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RESEARCH ARTICLE

Estimations on upper and lower bounds of solutions to a class of tensor complementarity problems

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Abstract We introduce a class of structured tensors, called generalized row strictly diagonally dominant tensors, and discuss some relationships between it and several classes of structured tensors, including nonnegative tensors, B-tensors, and strictly copositive tensors. In particular, we give estimations on upper and lower bounds of solutions to the tensor complementarity problem (TCP) when the involved tensor is a generalized row strictly diagonally dominant tensor with all positive diagonal entries. The main advantage of the results obtained in this paper is that both bounds we obtained depend only on the tensor and constant vector involved in the TCP; and hence, they are very easy to calculate.

Keywords Tensor complementarity problem (TCP), generalized row strictly diagonally dominant tensor, upper and lower bounds of solutions
MSC 15A48, 15A69, 65F10, 65H10, 65N22

1 Introduction

In recent years, the tensor complementarity problem (TCP), which firstly appeared in [21], has been a hot topic in the optimization field. It has been shown that a multi-person noncooperative game can be reformulated as a TCP, and the one-to-one correspondence between the solutions of these two problems has been established [14]. Up to now, a large number of theoretical results for the TCP have been obtained in the literature, including nonemptiness and/or compactness of the solution set [4,6,8,9,15,27–29], existence of a unique solution [2,5,18,26,27,30], error bound theory [13,16,35], strict feasibility of the problem [10,26], convexity of the solution set [3], stability of solutions and continuity of solution maps [1,12], and so on. Moreover, several numerical methods for

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solving the TCP have also been proposed in the literature, see [7,11,14,17,31,32, 34] for details.

When the solution set of a TCP is bounded, a natural question is that *how to estimate upper and lower bounds of solutions to the TCP*. Song and Yu [25] considered such a problem for the TCP with a strictly semi-positive tensor, and they gave estimations of upper and lower bounds of solutions to this class of TCPs with the help of norms of two operators. Generally, however, it is difficult to calculate these two norms of operators. Recently, Song and Qi [23] investigated these two norms of operators, and gave checkable upper bounds of them in terms of the tensor involved in the TCP; and hence, a checkable lower bound of solutions to the TCP with a strictly semi-positive tensor was obtained. More recently, Song and Mei [20] considered the TCP with a B-tensor, and they gave estimations of lower bounds of solutions to the problem, which is easy to calculate.

Motivated by the papers mentioned above, we investigate in this paper a class of structured tensors, called *generalized row strictly diagonally dominant tensors with all positive diagonal entries*, which is a subclass of strictly semi-positive tensors and an extension of B-tensors. We discuss some relationships between it and several classes of known structured tensors used in the literature. In particular, we give estimations of upper and lower bounds of solutions to the TCP when the involved tensor is a generalized row strictly diagonally dominant tensor with all positive diagonal entries. Both upper and lower bounds we obtained are easy checkable since they depend only on the tensor and constant vector involved in the TCP.

The rest of this paper is organized as follows. In Section 2, we recall some basic symbols, concepts, and results. In Section 3, we first introduce a class of structured tensors and discuss some relationships between it and several known structured tensors; and then, we investigate bounds of solutions to the TCP with the introduced tensor. Conclusions are given in the last section.

2 Preliminaries

An m-th order n-dimensional real tensor $\mathscr{A} = (a_{i_1 i_2 \cdots i_m})$ is a multi-array of real entries $a_{i_1 i_2 \cdots i_m} \in \mathbb{R}$, where $i_j \in [n] := \{1, 2, \ldots, n\}$ for $j \in [m] := \{1, 2, \ldots, m\}$; and if all its entries are nonnegative, then we call it a nonnegative tensor. We denote the set of all m-th order n-dimensional real tensors by $\mathbb{R}^{[m,n]}$. For any $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$, every $a_{i_1 i_2 \cdots i_m}$ is said to be a diagonal entry when $i_1 = i_2 = \cdots = i_m$; and an off-diagonal entry otherwise. For any $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ and $x = (x_1, x_2, \ldots, x_n)^\top \in \mathbb{R}^n$, $\mathscr{A} x^{m-1} \in \mathbb{R}^n$ is defined by

$$(\mathscr{A} x^{m-1})_i := \sum_{i_2, i_3, \ldots, i_m = 1}^{n} a_{i i_2 i_3 \cdots i_m} x_{i_2} x_{i_3} \cdots x_{i_m}, \quad \forall\, i \in [n].$$

