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RESEARCH ARTICLE

Relative homological dimensions in recollements of triangulated categories

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Abstract Let $\mathscr{E}$ be a proper class of triangles in a triangulated category $\mathscr{C}$, and let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of triangulated categories. Based on Beligiannis's work, we prove that $\mathscr{A}$ and $\mathscr{C}$ have enough $\mathscr{E}$-projective objects whenever $\mathscr{B}$ does. Moreover, in this paper, we give the bounds for the $\mathscr{E}$-global dimension of $\mathscr{B}$ in a recollement $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ by controlling the behavior of the $\mathscr{E}$-global dimensions of the triangulated categories $\mathscr{A}$ and $\mathscr{C}$. In particular, we show that the finiteness of the $\mathscr{E}$-global dimensions of triangulated categories is invariant with respect to the recollements of triangulated categories.

Keywords Triangulated category, proper class of triangles, recollement, $\mathscr{E}$-global dimension, derived category
MSC 16E10, 18E30, 18G25

1 Introduction

The triangulated categories were introduced by Verdier [25]. The theory of the triangulated categories is becoming increasingly an important tool for studying many branches of mathematics such as algebraic geometry, stable homotopy theory, and representation theory. Paralleling the homological algebra in an exact category in the sense of Quillen, Beligiannis [7] introduced and investigated a homological theory in the triangulated categories. Let $\mathscr{C}$ be a triangulated category with triangles Δ. By specifying a class of triangles $\mathscr{E} \subseteq \Delta$, called a proper class of triangles, Beligiannis introduced the definitions of the $\mathscr{E}$-projective objects, $\mathscr{E}$-projective and $\mathscr{E}$-global dimension, etc. Similar to homological theory in the module categories, the $\mathscr{E}$-global dimension was used to measure how far away a triangulated category is from being a semi-simple category. Later, this theory has been paid more attentions and developed much further. For details, we refer to [3,4,23].

Recollements of the triangulated categories were introduced by Beilinson

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et al. [6]. They are widely used in algebraic geometry and representation theory, see [1,14,15]. The recollements of abelian categories appeared in the construction of perverse sheaves by MacPherson and Vilonen [16]. Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of triangulated categories. Roughly speaking, we can view a recollement as a short exact sequence of triangulated categories,

$$0 \to \mathscr{A} \xrightarrow{i} \mathscr{B} \xrightarrow{e} \mathscr{C} \to 0,$$

that is, the triangle functor i is fully faithful and e induces a triangle equivalence

$$\mathscr{B}/\mathrm{Im}(i) \cong \mathscr{C}.$$

By decomposing the middle term $\mathscr{B}$ into smaller and possibly concise outer terms, it is convenient for us to investigate some homological properties, such as the homological smoothness, the Gorensteinness, see [10,11,22]. Moreover, the language of recollements provides us useful reduction techniques to calculate homological invariants or deal with some homological conjectures, for example, K-theory, Hochschild homology and cohomology, the finitistic dimension conjecture, the Cartan determinant conjecture, and so on, see [2,8–10,12,22].

As is well known, the properties of recollements also offer an efficacious method in the calculation of the bounds of homological dimensions. Along recollements, Qin [21] gave the bounds of the self-injective dimensions and the ϕ-dimensions of algebras inductively. Using the language of recollements of abelian categories, Psaroudakis [20] provided the bounds of a series of homological dimensions, such as the global dimensions, the representation dimensions, Finitistic dimensions, and the dimensions of the bounded derived categories of algebras.

In contrast with the global dimensions of rings, there are no more results on the behavior of $\mathscr{E}$-global dimensions of the triangulated categories under the recollements. Therefore, in this note, we aim to establish a relation between the $\mathscr{E}$-global dimension and the recollement and give the bounds of $\mathscr{E}$-global dimensions of the triangulated categories.

Now, we present our main result of this paper. For the notations in the following result, we refer to Sections 2 and 3.

Theorem A *Let $\mathscr{R}_{\mathrm{tr}} = (\mathscr{A},\mathscr{B},\mathscr{C},q,i,p,l,e,r)$ be an $\mathscr{X}_{\mathscr{B}}$-invariant recollement such that $\mathscr{X}_{\mathscr{C}} = e(\mathscr{X}_{\mathscr{B}})$ and $\mathscr{X}_{\mathscr{A}} = q(\mathscr{X}_{\mathscr{B}})$. Assume that the triangle*

$$le(X) \to X \to iq(X) \to \Sigma le(X) \tag{1.1}$$

is split for any $X \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{B}}))$. Then

$$\mathscr{E}(\mathscr{X}_{\mathscr{A}})\text{-gl.dim}\,\mathscr{A} < \infty, \quad \mathscr{E}(\mathscr{X}_{\mathscr{C}})\text{-gl.dim}\,\mathscr{C} < \infty,$$

if and only if

$$\mathscr{E}(\mathscr{X}_{\mathscr{B}})\text{-gl.dim}\,\mathscr{B} < \infty.$$

Moreover,

$$\begin{aligned}&\max\{\mathscr{E}(\mathscr{X}_{\mathscr{A}})\text{-gl.dim}\,\mathscr{A}, \mathscr{E}(\mathscr{X}_{\mathscr{C}})\text{-gl.dim}\,\mathscr{C}\}\\ &\quad\leqslant \mathscr{E}(\mathscr{X}_{\mathscr{B}})\text{-gl.dim}\,\mathscr{B}\\ &\quad\leqslant \mathscr{E}(\mathscr{X}_{\mathscr{A}})\text{-gl.dim}\,\mathscr{A}+\mathscr{E}(\mathscr{X}_{\mathscr{C}})\text{-gl.dim}\,\mathscr{C}+1.\end{aligned}$$

This article is organized as follows. In Section 2, we will recall the relevant notations and notions about the homological theory in the triangulated categories. Note that there are no suitable proper classes of triangles defined in the outer terms $\mathscr{A}$ and $\mathscr{C}$ although the middle term $\mathscr{B}$ of a recollement has proper class of triangles. For this purpose, using the subcategory $\mathscr{X}_{\mathscr{B}}$ of $\mathscr{B}$, we construct two suitable subcategories $\mathscr{X}_{\mathscr{A}}$ and $\mathscr{X}_{\mathscr{C}}$ in $\mathscr{A}$ and $\mathscr{C}$ in Section 3, respectively, such that all of them have enough $\mathscr{E}$-projective objects, see Proposition 2 below. The main results are proved in Section 4. In Section 5, we apply our main result to the derived categories and list some examples.

Some unexplained notations and terminologies can be referred to [3,4,7,20, 23].

2 Preliminaries

Throughout this paper, $\mathscr{C}=(\mathscr{C},\Sigma,\Delta)$ always denotes a triangulated category, where $\mathscr{C}$ is an additive category, $\Sigma\colon \mathscr{C}\to\mathscr{C}$ is an automorphism of $\mathscr{C}$ called a suspension functor, and Δ is a class of diagrams in $\mathscr{C}$ of the form

$$A\to B\to C\to \Sigma A,$$

called a (distinguished) triangle, satisfying axioms (Tr1)–(Tr4), see [25, Chapitre II, Définition 1.1] or [18, Chapter 1]. (Tr4) is also said to be the *octahedral axiom.*

Proposition 1 ([7, Section 2], [17]) *If the triple* $\mathscr{C}=(\mathscr{C},\Sigma,\Delta)$ *satisfies all the axioms of a triangulated category except possibly of* (Tr4)*, then the octahedral axiom is equivalent to each of the following.*

(i) **Base Change**. *For any triangle* $A\xrightarrow{f}B\xrightarrow{g}C\xrightarrow{h}\Sigma A\in\Delta$ *and any morphism* $\varepsilon\colon E\to C$*, there exists a commutative diagram*

$$\begin{array}{ccccccc}
0 & \longrightarrow & M & = & M & \longrightarrow & 0\\
\downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\delta} & & \downarrow\\
A & \xrightarrow{f'} & G & \xrightarrow{g'} & E & \xrightarrow{h'} & \Sigma A\\
\| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\varepsilon} & & \|\\
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A\\
\downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\xi} & & \downarrow\\
0 & \longrightarrow & \Sigma M & = & \Sigma M & \longrightarrow & 0
\end{array} \tag{2.1}$$

in which all horizontal and vertical are triangles in Δ.

