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RESEARCH ARTICLE

Castelnuovo-Mumford regularity and projective dimension of a squarefree monomial ideal

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Abstract Let $S = K[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n variables over a field K, and let I be a squarefree monomial ideal minimally generated by the monomials $u_1, u_2, \ldots, u_m$. Let w be the smallest number t with the property that for all integers $1 \leqslant i_1 < i_2 < \cdots < i_t \leqslant m$ such that $\mathrm{lcm}(u_{i_1}, u_{i_2}, \ldots, u_{i_t}) = \mathrm{lcm}(u_1, u_2, \ldots, u_m)$. We give an upper bound for Castelnuovo-Mumford regularity of I by the bigsize of I. As a corollary, the projective dimension of I is bounded by the number w.

Keywords Castelnuovo-Mumford regularity, projective dimension, squarefree monomial ideals
MSC 13F55, 13C10, 13C15

1 Introduction

Let $S = K[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n variables over a field K, and let I be an ideal of S. The (Castelnuovo-Mumford) regularity of an ideal I, denoted by $\mathrm{reg}(I)$, is defined to be the minimal number r such that the i-th syzygy module of I is generated by elements of degree $\leqslant i+r$ for all $i \geqslant 0$. It can be considered as a refined notion of the maximal degree of minimal generators of I as a measure of the complexity of Gröbner basis computation. On the other hand, the regularity provides the relationship between the local cohomology and the syzygy module of I, and directly links to the geometric degree, the dimension of I or S/I, and other invariant. We can refer to some papers, such as [1–3,5,6,13,16], for the background and some important development of the regularity of I. In particular, there are two conjectures (see [1,6]) related to the

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regularity of I:

$$\operatorname{reg}(I) \leqslant \text{geom-deg}(I),$$

$$\operatorname{reg}(I) \leqslant \deg(S/I) - \operatorname{codim}(S/I) + 1.$$

In fact, under the special conditions that I is a (squarefree) monomial ideal, geom-deg(I), $\deg(S/I) - \operatorname{codim}(S/I) + 1$, and so on, provide the bounds of the regularity of I (see [7,11]).

The notions of the size and bigsize of a monomial ideal were introduced by Lyubeznik [12] and Popescu [14], respectively. Lyubeznik used the size of a monomial ideal to study its arithmetical rank. One result was given by him that

$$\operatorname{projdim}(S/I) \leqslant \operatorname{ara}(I)$$

if I is a squarefree monomial ideal ([12, Proposition 3]). While the bigsize of a monomial ideal was used firstly by Popescu to consider the Stanley Conjecture.

Let I be a squarefree monomial ideal minimally generated by the monomials $u_1, u_2, \ldots, u_m$. We prove that the regularity of I can be bounded by $\operatorname{bigsize}_S(I) + 1$ ($\operatorname{bigsize}_S(I)$ denotes the bigsize of I). As a corollary, the projective dimension of a squarefree monomial ideal I is bounded by the number w. Here, w is the smallest number t with the property that for all integers $1 \leqslant i_1 < i_2 < \cdots < i_t \leqslant m$ such that

$$\operatorname{lcm}(u_{i_1}, u_{i_2}, \ldots, u_{i_t}) = \operatorname{lcm}(u_1, u_2, \ldots, u_m).$$

2 Results

Throughout this paper, Let $S = K[x_1, x_2, \ldots, x_n]$. Let $I \subset S$ be a squarefree monomial ideal, and let $I = \cap_{i=1}^{s} P_i$ be its presentation as an irredundant intersection of prime monomial ideals. It is well known that the set $\{P_1, P_2, \ldots, P_s\}$ is determined uniquely by I.

The following result is useful for the computation of the regularity of a graded finitely generated S-modules.

Lemma 1 [5, Corollary 20.19] *Let*

$$0 \to L \to M \to N \to 0$$

be a short exact sequence of graded finitely generated S-modules. Then

(i) $\operatorname{reg}(L) \leqslant \max\{\operatorname{reg}(M), \operatorname{reg}(N) + 1\}$, *the equality holds if* $\operatorname{reg}(M) \neq \operatorname{reg}(N)$;

(ii) $\operatorname{reg}(M) \leqslant \max\{\operatorname{reg}(L), \operatorname{reg}(N)\}$, *the equality holds if* $\operatorname{reg}(N) \neq \operatorname{reg}(L) - 1$ *or if* $L_n = 0$ *for* $n \gg 0$;

(iii) $\operatorname{reg}(N) \leqslant \max\{\operatorname{reg}(M), \operatorname{reg}(L) - 1\}$, *the equality holds if* $\operatorname{reg}(M) \neq \operatorname{reg}(L)$.

