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RESEARCH ARTICLE

Bounds of weighted multilinear Hardy-Cesàro operators in p-adic functional spaces

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Abstract We introduce the p-adic weighted multilinear Hardy-Cesàro operator. We also obtain the necessary and sufficient conditions on weight functions to ensure the boundedness of that operator on the product of Lebesgue spaces, Morrey spaces, and central bounded mean oscillation spaces. In each case, we obtain the corresponding operator norms. We also characterize the good weights for the boundedness of the commutator of weighted multilinear Hardy-Cesàro operator on the product of central Morrey spaces with symbols in central bounded mean oscillation spaces.

Keywords Weighted multilinear Hardy-Cesàro operator, bounded mean oscillation (BMO), commutator, p-adic analysis
MSC 42B35, 46E30, 42B15, 42B30

1 Introduction

Theories of functions and operators from $\mathbb{Q}_p^n$ into $\mathbb{R}$ or $\mathbb{C}$ play an important role in the theory of dynamical systems, in the stochastic analysis, in the p-adic quantum mechanics, and in p-adic analysis [1,4,7,15,19,20,23,24]. p-adic analysis and non-Archimedean geometry can be used not only for the description of geometry at small distances, but also for describing chaotic behavior of complicated systems such as spin glasses and fractals in the framework of traditional theoretical and mathematical physics (see [16,17,23,24] and references therein). As far as we know, the studies of the p-adic Hardy operators and p-adic Hausdorff operators are also useful for p-adic analysis [4,7,8,13,22, 25,26].

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The weighted Hardy averaging operators are defined for measurable functions on $\mathbb{Q}_p$ by

$$U_\psi^p f(x) = \int_{\mathbb{Z}_p^\star} f(tx)\psi(t)\mathrm{d}t, \quad x \in \mathbb{Q}_p^d, \tag{1.1}$$

where $\mathbb{Z}_p^\star$ is the ring of p-adic non-zero integers, and $\mathrm{d}t$ is the Haar measure on $\mathbb{Q}_p$. Rim and Lee [22] considered the problem of characterizing function ψ on $\mathbb{Z}_p^\star$ so that we have inequalities

$$\|U_\psi^p f\|_X \leqslant C\|f\|_X,$$

where X is a p-adic Lebesgue or bounded mean oscillation (BMO) space. The corresponding best constants C are also obtained by these authors. Such results has been extended by Hung [13], where he considered the above problems for a more general class of p-adic weighted Hardy averaging operators, which are called p-adic Hardy-Cesàro operators,

$$U_{\psi,s}^p f(x) = \int_{\mathbb{Z}_p^\star} f(s(t)x)\psi(t)\mathrm{d}t. \tag{1.2}$$

Here, $s\colon \mathbb{Z}_p^\star \to \mathbb{Q}_p$ and $\psi\colon \mathbb{Z}_p^\star \to [0,\infty)$ are measurable functions. The characterizations on function $\psi(t)$, under certain conditions on $s(t)$, so that

$$\|U_{\psi,s}^p f\|_X \leqslant C\|f\|_X, \quad \forall\, f \in X,$$

where X is a p-adic Lebesgue or BMO space, are obtained. The best constants C in the above inequalities are worked out, too. It is interesting to notice that, by applying the boundedness of $U_{\psi,s}$ on p-adic weighted Lebesgue spaces, Hung gave a relation between p-adic Hardy operators and discrete Hardy inequalities on the real field.

Our aim of this paper is to study a family of p-adic weighted multilinear Hardy averaging operators, which was considered very recently by Hung and Ky [14] and by us [6], on the real case. We define the p-adic weighted multilinear Hardy-Cesàro operator $U_{\psi,\boldsymbol{s}}^{p,m,n}$ as follows.