(ii) **Cobase Change**. *For any triangle* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \in \Delta$ *and any morphism* $\alpha\colon A \to D$, *there exists a commutative diagram*

$$\begin{array}{ccccccc}
0 & \longrightarrow & N & = & N & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \delta} & & \downarrow \\
\Sigma^{-1}(C) & \xrightarrow{-\Sigma^{-1}(h)} & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\| & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \| \\
\Sigma^{-1}(C) & \xrightarrow{-\Sigma^{-1}(h')} & D & \xrightarrow{f'} & F & \xrightarrow{g'} & C \\
\downarrow & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \vartheta} & & \downarrow \\
0 & \longrightarrow & \Sigma N & = & \Sigma N & \longrightarrow & 0
\end{array} \tag{2.2}$$

in which all horizontal and vertical are triangles in Δ.

Definition 1 [7, Section 2] Let $\mathscr{C}$ be a triangulated category. Suppose that a class $\mathscr{E}$ of triangles is contained in Δ.

(1) $\mathscr{E}$ is *closed under base change* if for any triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \in \mathscr{E}$ and any morphism $\varepsilon\colon E \to C$ as in diagram (2.1), the triangle $A \xrightarrow{f'} G \xrightarrow{g'} E \xrightarrow{h'} \Sigma A \in \mathscr{E}$.

(2) $\mathscr{E}$ is *closed under cobase change* if for any triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \in \mathscr{E}$ and any morphism $\alpha\colon A \to D$ as in diagram (2.2), the triangle $D \xrightarrow{f'} F \xrightarrow{g'} C \xrightarrow{h'} \Sigma D \in \mathscr{E}$.

(3) $\mathscr{E}$ is *closed under suspension* if for any triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \in \mathscr{E}$ and any $i \in \mathbb{Z}$, the triangle $\Sigma^i A \xrightarrow{(-1)^i\Sigma^i f} \Sigma^i B \xrightarrow{(-1)^i\Sigma^i g} \Sigma^i C \xrightarrow{(-1)^i\Sigma^i h} \Sigma^{i+1} A \in \mathscr{E}$.

(4) $\mathscr{E}$ is *saturated* if in diagram (2.1), the third vertical and the second horizontal triangles are in $\mathscr{E}$, then the triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \in \mathscr{E}$.

A triangle $\mathscr{T}\colon A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ is said to be *split* if $h = 0$. Moreover, if $\mathscr{T}$ is split, then $B \cong A \oplus C$. The subclass of Δ consisting of the split triangles will be denoted by Δ_0.

Definition 2 [7, Section 2] A class $\mathscr{E} \subseteq \Delta$ is called a *proper class of triangles* if the following conditions hold:

(1) $\mathscr{E}$ is closed under isomorphisms, finite coproducts, and $\Delta_0 \subseteq \mathscr{E} \subseteq \Delta$;

(2) $\mathscr{E}$ is closed under suspensions and is saturated;

(3) $\mathscr{E}$ is closed under base change and cobase change.

Example 1 [7, Example 2.3 (4)] If $\mathscr{X} \subseteq \mathscr{C}$ is a class of objects satisfying $\Sigma\mathscr{X} = \mathscr{X}$, then there is a proper class $\mathscr{E}(\mathscr{X})$ of triangles in $\mathscr{C}$, as follows.

A triangle $A \to B \to C \to \Sigma A$ lies in $\mathscr{E}(\mathscr{X})$ if and only if the induced sequence

$$0 \to \mathrm{Hom}_{\mathscr{C}}(X, A) \to \mathrm{Hom}_{\mathscr{C}}(X, B) \to \mathrm{Hom}_{\mathscr{C}}(X, C) \to 0$$

is exact in $\mathscr{A}b$ for any $X \in \mathscr{X}$.

Definition 3 [7, Definition 4.1] Let $\mathscr{C}$ be a triangulated category, and let $\mathscr{E}$ be a proper class of triangles in $\mathscr{C}$. An object $P \in \mathscr{C}$ is called *$\mathscr{E}$-projective*, if for any triangle $A \to B \to C \to \Sigma A \in \mathscr{E}$, the induced sequence

$$0 \to \mathrm{Hom}_{\mathscr{C}}(P, A) \to \mathrm{Hom}_{\mathscr{C}}(P, B) \to \mathrm{Hom}_{\mathscr{C}}(P, C) \to 0$$

is exact in $\mathscr{A}b$.

By the definition, we may conclude that the full subcategory $\mathscr{P}(\mathscr{E})$ consisting of all $\mathscr{E}$-projective objects in $\mathscr{C}$ is full, additive, closed under isomorphisms and direct summands, and Σ-stable (i.e., $\Sigma\mathscr{P}(\mathscr{E}) = \mathscr{P}(\mathscr{E})$). Recall from [7] that the triangulated category $\mathscr{C}$ *has enough $\mathscr{E}$-projective objects* if for any object A in $\mathscr{C}$, there exists a triangle $K \to P \to A \to \Sigma K \in \mathscr{E}$ with $P \in \mathscr{P}(\mathscr{E})$. Assume that $\mathscr{C}$ has enough $\mathscr{E}$-projective objects. Then, from [7, Lemma 4.2], a triangle $D \to E \to F \to \Sigma D \in \mathscr{E}$ if and only if the induced sequence

$$0 \to \mathrm{Hom}_{\mathscr{C}}(P, D) \to \mathrm{Hom}_{\mathscr{C}}(P, E) \to \mathrm{Hom}_{\mathscr{C}}(P, F) \to 0$$

is exact in $\mathscr{A}b$ for any $P \in \mathscr{P}(\mathscr{E})$.

Recall that an *$\mathscr{E}$-exact complex* $X^\bullet \to A$ with $A \in \mathscr{C}$ is a sequence

$$\cdots \to X^{n+1} \xrightarrow{d^{n+1}} X^n \to \cdots X^1 \to X^0 \xrightarrow{d^0} A \to 0$$

in $\mathscr{C}$, such that for any $n \geqslant 0$, there are triangles $K^{n+1} \xrightarrow{g^n} X^n \xrightarrow{f^n} K^n \xrightarrow{h^n} \Sigma K^{n+1} \in \mathscr{E}$ and the differential $d^n = g^{n-1} f^n$, where $K_0 = A$ and $d^0 = f^0$. If every component X^n of $X^\bullet$ is an $\mathscr{E}$-projective object in $\mathscr{C}$, then the $\mathscr{E}$-exact complex $X^\bullet \to A$ is said to be an *$\mathscr{E}$-projective resolution* of A. We call the object K_1 in the $\mathscr{E}$-projective resolution of A a *first $\mathscr{E}$-syzygy* of A. An *n-th $\mathscr{E}$-syzygy* of A is defined by induction.

Definition 4 [7] Let $\mathscr{C}$ be a triangulated category, and let $\mathscr{E}$ be a proper class of triangles in $\mathscr{C}$. The *$\mathscr{E}$-projective dimension* $\mathscr{E}\text{-pd}\, A$ of an object $A \in \mathscr{C}$ is defined inductively as follows.

(1) If $A \in \mathscr{P}(\mathscr{E})$, then $\mathscr{E}\text{-pd}\, A = 0$.

(2) Assume that $\mathscr{E}\text{-pd}\, A > 0$. $\mathscr{E}\text{-pd}\, A \leqslant n$ if there exists a triangle $K \to P \to A \to \Sigma K \in \mathscr{E}$ with $P \in \mathscr{P}(\mathscr{E})$ and $\mathscr{E}\text{-pd}\, K \leqslant n-1$.

(3) $\mathscr{E}\text{-pd}\, A = n$ if $\mathscr{E}\text{-pd}\, A \leqslant n$ and $\mathscr{E}\text{-pd}\, A \nleqslant n-1$. $\mathscr{E}\text{-pd}\, A = \infty$ if $\mathscr{E}\text{-pd}\, A \neq n$ for all $n \geqslant 0$.

(4) The $\mathscr{E}$-global dimension $\mathscr{E}\text{-gl.dim}\,\mathscr{C}$ of $\mathscr{C}$ is $\mathscr{E}\text{-gl.dim}\,\mathscr{C} = \sup\{\mathscr{E}\text{-pd}\, A \mid A \in \mathscr{C}\}$.