Let $I_1, I_2, \ldots, I_d$ be some monomial complete intersections. Chardin et al. [4] proved that

$$\operatorname{reg}(I_1 \cap I_2 \cap \cdots \cap I_d) \leqslant \operatorname{reg}(I_1) + \operatorname{reg}(I_2) + \cdots + \operatorname{reg}(I_d).$$

In fact, it was proved by Herzog [8] that this result holds in case that $I_1, I_2, \ldots, I_d$ are arbitrary monomial ideals. Note that $\operatorname{reg}(P) = 1$ for P an ideal generated by some variables, and that

$$\operatorname{reg}(S/I) = \operatorname{reg}(I) - 1.$$

The following lemma is an immediate consequence of their result. Here, we directly deduce it from Lemma 1. This technique of the proof gives us a hint that we could get an upper bound from the number of some particular subset of the minimal prime ideals appearing in the primary decomposition of a squarefree monomial ideal.

Lemma 2 *Let $I \subset S$ be a squarefree monomial ideal, and let $I = \cap_{i=1}^{s} P_i$ be its presentation as an irredundant intersection of prime monomial ideals. Then*

$$\operatorname{reg}(S/I) \leqslant s - 1.$$

Proof We use induction on s.

If $s = 1$, then

$$\operatorname{reg}(S/I) = \operatorname{reg}(S/P_1) = 0 \leqslant s - 1.$$

Note that

$$\bigcap_{i=1}^{s-1} P_i + P_s = \bigcap_{i=1}^{s-1} (P_i + P_s)$$

since these ideals are all monomial ideals. Then there exists a short exact sequence

$$0 \longrightarrow S/I \longrightarrow S/\bigcap_{i=1}^{s-1} P_i \oplus S/P_s \longrightarrow S/\bigcap_{i=1}^{s-1}(P_i + P_s) \longrightarrow 0.$$

Note that

$$\operatorname{reg}(S/P_i) = 0, \quad i \in \{1, 2, \ldots, s\}.$$

Then, by induction and Lemma 1, we have

$$\begin{aligned} \operatorname{reg}(S/I) &\leqslant \max\left\{\operatorname{reg}\left(S/\bigcap_{i=1}^{s-1} P_i \oplus S/P_s\right), \operatorname{reg}\left(S/\bigcap_{i=1}^{s-1}(P_i + P_s)\right) + 1\right\} \\ &\leqslant \max\left\{\operatorname{reg}\left(S/\bigcap_{i=1}^{s-1} P_i\right), \operatorname{reg}(S/P_s), \operatorname{reg}\left(S/\bigcap_{i=1}^{s-1}(P_i + P_s)\right) + 1\right\} \\ &\leqslant s - 1. \end{aligned}$$

□

In order to present our main result in this paper, we recall the notion of size and bigsize of a monomial ideal. Let $I \subset S$ be a squarefree monomial, and let $I = \cap_{i=1}^{s} P_i$ be an irredundant intersection of prime monomial ideals. The *size of* I, denoted by $\mathrm{size}_S(I)$, is the number

$$v + n - \mathrm{height}\left(\sum_{j=1}^{s} P_j\right) - 1,$$

where

$$v = \min\left\{t \,\Big|\, \sum_{k=1}^{t} P_{i_k} = \sum_{j=1}^{s} P_j \text{ holds for some integers } i_1 < i_2 < \cdots < i_t\right\}.$$

Replacing in the previous definition of v "for some integers $i_1 < i_2 < \cdots < i_t$" by "for any integers $i_1 < i_2 < \cdots < i_t$", one obtains the definition of *bigsize of* I, which is denoted by $\mathrm{bigsize}_S(I)$. The corresponding number v is denoted by $\text{b-size}_S(I)$. Clearly,

$$\mathrm{size}_S(I) \leqslant \mathrm{bigsize}_S(I) \leqslant s.$$

When

$$\sum_{i=1}^{s} P_i = (x_1, x_2, \ldots, x_n),$$

we have

$$\mathrm{bigsize}_S(I) = \text{b-size}_S(I) - 1.$$

Theorem 3 *Let $S = K[x_1, x_2, \ldots, x_n] = K[X]$. Let $I \subset S$ be a squarefree monomial ideal. Then*