Definition 1.1 Let m and n be positive integer numbers, and let $\psi\colon (\mathbb{Z}_p^\star)^n \to [0,\infty)$, $\boldsymbol{s} = (s_1,\dots,s_m)\colon (\mathbb{Z}_p^\star)^n \to \mathbb{Q}_p^m$ be measurable. The p-adic weighted multilinear Hardy-Cesàro operator $U_{\psi,\boldsymbol{s}}^{p,m,n}$, which define on $\boldsymbol{f} = (f_1,\dots,f_m)\colon \mathbb{Q}_p^d \to \mathbb{C}^m$, vector of measurable functions, by

$$U_{\psi,\boldsymbol{s}}^{p,m,n}(f_1,\dots,f_m)(x) = \int_{(\mathbb{Z}_p^\star)^n} \prod_{k=1}^m f_k(s_k(t)x)\psi(t)\mathrm{d}t. \tag{1.3}$$

When $m = n = 1$, $U_{\psi,\boldsymbol{s}}^{p,m,n}$ is reduced to $U_{\psi,s}^p$ as defined before. In this paper, we establish the sharp bounds of $U_{\psi,\boldsymbol{s}}^{p,m,n}$ on the product of weighted Lebesgue spaces and weighted central Morrey spaces. We also consider the problem of characterizing weights so that the commutators of $U_{\psi,\boldsymbol{s}}^{p,2,n}$ are bounded on the central Morrey spaces.

This paper is organized as follows. In Section 2, we give the notation and definitions that we shall use in the sequel. We define the weighted Morrey spaces $L_\omega^{q,\lambda}(\mathbb{Q}_p^d)$, the weighted central Morrey spaces $\dot{B}_\omega^{q,\lambda}(\mathbb{Q}_p^d)$, and the p-adic weighted central BMO spaces $CMO_\omega^{q,\lambda}(\mathbb{Q}_p^d)$. In Section 3, we state the main results on the boundedness of $U_{\psi,\boldsymbol{s}}^{p,m,n}$ on the above weighted spaces. We also work out the norms of $U_{\psi,\boldsymbol{s}}^{p,m,n}$ on such spaces. In Section 4, we obtain the sufficient and necessary results for the boundedness of commutator operators of $U_{\psi,\boldsymbol{s}}^{p,2,n}$, with symbols in the weighted central BMO spaces, on the weighted central Morrey spaces.

2 Basic notions and lemmas

Let p be a prime number, and let $r \in \mathbb{Q}^\star$. Write

$$r = p^\gamma \frac{a}{b},$$

where a and b are integers not divisible by p. Define the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}$ by $|r|_p = p^{-\gamma}$ and $|0|_p = 0$. The absolute value $|\cdot|_p$ gives a metric on $\mathbb{Q}$ defined by

$$d_p(x, y) = |x - y|_p.$$

We denote by $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ with respect to the metric d. $\mathbb{Q}_p$ with natural operations and topology induced by the metric d_p is a locally compact, non-discrete, complete, and totally disconnected field. A non-zero element x of $\mathbb{Q}_p$, is uniquely represented as a canonical form

$$x = p^\gamma(x_0 + x_1 p + x_2 p^2 + \cdots),$$

where $x_j \in \mathbb{Z}/p\mathbb{Z}$ and $x_0 \neq 0$. We then have $|x|_p = p^{-\gamma}$. Define

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \colon |x|_p \leqslant 1\}, \quad \mathbb{Z}_p^\star = \mathbb{Z}_p \setminus \{0\}.$$

$\mathbb{Q}_p^n = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p$ contains all n-tuples of $\mathbb{Q}_p$. The norm on $\mathbb{Q}_p^n$ is

$$|x|_p = \max_{1 \leqslant k \leqslant n} |x_k|_p, \quad x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n.$$

The space $\mathbb{Q}_p^n$ is a complete metric locally compact and totally disconnected space. For each $a \in \mathbb{Q}_p$ and $x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n$, we denote

$$ax = (ax_1, \ldots, ax_n).$$

For $\gamma \in \mathbb{Z}$, we denote B_γ as a γ-ball of $\mathbb{Q}_p^n$ with center at 0, containing all x with $|x|_p \leqslant p^\gamma$, and $S_\gamma = B_\gamma \setminus B_{\gamma-1}$ its boundary. Also, for $a \in \mathbb{Q}_p^d$, $B_\gamma(a)$ consists of all x with $x - a \in B_\gamma$, and $S_\gamma(a)$ consists of all x with $x - a \in S_\gamma$.