For any given tensor $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ and for any $i \in [n]$, if there is no

$a_{ii_2i_3\cdots i_m} \geqslant 0$ for all $(i_2, i_3, \ldots, i_m) \neq (i, i, \ldots, i)$, then we define $r_i(\mathscr{A})_+ := 0$, and if there is no $a_{ii_2i_3\cdots i_m} < 0$ for all $(i_2, i_3, \ldots, i_m) \neq (i, i, \ldots, i)$, then we define $r_i(\mathscr{A})_- := 0$; otherwise, we define

$$r_i(\mathscr{A})_+ := \sum_{a_{ii_2i_3\cdots i_m} \geqslant 0,\, (i_2,i_3,\ldots,i_m) \neq (i,i,\ldots,i)} a_{ii_2i_3\cdots i_m} \tag{1}$$

and

$$r_i(\mathscr{A})_- := \sum_{a_{ii_2i_3\cdots i_m} < 0,\, (i_2,i_3,\ldots,i_m) \neq (i,i,\ldots,i)} |a_{ii_2i_3\cdots i_m}|. \tag{2}$$

Given a tensor $\mathscr{A} \in \mathbb{R}^{[m,n]}$ and a vector $q \in \mathbb{R}^n$, the tensor complementarity problem, denoted by $\mathrm{TCP}(\mathscr{A}, q)$, is to find a vector $x \in \mathbb{R}^n$ such that

$$x \geqslant 0, \quad \mathscr{A}x^{m-1} + q \geqslant 0, \quad x^\top(\mathscr{A}x^{m-1} + q) = 0.$$

Definition 1 [24] A tensor $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is said to be strictly semi-positive if and only if for each $x \geqslant 0$ and $x \neq 0$, there exists an index $k \in [n]$ such that

$$x_k > 0, \quad (\mathscr{A}x^{m-1})_k > 0.$$

Theorem 1 [22] *A tensor $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is strictly semi-positive if and only if the* $\mathrm{TCP}(\mathscr{A}, q)$ *has a unique solution for every $q \geqslant 0$.*

Definition 2 [21] A tensor $\mathscr{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called a B-tensor if and only if

$$\sum_{i_2,i_3,\ldots,i_m=1}^{n} a_{ii_2i_3\cdots i_m} > 0, \quad \forall\, i \in [n],$$

and

$$\frac{1}{n^{m-1}}\left(\sum_{i_2,i_3,\ldots,i_m=1}^{n} a_{ii_2i_3\cdots i_m}\right) > a_{ij_2j_3\cdots j_m}, \quad \forall\, (j_2, j_3, \ldots, j_m) \neq (i, i, \ldots, i).$$

Theorem 2 [21] *If $\mathscr{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ is a B-tensor, then for each $i \in [n]$,*

(a) $a_{ii\cdots i} > \sum_{a_{ii_2i_3\cdots i_m}<0} |a_{ii_2i_3\cdots i_m}|$;

(b) $a_{ii\cdots i} > |a_{ij_2j_3\cdots j_m}|$ *for all $(j_2, j_3, \ldots, j_m) \neq (i, i, \ldots, i)$ with $j_2, j_3, \ldots, j_m \in [n]$.*

Definition 3 A tensor $\mathscr{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called to be row strictly diagonally dominant if and only if

$$|a_{ii\cdots i}| > \sum_{(i_2,i_3,\ldots,i_m) \neq (i,i,\ldots,i)} |a_{ii_2i_3\cdots i_m}|, \quad \forall\, i \in [n].$$

If $\mathscr{A}$ additionally satisfies that $a_{ii\cdots i} > 0$ for all $i \in [n]$, then we call $\mathscr{A}$ a row strictly diagonally dominant tensor with all positive diagonal entries.

When $m = 2$, a row strictly diagonally dominant tensor reduces to a row strictly diagonally dominant matrix. The row strictly diagonally dominant tensor was discussed in [6] and the row strictly diagonally dominant tensor with all positive diagonal entries was discussed in [21].