$$\mathrm{reg}(S/I) \leqslant \text{b-size}_S(I) - 1.$$

In particular,

$$\mathrm{reg}(S/I) \leqslant \mathrm{bigsize}_S(I).$$

Proof Let $I = \cap_{i=1}^{s} P_i$ be its presentation as an irredundant intersection of prime monomial ideals. We prove the result by using induction on $\text{b-size}_S(I)$. We may assume that

$$\sum_{i=1}^{s} P_i = \mathbf{m} = (x_1, x_2, \ldots, x_n),$$

the graded maximal ideal of S. Indeed, let

$$Z = \left\{x_i \notin \sum_{i=1}^{s} P_i\right\}, \quad T = K[X \setminus Z], \quad J = I \cap T.$$

Then the sum of the associated prime ideals of J is the graded maximal ideal of T,

$$\operatorname{reg}(S/I) = \operatorname{reg}(T/J),$$

and by the definition of b-size and bigsize,

$$\text{b-size}_S(I) = \text{b-size}_T(J), \quad \operatorname{bigsize}_S(I) = \operatorname{bigsize}_T(J) + |Z|.$$

If $\text{b-size}_S(I) = 1$, then

$$I = P_1 = \mathbf{m},$$

so $\operatorname{reg}(S/I) = 0$ and the result is clear.

If $\text{b-size}_S(I) = 2$, then

$$S/\Big(\bigcap_{i=1}^{s-1} P_i + P_s\Big) = S/\bigcap_{i=1}^{s-1}(P_i + P_s) = S/\mathbf{m}.$$

By Lemma 1, the short exact sequence

$$0 \longrightarrow S/I \longrightarrow S/\bigcap_{i=1}^{s-1} P_i \oplus S/P_s \longrightarrow S/\Big(\bigcap_{i=1}^{s-1} P_i + P_s\Big) \longrightarrow 0$$

implies that

$$\begin{aligned}
\operatorname{reg}(S/I) &\leqslant \max\Big\{\operatorname{reg}\Big(S/\bigcap_{i=1}^{s-1} P_i \oplus S/P_s\Big), \operatorname{reg}\Big(S/\Big(\bigcap_{i=1}^{s-1} P_i + P_s\Big)\Big) + 1\Big\} \\
&\leqslant \max\Big\{\operatorname{reg}\Big(S/\bigcap_{i=1}^{s-1} P_i\Big), \operatorname{reg}(S/P_s), \operatorname{reg}(S/\mathbf{m}) + 1\Big\} \\
&\leqslant \max\Big\{\operatorname{reg}\Big(S/\bigcap_{i=1}^{s-1} P_i\Big), 1\Big\}.
\end{aligned}$$

When

$$s - 1 \geqslant \text{b-size}_S(I) = 2,$$

noting that

$$\text{b-size}_S\Big(\bigcap_{i=1}^{s-1} P_i\Big) = 2,$$

we replace the above ideal I by $\cap_{i=1}^{s-1} P_i$ and repeat the above process. Then

$$\operatorname{reg}(S/I) \leqslant \max\Big\{\operatorname{reg}\Big(S/\bigcap_{i=1}^{s-1} P_i\Big), 1\Big\} \leqslant 1 = \text{b-size}_S(I) - 1.$$

Now, we let

$$t = \text{b-size}_S(I) > 2,$$

and assume that the result holds for any squarefree monomial ideal with smaller b-size. Clearly, $t \leqslant s$.

By Lemma 1, the short exact sequence

$$0 \longrightarrow S/I \longrightarrow S/\bigcap_{i=1}^{s-1} P_i \oplus S/P_s \longrightarrow S/\Big(\bigcap_{i=1}^{s-1} P_i + P_s\Big) \longrightarrow 0$$

implies that

$$\begin{aligned}\operatorname{reg}(S/I) &\leqslant \max\Big\{\operatorname{reg}\Big(S/\bigcap_{i=1}^{s-1} P_i \oplus S/P_s\Big), \operatorname{reg}\Big(S/\Big(\bigcap_{i=1}^{s-1} P_i + P_s\Big)\Big) + 1\Big\}\\ &\leqslant \max\Big\{\operatorname{reg}\Big(S/\bigcap_{i=1}^{s-1} P_i\Big), \operatorname{reg}(S/P_s), \operatorname{reg}\Big(S/\bigcap_{i=1}^{s-1}(P_i + P_s)\Big) + 1\Big\}.\end{aligned}$$