Since $\mathbb{Q}_p^d$ is a locally-compact commutative group with respect to addition, there exists the Haar measure $\mathrm{d}x$ on the additive group of $\mathbb{Q}_p^d$ normalized by

$$\int_{B_0} \mathrm{d}x = 1.$$

Then

$$\mathrm{d}(ax) = |a|_p^d \mathrm{d}x,\ \forall\, a \in \mathbb{Q}_p^\star, \quad |B_\gamma(x)| = p^{d\gamma}, \quad |S_\gamma(x)| = p^{d\gamma}(1 - p^{-d}).$$

We shall consider the class of weights $\mathscr{W}_\alpha$, which consists of all nonnegative locally integrable function ω on $\mathbb{Q}_p^d$ so that

$$\omega(tx) = |t|_p^\alpha \omega(x),\ \forall\, x \in \mathbb{Q}_p^d, \forall\, t \in \mathbb{Q}_p^\star, \quad 0 < \int_{S_0} \omega(x)\mathrm{d}x < \infty.$$

It is easy to see that $\omega(x) = |x|_p^\alpha$ is in $\mathscr{W}_\alpha$ if and only if $\alpha > -d$.

Let ω be any weight function on $\mathbb{Q}_p^d$, that is a nonnegative, locally integrable function from $\mathbb{Q}_p^d$ into $\mathbb{R}$. Let $L_\omega^r(\mathbb{Q}_p^d)$ $(1 \leqslant r < \infty)$ be the space of complex-valued functions f on $\mathbb{Q}_p^d$ so that

$$\|f\|_{L_\omega^r(\mathbb{Q}_p^d)} = \left(\int_{\mathbb{Q}_p^d} |f(x)|^r \omega(x)\mathrm{d}x \right)^{1/r} < \infty.$$

For further readings on p-adic analysis, see [23,24]. Here, some often used computational principles are worth mentioning at the outset. First, for $f \in L_\omega^1(\mathbb{Q}_p)$, we can write

$$\int_{\mathbb{Q}_p^d} f(x)\omega(x)\mathrm{d}x = \sum_{\gamma\in\mathbb{Z}} \int_{S_\gamma} f(y)\omega(y)\mathrm{d}y. \tag{2.1}$$

Second, we also often use the fact that

$$\int_{\mathbb{Q}_p^d} f(ax)\mathrm{d}x = \frac{1}{|a|_p^d} \int_{\mathbb{Q}_p^d} f(x)\mathrm{d}x, \quad \forall\, a \in \mathbb{Q}_p^d \setminus \{0\}, \forall\, f \in L^1(\mathbb{Q}_p^d). \tag{2.2}$$

It is well known that Morrey spaces are useful to study the local behavior of solutions to second-order elliptic partial differential equations and the boundedness of Hardy-Littlewood maximal operator, the fractional integral operators, and singular integral operators. We notice that the weighted Morrey spaces in Eucliden settings were introduced by Komori and Shirai [18], where they used them to study the boundedness of some important classical operators in harmonic analysis such as Hardy-Littlewood maximal operators and Calderón-Zygmund operators. Their p-adic versions are given in the followings.

Definition 2.1 Let ω be a weight function on $\mathbb{Q}_p^d$, $1 \leqslant q \leqslant \infty$, and let λ be a real number such that $-1/q \leqslant \lambda < \infty$. The weighted p-adic Morrey space $L^{q,\lambda}_{\omega}(\mathbb{Q}_p^d)$ is defined by the set of all functions $f \in L^q_{\omega,\mathrm{loc}}(\mathbb{Q}_p^d)$ so that

$$\|f\|_{L^{q,\lambda}_{\omega}(\mathbb{Q}_p^d)} < \infty,$$

where

$$\|f\|_{L^{q,\lambda}_{\omega}(\mathbb{Q}_p^d)} = \sup_{\gamma\in\mathbb{Z}} \sup_{a\in\mathbb{Q}_p^d} \left(\frac{1}{\omega(B_\gamma(a))^{1+\lambda q}} \int_{B_\gamma(a)} |f(x)|^q \omega(x) \mathrm{d}x \right)^{1/q}. \tag{2.3}$$

With the norm $\|\cdot\|_{L^{q,\lambda}_{\omega}(\mathbb{Q}_p^d)}$, $L^{q,\lambda}_{\omega}(\mathbb{Q}_p^d)$ becomes a Banach space. From Definition 2.1, it is easy to get

$$L^{q,-1/q}_{\omega}(\mathbb{Q}_p^d) = L^q_{\omega}(\mathbb{Q}_p^d).$$

Here, we restrict our consideration in case when λ belongs to $[-1/q, \infty)$ since the fact that $L^{q,\lambda}_{\omega}(\mathbb{Q}_p^d) = \{0\}$ for any $\lambda < -1/q$. One of useful example for functions from p-adic weighted Morrey spaces is given in the following lemma.