It is well known that $\mathscr{E}\text{-pd}\, A \leqslant n$ if and only if there is a finite $\mathscr{E}$-projective resolution

$$0 \to P^n \to P^{n-1} \to \cdots \to P^1 \to P^0 \to A \to 0,$$

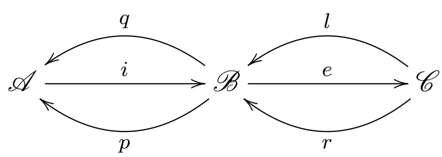
where P^i is an $\mathscr{E}$-projective object for $i = 0, 1, \ldots, n$. More results about the $\mathscr{E}$-projective objects and $\mathscr{E}$-global dimension of $\mathscr{C}$ may be referred to [7].

In general, it is not easy to find a proper class $\mathscr{E}$ of triangles in a triangulated category $\mathscr{C}$ such that $\mathscr{C}$ has enough $\mathscr{E}$-projective objects. Thanks to Beligiannis, we list one of these examples which will be used frequently in what follows.

Example 2 [7, Lemma 8.1] Let $\mathscr{C}$ be a triangulated category which admits infinite coproducts, and let $\mathscr{X}$ be a full subcategory of $\mathscr{C}$ which is closed under suspensions and contains only a set of isomorphism classes of objects. Then $\mathscr{X}$ induces a proper class $\mathscr{E}$ of triangles in $\mathscr{C}$, see Example 1 and $\mathscr{C}$ has enough $\mathscr{E}$-projective objects.

Now, we recall the definition of recollement of triangulated categories.

Definition 5 [6] Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be triangulated categories. $\mathscr{B}$ is said to be a *recollement* of $\mathscr{A}$ and $\mathscr{C}$ if there are six triangle functors as in the diagram



such that

(1) (q, i), (i, p), and (l, e), (e, r) are adjoint pairs;

(2) i, l, and r are fully faithful functors;

(3) $ei = 0$ (and thus, also $ql = 0$ and $pr = 0$);

(4) for each $X \in \mathscr{B}$, there are triangles

$$le(X) \to X \to iq(X) \to \Sigma le(X)$$

and

$$ip(X) \to X \to re(X) \to \Sigma ip(X)$$

in $\mathscr{B}$, where the arrows to and from X are the counit and the unit morphisms, respectively.

In this paper, we always assume that all the triangulated categories are not trivial and admit coproducts unless stated otherwise. The recollement in Definition 5 is denoted simply by $(\mathscr{A}, \mathscr{B}, \mathscr{C})$. In what follows, all modules are right modules. We denote by $\operatorname{Mod} R$ the category of right R-modules, where R is a ring with unit.

3 $\mathscr{E}$-Projective objects in recollments

In this section, we aim to give the definition of the invariant recollement of triangulated categories and study the $\mathscr{E}$-projective objects in a recollement. We fix some notations as follows. Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of triangulated categories, and let $\mathscr{X}_{\mathscr{B}}$ always be a full, additive subcategory of $\mathscr{B}$ and closed under isomorphisms, direct summands. For triangle functors $q\colon \mathscr{B} \to \mathscr{A}$ and $i\colon \mathscr{A} \to \mathscr{B}$, there is a full, additive subcategory of $\mathscr{A}$ as follows:

$$\mathscr{X}_{\mathscr{A}} = \{q(X) \mid iq(X) \in \mathscr{X}_{\mathscr{B}},\ X \in \mathscr{X}_{\mathscr{B}}\}.$$

Similarly, we obtain a full, additive subcategory of $\mathscr{C}$:

$$\mathscr{X}_{\mathscr{C}} = \{e(X) \mid le(X) \in \mathscr{X}_{\mathscr{B}},\ X \in \mathscr{X}_{\mathscr{B}}\},$$

associated with the functors $l\colon \mathscr{C} \to \mathscr{B}$ and $e\colon \mathscr{B} \to \mathscr{C}$.

Definition 6 Let $\mathscr{R}_{\text{tr}} = (\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of triangulated categories. Assume that $\mathscr{X}_{\mathscr{B}} \neq 0$ is a full, additive subcategory of $\mathscr{B}$. $\mathscr{R}_{\text{tr}}$ is said to be a *left $\mathscr{X}_{\mathscr{B}}$-invariant recollement* if the subcategory $\mathscr{X}_{\mathscr{A}} \neq 0$. $\mathscr{R}_{\text{tr}}$ is said to be a *right $\mathscr{X}_{\mathscr{B}}$-invariant recollement* if the subcategory $\mathscr{X}_{\mathscr{C}} \neq 0$. $\mathscr{R}_{\text{tr}}$ is *$\mathscr{X}_{\mathscr{B}}$-invariant* if it is both left and right $\mathscr{X}_{\mathscr{B}}$-invariant.

Remark 1 If the subcategory $\mathscr{X}_{\mathscr{A}} \neq 0$ (or, $\mathscr{X}_{\mathscr{C}} \neq 0$), it means that there is an object $X \in \mathscr{B}$ such that $iq(X)$ (or, $le(X)$) still falls into $\mathscr{B}$. So, we call it 'invariant'.

Example 3 Let $\mathscr{R}_{\text{tr}} = (\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of triangulated categories. If one takes $\mathscr{X}_{\mathscr{B}} = \operatorname{Im}(i)$, then, clearly, $\mathscr{X}_{\mathscr{A}} = \mathscr{A}$ and $\mathscr{X}_{\mathscr{C}} = 0$, since $\operatorname{Im}(i) = \operatorname{Ker}(e)$ and i is a fully faithful functor. Thus, $\mathscr{R}_{\text{tr}}$ is left $\mathscr{X}_{\mathscr{B}}$-invariant. If one takes $\mathscr{X}_{\mathscr{B}} = \operatorname{Im}(l)$, then, clearly, $\mathscr{X}_{\mathscr{A}} = 0$ and $\mathscr{X}_{\mathscr{C}} = \mathscr{C}$, since $\operatorname{Im}(l) = \operatorname{Ker}(q)$ and l is a fully faithful functor. In this case, $\mathscr{R}_{\text{tr}}$ is right $\mathscr{X}_{\mathscr{B}}$-invariant.

Recall from [19, Definition 4.3.1] that a triangulated category $\mathscr{D}$ is *generated* by a class $\mathscr{Q}$ of objects in $\mathscr{D}$, that is, an object M of $\mathscr{D}$ is zero whenever $\operatorname{Hom}_{\mathscr{D}}(\Sigma^n Q, M) = 0$ for every object Q of $\mathscr{Q}$ and every $n \in \mathbb{Z}$. Note that it coincides with the definition as in [24, Definition 5.2]. In this case, the objects of $\mathscr{Q}$ are said to be *generators*. P is a *compact object* in $\mathscr{D}$ if the functor $\operatorname{Hom}_{\mathscr{D}}(P, \text{-})$ commutes with coprduct in $\mathscr{D}$. For example, the algebra A is a compact generator in the derived category $D(\operatorname{Mod} A)$.

Example 4 Let

$$\mathscr{R}_{\text{tr}} = (D^b(\operatorname{Mod} A), D^b(\operatorname{Mod} B), D^b(\operatorname{Mod} C))$$

be a recollement of bounded derived categories with respect to Artin algebras A, B, and C. Let $\mathscr{X}_{\mathscr{B}} = K^b(\mathscr{P}(B))$ be the homotopy category of bounded complexes of projective modules. For any $P^\bullet \in K^b(\mathscr{P}(B))$, we have $e(P^\bullet) \in K^b(\mathscr{P}(C))$, $q(P^\bullet) \in K^b(\mathscr{P}(A))$ by [5, Lemma 1.2.1] or [26, Lemma 3.1]. Moreover, since (l, e) and (q, i) are adjoint pairs, we obtain that $le(P^\bullet)$ and $iq(P^\bullet)$ are in $K^b(\mathscr{P}(B))$. Note that $B \in K^b(\mathscr{P}(B))$, $q(B)$, and $e(B)$ are compact generators in $D^b(\operatorname{Mod} B)$, $D^b(\operatorname{Mod} A)$, and $D^b(\operatorname{Mod} C)$, respectively. Hence, $\mathscr{X}_{\mathscr{A}} \neq 0$ and $\mathscr{X}_{\mathscr{C}} \neq 0$. It follows that $\mathscr{R}_{\text{tr}}$ is $K^b(\mathscr{P}(B))$-invariant.