Note that

$$\text{b-size}_S\Big(\bigcap_{i=1}^{s-1}(P_i + P_s)\Big) \leqslant t - 1.$$

So by induction,

$$\operatorname{reg}\Big(S/\bigcap_{i=1}^{s-1}(P_i + P_s)\Big) \leqslant t - 2.$$

Then

$$\operatorname{reg}(S/I) \leqslant \max\Big\{\operatorname{reg}\Big(S/\bigcap_{i=1}^{s-1} P_i\Big), t - 1\Big\}.$$

When $s - 1 > t$, noting that

$$\text{b-size}_S\Big(\bigcap_{i=1}^{s-1} P_i\Big) \leqslant t,$$

we replace the above ideal I by $\cap_{i=1}^{s-1} P_i$ and repeat the above process until $s - 1 \leqslant t$. Then

$$\operatorname{reg}(S/I) \leqslant \max\Big\{\operatorname{reg}\Big(S/\bigcap_{i=1}^{s-1} P_i\Big), t - 1\Big\} \leqslant t - 1 = \text{b-size}_S(I) - 1.$$

(The last inequality holds by Lemma 2.) This completes the proof. □

Example 4 Let $S = K[x_1, x_2, x_3, x_4, x_5]$, and let

$$\begin{aligned} I &= (x_1x_3, x_1x_4, x_2x_5, x_2x_3)\\ &= (x_1, x_2) \cap (x_1, x_3, x_5) \cap (x_3, x_4, x_5) \cap (x_2, x_3, x_4).\end{aligned}$$

Then

$$\operatorname{size}_S(I) = 1, \quad \operatorname{bigsize}_S(I) = 2.$$

By Theorem 3,

$$\operatorname{reg}(S/I) \leqslant 2.$$

If $\operatorname{reg}(I) = 2$, then I has a linear resolution. This is impossible. So

$$\operatorname{reg}(S/I) = 2.$$

This tells us that the upper bound $\operatorname{bigsize}_S(I)$ of $\operatorname{reg}(S/I)$ cannot be refined by $\operatorname{size}_S(I)$.

Remark 5 Let $I \subset S$ be a squarefree monomial ideal. It is well known that

$$\dim(S/I) \geqslant \operatorname{reg}(S/I)$$

(see [15, Chap. 2, Lemma 2.5]). In addition, some upper bounds of $\operatorname{reg}(I)$ were provided by Frühbis-Krüger and Terai [7], Hoa and Trung [11], and so on. Theorem 3 also provides an upper bound for the regularity of I.

Example 6 Let $S = K[x_1, x_2, x_3, x_4, x_5]$, and let

$$I = (x_1x_4, x_2x_4x_5, x_3x_4x_5) = (x_1, x_2, x_3) \cap (x_4) \cap (x_1, x_5).$$

Then

$$\operatorname{bigsize}_S(I) = 2, \quad \dim_S(S/I) = 4, \quad \operatorname{depth}_S(S/I) = 2.$$

By Theorem 3, $\operatorname{reg}(S/I) \leqslant 2$, and so $\operatorname{reg}(S/I) = 2$. Thus, the bound in Theorem 3 is tight in some cases. In particular, [7, Theorem 4.1 or 4.3] is not applicable in this situation.

Example 7 Let $S = K[x_1, x_2, \ldots, x_7]$, and let

$$\begin{aligned} I &= (x_1x_3, x_2x_4, x_5x_6x_7) \\ &= (x_1, x_2, x_5) \cap (x_2, x_3, x_5) \cap (x_1, x_4, x_5) \cap (x_1, x_4, x_6) \\ &\quad \cap (x_3, x_4, x_5) \cap (x_3, x_4, x_6) \cap (x_1, x_4, x_7) \cap (x_1, x_2, x_6) \\ &\quad \cap (x_1, x_2, x_7) \cap (x_2, x_3, x_6) \cap (x_2, x_3, x_7) \cap (x_3, x_4, x_7). \end{aligned}$$

Then

$$\operatorname{size}_S(I) = 2, \quad \operatorname{bigsize}_S(I) = 6, \quad \operatorname{reg}(S/I) = 4, \quad \dim_S(S/I) = 4.$$

This tells us that Theorem 3 provides the upper bound $\operatorname{bigsize}_S(I)$ for $\operatorname{reg}(I)$ is bad sometimes.