Lemma 2.1 *Let $1 < q < \infty$, $-1/q \leqslant \lambda \leqslant 0$, and $\omega \in \mathscr{W}_\alpha$, where $\alpha > -d$. If $f_0(x) = |x|_p^{(d+\alpha)\lambda}$, then $f_0 \in L^{q,\lambda}_{\omega}(\mathbb{Q}_p^d)$ and $\|f_0\|_{L^{q,\lambda}_{\omega}(\mathbb{Q}_p^d)} > 0$.*

Proof Let $a \in \mathbb{Q}_p^n$ and $\gamma \in \mathbb{Z}$, and put

$$I_{a,\gamma} = \frac{1}{\omega(B_\gamma(a))^{1+\lambda q}} \int_{B_\gamma(a)} |f_0(x)|^q \omega(x) \mathrm{d}x.$$

Since $f_0(x) > 0$ almost everywhere $x \in \mathbb{Q}_p^n$, it is enough to prove $I_{a,\gamma} \leqslant C$, where C is a positive constant which does not depend on a, γ. We consider two cases.

Case 1 $|a|_p = p^{\gamma'} > p^\gamma$.

For each $x \in B_\gamma(a)$, we have

$$|x|_p = \max\{|a|_p, |x-a|_p\} = |a|_p.$$

This implies $B_\gamma(a) \subset S_{\gamma'}$. As a consequence, we have

$$\begin{aligned} I_{a,\gamma} &= \frac{1}{\omega(B_\gamma(a))^{1+\lambda q}} \int_{B_\gamma(a)} |x|^{(n+\alpha)\lambda q} \omega(x) \mathrm{d}x \\ &= (|a|_p^{-(n+\alpha)} \omega(B_\gamma(a)))^{-\lambda q} \\ &\leqslant (|a|_p^{-(n+\alpha)} \omega(S_{\gamma'}))^{-\lambda q} \\ &= (\omega(S_0))^{-\lambda q} \\ &< \infty. \end{aligned}$$

Case 2 $|a|_p \leqslant p^\gamma$.

In this case, $B_\gamma(a) = B_\gamma$. Similarly, we get

$$I_{a,\gamma} \leqslant (p^{-(n+\alpha)\gamma}\omega(B_\gamma))^{-\lambda q} = (\omega(B_0))^{-\lambda q}.$$

Thus, we obtain

$$I_{a,\gamma} \leqslant \max\{(\omega(S_0))^{-\lambda q}, (\omega(B_0))^{-\lambda q}\}, \quad \forall\, (a,\gamma) \in \mathbb{Q}_p^n \times \mathbb{Z}.$$

This completes the proof of the lemma. □

We also consider here λ-central bounded mean oscillation spaces, which are very close to Morrey spaces. The class of such spaces turn out to be useful to study the continuity of Hardy operators ([12,14]).

Definition 2.2 Let λ and q be real numbers so that $1 < q < \infty$. We define the p-adic weighted central Morrey space $\dot{B}_\omega^{q,\lambda}(\mathbb{Q}_p^d)$ by the set of all functions f on $\mathbb{Q}_p^d$ satisfying $f \in L^q_{\omega,\mathrm{loc}}(\mathbb{Q}_p^d)$ such that $\|f\|_{\dot{B}_\omega^{q,\lambda}(\mathbb{Q}_p^d)} < \infty$, where

$$\|f\|_{\dot{B}_\omega^{q,\lambda}(\mathbb{Q}_p^d)} = \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{\omega(B_\gamma)^{1+\lambda q}} \int_{B_\gamma} |f(x)|^q \omega(x)\mathrm{d}x\right)^{1/q}. \tag{2.4}$$