Recall that a *skeletally small category* is one in which the collection of isomorphism classes of objects is a set. Now, we assume that $\mathscr{X}_{\mathscr{B}}$ is a full skeletally small, additive subcategory of triangulated category $\mathscr{B}$ and closed under isomorphisms, direct summands, and Σ-stable. It is not difficult to verify that $\mathscr{X}_{\mathscr{A}}$ and $\mathscr{X}_{\mathscr{C}}$ are also skeletally small, closed under isomorphisms, and Σ-stable. Moreover, if $\mathscr{X}_{\mathscr{C}} = e(\mathscr{X}_{\mathscr{B}})$ and $\mathscr{X}_{\mathscr{A}} = q(\mathscr{X}_{\mathscr{B}})$, then $\mathscr{X}_{\mathscr{A}}$ and $\mathscr{X}_{\mathscr{C}}$ are closed under direct summands. In fact, these conditions can be realized, see Proposition 6 or Example 7 below, and are important to the proof of our main result in Section 4.

Example 5 [7,13] Let R be a ring with unit, and let $D(\operatorname{Mod} R)$ be the unbounded derived category of $\operatorname{Mod} R$. Then $K^b(\operatorname{proj} R)$ (the homotopy category of bounded complexes over finitely generated projective R-modules) and $\mathscr{P}^b_{CE}(\operatorname{mod} R)$ (all bounded Cartan-Eilenberg projective complexes which are homotopy equivalent to complexes having finitely generated projective components and zero differentials) are full skeletally small, additive subcategories of triangulated category $D(\operatorname{Mod} R)$ and closed under isomorphisms, direct summands, and Σ-stable.

Let $\mathscr{C}$ be a triangulated category, and let $\mathscr{E}$ be a proper class of triangles in $\mathscr{C}$. If the triangle $A \overset{f}{\to} B \overset{g}{\to} C \overset{h}{\to} \Sigma A \in \mathscr{E}$, then the morphism $h\colon C \to \Sigma A$ is called an *$\mathscr{E}$-phantom map*. We denote by $Ph_{\mathscr{E}}(\mathscr{C})$ the class of $\mathscr{E}$-phantom map.

Let $\mathscr{X}$ be a class of objects of a triangulated category $\mathscr{C}$ satisfying $\Sigma\mathscr{X} = \mathscr{X}$. Then there is a proper class $\mathscr{E}(\mathscr{X})$ of triangles, which is described as in Example 1.

Recall that the *Jacobson radical* $\mathscr{J}(\mathscr{C})$ of an additive category $\mathscr{C}$ is the ideal in $\mathscr{C}$ defined by

$$\mathscr{J}(\mathscr{C})(A,B) = \{f\colon A \to B \mid \forall\, g\colon B \to A, \text{ morphism } id_A - gf \text{ is invertible}\}.$$

Definition 7 [7] A proper class of triangles $\mathscr{E}$ in $\mathscr{C}$ *projectively generates* $\mathscr{C}$, if $Ph_{\mathscr{E}}(\mathscr{C}) \subseteq \mathscr{J}(\mathscr{C})$ and $\mathscr{C}$ has enough $\mathscr{E}$-projectives.

Now, we can describe the projective objects in the triangulated categories in the recollement.

Proposition 2 *Let $\mathscr{R}_{\rm tr} = (\mathscr{A}, \mathscr{B}, \mathscr{C})$ be an $\mathscr{X}_{\mathscr{B}}$-invariant recollement such that $\mathscr{X}_{\mathscr{C}} = e(\mathscr{X}_{\mathscr{B}})$ and $\mathscr{X}_{\mathscr{A}} = q(\mathscr{X}_{\mathscr{B}})$. Then*

(1) *$\mathscr{B}$ has enough $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-projective objects and $\mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{B}})) = \operatorname{Add} \mathscr{X}_{\mathscr{B}}$;*

(2) *$\mathscr{A}$ has enough $\mathscr{E}(\mathscr{X}_{\mathscr{A}})$-projective objects and $\mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{A}})) = \operatorname{Add} \mathscr{X}_{\mathscr{A}}$;*

(3) *$\mathscr{C}$ has enough $\mathscr{E}(\mathscr{X}_{\mathscr{C}})$-projective objects and $\mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{C}})) = \operatorname{Add} \mathscr{X}_{\mathscr{C}}$.*

Moreover, if $\mathscr{X}_{\mathscr{B}}$ generates $\mathscr{B}$, then $\mathscr{E}(\mathscr{X}_{\mathscr{A}})$, $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$, and $\mathscr{E}(\mathscr{X}_{\mathscr{C}})$ projectively generate $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$, respectively.

Proof Statements (1)–(3) follow from [7, Lemma 8.1]. By [7, Lemma 8.2], it remains to show that $\mathscr{E}(\mathscr{X}_{\mathscr{A}})$ generates $\mathscr{A}$ and $\mathscr{E}(\mathscr{X}_{\mathscr{C}})$ generates $\mathscr{C}$. Indeed, assume that $A \in \mathscr{A}$ such that $\mathrm{Hom}_{\mathscr{A}}(X, A) = 0$ for any $X \in \mathscr{X}_{\mathscr{A}}$. Since (q, i) is an adjoint pair, we have

$$\mathrm{Hom}_{\mathscr{B}}(Y, i(A)) \cong \mathrm{Hom}_{\mathscr{A}}(q(Y), A), \quad \forall\, Y \in \mathscr{X}_{\mathscr{B}}.$$

By the assumption on $\mathscr{X}_{\mathscr{B}}$ and $\mathscr{X}_{\mathscr{A}} = q(\mathscr{X}_{\mathscr{B}})$, it yields that $i(A) = 0$ and so, $A = 0$ since i is a fully faithful functor. Now, assume that $C \in \mathscr{C}$ such that $\mathrm{Hom}_{\mathscr{C}}(X, C) = 0$ for any $X \in \mathscr{X}_{\mathscr{C}}$. Since (e, r) is an adjoint pair, we have

$$\mathrm{Hom}_{\mathscr{B}}(Y, r(C)) \cong \mathrm{Hom}_{\mathscr{C}}(e(Y), C), \quad \forall\, Y \in \mathscr{X}_{\mathscr{B}}.$$

Hence, it implies that $r(C) = 0$ and so, $C = 0$ because r is a fully faithful functor. The results come from [7, Lemma 8.2(2)]. □

Remark 2 Indeed, under the above conditions, all of the triangulated categories $\mathscr{A}, \mathscr{B}$, and $\mathscr{C}$ have enough $\mathscr{E}$-injective objects by [7, Theorem 8.6].

Lemma 1 *Let $\mathscr{R}_{\mathrm{tr}} = (\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement. Then the following results hold*:

(1) *if $\mathscr{R}_{\mathrm{tr}}$ is left $\mathscr{X}_{\mathscr{B}}$-invariant such that $\mathscr{X}_{\mathscr{A}} = q(\mathscr{X}_{\mathscr{B}})$, then the functor i sends all triangles in $\mathscr{E}(\mathscr{X}_{\mathscr{A}})$ to $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$*;

(2) *if $\mathscr{R}_{\mathrm{tr}}$ is right $\mathscr{X}_{\mathscr{B}}$-invariant such that $\mathscr{X}_{\mathscr{C}} = e(\mathscr{X}_{\mathscr{B}})$, then the functor e sends all triangles in $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$ to $\mathscr{E}(\mathscr{X}_{\mathscr{C}})$.*

Proof (1) For any triangle $A \to B \to C \to \Sigma A$ in $\mathscr{E}(\mathscr{X}_{\mathscr{A}})$, it suffices to show that for any $X \in \mathscr{X}_{\mathscr{B}}$, the induced sequence

$$0 \to \mathrm{Hom}_{\mathscr{B}}(X, i(A)) \to \mathrm{Hom}_{\mathscr{B}}(X, i(B)) \to \mathrm{Hom}_{\mathscr{B}}(X, i(C)) \to 0$$

is exact. Since $\mathscr{X}_{\mathscr{A}} = q(\mathscr{X}_{\mathscr{B}})$, the induced sequence

$$0 \to \mathrm{Hom}_{\mathscr{B}}(q(X), A) \to \mathrm{Hom}_{\mathscr{B}}(q(X), B) \to \mathrm{Hom}_{\mathscr{B}}(q(X), C) \to 0$$

is exact. The result comes from the fact that (q, i) is an adjoint pair.