Definition 4 [19] Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$. $\mathscr{A}$ is said to be

(a) copositive if $x^\top \mathscr{A} x^{m-1} \geqslant 0$ for any $x \in \mathbb{R}^n_+$;

(b) strictly copositive if $x^\top \mathscr{A} x^{m-1} > 0$ for any $x \in \mathbb{R}^n_+ \backslash \{0\}$.

3 Main results

3.1 A class of structured tensors

In this subsection, we introduce a class of structured tensors, and discuss some relationships between it and several classes of known structured tensors.

Definition 5 A tensor $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is said to be generalized row strictly diagonally dominant if and only if for all $i \in [n]$,

$$|a_{ii\cdots i}| - r_i(\mathscr{A})_- > 0,$$

where $r_i(\mathscr{A})_-$ is defined by (2). If $\mathscr{A}$ additionally satisfies $a_{ii\cdots i} > 0$ for all $i \in [n]$, then we call $\mathscr{A}$ a generalized row strictly diagonally dominant tensor with all positive diagonal entries.

From Definition 5, we have the following proposition easily.

Proposition 1 *Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$. Then the following results hold.*

(a) *If $\mathscr{A}$ is a nonnegative tensor with all positive diagonal entries, then it is a generalized row strictly diagonally dominant tensor with all positive diagonal entries.*

(b) *If $\mathscr{A}$ is a B-tensor, then it is a generalized row strictly diagonally dominant tensor with all positive diagonal entries.*

(c) *If $\mathscr{A}$ is a row strictly diagonally dominant tensor (with all positive diagonal entries), then it is a generalized row strictly diagonally dominant tensor (with all positive diagonal entries).*

Generally, the inverses of three conclusions in Proposition 1 are not true, which can be seen by the following example.

Example 1 Let $\mathscr{A} = (a_{ijk}) \in \mathbb{R}^{[3,3]}$, where

$$a_{111} = a_{222} = a_{333} = 2, \quad a_{122} = 3, \quad a_{211} = -1,$$

and other entries are zero.

It is obvious that $\mathscr{A}$ given in Example 1 is a generalized row strictly diagonally dominant tensor with all positive diagonal entries, but not a non-negative tensor with all positive diagonal entries. Moreover, $\mathscr{A}$ is not a row strictly diagonally dominant tensor with all positive diagonal entries because $|a_{111}| = a_{111} < a_{122}$; and $\mathscr{A}$ is not a B-tensor because

$$\frac{1}{3^2}\left(\sum_{j,k=1}^{3} a_{1jk}\right) = \frac{1}{9}\,(a_{111} + a_{122}) = \frac{5}{9} < 3 = a_{122}.$$

Recall that $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is called a Z-tensor if and only if all its off-diagonal entries are non-positive [33]. Based on properties of Z-tensors, we have the following results.

Proposition 2 *Let $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a Z-tensor. Then the following statements are equivalent*:

(a) *$\mathscr{A}$ is a B-tensor*;

(b) *for each $i \in [n]$, $\sum_{i_2,i_3,\ldots,i_m=1}^{n} a_{i i_2 i_3 \cdots i_m} > 0$*;

(c) *$\mathscr{A}$ is row strictly diagonally dominant with all positive diagonal entries*;

(d) *$\mathscr{A}$ is generalized row strictly diagonally dominant with all positive diagonal entries.*

Proof Since '(a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c)' were proved in [21], and '(c) $\Rightarrow$ (d)' holds directly from Proposition 1 (c), it is sufficient to show '(d) $\Rightarrow$ (b)'. In fact, if $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is a generalized row strictly diagonally dominant tensor with all positive diagonal entries, then it follows from Definition 5 that $a_{ii\cdots i} > r_i(\mathscr{A})_-$ for each $i \in [n]$, that is, $\sum_{i_2,i_3,\ldots,i_m=1}^{n} a_{i i_2 i_3 \cdots i_m} > 0$ for each $i \in [n]$, which means that '(d) $\Rightarrow$ (b)' holds. □