It is clear that $L_\omega^{q,\lambda}(\mathbb{Q}_p^d)$ is continuously embedded in $\dot{B}_\omega^{q,\lambda}(\mathbb{Q}_p^d)$ for all $q \in (1,\infty)$ and $\lambda \in \mathbb{R}$. Moreover, $\dot{B}_\omega^{q,\lambda}(\mathbb{Q}_p^d)$ is a Banach space and reduces to $\{0\}$ when $\lambda < -1/q$. We remark that if $1 < q_1 < q_2 < \infty$, then

$$\dot{B}_\omega^{q_2,\lambda}(\mathbb{Q}_p^d) \subset \dot{B}_\omega^{q_1,\lambda}(\mathbb{Q}_p^d), \quad \forall\, \lambda \in \mathbb{R}.$$

Indeed, this follows immediately from Hölder's inequality. On the other hand, while $b_0(x) = \log |x|_p$ belongs to BMO space (see [13, Lemma 6.1]), it is not hard to see that $b_0(x) \notin \dot{B}_\omega^{q,\lambda}(\mathbb{Q}_p^d)$.

Lemma 2.2 *Let $1 < q < \infty$, $-1/q \leqslant \lambda < 0$, and $\omega \in \mathscr{W}_\alpha$, $\alpha > -d$. Then the function $f_0(x) = |x|_p^{(d+\alpha)\lambda}$ belongs to $\dot{B}_\omega^{q,\lambda}(\mathbb{Q}_p^d)$.*

Proof From Lemma 2.1, f_0 belongs to $L_\omega^{q,\lambda}(\mathbb{Q}_p^d)$. Since $L_\omega^{q,\lambda}(\mathbb{Q}_p^d)$ is continuously included in $B_\omega^{q,\lambda}(\mathbb{Q}_p^d)$, we get $f_0 \in B_\omega^{q,\lambda}(\mathbb{Q}_p^d)$. □

The spaces of bounded central mean oscillation CMO^q appears naturally when considering the dual spaces of the homogeneous Herz type Hardy spaces (see [2,3,21] for the settings in $\mathbb{T}$ and $\mathbb{R}^d$). The p-adic setting of such spaces is as follows.

Definition 2.3 Let $\lambda < 1/d$ and $q \in (1,\infty)$ be two real numbers. The p-adic weighted space $CMO_\omega^{q,\lambda}(\mathbb{Q}_p^d)$ is defined as the set of all functions $f \in L^q_{\omega,\mathrm{loc}}(\mathbb{Q}_p^d)$ such that

$$\|f\|_{CMO_\omega^{q,\lambda}(\mathbb{Q}_p^d)} = \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{\omega(B_\gamma)^{1+\lambda q}} \int_{B_\gamma} |f(x) - f_{\omega,\gamma}|^q \omega(x)\mathrm{d}x\right)^{1/q} < \infty. \tag{2.5}$$

It is clear that, $CMO^{q,\lambda}_{\omega}(\mathbb{Q}_p^d)$ becomes a Banach space if we identify functions that differ in a constant. We denote $CMO^{q}_{\omega}(\mathbb{Q}_p^d)$ by $CMO^{q,0}_{\omega}(\mathbb{Q}_p^d)$. On the other hand, it follows from Definition 2.3 that, $B^{q,\lambda}_{\omega}(\mathbb{Q}_p^d)$ are Banach spaces continuously included in $CMO^{q}_{\omega}(\mathbb{Q}_p^d)$ spaces. By a simple argument, one can see that $CMO^{q,\lambda}_{\omega}(\mathbb{Q}_p^d)$ reduces to the constant functions when $\lambda < -1/q$. In this work, we study multilinear operators defined on products of functional spaces and we seek conditions to have the boundedness of operators on certain products of Banach spaces. We will use the notation

$$\|T\|_{X_1\times\cdots\times X_m\to X} = \sup_{\|f_j\|_{X_j}=1,\, 1\leqslant j\leqslant m} \|T(f_1,\dots,f_m)\|_X.$$

Throughout the whole paper, the letter C will indicate an absolute constant, probably different at different occurrences. The symbol $f \lesssim g$ means that $f \leqslant Cg$. With χ_E we will denote the characteristic function of a set E. With $|A|$ we will denote the Haar measure of a measurable set E, and E^c will be the set $\mathbb{Q}^d \setminus E$. With $\omega(E)$ we will denote by $\int_E \omega(x)\mathrm{d}x$.

