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RESEARCH ARTICLE

On $c^\sharp$-normal subgroups in finite groups

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Abstract A subgroup H of a finite group G is called a $c^\sharp$-normal subgroup of G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K$ is a CAP-subgroup of G. In this paper, we investigate the influence of fewer $c^\sharp$-normal subgroups of Sylow p-subgroups on the p-supersolvability, p-nilpotency, and supersolvability of finite groups. We obtain some new sufficient and necessary conditions for a group to be p-supersolvable, p-nilpotent, and supersolvable. Our results improve and extend many known results.

Keywords Finite group, $c^\sharp$-normal, p-supersolvable, p-nilpotent, supersolvable
MSC 20D10

1 Introduction

All groups considered will be finite. For a group G, $\pi(G)$ will denote the set of all prime divisors of the order of G.

As we know, the normality of subgroups of a group has been investigated by many scholars. Thereinto, the cover-avoidance property is a generalization of normality. A subgroup H of a group G is said to be a CAP-subgroup of G (have the cover-avoidance property) if H either covers or avoids any G-chief factor A/B, namely, either $HA = AH$ or $H\cap A = H\cap B$. This concept was introduced by Gaschutz [6] and has been studied extensively by some scholars. For example, the interested readers can refer to [1,4]. As another generalization of normality, the c-normality of subgroups was introduced by Wang [9]: a subgroup H of a group G is said to be a c-normal in G if there exists a normal subgroup K of G such that

$$G = HK, \quad H \cap K \leqslant H_G,$$

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where H_G is the core of H in G. Since then, a number of scholars continued the study of influence of c-normality together with its generalization on the structure of groups; see, for instance, [2,7,11,13,14,16,17,19]. As a common generalization of cover-avoidance property (CAP) and c-normality, Wei and Wang introduced the following concept of $c^\sharp$-normality (refer to [12] or [10]).

Definition 1.1 A subgroup H of a group G is said to be a $c^\sharp$-normal subgroup of G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K$ is a CAP-subgroup of G.

It is clear that both CAP-subgroup and c-normal subgroup are $c^\sharp$-normal subgroups, but the converse is not true, see, e.g., [10]. In order to use fewer $c^\sharp$-normal subgroups to characterize the structure of a group, we employ the following definition (refer to [8]).

Definition 1.2 Given a prime p and a p-group P, assume $|P/\Phi(P)| = p^d$. Then, given a set

$$\mathscr{M}_d(P) = \{P_1, P_2, \ldots, P_d\}$$

of d maximal subgroups of P, we say that $\mathscr{M}_d(P)$ is a minimal system of maximal subgroups of P if

$$\bigcap_{i=1}^{d} P_i = \Phi(P).$$

In this paper, we try to use the $c^\sharp$-normality of maximal subgroups of P in $\mathscr{M}_d(P)$ to characterize the structure of a group G. We obtain some new sufficient and necessary conditions for a group to be p-supersolvable, p-nilpotent, and supersolvable. Our results improve and extend many related known results.

2 Preliminaries

Lemma 2.1 ([12, Lemma 1.2.6] or [10, Lemma 2.5]) *Let G be a group, let H be a subgroup of G, and let N be a normal subgroup of G.*

(1) *If $N \leqslant H$, then H is $c^\sharp$-normal in G if and only if H/N is $c^\sharp$-normal in G/N.*

(2) *Let π be a set of primes, let H be a π-subgroup of G, and let N be a normal π'-subgroup of G. If H is $c^\sharp$-normal in G, then HN/N is $c^\sharp$-normal in G/N.*

(3) *Let L be a subgroup of G such that $H \leqslant \Phi(L)$. If H is $c^\sharp$-normal in G, then H is a CAP-subgroup of G.*

Lemma 2.2 [3, Theorem A.9.2] *Let G be a group, let N be a normal subgroup of G, and let H be a subgroup of G. If $N \leqslant \Phi(H)$, then $N \leqslant \Phi(G)$.*

Lemma 2.3 [18, p. 180] *Let G be a π-separable group. If $O_{\pi'}(G) = 1$, then*

$$C_G(O_\pi(G)) \leqslant O_\pi(G).$$

Lemma 2.4 [3, Theorem A.11.1] *Let N be a normal abelian subgroup of a group G, and let $N \leqslant M \leqslant G$ such that $(|N|, |G : M|) = 1$. If a complement subgroup of N in M exists, then N possesses a complement subgroup in G.*