Proposition 3 *Let $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a generalized row strictly diagonally dominant tensor satisfying $a_{ii\cdots i} > \sum_{j=1}^{n} r_j(\mathscr{A})_-$ for each $i \in [n]$. Then $\mathscr{A}$ is strictly copositive.*

Proof Suppose that $x \in \mathbb{R}^n_+\backslash\{0\}$ and $\|x\|_\infty = x_k > 0$ with $k \in [n]$. Then we have

$$\begin{aligned} x^\top \mathscr{A} x^{m-1} &= \sum_{i_1,i_2,\ldots,i_m=1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \\ &= x_k^m \sum_{i_1,i_2,\ldots,i_m=1}^{n} a_{i_1 i_2 \cdots i_m} \frac{x_{i_1} x_{i_2} \cdots x_{i_m}}{x_k^m} \\ &\geqslant x_k^m \left(\sum_{i=1}^{n} a_{ii\cdots i} \frac{x_i^m}{x_k^m} + \sum_{a_{j_1 j_2 \cdots j_m} < 0} a_{j_1 j_2 \cdots j_m} \frac{x_{j_1} x_{j_2} \cdots x_{j_m}}{x_k^m} \right) \\ &\geqslant x_k^m \left(a_{kk\cdots k} - \sum_{j=1}^{n} r_j(\mathscr{A})_- \right) \\ &> 0, \end{aligned}$$

which implies that $\mathscr{A}$ is a strictly copositive tensor. □

Theorem 3 *Let $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a generalized row strictly diagonally dominant tensor with all positive diagonal entries. Then $\mathscr{A}$ is strictly semi-positive.*

Proof For any $x \geqslant 0$ and $x \neq 0$, we assume $x_k = \|x\|_\infty > 0$. Then we have

$$\begin{aligned}
(\mathscr{A}x^{m-1})_k &= \sum_{i_2,i_3,\ldots,i_m=1}^{n} a_{ki_2i_3\cdots i_m} x_{i_2} x_{i_3} \cdots x_{i_m} \\
&= x_k^{m-1} \sum_{i_2,i_3,\ldots,i_m=1}^{n} a_{ki_2i_3\cdots i_m} \frac{x_{i_2} x_{i_3} \cdots x_{i_m}}{x_k^{m-1}} \\
&= \|x\|_\infty^{m-1} \Bigg(a_{kk\cdots k} + \sum_{(i_2,i_3,\ldots,i_m)\neq(k,k,\ldots,k)} a_{ki_2i_3\cdots i_m} \frac{x_{i_2} x_{i_3} \cdots x_{i_m}}{\|x\|_\infty^{m-1}} \Bigg) \\
&\geqslant \|x\|_\infty^{m-1} \Bigg(a_{kk\cdots k} - \sum_{a_{ki_2i_3\cdots i_m}<0} |a_{ki_2i_3\cdots i_m}| \Bigg) \\
&= \|x\|_\infty^{m-1} \Big(a_{kk\cdots k} - r_k(\mathscr{A})_- \Big) \\
&> 0,
\end{aligned}$$

which implies that $\mathscr{A}$ is strictly semi-positive. □

The following example shows that a strictly semi-positive tensor may not be generalized row strictly diagonally dominant.

Example 2 Let $\mathscr{A} \in \mathbb{R}^{[3,2]}$, where

$$a_{111} = a_{222} = a_{211} = 1, \quad a_{122} = -2,$$

and other entries are zero.

Clearly, the tensor $\mathscr{A}$ in Example 2 is not generalized row strictly diagonally dominant. However, for any $x = (\sigma, \tau)^\top \in \mathbb{R}^2_+ \backslash \{0\}$, it follows that

$$(\mathscr{A}x^2)_1 = \sigma^2 - 2\tau^2, \quad (\mathscr{A}x^2)_2 = \sigma^2 + \tau^2,$$

which implies that $\mathscr{A}$ is strictly semi-positive.

Since the solution set of the TCP with a strictly semi-positive tensor is nonempty and compact [23,24], it follows from Theorem 3 that the TCP has a nonempty and compact solution set when the involved tensor is a generalized row strictly diagonally dominant tensor with all positive diagonal entries. In the following subsection, we discuss the bounds of solutions of such a TCP.