(2) The proof is similar to (1). □

Proposition 3 *Let $\mathscr{R}_{\mathrm{tr}} = (\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement. Then the following results hold*:

(1) *if $\mathscr{R}_{\mathrm{tr}}$ is a left $\mathscr{X}_{\mathscr{B}}$-invariant recollement such that $\mathscr{X}_{\mathscr{A}} = q(\mathscr{X}_{\mathscr{B}})$, then*

$$i(P^\bullet)\colon \cdots \to i(P^{n+1}) \to i(P^n) \to \cdots \to i(P^1) \to i(P^0) \to i(A) \to 0$$

is an $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-projective resolution of $i(A)$ in $\mathscr{B}$, where

$$P^\bullet\colon \cdots \to P^{n+1} \to P^n \to \cdots \to P^1 \to P^0 \to A \to 0$$

is an $\mathscr{E}(\mathscr{X}_{\mathscr{A}})$-projective resolution of A in $\mathscr{A}$;

(2) *if $\mathscr{R}_{\mathrm{tr}}$ is a right $\mathscr{X}_{\mathscr{B}}$-invariant recollement such that $\mathscr{X}_{\mathscr{C}} = e(\mathscr{X}_{\mathscr{B}})$, then*

$$e(Q^\bullet)\colon \cdots \to e(Q^{n+1}) \to e(Q^n) \to \cdots \to e(Q^1) \to e(Q^0) \to e(B) \to 0$$

is an $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$*-projective resolution of* $e(B)$ *in* $\mathscr{C}$*, where*

$$Q^{\bullet}\colon\ \cdots \to Q^{n+1} \to Q^{n} \to \cdots \to Q^{1} \to Q^{0} \to B \to 0$$

is an $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$*-projective resolution of* B *in* $\mathscr{B}$*.*

Proof Since (i, p) and (e, r) are adjoint pairs, i and e preserve coproducts. Thus, for any $P \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{A}})) = \mathrm{Add}\,\mathscr{X}_{\mathscr{A}}$ and $Q \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{B}})) = \mathrm{Add}\,\mathscr{X}_{\mathscr{B}}$, we have $i(P) \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{B}}))$ and $e(Q) \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{C}}))$. By Lemma 1, we have the results. □

Corollary 1 *Let* $\mathscr{R}_{\mathrm{tr}} = (\mathscr{A}, \mathscr{B}, \mathscr{C})$ *be a left* $\mathscr{X}_{\mathscr{B}}$*-invariant recollement such that* $\mathscr{X}_{\mathscr{A}} = q(\mathscr{X}_{\mathscr{B}})$*. If* $\mathscr{E}(\mathscr{X}_{\mathscr{A}})$-gl.dim $\mathscr{A} < \infty$*, then*

$$\mathscr{E}(\mathscr{X}_{\mathscr{B}})\text{-pd}(iq(M)) \leqslant \mathscr{E}(\mathscr{X}_{\mathscr{A}})\text{-gl.dim}\,\mathscr{A}, \quad \forall\, M \in \mathscr{B}.$$

Proof Assume that $\mathscr{E}(\mathscr{X}_{\mathscr{A}})$-gl.dim $\mathscr{A} = n$. It is clearly holds for $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-projective objects. For any non-$\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-projective object $M \in \mathscr{B}$, we claim that

$$\mathscr{E}(\mathscr{X}_{\mathscr{B}})\text{-pd}(iq(M)) \leqslant n. \tag{3.1}$$

Note that $q(M) \in \mathscr{A}$. We have a finite $\mathscr{E}(\mathscr{X}_{\mathscr{A}})$-projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to q(M) \to 0$$

with each $P_i \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{A}}))$. By Proposition 3 (1), we have a finite $\mathscr{E}(\mathscr{X}_{\mathscr{A}})$-projective resolution

$$0 \to i(P_n) \to i(P_{n-1}) \to \cdots \to i(P_1) \to i(P_0) \to iq(M) \to 0$$

with each $i(P_i) \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{B}}))$. Therefore, (3.1) holds. □

4 Proof of main result

Keep the definitions of $\mathscr{X}_{\mathscr{C}}$ and $\mathscr{X}_{\mathscr{A}}$ in Section 3. In this section, we give the proof of the main result (Theorem A), which is a combination of Theorems 1 and 2 below. Assume that $\mathscr{X}_{\mathscr{B}} \neq 0$ is a full skeletally small, additive subcategory of the triangulated category $\mathscr{B}$ and closed under isomorphisms, direct summands, and Σ-stable. It is well known that Schanuel's Lemma plays an important role in the theory of homological dimension in the categories of modules. In order to discuss the homological dimension in the triangulated categories, we need the relative version of Schanuel's Lemma.

Lemma 2 *Let* $n \geqslant 0$*, and let*

$$0 \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

and

$$0 \to G_{n+1} \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$$

be $\mathscr{E}$-exact complexes with $P_i, G_i \in \mathscr{P}(\mathscr{E})$ for $0 \leqslant i \leqslant n$. Then

$$P_{n+1} \oplus G_n \oplus P_{n-1} \oplus \cdots \cong G_{n+1} \oplus P_n \oplus G_{n-1} \oplus \cdots .$$

Proof By induction on n. For $n = 0$, by the definition of $\mathscr{E}$-exact complexes, the triangles $P_1 \to P_0 \to A \to \Sigma P_1$ and $G_1 \to G_0 \to A \to \Sigma G_1$ are in $\mathscr{E}$. Since $P_0, G_0 \in \mathscr{P}(\mathscr{E})$, we have

$$G_1 \oplus P_0 \cong P_1 \oplus G_0$$

by [7, Proposition 4.4]. Now, we assume that $n > 0$. Since the triangles $K_2 \to P_1 \oplus G_0 \to K_1 \oplus G_0 \to \Sigma K_2$ and $L_2 \to G_1 \oplus P_0 \to L_1 \oplus P_0 \to \Sigma L_2$ are in $\mathscr{E}$, we have the $\mathscr{E}$-exact complexes

$$0 \to P_{n+1} \to P_n \to \cdots \to P_1 \oplus G_0 \to K_1 \oplus G_0 \to 0$$

and

$$0 \to G_{n+1} \to G_n \to \cdots \to G_2 \to G_1 \oplus P_0 \to L_1 \oplus P_0 \to 0,$$

where L_2, L_1 and K_2, K_1 are $\mathscr{E}$-syzygies of A. Observing that $K_1 \oplus G_0 \cong L_1 \oplus P_0$ by the above discussion, we obtain the result by induction hypothesis. □

Theorem 1 *Let $\mathscr{R}_{\rm tr} = (\mathscr{A}, \mathscr{B}, \mathscr{C})$ be an $\mathscr{X}_{\mathscr{B}}$-invariant recollement. Assume $\mathscr{X}_{\mathscr{C}} = e(\mathscr{X}_{\mathscr{B}})$ and $\mathscr{X}_{\mathscr{A}} = q(\mathscr{X}_{\mathscr{B}})$. If $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-gl.dim $\mathscr{B} < \infty$, then*

$$\mathscr{E}(\mathscr{X}_{\mathscr{A}})\text{-gl.dim}\,\mathscr{A} < \infty, \quad \mathscr{E}(\mathscr{X}_{\mathscr{C}})\text{-gl.dim}\,\mathscr{C} < \infty.$$

Moreover,

$$\max\{\mathscr{E}(\mathscr{X}_{\mathscr{A}})\text{-gl.dim}\,\mathscr{A}, \mathscr{E}(\mathscr{X}_{\mathscr{C}})\text{-gl.dim}\,\mathscr{C}\} \leqslant \mathscr{E}(\mathscr{X}_{\mathscr{B}})\text{-gl.dim}\,\mathscr{B}.$$

Proof Assume $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-gl.dim $\mathscr{B} = n$. For each object $A \in \mathscr{A}$, we have $i(A) \in \mathscr{B}$ and thus, there exists a finite $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to i(A) \to 0.$$

On the other hand, since $\mathscr{A}$ has enough $\mathscr{E}(\mathscr{X}_{\mathscr{A}})$-projective objects, there exists an $\mathscr{E}(\mathscr{X}_{\mathscr{A}})$-exact complex

$$0 \to K_n \to Q_{n-1} \to \cdots \to Q_1 \to Q_0 \to A \to 0$$

with $Q_i \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{A}}))$. By Lemma 1 (1), we have an $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-exact complex

$$0 \to i(K_n) \to i(Q_{n-1}) \to \cdots \to i(Q_1) \to i(Q_0) \to i(A) \to 0$$

with each $i(Q_j) \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{B}}))$. From Lemma 2, we learn that

$$i(K_n) \oplus P_{n-1} \oplus i(Q_{n-2}) \oplus \cdots \cong P_n \oplus i(Q_{n-1}) \oplus P_{n-2} \oplus \cdots \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{B}})).$$

Thus, $i(K_n) \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{B}}))$ and so $K_n \cong qi(K_n) \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{A}}))$. Therefore, $\mathscr{E}(\mathscr{X}_{\mathscr{A}})$-pd $A \leqslant n$ and so, $\mathscr{E}(\mathscr{X}_{\mathscr{A}})$-gl.dim $\mathscr{A} \leqslant n$.