Lemma 2.5 [11, Lemma 3] *Let $H \neq 1$ be a solvable normal subgroup of a group G. If every minimal normal subgroup of G which is contained in H is not contained in $\Phi(G)$, then the Fitting subgroup $F(H)$ of H is the direct product of minimal normal subgroups of G which are contained in H.*

Lemma 2.6 ([14, Lemma 2.8] or [15, Lemma 2.2]) *Let G be a group, and let p be a prime divisor of $|G|$ with $(|G|, p-1) = 1$.*

(1) *If N is normal in G of order p, then $N \leqslant Z(G)$.*

(2) *If G has cyclic Sylow p-subgroups, then G is p-nilpotent.*

(3) *If $M \leqslant G$ and $|G : M| = p$, then $M \trianglelefteq G$.*

3 Main results

Theorem 3.1 *Let G be a p-solvable group, and let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$. Then G is p-supersolvable if and only if every member in some fixed $\mathscr{M}_d(P)$ is $c^\sharp$-normal in G.*

Proof If G is p-supersolvable, then any p-subgroup of G is a CAP-subgroup of G.

Conversely, suppose that every member in some fixed $\mathscr{M}_d(P)$ is $c^\sharp$-normal in G. We will show that G is p-supersolvable. Let G be a counter-example of minimal order, and let $\mathscr{M}_d(P) = \{P_1, P_2, \ldots, P_d\}$. By hypotheses, P_i is $c^\sharp$-normal in G, and hence, there exists $K_i \trianglelefteq G$ such that $G = P_iK_i$ and $P_i \cap K_i$ is a CAP-subgroup of G, $i = 1, 2, \ldots, d$. Furthermore, we have the following four claims.

(1) $O_{p'}(G) = 1$.

It follows from Lemma 2.1 and the choice of G.

(2) $\mathrm{Core}_G(\Phi(P)) = 1$; in particular, $\Phi(O_p(G)) = 1$.

Since the class of p-supersolvable groups is a saturated formation, by Lemma 2.2, we can assume without loss of generality that $\mathrm{Core}_G(\Phi(P)) = 1$. In particular, $\Phi(O_p(G)) = 1$.

(3) Every minimal normal subgroup of G contained in $O_p(G)$ is of order p.

Since G is p-solvable, $O_p(G) \neq 1$ by (1). Let N be a minimal normal subgroup of G contained in $O_p(G)$. If for each i, $P_i \cap K_i$ covers $N/1$, namely,

$$(P_i \cap K_i)N = P_i \cap K_i,$$

then

$$N \leqslant P_i \cap K_i.$$

Consequently,

$$N \leqslant \bigcap_{i=1}^{d} P_i = \Phi(P),$$

which is contrary to (2). Hence, there exists some j such that $P_j \cap K_j$ avoids $N/1$, that is,

$$P_j \cap K_j \cap N = 1.$$

By the minimal normality of N in G, either $K_j \cap N = 1$ or $K_j \cap N = N$. If $K_j \cap N = 1$, then NK_j/K_j is minimal normal in G/K_j. But $G = P_jK_j$ implies that G/K_j is a p-group, so $N \cong NK_j/K_j$ is of order p. If $K_j \cap N = N$, then

$$P_j \cap K_j \cap N = P_j \cap N = 1.$$

As $NP_j = P$, we also get $|N| = p$ and (3) follows.

(4) The counter-example does not exist.

Since G is p-solvable, by (1), (2), and Lemma 2.3, we have

$$C_G(O_p(G)) = O_p(G).$$

Now, we claim that

$$O_p(G) \cap \Phi(G) = 1.$$

If not, let N be a minimal normal subgroup of G contained in $O_p(G) \cap \Phi(G)$. Then N is of order p by (3), so N is complemented in P by (2). By applying Lemma 2.4, N is complemented in G, which is contrary to $N \leqslant \Phi(G)$. So $O_p(G) \cap \Phi(G) = 1$. In view of Lemma 2.5,

$$O_p(G) = N_1 \times N_2 \times \cdots \times N_s,$$

where $N_i \trianglelefteq G$ and $|N_i| = p$ $(i = 1, 2, \ldots, s)$. Since $G/C_G(N_i) \lesssim \mathrm{Aut}(N_i)$ and $\mathrm{Aut}(N_i)$ is abelian, $G/C_G(N_i)$ is abelian. Thus,

$$G/\bigcap_{i=1}^{s} C_G(N_i) = G/C_G(O_p(G))$$

is also abelian, namely, $G/O_p(G)$ is abelian. Now, every chief factor of G below $O_p(G)$ is of order p, and hence G is p-supersolvable. This is the final contradiction.