3 Bounds of $U^{p,m,n}_{\psi,s}$ on product of Lebesgue spaces and spaces of Morrey type

Let X be $L^q_{\omega}(\mathbb{Q}_p^d)$ or $L^{q,\lambda}_{\omega}(\mathbb{Q}_p^d)$. Our purpose in this section is to characterize conditions on functions $\psi(t)$ and $s_1(t),\dots,s_m(t)$ such that

$$\|U^{p,m,n}_{\psi,\boldsymbol{s}}(f_1,\dots,f_m)\|_{X\times\cdots\times X} \leqslant C\prod_{k=1}^{m}\|f_k\|_X \tag{3.1}$$

holds for any $f_1,\dots,f_k$ and the best constant C is obtained. The main results of this section are Theorems 3.1–3.3.

In this and the next sections, if not explicitly stated otherwise, q,α,q_i,α_j are real numbers, $1\leqslant q<\infty$, $1\leqslant q_j<\infty$, $\alpha_j>-d$ for each $j=1,\dots,m$ so that

$$\frac{1}{q}=\frac{1}{q_1}+\cdots+\frac{1}{q_m}, \tag{3.2}$$

and

$$\alpha=\frac{q\alpha_1}{q_1}+\cdots+\frac{q\alpha_m}{q_m}. \tag{3.3}$$

For the weights $\omega_k\in\mathscr{W}_{\alpha_k}$, $k=1,\dots,m$, set

$$\omega(x)=\prod_{k=1}^{m}\omega_k^{q/q_k}(x). \tag{3.4}$$

It is obvious that $\omega\in\mathscr{W}_{\alpha}$.

Definition 3.1 We say that $(\omega_1,\ldots,\omega_m)$ satisfies the $\mathscr{W}_{\boldsymbol{\alpha}}$ condition if

$$\omega(S_0) \geqslant \prod_{k=1}^{m} \omega_k(S_0)^{q/q_k}. \tag{3.5}$$

For example, $(\omega_1,\ldots,\omega_m)$, where $\omega_k(x) = |x|_p^{\alpha_k}$ for $k = 1,\ldots,m$, satisfies the $\mathscr{W}_{\boldsymbol{\alpha}}$ condition.

Throughout this paper, $s_1,\ldots,s_m$ are measurable functions from $(\mathbb{Z}_p^{\star})^n$ into $\mathbb{Q}_p$ and we denote by $\boldsymbol{s}$ the vector $(s_1,\ldots,s_m)$.

Theorem 3.1 *Assume that* $(\omega_1,\ldots,\omega_m)$ *satisfies* $\mathscr{W}_{\boldsymbol{\alpha}}$ *condition and there exists a constant* $\beta > 0$ *such that*

$$|s_k(t_1,\ldots,t_n)|_p \geqslant \min\{|t_1|_p^{\beta},\ldots,|t_n|_p^{\beta}\} \quad \text{a.e. } t = (t_1,\ldots,t_n) \in (\mathbb{Z}_p^{\star})^n, \tag{3.6}$$

for every $k = 1,\ldots,m$. *Then there exists a constant* C *such that the inequality*

$$||U_{\psi,\boldsymbol{s}}^{p,m,n}(f_1,\ldots,f_m)||_{L^q_\omega(\mathbb{Q}_p^d)} \leqslant C \prod_{k=1}^{m} \|f_k\|_{L^{q_k}_{\omega_k}(\mathbb{Q}_p^d)} \tag{3.7}$$

holds for any measurable $f_1,\ldots,f_m$ *if and only if*

$$\mathscr{A} := \int_{(\mathbb{Z}_p^{\star})^n} \prod_{k=1}^{m} |s_k(t)|_p^{-(d+\alpha_k)/q_k} \psi(t)\mathrm{d}t < \infty. \tag{3.8}$$

Moreover, $\mathscr{A}$ *is the best constant* C *in* (3.7).