As before, let $S = K[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n variables over a field K, and let Δ be a simplicial complex on $[n]$. For each subset $F \subset [n]$, we set

$$x_F = \prod_{i \in F} x_i.$$

Recall that the Stanley-Reisner ideal of Δ is the ideal I_Δ of S which is generated by those squarefree monomials x_F with $F \notin \Delta$. One sets

$$K[\Delta] = S/I_\Delta.$$

Then the Alexander dual of Δ is defined to be the simplicial complex

$$\Delta^\vee = \{[n] \setminus F \colon F \notin \Delta\}.$$

Clearly, one has

$$(\Delta^\vee)^\vee = \Delta.$$

Let

$$I_\Delta = P_{F_1} \cap P_{F_2} \cap \cdots \cap P_{F_m}$$

be the standard primary decomposition of I_Δ. Here,

$$P_G = (\{x_i\}_{i \in G}), \quad G \subset [n].$$

Then $\{x_{F_1}, x_{F_2}, \ldots, x_{F_m}\}$ is the minimal monomial set of generators of $I_{\Delta^\vee}$. It is well known that (see [17])

$$\operatorname{projdim}(I_\Delta) = \operatorname{reg}(K[\Delta^\vee]).$$

Let I be a squarefree monomial ideal minimally generated by the monomials $u_1, u_2, \ldots, u_m$. Let

$$G(I) = \{u_1, u_2, \ldots, u_m\}.$$

Recently, the notion of the big cosize of I was defined by Herzog et al. [10]. Let w be the smallest number t with the property that for all integers $1 \leqslant i_1 < i_2 < \cdots < i_t \leqslant m$,

$$\operatorname{lcm}(u_{i_1}, u_{i_2}, \ldots, u_{i_t}) = \operatorname{lcm}(u_1, u_2, \ldots, u_m).$$

Then the number $\deg \operatorname{lcm}(u_1, u_2, \ldots, u_m) - w$ is called the *big cosize of* I, denoted by $\operatorname{bigcosize}_S(I)$. We denote the corresponding number w by $\operatorname{b-cosize}_S(I)$.

It is well known that $\operatorname{projdim}(M) \leqslant n$ for any finitely generated S-module M. Lyubeznik also gave a result that

$$\operatorname{projdim}(S/I) \leqslant \operatorname{ara}(I)$$

if I is a squarefree monomial ideal. So the following result relates these two results in case that I is a squarefree monomial ideal.

Corollary 8 *Let I be a squarefree monomial ideal of S. Then*

$$\operatorname{projdim}(S/I) \leqslant \operatorname{b-cosize}_S(I).$$

Proof Let Δ be the simplicial complex with the property that $I = I_\Delta$. Note that

$$\text{b-cosize}_S(I_\Delta) = \text{b-size}_S(I_{\Delta^\vee}).$$

Then, by Theorem 3,

$$\text{projdim}(I_\Delta) = \text{reg}(K[\Delta^\vee]) \leqslant \text{b-size}_S(I_{\Delta^\vee}) - 1 = \text{b-cosize}_S(I_\Delta) - 1. \qquad \square$$

Let u be a monomial in S. Set

$$\text{m}(u) = \max\{i \mid x_i \mid u\}.$$

It shows that the generators of a squarefree stable monomial ideal have the following property.

Corollary 9 *Let I be a squarefree stable ideal of S. Then for any $u \in G(I)$,*

$$\text{m}(u) - \deg(u) + 1 \leqslant \text{b-cosize}_S(I).$$

Proof Note that, for the squarefree stable ideal I,

$$\text{projdim}(I) = \max\{\text{m}(u) - \deg(u) \mid u \in G(I)\}$$

(see [9, Corollary 7.4.2]). Then the result follows. $\square$

Corollary 10 *Let I be a squarefree monomial ideal of S. Then*

$$\text{grade}(I, S) \leqslant \text{b-cosize}_S(I).$$

Proof By using the Auslander-Buchsbaum formula and Corollary 8, we have

$$\begin{aligned} \text{b-cosize}_S(I) - 1 &\geqslant \text{projdim}(I) \\ &= n - \text{depth}(I) \\ &= \dim(S) - \text{depth}(S/I) - 1 \\ &\geqslant \dim(S) - \dim(S/I) - 1 \\ &\geqslant \text{grade}(I, S) - 1. \end{aligned}$$

Then the result follows. $\square$

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