The proof is complete. □

If p is some special prime, then the condition that G is p-solvable in Theorem 3.1 can be removed. In fact, we have the following result.

Theorem 3.2 *Let G be a group, and let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if every member in some fixed $\mathscr{M}_d(P)$ is $c^\sharp$-normal in G.*

Proof Suppose that G is p-nilpotent. Then G is p-supersolvable, and hence, every member in some fixed $\mathscr{M}_d(P)$ is $c^\sharp$-normal in G by Theorem 3.1.

Conversely, suppose that every member in some fixed $\mathscr{M}_d(P)$ is $c^\sharp$-normal in G. We will show that G is a p-nilpotent group. Let G be a counter-example of minimal order and let $\mathscr{M}_d(P) = \{P_1, P_2, \dots, P_d\}$. Since P_i is $c^\sharp$-normal in G, there exists $K_i \trianglelefteq G$ such that $G = P_iK_i$ and $P_i \cap K_i$ is a CAP-subgroup of G, $i = 1, 2, \dots, d$. With the similar arguments as in the proof of Theorem 3.1, we have the following five claims.

(1) $O_{p'}(G) = 1$.

(2) $\mathrm{Core}_G(\Phi(P)) = 1$.

(1) and (2) are obvious.

(3) Every minimal normal subgroup of G is contained in $O_p(G)$.

Let N be a minimal normal subgroup of G. Because $O_{p'}(G) = 1$, we have $p \mid |N|$. If for some i, $N \cap K_i = 1$, then

$$N \cong NK_i/K_i \leqslant G/K_i,$$

and hence, N is a p-group and $N \leqslant O_p(G)$. Now, we assume $N \cap K_i = N$ for each i. Then $N \leqslant K_i$. Since $P_i \cap K_i$ is a CAP-subgroup of G, it either covers or avoids $N/1$. If $P_i \cap K_i$ cover $N/1$, then

$$(P_i \cap K_i)N = P_i \cap K_i,$$

and, of course, $N \leqslant O_p(G)$. If $P_i \cap K_i$ avoids $N/1$, then

$$(P_i \cap K_i) \cap N = 1,$$

that is, $P_i \cap N = 1$. Thus, $|N|_p = p$ and consequently, N is p-nilpotent by Lemma 2.6. By (1), N is a p-group, thereby, $N \leqslant O_p(G)$ and (3) follows.

(4) Every minimal normal subgroup of G is of order p.

Let N be a minimal normal subgroup of G. By (3), $N \leqslant O_p(G)$. If for some i, $N \cap K_i = 1$, then

$$N \cong NK_i/K_i.$$

However, NK_i/K_i is minimal normal in the p-group G/K_i, hence,

$$|N| = |NK_i/K_i| = p.$$

Now, we assume $N \leqslant K_i$ for each i. If for some j, $(P_j \cap K_j) \cap N = 1$, then

$$P_j \cap N = 1, \quad |N| = p.$$

So assume $(P_i \cap K_i)N = P_i \cap K_i$ for each i. Then $N \leqslant P_i \cap K_i$, and hence,

$$N \leqslant \bigcap_{i=1}^{d} P_i = \Phi(P),$$

which is contrary to (2).

(5) The counter-example does not exist.

Let $N_1, N_2, \ldots, N_s$ be all minimal normal subgroups of G. By (4), N_i is of order p. Moreover, N_i is complemented in P by (2), so N_i has a complement M_i in G by applying Lemma 2.4. In view of Lemma 2.6, $N_i \leqslant Z(G)$, hence $M_i \trianglelefteq G$, where $i = 1, 2, \ldots, s$. Now, let M be a supplement of $N_1 N_2 \cdots N_s$ to G with order as small as possible. Assume $O_p(G) \cap M \neq 1$. Since $O_p(G) \cap M \trianglelefteq G$, we can take a minimal normal subgroup N of G contained in $O_p(G) \cap M$. Then $N = N_j$ for some j, and so

$$G = NM_j, \quad M = N(M \cap M_j).$$

Furthermore,

$$G = (N_1 N_2 \cdots N_s)(M \cap M_j).$$

The choice of M implies that $M \cap M_j = M$, and hence,

$$N \leqslant M \leqslant M_j,$$

which is impossible. This proves that $O_p(G) \cap M = 1$. Since $1 < M \trianglelefteq G$, there exists some k such that N_k is contained in M. This is contrary to $O_p(G) \cap M = 1$.