Proof As we noted above, $\omega \in \mathscr{W}_{\boldsymbol{\alpha}}$. First, suppose that $\mathscr{A}$ is finite. Let $f_k \in L^{q_k}_{\omega_k}(\mathbb{Q}_p^d)$. Using Minkowski's inequality, Hölder's inequality, and p-adic change of variable (2.2), we have

$$\begin{aligned}
&\|U_{\psi,\boldsymbol{s}}^{p,m,n}(f_1,\ldots,f_m)\|_{L^q_\omega(\mathbb{Q}_p^d)}\\
&\leqslant \left(\int_{\mathbb{Q}_p^d}\left(\int_{(\mathbb{Z}_p^{\star})^n}\prod_{k=1}^{m}|f_k(s_k(t)x)|\psi(t)\mathrm{d}t\right)^{q}\omega(x)\mathrm{d}x\right)^{1/q}\\
&\leqslant \int_{(\mathbb{Z}_p^{\star})^n}\left(\int_{\mathbb{Q}_p^d}\prod_{k=1}^{m}|f_k(s_k(t)x)|^{q}\omega(x)\mathrm{d}x\right)^{1/q}\psi(t)\mathrm{d}t\\
&\leqslant \int_{(\mathbb{Z}_p^{\star})^n}\prod_{k=1}^{m}\left(\int_{\mathbb{Q}_p^d}|f_k(s_k(t)x)|^{q_k}\omega_k(x)\mathrm{d}x\right)^{1/q_k}\psi(t)\mathrm{d}t\\
&= \mathscr{A}\prod_{k=1}^{m}\|f_k\|_{L^{q_k}_{\omega_k}(\mathbb{Q}_p^d)}\\
&< \infty.
\end{aligned}$$

Thus, $U_{\psi,\boldsymbol{s}}^{p,m,n}$ is bounded from $L^{q_1}_{\omega_1}(\mathbb{Q}_p^d)\times\cdots\times L^{q_m}_{\omega_m}(\mathbb{Q}_p^d)$ to $L^q_\omega(\mathbb{Q}_p^d)$ and the best constant C in (3.7) satisfies

$$C \leqslant \mathscr{A}. \tag{3.9}$$

In order to prove the converse, we first need the following lemma.

Lemma 3.1 *Let $\omega \in \mathscr{W}_\alpha$, $\alpha > -d$, and $\gamma > 0$. Then the functions*

$$f_{r,\gamma}(x) = \begin{cases} 0, & |x|_p < 1, \\ |x|_p^{-\frac{d+\alpha}{r} - \frac{1}{\gamma^2}}, & |x|_p \geqslant 1, \end{cases}$$

belong to $L^r_\omega(\mathbb{Q}_p^d)$ and

$$\|f_{r,\gamma}\|_{L^r_\omega(\mathbb{Q}_p^d)} = \Big(\frac{\omega(S_0)}{1 - p^{-r/\gamma^2}}\Big)^{1/r} > 0. \tag{3.10}$$

Proof From the formula for $f_{r,\gamma}$, we see that

$$\begin{aligned} \|f_{r,\gamma}\|^r_{L^r_\omega(\mathbb{Q}_p^d)} &= \int_{\mathbb{Q}_p^d} |f_{r,\gamma}|^r \omega(x) \mathrm{d}x \\ &= \int_{|x|_p \geqslant 1} |x|_p^{-(d+\alpha+\frac{r}{\gamma^2})} \omega(x) \mathrm{d}x \\ &= \sum_{k=0}^{\infty} \int_{S_k} p^{-k(d+\alpha+\frac{r}{\gamma^2})} \omega(x) \mathrm{d}x \\ &= \sum_{k=0}^{\infty} \int_{S_0} p^{-k(d+\alpha+\frac{r}{\gamma^2})} p^{k\alpha+kd} \omega(y) \mathrm{d}y \\ &= \sum_{k=0}^{\infty} p^{-kr/\gamma^2} \omega(S_0) \\ &= \frac{1}{1 - p^{-r/\gamma^2}} \, \omega(S_0) \\ &< \infty. \end{aligned}$$

Thus, $f_{r,\gamma} \in L^r_\omega(\mathbb{Q}_p^d)$ for each γ and (3.10) holds. $\square$

Now, assume that $U^{p,m,n}_{\psi,\boldsymbol{s}}$ is defined as a bounded operator from $L^{q_1}_{\omega_1}(\mathbb{Q}_p^d) \times \cdots \times L^{q_m}_{\omega_m}(\mathbb{Q}_p^d)$ to $L^q_\omega(\mathbb{Q}_p^d)$. Let γ be an arbitrary positive number and for each $k = 1, \ldots, m$, we set $\gamma_k := \sqrt{q_k/q}\,\gamma$ and

$$f_{q_k,\gamma_k}(x) = \begin{cases} 0, & |x|_p \leqslant 1, \\ |x|_p^{-\frac{d+\alpha_k}{q_k} - \frac{1}{\gamma_k^2}}, & |x|_p \geqslant 1. \end{cases}$$