3.2 Estimations of bounds of solutions to TCP

Suppose that $\mathscr{A}$ is a generalized row strictly diagonally dominant tensor with all positive diagonal entries. Then it follows from Theorems 3 and 1 that $0 \in \mathbb{R}^n$ is the unique solution of the TCP$(\mathscr{A}, q)$ when $q \in \mathbb{R}^n_+$. Thus, we consider the

TCP($\mathscr{A}, q$) with $q \in \mathbb{R}^n \backslash \mathbb{R}^n_+$, i.e., there exists $k \in [n]$ such that $q_k < 0$. In the following, we denote

$$\Omega(q) := \{i \in [n] \colon q_i < 0\}, \quad \forall\, q \in \mathbb{R}^n \backslash \mathbb{R}^n_+. \tag{3}$$

Lemma 1 *Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$ be a generalized row strictly diagonally dominant tensor with all positive diagonal entries, $q \in \mathbb{R}^n \backslash\, \mathbb{R}^n_+$ be any given vector, and $\Omega(q)$ be defined by* (3). *If $x \in \mathbb{R}^n$ is a solution of the* TCP($\mathscr{A}, q$) *with $\|x\|_\infty = x_k$, then $k \in \Omega(q)$ and*

$$(\mathscr{A}x^{m-1})_k + q_k = 0.$$

Proof Suppose $k \notin \Omega(q)$. Then $q_k \geqslant 0$. Since 0 is not a solution of the TCP($\mathscr{A}, q$) when $q \notin \mathbb{R}^n_+$, it follows that $x_k = \|x\|_\infty > 0$. Thus, we have

$$\begin{aligned}
\frac{(\mathscr{A}x^{m-1} + q)_k}{\|x\|_\infty^{m-1}} &= \frac{(\mathscr{A}x^{m-1})_k}{\|x\|_\infty^{m-1}} + \frac{q_k}{\|x\|_\infty^{m-1}} \\
&\geqslant \frac{(\mathscr{A}x^{m-1})_k}{\|x\|_\infty^{m-1}} \\
&= \sum_{i_2, i_3, \ldots, i_m = 1}^{n} a_{k i_2 i_3 \cdots i_m} \frac{x_{i_2} x_{i_3} \cdots x_{i_m}}{\|x\|_\infty^{m-1}} \\
&= a_{kk\cdots k} \frac{x_k^{m-1}}{x_k^{m-1}} + \sum_{(i_2, i_3, \ldots, i_m) \neq (k,k,\ldots,k)} a_{k i_2 i_3 \cdots i_m} \frac{x_{i_2} x_{i_3} \cdots x_{i_m}}{\|x\|_\infty^{m-1}} \\
&\geqslant a_{kk\cdots k} - \sum_{a_{k i_2 \cdots i_m} < 0,\, (i_2, i_3, \ldots, i_m) \neq (i,i,\ldots,i)} |a_{k i_2 \cdots i_m}| \\
&= a_{kk\cdots k} - r_k(\mathscr{A})_- \\
&> 0.
\end{aligned}$$

Then we have

$$x_k(\mathscr{A}x^{m-1} + q)_k > 0,$$

which contradicts the fact that x solves the TCP($\mathscr{A}, q$). Thus, it holds that $k \in \Omega(q)$. Moreover, we have $(\mathscr{A}x^{m-1})_k + q_k = 0$ because $x_k > 0$ and $x_k(\mathscr{A}x^{m-1} + q)_k = 0$. □

Based on Lemma 1, we further give the bounds of solutions to the TCP($\mathscr{A}, q$) as follows.