The proof is complete. □

Theorem 3.3 *Let G be a group, and let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and every member in some fixed $\mathscr{M}_d(P)$ is $c^\sharp$-normal in G.*

Proof We only need to prove the 'if' part by Theorem 3.1. Let G be a counter-example of minimal order, and let $\mathscr{M}_d(P) = \{P_1, P_2, \ldots, P_d\}$. By hypotheses, there exists $K_i \trianglelefteq G$ such that $G = P_i K_i$ and $P_i \cap K_i$ is a CAP-subgroup of G, $i = 1, 2, \ldots, d$. Then we have the following five claims.

(1) $O_{p'}(G) = 1$.

(2) $\mathrm{Core}_G(\Phi(P)) = 1$.

(1) and (2) are obvious.

(3) Every minimal normal subgroup of G is contained in $O_p(G)$.

Let N be a minimal normal subgroup of G. By (1), $p \mid |N|$. Consider N and K_i. If for some i, $N \cap K_i = 1$, then

$$N \cong NK_i/K_i \leqslant G/K_i,$$

and hence, N is a p-group and $N \leqslant O_p(G)$. Now, assume $N \cap K_i = N$, namely, $N \leqslant K_i$ for each i. Since $P_i \cap K_i$ is a CAP-subgroup of G, it either covers or avoids $N/1$. If $P_i \cap K_i$ cover $N/1$, then

$$(P_i \cap K_i)N = P_i \cap K_i,$$

and, of course, $N \leqslant O_p(G)$. If $P_i \cap K_i$ avoids $N/1$, then

$$(P_i \cap K_i) \cap N = 1,$$

that is, $P_i \cap N = 1$. Thus, $|P \cap N| = p$. If $N = G$, then P is cyclic of order p, and so $C_G(P) = N_G(P)$ by the p-nilpotency of $N_G(P)$. By the well-known Burnside theorem, G is p-nilpotent, a contradiction. Hence, $N < G$. Now, write $G_0 = PN$. Clearly,

$$|PN| = \frac{|P|\,|N|}{|P \cap N|} = |P_i|\,|N| = |P_iN|,$$

hence, $G_0 = P_iN$ with $P_i \cap N = 1$. Moreover, $N_{G_0}(P)$ is p-nilpotent, thereby G_0 satisfies the hypotheses of the theorem. If $G_0 < G$, then G_0 is p-nilpotent by the choice of G. Of course, N is also p-nilpotent. It follows from (1) that N is a p-group and $N \leqslant O_p(G)$. Now, assume $G_0 = G$ and set $G_1 = N_G(P \cap N)$. Obviously,

$$P \leqslant G_1, \quad G_1 = P_iN \cap G_1 = P_i(N \cap G_1).$$

Again, $N_{G_1}(P)$ is p-nilpotent, hence G_1 satisfies the hypotheses of the theorem. If $G_1 < G$, then $G_1 = N_G(P \cap N)$ is p-nilpotent; of course, $N_N(P \cap N)$ is also p-nilpotent. This implies that

$$C_N(P \cap N) = N_N(P \cap N),$$

and so N is p-nilpotent. Similarly, we have $N \leqslant O_p(G)$. If $G_1 = G$, then $P \cap N \trianglelefteq G$. The minimal normality of N implies that $P \cap N = N$, and hence, $N \leqslant O_p(G)$ and (3) follows.

(4) Every minimal normal subgroup of G is of order p.

Let N be a minimal normal subgroup of G. By (3), $N \leqslant O_p(G)$. If for some i, $N \cap K_i = 1$, then $N \cong NK_i/K_i$. Moreover, NK_i/K_i is minimal normal in the p-group G/K_i, and hence

$$|N| = |NK_i/K_i| = p.$$

Now, we assume $N \leqslant K_i$ for each i. If for some j, $(P_j \cap K_j) \cap N = 1$, then $P_j \cap N = 1$ and $|N| = p$. So assume $(P_i \cap K_i)N = P_i \cap K_i$ for each i. Then $N \leqslant P_i \cap K_i$, and hence, $N \leqslant \cap_{i=1}^{d} P_i = \Phi(P)$, which is contrary to (2).

(5) The counter-example does not exist.