For any $C \in \mathscr{A}$, since $l(C) \in \mathscr{B}$, there is a finite $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-projective resolution

$$0 \to Q_n \to Q_{n-1} \to \cdots \to Q_1 \to Q_0 \to l(C) \to 0,$$

where $Q_i \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{B}}))$. By Proposition 3 (2), $\mathscr{E}(\mathscr{X}_{\mathscr{C}})$-pd $C \leqslant n$ and hence, $\mathscr{E}(\mathscr{X}_{\mathscr{C}})$-gl.dim $\mathscr{C} \leqslant n$. □

If we weaken the conditions in Theorem 1, then there are two results as follows.

Proposition 4 *Let $\mathscr{R}_{\rm tr} = (\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a left $\mathscr{X}_{\mathscr{B}}$-invariant recollement. If $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-gl.dim $\mathscr{B} < \infty$, then the following statements are equivalent*:

(1) $\mathscr{E}(\mathscr{X}_{\mathscr{A}})$-gl.dim $\mathscr{A} \leqslant \mathscr{E}(\mathscr{X}_{\mathscr{B}})$-gl.dim $\mathscr{B}$.

(2) $\mathscr{E}(\mathscr{X}_{\mathscr{A}})$-pd$(q(X)) \leqslant \mathscr{E}(\mathscr{X}_{\mathscr{B}})$-gl.dim $\mathscr{B}$ *for any object* $X \in \mathscr{B}$.

Proof (1) ⇒ (2) It is obvious.

(2) ⇒ (1) Assume that $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-gl.dim $\mathscr{B} = n$. For any $A \in \mathscr{A}$, we have $i(A) \in \mathscr{B}$. Note that (q,i) is an adjoint pair and the functor i is fully faith. It follows from assumption (2) that $\mathscr{E}(\mathscr{X}_{\mathscr{A}})$-pd $A \leqslant n$ and so $\mathscr{E}(\mathscr{X}_{\mathscr{A}})$-gl.dim $\mathscr{A} \leqslant n$. □

Proposition 5 *Let $\mathscr{R}_{\rm tr} = (\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a right $\mathscr{X}_{\mathscr{B}}$-invariant recollement. If $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-gl.dim $\mathscr{B} < \infty$, then the following statements are equivalent*:

(1) $\mathscr{E}(\mathscr{X}_{\mathscr{C}})$-gl.dim $\mathscr{C} \leqslant \mathscr{E}(\mathscr{X}_{\mathscr{B}})$-gl.dim $\mathscr{B}$.

(2) $\mathscr{E}(\mathscr{X}_{\mathscr{C}})$-pd$(e(X)) \leqslant \mathscr{E}(\mathscr{X}_{\mathscr{B}})$-gl.dim $\mathscr{B}$ *for any object* $X \in \mathscr{B}$.

Proof (1) ⇒ (2) It is obvious.

(2) ⇒ (1) Assume that $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-gl.dim $\mathscr{B} = n$. For any $C \in \mathscr{C}$, we have $l(C) \in \mathscr{B}$. Note that (l,e) is an adjoint pair and the functor l is fully faith. It follows from assumption (2) that $\mathscr{E}(\mathscr{X}_{\mathscr{C}})$-pd $C \leqslant n$ and so $\mathscr{E}(\mathscr{X}_{\mathscr{C}})$-gl.dim $\mathscr{C} \leqslant n$. □

Before considering the upper bound, we need the following concepts.

By definition, a triangle $A \to B \to C \to \Sigma A$ in a triangulated category $\mathscr{C}$ is said to be $\mathrm{Hom}_{\mathscr{C}}(\mathscr{X}, \text{-})$-*exact* for a class of objects $\mathscr{X}$ of $\mathscr{C}$ provided that there is a short exact sequence

$$0 \to \mathrm{Hom}_{\mathscr{C}}(X, A) \to \mathrm{Hom}_{\mathscr{C}}(X, B) \to \mathrm{Hom}_{\mathscr{C}}(X, C) \to 0$$

for any $X \in \mathscr{X}$. For example, all triangles in $\mathscr{E}$ are $\mathrm{Hom}_{\mathscr{C}}(\mathscr{P}(\mathscr{E}), \text{-})$-exact.

Lemma 3 *Let $\mathscr{X}$ be a class of objects in $\mathscr{C}$, and let $A \to B \to C \to \Sigma A$ be a triangle. Then the following results hold*:

(1) *if $\Sigma\mathscr{X} \subseteq \mathscr{X}$, then the triangle is $\mathrm{Hom}_{\mathscr{C}}(\mathscr{X}, \text{-})$-exact if and only if the induced map*

$$\mathrm{Hom}_{\mathscr{C}}(X, B) \to \mathrm{Hom}_{\mathscr{C}}(X, C)$$

is an epimorphism for each $X \in \mathscr{X}$;

(2) *if* $\Sigma^{-1}\mathscr{X} \subseteq \mathscr{X}$*, then the triangle is* $\mathrm{Hom}_{\mathscr{C}}(\mathscr{X}, \text{-})$*-exact if and only if the induced map*

$$\mathrm{Hom}_{\mathscr{C}}(X, A) \to \mathrm{Hom}_{\mathscr{C}}(X, B)$$

is a monomorphism for each $X \in \mathscr{X}$*.*

Proof (1) The necessity is clear. We only need to show the sufficiency. For each $X \in \mathscr{X}$, applying the functor $\mathrm{Hom}_{\mathscr{C}}(X, \text{-})$ to the triangle $A \to B \to C \to \Sigma A$, we have a long exact sequence

$$\begin{aligned} \cdots \to \mathrm{Hom}_{\mathscr{C}}(X, \Sigma^{-1}B) &\to \mathrm{Hom}_{\mathscr{C}}(X, \Sigma^{-1}C) \\ \to \mathrm{Hom}_{\mathscr{C}}(X, A) &\to \mathrm{Hom}_{\mathscr{C}}(X, B) \to \cdots. \end{aligned}$$

By the assumption, it suffices to show that the induced map

$$\mathrm{Hom}_{\mathscr{C}}(X, A) \to \mathrm{Hom}_{\mathscr{C}}(X, B)$$

is a monomorphism, or equivalently, the induced map

$$\mathrm{Hom}_{\mathscr{C}}(X, \Sigma^{-1}B) \to \mathrm{Hom}_{\mathscr{C}}(X, \Sigma^{-1}C)$$

is an epimorphism. Note that $\Sigma\mathscr{X} \subseteq \mathscr{X}$ and hence, the induced map

$$\mathrm{Hom}_{\mathscr{C}}(\Sigma X, B) \to \mathrm{Hom}_{\mathscr{C}}(\Sigma X, C)$$

is an epimorphism. Since Σ is an automorphism of $\mathscr{C}$, the result follows.

(2) The proof is a dual of (1). □

Now, we are in position to show the converse of Theorem 1.