Theorem 4 *Let $\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a generalized row strictly diagonally dominant tensor with all positive diagonal entries, and $q \in \mathbb{R}^n \backslash \mathbb{R}^n_+$ be any given vector. If $x \in \mathbb{R}^n$ is a solution of the* TCP($\mathscr{A}, q$), *then it holds that*

$$\min_{i \in \Omega(q)} \frac{-q_i}{a_{ii\cdots i} + r_i(\mathscr{A})_+} \leqslant \|x\|_\infty^{m-1} \leqslant \max_{i \in \Omega(q)} \frac{-q_i}{a_{ii\cdots i} - r_i(\mathscr{A})_-}, \tag{4}$$

where $r_i(\mathscr{A})_+$, $r_i(\mathscr{A})_-$, and $\Omega(q)$ are defined by (1), (2), *and* (3), *respectively.*

Proof From Lemma 1, there exists $k \in \Omega(q)$ such that $x_k = \|x\|_\infty > 0$ and $(\mathscr{A}x^{m-1})_k + q_k = 0$. Thus, we have

$$\begin{aligned} 0 &< \frac{-q_k}{\|x\|_\infty^{m-1}} \\ &= \Big(\mathscr{A}\Big(\frac{x}{\|x\|_\infty}\Big)^{m-1}\Big)_k \\ &= \sum_{i_2,i_3,\ldots,i_m=1}^{n} a_{ki_2i_3\cdots i_m} \frac{x_{i_2}x_{i_3}\cdots x_{i_m}}{\|x\|_\infty^{m-1}} \\ &= \sum_{a_{ki_2i_3\cdots i_m}<0} a_{ki_2i_3\cdots i_m} \frac{x_{i_2}x_{i_3}\cdots x_{i_m}}{\|x\|_\infty^{m-1}} + \sum_{a_{kj_2j_3\cdots j_m}\geqslant 0} a_{kj_2j_3\cdots j_m} \frac{x_{j_2}x_{j_3}\cdots x_{j_m}}{\|x\|_\infty^{m-1}} \\ &\leqslant \sum_{a_{kj_2j_3\cdots j_m}\geqslant 0} a_{kj_2j_3\cdots j_m} \\ &= a_{kk\cdots k} + r_k(\mathscr{A})_+. \end{aligned} \tag{5}$$

Moreover, we have

$$\begin{aligned} \frac{-q_k}{\|x\|_\infty^{m-1}} &= \Big(\mathscr{A}\Big(\frac{x}{\|x\|_\infty}\Big)^{m-1}\Big)_k \\ &= \sum_{i_2,i_3,\ldots,i_m=1}^{n} a_{ki_2i_3\cdots i_m} \frac{x_{i_2}x_{i_3}\cdots x_{i_m}}{\|x\|_\infty^{m-1}} \\ &= a_{kk\cdots k}\frac{x_k^{m-1}}{\|x\|_\infty^{m-1}} + \sum_{a_{ki_2i_3\cdots i_m}<0} a_{ki_2i_3\cdots i_m} \frac{x_{i_2}x_{i_3}\cdots x_{i_m}}{\|x\|_\infty^{m-1}} \\ &\quad + \sum_{a_{kj_2j_3\cdots j_m}\geqslant 0,\,(j_2,j_3,\ldots,j_m)\neq(k,k,\ldots,k)} a_{kj_2j_3\cdots j_m} \frac{x_{j_2}x_{j_3}\cdots x_{j_m}}{\|x\|_\infty^{m-1}} \\ &\geqslant a_{kk\cdots k} + \sum_{a_{ki_2i_3\cdots i_m}<0} a_{ki_2i_3\cdots i_m} \frac{x_{i_2}x_{i_3}\cdots x_{i_m}}{\|x\|_\infty^{m-1}} \\ &\geqslant a_{kk\cdots k} - \sum_{a_{ki_2i_3\cdots i_m}<0} |a_{ki_2i_3\cdots i_m}| \\ &= a_{kk\cdots k} - r_k(\mathscr{A})_- \\ &> 0. \end{aligned} \tag{6}$$

Thus, it follows from (5) and (6) that

$$\frac{-q_k}{a_{kk\cdots k} + r_k(\mathscr{A})_+} \leqslant \|x\|_\infty^{m-1} \leqslant \frac{-q_k}{a_{kk\cdots k} - r_k(\mathscr{A})_-}.$$

This implies that (4) holds. □

It is easy to see from (1) and (2) that both $r_i(\mathscr{A})_+$ and $r_i(\mathscr{A})_-$ depend only on the entries of $\mathscr{A}$; and hence, they are easy to calculate. Thus, when

$\mathscr{A} \in \mathbb{R}^{[m,n]}$ is a generalized row strictly diagonally dominant tensor with all positive diagonal entries, both upper and lower bounds given in Theorem 4 are easy to calculate. Obviously, the smaller the number of elements in the set $\Omega(q)$, the easier these two bounds to calculate. Particularly, if the set $\Omega(q)$ contains only one element, then we have the following result.

