 = \theta_1\theta_1^* - \frac{12}{h_4(p_1 - p_1^*)^2 + h_5(q_1 - q_1^*)^2} > 0. \tag{10}$$

Putting

$$p_1 = a + b\mathrm{i}, \quad q_1 = d + g\mathrm{i},$$

and substituting (10) into

$$u = \frac{12h_2}{h_1}(\log f_2)_{xx},$$

we get the first-order lump wave solutions of Eq. (1), given by

$$\begin{aligned} u &= \frac{12h_2}{h_1}\frac{\partial^2}{\partial x^2}\log\left[(x' + ay' + dz')^2 + (by' + gz')^2 + \frac{3}{h_4 b^2 + h_5 g^2}\right] \\ &= \frac{24h_2}{h_1}\frac{-(x' + ay' + dz')^2 + (by' + gz')^2 + \frac{3}{h_4b^2+h_5g^2}}{[(x' + ay' + dz')^2 + (by' + gz')^2 + \frac{3}{h_4b^2+h_5g^2}]^2}, \end{aligned} \tag{11}$$

with

$$x' = x + [h_4(a^2 + b^2) + h_5(d^2 + g^2) - h_3]t,$$
$$y' = y - 2h_4 a t,$$
$$z' = z - 2h_5 d t.$$

The rational solution (11) is a permanent lump solution, decaying as $O(1/x^2, 1/y^2, 1/z^2)$ for $|x|, |y|, |z| \to \infty$ and moving with the velocity

$$v_x = h_3 - h_4(a^2 + b^2) - h_5(d^2 + g^2), \quad v_y = 2h_4 a, \quad v_z = 2h_5 d.$$

As shown in Figs. 1 and 2, in order to observe the characteristics of solution (11) clearly, its evolution plots are drawn by choosing suitable parameters.

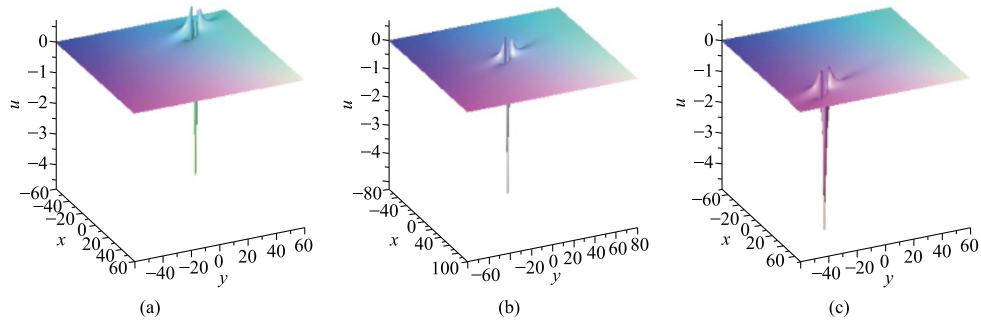

Fig. 1 Evolution plots of first-order lump wave solution (11) by taking $z=1$, $h_2=-1/2$, $h_1=h_3=h_4=h_5=a=b=d=g=1$ at time (a) $t=-10$, (b) $t=0$, (c) $t=10$

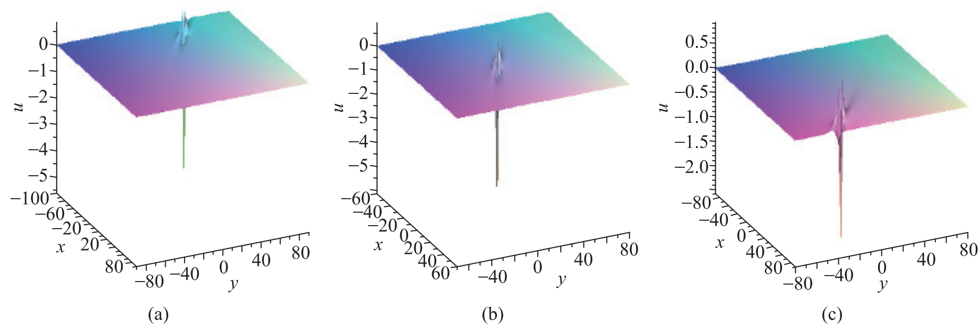

Fig. 2 Evolution plots of first-order lump wave solution (11) by taking $z=x$, $h_2=-1/2$, $h_1=h_3=h_4=h_5=a=b=d=g=1$ at time (a) $t=-10$, (b) $t=0$, (c) $t=10$