Theorem 2 *Let* $\mathscr{R}_{\mathrm{tr}} = (\mathscr{A}, \mathscr{B}, \mathscr{C})$ *be an* $\mathscr{X}_{\mathscr{B}}$*-invariant recollement such that* $\mathscr{X}_{\mathscr{C}} = e(\mathscr{X}_{\mathscr{B}})$ *and* $\mathscr{X}_{\mathscr{A}} = q(\mathscr{X}_{\mathscr{B}})$*. Assume that the triangle*

$$le(X) \to X \to iq(X) \to \Sigma le(X)$$

is split for any $X \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{B}}))$*. If*

$$\mathscr{E}(\mathscr{X}_{\mathscr{A}})\text{-gl.dim}\,\mathscr{A} < \infty, \quad \mathscr{E}(\mathscr{X}_{\mathscr{C}})\text{-gl.dim}\,\mathscr{C} < \infty,$$

then

$$\mathscr{E}(\mathscr{X}_{\mathscr{B}})\text{-gl.dim}\,\mathscr{B} \leqslant \mathscr{E}(\mathscr{X}_{\mathscr{A}})\text{-gl.dim}\,\mathscr{A} + \mathscr{E}(\mathscr{X}_{\mathscr{C}})\text{-gl.dim}\,\mathscr{C} + 1.$$

Proof Assume

$$\mathscr{E}(\mathscr{X}_{\mathscr{A}})\text{-gl.dim}\,\mathscr{A} = m, \quad \mathscr{E}(\mathscr{X}_{\mathscr{C}})\text{-gl.dim}\,\mathscr{C} = n.$$

For any non-$\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-projective object $M \in \mathscr{B}$, we have a finite $\mathscr{E}(\mathscr{X}_{\mathscr{C}})$-projective resolution of $e(M)$:

$$0 \to Q_n \to Q_{n-1} \to \cdots \to Q_1 \to Q_0 \to e(M) \to 0.$$

On the other hand, there is an $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-exact complex

$$0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0,$$

where $P_i \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{B}}))$, since $\mathscr{B}$ has enough $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-projective objects by Proposition 2. Thus, applying the functor e to the projective resolution of $le(M)$, we obtain an $\mathscr{E}(\mathscr{X}_{\mathscr{C}})$-exact complex

$$0 \to e(K_n) \to e(P_{n-1}) \to \cdots \to e(P_1) \to e(P_0) \to e(M) \to 0,$$

where $e(P_i) \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{C}}))$. By Lemma 2, it follows that

$$e(K_n) \oplus Q_{n-1} \oplus e(P_{n-2}) \oplus \cdots \cong Q_n \oplus e(P_{n-1}) \oplus Q_{n-2} \oplus \cdots \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{C}})).$$

Thus, $e(K_n) \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{C}}))$ and so, $le(K_n) \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{B}}))$.

Next, we claim that $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-pd$(K_n) \leqslant m+1$. Since $\mathscr{B}$ has enough $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-projective objects, we have a triangle $K_{n+1} \to P_{n+1} \to K_n \to \Sigma K_{n+1}$ in $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$ with $P_{n+1} \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{B}}))$. The discussion on the projective dimension of K_n can be divided into two cases.

Case 1 If $e(K_{n+1}) = 0$, then K_{n+1} lies in $\operatorname{Ker} e = \operatorname{Im} i$. Thus, there is an object $Y \in \mathscr{A}$ such that $i(Y) = K_{n+1}$. Since $\mathscr{E}(\mathscr{X}_{\mathscr{A}})$-gl.dim $\mathscr{A} = m$, there is a finite $\mathscr{E}(\mathscr{X}_{\mathscr{A}})$-projective resolution of Y:

$$0 \to Q'_m \to Q'_{m-1} \to \cdots \to Q'_1 \to Q'_0 \to Y \to 0.$$

Then, there is a finite $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-projective resolution of K_{n+1}:

$$0 \to i(Q'_m) \to i(Q'_{m-1}) \to \cdots \to i(Q'_1) \to i(Q'_0) \to K_{n+1} \to 0.$$

Hence, we know that $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-pd $K_{n+1} \leqslant m$. It is easy to see that $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-pd K_n $\leqslant m+1$.

Case 2 Otherwise, we obtain a split triangle $e(K_{n+1}) \to e(P_{n+1}) \to e(K_n) \to \Sigma e(K_{n+1})$ in $\mathscr{E}(\mathscr{X}_{\mathscr{C}})$ with $e(P_{n+1}) \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{C}}))$. Hence, the triangle

$$le(K_{n+1}) \to le(P_{n+1}) \to le(K_n) \to \Sigma le(K_{n+1})$$

is spilt in $\mathscr{B}$ and $le(K_{n+1})$ lies in $\mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{B}}))$. Then we have the following commutative diagram:

$$\begin{array}{ccccccc}
le(K_{n+1}) & \longrightarrow & K_{n+1} & \longrightarrow & iq(K_{n+1}) & \longrightarrow & \Sigma le(K_{n+1}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
le(P_{n+1}) & \longrightarrow & P_{n+1} & \longrightarrow & iq(P_{n+1}) & \longrightarrow & \Sigma le(P_{n+1}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
le(K_n) & \longrightarrow & K_n & \longrightarrow & iq(K_n) & \longrightarrow & \Sigma le(K_n) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma le(K_{n+1}) & \longrightarrow & \Sigma K_{n+1} & \longrightarrow & \Sigma iq(K_{n+1}) & \longrightarrow & \Sigma^2 le(K_{n+1})
\end{array}$$

Assume that P lies in $\mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{B}}))$. Applying the functor $\mathrm{Hom}_{\mathscr{B}}(P,\text{-})$ to the above diagram, we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \mathrm{Hom}_{\mathscr{B}}(P, le(K_{n+1})) & \to & \mathrm{Hom}_{\mathscr{B}}(P, K_{n+1}) & \to & \mathrm{Hom}_{\mathscr{B}}(P, iq(K_{n+1})) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}_{\mathscr{B}}(P, le(P_{n+1})) & \to & \mathrm{Hom}_{\mathscr{B}}(P, P_{n+1}) & \to & \mathrm{Hom}_{\mathscr{B}}(P, iq(P_{n+1})) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \mathrm{Hom}_{\mathscr{B}}(P, le(K_n)) & \to & \mathrm{Hom}_{\mathscr{B}}(P, K_n) & \to & \mathrm{Hom}_{\mathscr{B}}(P, iq(K_n)) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & \vdots & &
\end{array}$$

where the first, second verticals and the second horizontal are short exact sequences. Thus, we infer that the morphism

$$\mathrm{Hom}_{\mathscr{B}}(P, le(K_{n+1})) \to \mathrm{Hom}_{\mathscr{B}}(P, K_{n+1})$$

is a monomorphism. By Lemma 3, the triangle

$$le(K_{n+1}) \to K_{n+1} \to iq(K_{n+1}) \to \Sigma le(K_{n+1})$$

lies in $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$. Note that $q(K_{n+1})$ lies in $\mathscr{A}$ and $\mathscr{E}(\mathscr{X}_{\mathscr{A}})$-gl.dim $\mathscr{A} = m$. Hence, we infer that $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-pd $iq(K_{n+1}) \leqslant m$ and so, $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-pd $K_{n+1} \leqslant m$. It follows that $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-pd $K_n \leqslant m+1$.

Therefore, we know that $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-pd $M \leqslant n+m+1$ and so, $\mathscr{E}(\mathscr{X}_{\mathscr{B}})$-gl.dim $\mathscr{B}$ $\leqslant n+m+1$. □

Now, we can prove our main result.

Proof of Theorem A It follows from Theorems 1 and 2. □

Let $\mathscr{R}_{\mathrm{tr}} = (\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement. If $\mathscr{X}_{\mathscr{C}}$ and $\mathscr{X}_{\mathscr{A}}$ are full skeletally small, closed under isomorphisms, direct summands, and Σ-stable, additive subcategories of $\mathscr{A}$ and $\mathscr{C}$, respectively, then one can construct a full skeletally small additive subcategory $\mathscr{X}_{\mathscr{B}}$ of $\mathscr{B}$, where

$$\mathscr{X}_{\mathscr{B}} = \mathrm{add}\{l(\mathscr{X}_{\mathscr{C}}), i(\mathscr{X}_{\mathscr{A}})\}.$$

In this case, $\mathscr{R}_{\mathrm{tr}} = (\mathscr{A}, \mathscr{B}, \mathscr{C})$ is an $\mathscr{X}_{\mathscr{B}}$-invariant recollement such that $\mathscr{X}_{\mathscr{A}} = q(\mathscr{X}_{\mathscr{B}})$ and $\mathscr{X}_{\mathscr{C}} = e(\mathscr{X}_{\mathscr{B}})$. Moreover, it is easy to see that the triangle $le(X) \to X \to iq(X) \to \Sigma le(X)$ is split, for any $X \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{B}}))$. Thus, we have the following consequence.