Corollary 1 *Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$ be a generalized row strictly diagonally dominant tensor with all positive diagonal entries, $q \in \mathbb{R}^n \backslash \mathbb{R}^n_+$ be a vector, and $x \in \mathbb{R}^n$ be a solution of the* TCP$(\mathscr{A}, q)$. *If $\Omega(q) = \{k\}$ for some $k \in [n]$, then it holds that*

$$\frac{-q_k}{a_{kk\cdots k} + r_k(\mathscr{A})_+} \leqslant \|x\|_\infty^{m-1} \leqslant \frac{-q_k}{a_{kk\cdots k} - r_k(\mathscr{A})_-}.$$

Moreover, both bounds obtained in Theorem 4 are tight for some TCPs. For example, when $\Omega(q) = \{k\}$ for some $k \in [n]$ and $r_k(\mathscr{A})_+ = r_k(\mathscr{A})_- = 0$, both upper bound and lower bound in Corollary 1 are $-q_k/a_{kk\cdots k}$. The following example shows that such a TCP exists.

Example 3 Let $q = (1, -1)^\top$ and $\mathscr{A} = (a_{ijk}) \in \mathbb{R}^{[3,2]}$, where $a_{111} = a_{122} = a_{222} = 1$, and other entries are zero.

Let $\mathscr{A} \in \mathbb{R}^{[3,2]}$ and $q \in \mathbb{R}^2$ be given by Example 3. Consider the corresponding TCP$(\mathscr{A}, q)$, i.e., find a vector $x \in \mathbb{R}^2$ such that

$$x \geqslant 0, \quad \begin{cases} x_1^2 + x_2^2 + 1 \geqslant 0, \\ x_2^2 - 1 \geqslant 0, \end{cases}, \quad x_1(x_1^2 + x_2^2 + 1) + x_2(x_2^2 - 1) = 0. \tag{7}$$

We can verify that $x^* = (0, 1)^\top$ is the unique solution of the TCP (7) with $\|x^*\|_\infty^{m-1} = 1$. Moreover, it is easy to see that both upper bound and lower bound in Corollary 1 are 1 for the TCP (7).

4 Conclusions

In this paper, we investigated the relationships between the generalized row strictly diagonally dominant tensor and several known tensors studied in the literature. In particular, we obtained the checkable upper and lower bounds of solutions to the TCP when the involved tensor is a generalized row strictly diagonally dominant tensor with all positive diagonal entries. Moreover, the bounds we obtained are tight for some TCPs.

When the tensor involved in the TCP is a generalized row strictly diagonally dominant tensor with all positive diagonal entries, it is possible from Lemma 1 that a solution of the TCP can be found by solving a system of lower dimensional tensor equations. It is worth investigating how to design a numerical method to find a solution of the TCP in this way, since such a method can reduce the calculation cost.