3.2 Second-order lump-wave solutions

In this subsection, in order to get second-order lump wave solutions of Eq. (1), we rewrite equation (7) by taking $n=2$, $N=4$. Due to the expression f_4 is very complicated, for simplicity, we choose

$$q_1=q_2=q_3=q_4=q, \tag{12}$$

q is a real constant. Then f_4 can be written as

$$\begin{aligned} f_4 = {} & \theta_1\theta_2\theta_3\theta_4 + B_{12}\theta_3\theta_4 + B_{13}\theta_2\theta_4 + B_{14}\theta_2\theta_3 + B_{23}\theta_1\theta_4 \\ & + B_{24}\theta_1\theta_3 + B_{34}\theta_1\theta_2 + B_{12}B_{34} + B_{13}B_{24} + B_{14}B_{23}, \end{aligned} \tag{13}$$

with

$$\theta_i = x + p_i y + qz - (h_3 + h_4 p_i^2 + h_5 q^2)t, \quad i=1,2,3,4,$$

$$B_{ij} = \frac{12}{h_4(p_i-p_j)^2}, \quad 1 \leqslant i < j \leqslant 4.$$

Setting

$$p_3 = p_1^* = p_R - \mathrm{i}p_I, \quad p_4 = p_2^* = \lambda_R - \mathrm{i}\lambda_I, \tag{14}$$

$$\operatorname{Re} p_i > 0, \quad i = 1, 2,$$

and substituting (13) into the transformation

$$u = \frac{12h_2}{h_1} (\log f_4)_{xx},$$

we can obtain the second-order lump wave solutions of Eq. (1). It is worth pointing that f_4 is a positive function consisted of biquadratic and quadratic perfect square functions. Similarly, we also display evolution plots with time of the second-order lump solution for understanding dynamical behavior better in Figs. 3 and 4.

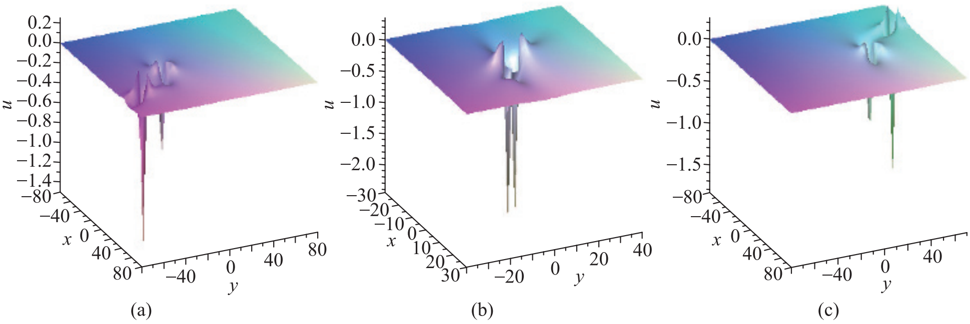

Fig. 3 Evolution plots of second-order lump wave solution by taking $z = 1$, $h_1 = 2$, $h_2 = -1/2$, $h_4 = -2$, $\lambda_R = 1/10$, $q = h_3 = h_5 = p_R = p_I = 1$, $\lambda_I = 1/2$ at time (a) $t = -10$, (b) $t = 0$, (c) $t = 10$

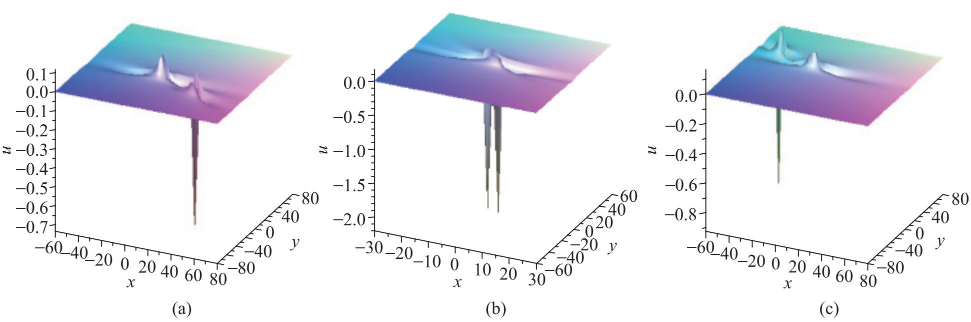

Fig. 4 Evolution plots of second-order lump wave solution by taking $z = x$, $h_1 = 2$, $h_2 = -1/2$, $h_4 = -2$, $\lambda_R = 1/10$, $q = h_3 = h_5 = p_R = p_I = 1$, $\lambda_I = 1/2$ at time (a) $t = -10$, (b) $t = 0$, (c) $t = 10$

4 Semi-rational solutions

Now, we are in a position to search the semi-rational solutions for Eq. (1). Here, by taking a long wave limit of the partial exponential functions in (4), a combination of polynomial and exponential functions can be derived, which also be called as semi-rational solutions or hybrid solutions. To illustrate the solution clearly, we will focus on the following two types of hybrid solutions.