Corollary 2 *Keep the notations $\mathscr{R}_{\mathrm{tr}} = (\mathscr{A}, \mathscr{B}, \mathscr{C})$, $\mathscr{X}_{\mathscr{A}}$, $\mathscr{X}_{\mathscr{B}}$, and $\mathscr{X}_{\mathscr{C}}$ as above. Then the finiteness of the $\mathscr{E}$-global dimensions of triangulated categories is invariant with respect to the recollement $\mathscr{R}_{\mathrm{tr}}$.*

5 Applications to derived categories

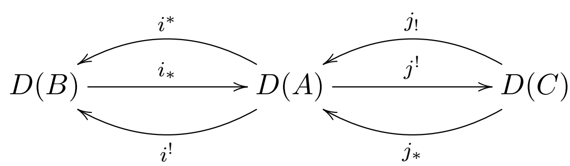
In this section, we aim to realize Theorem A on the derived categories of algebras. Let k be a field, and let A be a finite-dimensional k-algebra. For convenience, we denote by $D(A)$ the unbounded derived category of $\operatorname{Mod} A$.

Next, we apply our main result into unbounded derived categories of algebras.

Proposition 6 *Let k be a field. Assume that $A = \begin{bmatrix} B & 0 \\ M & C \end{bmatrix}$ is a triangular matrix algebra, where A, B are finite-dimensional k-algebras and M a finitely generated C-B-bimodule. Then there are three proper classes of triangles $\mathscr{E}_A$, $\mathscr{E}_B$, and $\mathscr{E}_C$ in $D(A)$, $D(B)$, and $D(C)$, respectively. Moreover,*

$$\begin{aligned} &\max\{\mathscr{E}_B\text{-gl.dim}\, D(B), \mathscr{E}_C\text{-gl.dim}\, D(C)\} \\ &\quad \leqslant \mathscr{E}_A\text{-gl.dim}\, D(A) \\ &\quad \leqslant \mathscr{E}_B\text{-gl.dim}\, D(B) + \mathscr{E}_C\text{-gl.dim}\, D(C) + 1. \end{aligned}$$

Proof Let $e = \begin{bmatrix} 0 & 0 \\ 0 & 1_C \end{bmatrix}$ be an orthogonal idempotent. Then, by [2, Example 3.4], there is a recollement



The triangle functors i^*, i_*, $j_!$, and $j^!$ can be listed as follows:

$$i^* = - \otimes_A^{\mathbf{L}} B_B, \quad j_! = - \otimes_C^{\mathbf{L}} eA_A,$$
$$i_* = - \otimes_B^{\mathbf{L}} B_A, \quad j^! = - \otimes_A^{\mathbf{L}} Ae_C.$$

Assume that complexes

$$X = \bigoplus_{i\in\mathbb{Z}} i_*(B)[-i] = \bigoplus_{i\in\mathbb{Z}} B[-i],$$
$$Y = \bigoplus_{i\in\mathbb{Z}} j_!(C)[-i] = \bigoplus_{i\in\mathbb{Z}} eA[-i],$$
$$Z = \bigoplus_{i\in\mathbb{Z}} (B \oplus eA)[-i].$$

Let

$$\mathscr{X}_A = \operatorname{add} Z, \quad \mathscr{X}_B = \operatorname{add} X, \quad \mathscr{X}_C = \operatorname{add} Y.$$

Clearly, they are skeletally small, closed under isomorphisms, direct summands, and Σ-stable additive full subcategories of $D(A)$, $D(B)$, and $D(C)$, respectively. In this case, we infer that

$$\mathscr{X}_B = i^*(\mathscr{X}_A), \quad \mathscr{X}_C = j^!(\mathscr{X}_A).$$

Moreover, by Proposition 2 and our construction, any object $P \in \mathscr{P}(\mathscr{E}(\mathscr{X}_{\mathscr{B}}))$ is a sum of $i_*i^*(P)$ and $j_!j^!(P)$. Hence, triangle (1.1) in Theorem A is split. Now, it is easy to see that the recollement $(D(B), D(A), D(C))$ is an $\mathscr{X}_A$-invariant recollement and satisfies the conditions of Theorem A. By Example 1, we can obtain three proper classes of triangles $\mathscr{E}_A$, $\mathscr{E}_B$, and $\mathscr{E}_C$ in $D(A)$, $D(B)$, and $D(C)$, respectively. The rest of the results comes from Theorem A. $\square$

Definition 8 [7] Let $\mathscr{C}$ be a triangulated category with a proper class of triangles $\mathscr{E}$ such that $\mathscr{C}$ has enough $\mathscr{E}$-projective objects. The triangulated category $\mathscr{C}$ is said to be a *hereditary category* with respect to $\mathscr{E}$ provided that $\mathscr{E}$-gl.dim $\mathscr{C} \leqslant 1$.

Example 6 Let k be a field, and let $\mathscr{E}$ be a proper class of triangles in the derived category $D(k)$. It is well known that

$$D(k) \cong \prod_{i\in\mathbb{Z}} \mathrm{Mod}\, k.$$

Then it is easy to check that all objects of the derived category $D(k)$ are $\mathscr{E}$-projective objects. Hence, by [7, Theorem 4.25], we know that $\mathscr{E}$-gl.dim $D(k) = 0$. That is, the derived category $D(k)$ is an $\mathscr{E}$-hereditary category.

Corollary 3 *Let k be a field. Assume that $A = \begin{bmatrix} k & 0 \\ M & k \end{bmatrix}$ is a triangular matrix algebra, where M is a finite-dimensional vector space. Then there is a proper class $\mathscr{E}$ of triangles in $D(A)$ such that $D(A)$ is an $\mathscr{E}$-hereditary category.*

Proof One can directly get the result from Proposition 6 and Example 6. $\square$

Example 7 Let k be a field. Assume that Λ is the path algebra over k given by the quiver

$$\underset{2}{\circ} \xrightarrow{\ \alpha\ } \underset{1}{\circ}$$

Then Λ can be viewed as a triangular matrix algebra $\begin{bmatrix} k & 0 \\ k & k \end{bmatrix}$. Then, from Corollary 3, we know that the derived category $D(\Lambda)$ is an $\mathscr{E}$-hereditary category.

Example 8 Let k be a field, and let Λ be the k-algebra given by the quiver

$$\underset{3}{\circ} \xrightarrow{\ \alpha\ } \underset{2}{\circ} \xrightarrow{\ \beta\ } \underset{1}{\circ}$$

with the relation $\alpha\beta = 0$. Then Λ is isomorphic to a triangular matrix algebra $\begin{bmatrix} k & 0 & 0 \\ k & k & 0 \\ 0 & k & k \end{bmatrix}$. Let $e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be an idempotent of the triangular matrix algebra. Then

$$\Lambda/\Lambda e\Lambda \cong \begin{bmatrix} k & 0 \\ k & k \end{bmatrix}, \quad e\Lambda e \cong k.$$

By Proposition 6, there are three proper classes of triangles $\mathscr{E}_\Lambda$, $\mathscr{E}_{\Lambda/\Lambda e\Lambda}$, and $\mathscr{E}_{e\Lambda e}$ in $D(\Lambda)$, $D(\Lambda/\Lambda e\Lambda)$, and $D(e\Lambda e)$, respectively. In addition, we know that

$$\mathscr{E}_\Lambda\text{-gl.dim}\, D(\Lambda) \leqslant \mathscr{E}_{\Lambda/\Lambda e\Lambda}\text{-gl.dim}\, D(\Lambda/\Lambda e\Lambda) + \mathscr{E}_{e\Lambda e}\text{-gl.dim}\, D(e\Lambda e) + 1.$$

From Examples 6 and 7, we have $\mathscr{E}_\Lambda$-gl.dim $D(\Lambda) \leqslant 2$.

Remark 3 Finally, we should remark that by the results provided in this paper, it is difficult to obtain that the derived categories of all finite-dimensional hereditary algebras are $\mathscr{E}$-hereditary categories. However, it is a question worthy of consideration.

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