Let L be a supplement of $O_p(G)$ to G with order as small as possible. We claim that $O_p(G) \cap L = 1$. In fact, if the claim is false, since $O_p(G) \cap L \trianglelefteq G$, we may take a minimal normal subgroup N of G contained in $O_p(G) \cap L$. Then $|N| = p$ by (4) and N is complemented in P by (2), which follows that N has a complement M in G by Lemma 2.4. Now,

$$L = L \cap NM = N(L \cap M),$$

hence,

$$G = O_p(G)(L \cap M).$$

The choice of L implies that $L \cap M = L$, namely, $L \leqslant M$. Thus,

$$N \leqslant O_p(G) \cap L \leqslant M,$$

which is impossible. This proves

$$O_p(G) \cap L = 1.$$

On the other hand, with the similar arguments as in the proof of Theorem 3.1, we see that

$$O_p(G) = N_1 \times N_2 \times \cdots \times N_s,$$

where $N_i \trianglelefteq G$ and $|N_i| = p$ $(i = 1, 2, \ldots, s)$. Hence,

$$O_p(G) \leqslant Z(P).$$

Since $P = O_p(G)(P \cap L)$ and $N_G(P)$ is p-nilpotent, $P \cap L \neq 1$. Let T be a minimal normal subgroup of G contained in $(P \cap L)^G$, the normal closure of $P \cap L$ in G. Then $T \leqslant O_p(G)$ by (3). However,

$$(P \cap L)^G = (P \cap L)^{O_p(G)L} = (P \cap L)^L \leqslant L,$$

hence $T \leqslant L$. This is contrary to $O_p(G) \cap L = 1$.

The proof is complete. □

Remark 3.4 The conditions that G is p-solvable in Theorem 3.1, $(|G|, p-1) = 1$ in Theorem 3.2, and $N_G(P)$ is p-nilpotent in Theorem 3.3 cannot be removed. For example, $G = A_5$ is a counter-example for $p = 5$.

As an application of Theorem 3.2, we have the following result.

Theorem 3.5 *Let G be a group. Then G is supersolvable if and only if every member in some fixed $\mathscr{M}_d(P)$ is $c^\sharp$-normal in G for every non-cyclic Sylow subgroup P of G.*

Proof Suppose that G is supersolvable. Then every chief factor of G is of prime order, and hence, every subgroup of G is a CAP-subgroup.

Conversely, suppose that every member in some fixed $\mathscr{M}_d(P)$ is $c^\sharp$-normal in G for every non-cyclic Sylow subgroup P of G. If p is the smallest prime dividing $|G|$, then G is p-nilpotent by Theorem 3.2. By the Odd Order Theorem, G is solvable. Now, if P is cyclic, then G is p-supersolvable. On the other hand, if P is non-cyclic, then G is p-supersolvable by Theorem 3.1. Thus, G is supersolvable. □

Remark 3.6 Fan et al. [5] introduced the concept of semi-CAP-subgroup which is also a common generalization of the concepts of CAP-subgroup and c-normal subgroup. Naturally, one may ask if the above theorems are true if the CAP-subgroup in Definition 1.1 is replaced by the semi-CAP-subgroup. Here we give a negative answer to this question.

For example, let $G = C_2 \times A_4$, where A_4 is the alternating group of degree 4 and $C_2 = \langle c \rangle$ is a cyclic group of order 2 with generator c. Then

$$A_4 = K_4 \rtimes C_3,$$

where $K_4 = \langle a, b\rangle$ is the Klein four group with generators a and b of order 2 and C_3 is a cyclic group of order 3. It is clear that $P = C_2K_4$ is a Sylow 2-subgroup of G and P is an elementary abelian 2-group. Take

$$P_1 = K_4, \quad P_2 = \langle a\rangle \times \langle bc\rangle, \quad P_3 = \langle ab\rangle \times \langle ac\rangle.$$

Then

$$\mathscr{M}_d(P) = \{P_1, P_2, P_3\}, \quad P_1 \cap P_2 \cap P_3 = 1.$$

It is clear that

$$\Gamma\colon 1 < \langle c\rangle < P < G$$

is a chief series of G and P_i either covers or avoids each chief factor in Γ. Hence, P_i is a semi-CAP-subgroup of G, where $i = 1, 2, 3$. However, G is not 2-supersolvable; of course, G is neither 2-nilpotent nor supersolvable. Hence, none of Theorems 3.1, 3.2, and 3.5 is true if the CAP-subgroups are replaced by the weaker semi-CAP-subgroups.

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