4.1 Hybrid solutions of first-order lump wave and single-soliton wave

Primarily, we think about the case of $N = 3$ under condition (6). Let

$$N = 3, \quad \eta_1^0 = \eta_2^0 = \mathrm{i}\pi,$$

and take $k_1, k_2 \to 0$ in Eq. (4). One obtains

$$f = (\theta_1\theta_2 + B_{12}) + (\theta_1\theta_2 + B_{12} + B_{13}\theta_1 + B_{23}\theta_1 + B_{12}B_{23})\mathrm{e}^{\eta_3},$$

with

$$B_{s3} = -\frac{12h_2k_3}{3h_2k_3^2 - h_4(p_s - p_3)^2 - h_5(q_s - q_3)^2}, \quad s = 1, 2,$$

where θ_1, θ_2 are from (8), B_{12} is given by (9), and η_3 is defined by (5). Then set

$$p_2 = p_1^* = a - b\mathrm{i}, \quad q_2 = q_1^* = d - g\mathrm{i}.$$

Indeed, the corresponding hybrid solutions u composed of first-order lump wave and single-soliton wave can be obtained.

To understand dynamical behaviors better, we give the three-dimensional plots of hybrid solutions consisted of first-order lump wave and single-soliton wave in different time by choosing appropriate parameters. As shown in Fig. 5, we can find that the lump moves and passes the soliton and in the interaction domain of the two waveforms the amplitude increases considerably.

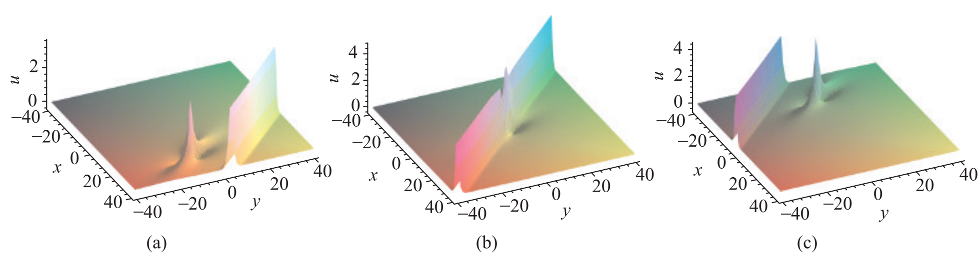

Fig. 5 Evolution plots in (x, y)-plane of hybrid solutions consisted of first-order lump wave and single-soliton wave by taking $z = 0$, $h_1 = -1$, $h_2 = -1/2$, $h_3 = -9/5$, $h_5 = -11/5$, $h_4 = a = b = d = g = p_3 = q_3 = 1$, $k_3 = -8/5$, $\eta_3^0 = 0$ at time (a) $t = -10$, (b) $t = 0$, (c) $t = 10$

4.2 Hybrid solutions of second-order lump wave and single-soliton wave

Here, we consider the situation of $N = 5$ to derive hybrid of second-order lump wave and single-soliton wave under condition (6). Put