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References

1. Bai X L, Huang Z H, Li X. Stability of solutions and continuity of solution maps of tensor complementarity problems. Asia-Pac J Oper Res, 2019, 36(2): 1940002 (19pp)
2. Bai X L, Huang Z H, Wang Y. Global uniqueness and solvability for tensor complementarity problems. J Optim Theory Appl, 2016, 170(1): 72–84
3. Balaji R, Palpandi K. Positive definite and Gram tensor complementarity problems. Optim Lett, 2018, 12(3): 639–648
4. Che M L, Qi L, Wei Y M. Positive-definite tensors to nonlinear complementarity problems. J Optim Theory Appl, 2016, 168(2): 475–487
5. Chen H, Qi L, Song Y. Column sufficient tensors and tensor complementarity problems. Front Math China, 2018, 13(2): 255–276
6. Ding W, Luo Z, Qi L. P-tensors, P_0-tensors, and their applications. Linear Algebra Appl, 2018, 555: 336–354
7. Du S, Zhang L. A mixed integer programming approach to the tensor complementarity problem. J Global Optim, 2019, 73(4): 789–800
8. Gowda M S. Polynomial complementarity problems. Pac J Optim, 2017, 13(2): 227–241
9. Gowda M S, Luo Z, Qi L, Xiu N H. Z-tensors and complementarity problems. arXiv: 1510.07933v2
10. Guo Q, Zheng M M, Huang Z H. Properties of S-tensors. Linear Multilinear Algebra, 2019, 67(4): 685–696
11. Han L X. A continuation method for tensor complementarity problems. J Optim Theory Appl, 2019, 180(3): 949–963
12. Hieu Vu T. On the R_0-tensors and the solution map of tensor complementarity problems. J Optim Theory Appl, 2019, 181(1): 163–183
13. Hu S, Huang Z H, Wang J. Error bounds for the solution set of quadratic complementarity problems. J Optim Theory Appl, 2018, 179(3): 983–1000
14. Huang Z H, Qi L. Formulating an n-person noncooperative game as a tensor complementarity problem. Comput Optim Appl, 2017, 66(3): 557–576
15. Huang Z H, Suo Y Y, Wang J. On Q-tensors. arXiv: 1509.03088v1
16. Ling L, He H, Ling C. On error bounds of polynomial complementarity problems with structured tensors. Optimization, 2018, 67(2): 341–358
17. Liu D, Li W, Vong S W. Tensor complementarity problems: the GUS-property and an algorithm. Linear Multilinear Algebra, 2018, 66(9): 1726–1749
18. Luo Z, Qi L, Xiu N. The sparsest solutions to Z-tensor complementarity problems. Optim Lett, 2017, 11(3): 471–482
19. Qi L. Symmetric nonnegative tensors and copositive tensors. Linear Algebra Appl, 2013, 439(1): 228–238
20. Song Y, Mei W. Structural properties of tensors and complementarity problems. J Optim Theory Appl, 2018, 176(2): 289–305
21. Song Y, Qi L. Properties of some classes of structured tensors. J Optim Theory Appl, 2015, 165(3): 854–873
22. Song Y, Qi L. Tensor complementarity problem and semi-positive tensors. J Optim Theory Appl, 2016, 169(3): 1069–1078
23. Song Y, Qi L. Strictly semi-positive tensors and the boundedness of tensor complementarity problems. Optim Lett, 2017, 11(7): 1407–1426
24. Song Y, Qi L. Properties of tensor complementarity problem and some classes of structured tensors. Ann Appl Math, 2017, 33(3): 308–323
25. Song Y, Yu G. Properties of solution set of tensor complementarity problem. J Optim Theory Appl, 2016, 170(1): 85–96

26. Tawhid M A, Rahmati, S. Complementarity problems over a hypermatrix (tensor) set. Optim Lett, 2018, 12(6): 1443–1454
27. Wang J, Hu S, Huang Z H. Solution sets of quadratic complementarity problems. J Optim Theory Appl, 2018, 176(1): 120–136
28. Wang X, Chen H, Wang Y. Solution structures of tensor complementarity problem. Front Math China, 2018, 13(4): 935–945
29. Wang Y, Huang Z H, Bai X L. Exceptionally regular tensors and tensor complementarity problems. Optim Methods Softw, 2016, 31(4): 815–828
30. Wang Y, Huang Z H, Qi L. Global uniqueness and solvability of tensor variational inequalities. J Optim Theory Appl, 2018, 177(1): 137–152
31. Xie S L, Li D H, Xu H R. An iterative method for finding the least solution to the tensor complementarity problem. J Optim Theory Appl, 2017, 175(1): 119–136
32. Zhang K, Chen H, Zhao P. A potential reduction method for tensor complementarity problems. J Ind Manag Optim, 2019, 15(2): 429–443
33. Zhang L, Qi L, Zhou G. M-tensors and some applications. SIAM J Matrix Anal Appl, 2014, 35(2): 437–452
34. Zhao X, Fan J. A semidefinite method for tensor complementarity problems. Optim Methods Softw, 2018, DOI: 10.1080/10556788.2018.1439489
35. Zheng M M, Zhang Y, Huang Z H. Global error bounds for the tensor complementarity problem with a P-tensor. J Ind Manag Optim, 2019, 15(2): 933–946