$$N = 5, \quad \eta_1^0 = \eta_2^0 = \eta_3^0 = \eta_4^0 = \mathrm{i}\pi,$$

and take $k_1, k_2, k_3, k_4 \to 0$ in Eq. (4). One gets

$$\begin{aligned} f = {} & (\theta_1\theta_2\theta_3\theta_4 + B_{12}\theta_3\theta_4 + B_{13}\theta_2\theta_4 + B_{14}\theta_2\theta_3 + B_{23}\theta_1\theta_4 + B_{24}\theta_1\theta_3 + B_{34}\theta_1\theta_2 \\ & + B_{12}B_{34} + B_{13}B_{24} + B_{14}B_{23}) + \mathrm{e}^{\eta_5}[\theta_1\theta_2\theta_3\theta_4 + B_{45}\theta_1\theta_2\theta_3 + B_{35}\theta_1\theta_2\theta_4 \\ & + B_{25}\theta_1\theta_3\theta_4 + B_{15}\theta_2\theta_3\theta_4 + (B_{35}B_{45} + B_{34})\theta_1\theta_2 + (B_{25}B_{45} + B_{24})\theta_1\theta_3 \\ & + (B_{25}B_{35} + B_{23})\theta_1\theta_4 + (B_{15}B_{45} + B_{14})\theta_2\theta_3 + (B_{15}B_{35} + B_{13})\theta_2\theta_4 \\ & + (B_{15}B_{25} + B_{12})\theta_3\theta_4 + (B_{25}B_{35}B_{45} + B_{23}B_{45} + B_{25}B_{34} + B_{24}B_{35})\theta_1 \\ & + (B_{15}B_{35}B_{45} + B_{14}B_{35} + B_{13}B_{45} + B_{15}B_{34})\theta_2 + (B_{15}B_{25}B_{45} + B_{14}B_{25} \\ & + B_{15}B_{24} + B_{12}B_{45})\theta_3 + (B_{15}B_{25}B_{35} + B_{15}B_{23} + B_{13}B_{25} + B_{12}B_{35})\theta_4 \\ & + B_{12}B_{34} + B_{13}B_{24} + B_{14}B_{23} + B_{12}B_{35}B_{45} + B_{13}B_{25}B_{45} + B_{14}B_{25}B_{35} \\ & + B_{15}B_{24}B_{35} + B_{15}B_{25}B_{34} + B_{15}B_{23}B_{45} + B_{15}B_{25}B_{35}B_{45}], \end{aligned} \tag{15}$$

with

$$B_{s5} = -\frac{12h_2k_5}{3h_2k_5^2 - h_4(p_s - p_5)^2}, \quad s = 1, 2, 3, 4,$$

where θ_i, B_{ij} are, respectively, from (8) and (9), and η_5 is given by (5). It is necessary to point out that we use (12) for simplicity in above results. Then, set p_3 and p_4 as in (14) with $p_R, p_I, \lambda_R, \lambda_I$, and q are all real constants. The corresponding hybrid solutions u defined by (4) with (15) are represented.

In what follows, the evolution plots of hybrid of second-order lump wave and single-soliton wave are respectively revealed at different time $t = -20$, $t = 0$, $t = 20$ by taking suitable parameters.

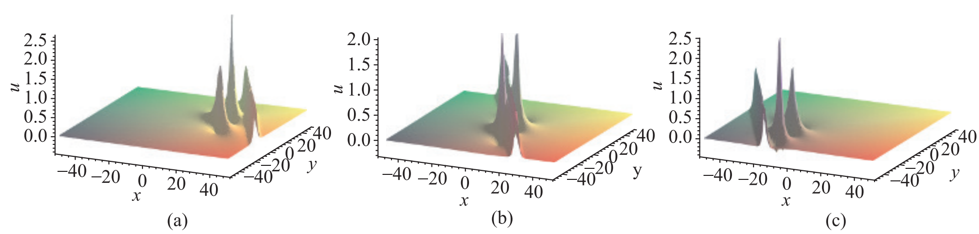

Fig. 6 Evolution plots in (x, y)-plane of hybrid solutions consisted of second-order lump wave and single-soliton wave by taking $z = 0$, $h_1 = -1$, $h_3 = -9/5$, $h_2 = h_4 = -1/2$, $h_5 = -11/5$, $p_R = p_I = \lambda_I = 1$, $\lambda_R = 1/2$, $k_5 = -4/5$, $p_5 = 9/5$, $q = \eta_5^0 = 0$ at time (a) $t = -20$, (b) $t = 0$, (c) $t = 20$

5 Conclusions and discussion

In this work, we have researched the generalized $(3+1)$-dimensional nonlinear wave equation in liquid with gas bubbles. Its bilinear form and soliton solutions have been constructed by employing Hirota's bilinear approach. Moreover, the lump wave solutions and the semi-rational solutions have also been obtained legitimately by employing the long wave limit method. Most importantly, the figures of first-order lump wave, second-order lump wave, and two types of hybrid solutions have been presented in Figs. 1–6 in order to better understand their behavior characteristics. In view of the obtained graphs, the propagation properties of the resulting solutions can be well represented for the $(3+1)$-dimensional nonlinear wave equation in a bubbly liquid.

The paper shows an effective and powerful method to seek exact solutions of NLEEs, which is worthy of further exploration to other models in mathematical physics and engineering. Finally, we hope that our results provided in this paper are helpful to comprehend the lump solutions and hybrid solutions for more models.

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