By Lemma 3.1, $f_{q_k,\gamma_k} \in L^{q_k}_{\omega_k}(\mathbb{Q}_p^d)$ and

$$\|f_{q_k,\gamma_k}\|_{L^{q_k}_{\omega_k}(\mathbb{Q}_p^d)} = \Big(\frac{\omega_k(S_0)}{1 - p^{-q_k/\gamma_k^2}}\Big)^{1/q_k} > 0.$$

We fix $x \in \mathbb{Q}_p^d$ such that $|x|_p \geqslant 1$ and set

$$E_x = \bigcap_{k=1}^{m} \{t \in (\mathbb{Z}_p^\star)^n \colon |s_k(t)x|_p > 1\}.$$

From the assumption (3.6), there exists a subset F of $(\mathbb{Z}_p^\star)^n$ has measure zero and E_x is contained in

$$\{t \in (\mathbb{Z}_p^\star)^n \colon |t|_p \geqslant |x|_p^{-1/\beta}\} \setminus F.$$

Consequently, we have

$$\begin{aligned}
&\|U_{\psi,\boldsymbol{s}}^{p,m,n}(f_{q_1,\gamma_1},\dots,f_{q_m,\gamma_m})\|^q_{L^q_\omega(\mathbb{Q}_p^d)} \\
&= \int_{\mathbb{Q}_p^d} \Big| \int_{(\mathbb{Z}_p^\star)^n} \prod_{k=1}^m f_{q_k,\gamma_k}(s_k(t)x)\psi(t)\mathrm{d}t\Big|^q \omega(x)\mathrm{d}x \\
&= \int_{|x|_p\geqslant 1} \Big(\prod_{k=1}^m |x|_p^{-\frac{d+\alpha_k}{q_k}-\frac{1}{\gamma_k^2}}\Big)^q \Big| \int_{E_x} \prod_{k=1}^m |s_k(t)|_p^{-\frac{d+\alpha_k}{q_k}-\frac{1}{\gamma_k^2}}\psi(t)\mathrm{d}t\Big|^q \omega(x)\mathrm{d}x \\
&\geqslant \int_{|x|_p\geqslant 1} |x|_p^{-d-\alpha-\frac{q}{\gamma^2}} \Big(\int_{E_x} \prod_{k=1}^m |s_k(t)|_p^{-\frac{d+\alpha_k}{q_k}-\frac{1}{\gamma_k^2}}\psi(t)\mathrm{d}t\Big)^q \omega(x)\mathrm{d}x \\
&\geqslant \int_{|x|_p\geqslant p^\gamma} |x|_p^{-d-\alpha-\frac{q}{\gamma^2}} \omega(x)\mathrm{d}x \Big(\int_{E_x} \prod_{k=1}^m |s_k(t)|_p^{-\frac{d+\alpha_k}{q_k}-\frac{1}{\gamma_k^2}}\psi(t)\mathrm{d}t\Big)^q \\
&= p^{-q/\gamma} \prod_{k=1}^m \|f_{q_k,\gamma_k}\|^q_{L^{q_k}_{\omega_k}(\mathbb{Q}_p^d)} \Big(\int_{E_x} \prod_{j=1}^m |s_j(t)|_p^{-\frac{d+\alpha_j}{q_j}-\frac{1}{\gamma_j^2}}\psi(t)\mathrm{d}t\Big)^q.
\end{aligned}$$

Here, we denote E by the set $\{t \in (\mathbb{Z}_p^\star)^n \colon |t|_p \geqslant p^{-\gamma/\beta}\}$. Assumption (3.6) implies $E_x \supset E$. Thus, we have the following inequality:

$$\begin{aligned}
\int_E \prod_{k=1}^m |s_k(t)|_p^{-\frac{d+\alpha_k}{q_k}-\frac{1}{\gamma_k^2}}\psi(t)\mathrm{d}t &\leqslant p^{1/\gamma} \frac{\|U_{\psi,\boldsymbol{s}}^{p,m,n}(f_{q_1,\gamma_1},\dots,f_{q_m,\gamma_m})\|}{\prod_{k=1}^m \|f_{q_k,\gamma_k}\|_{L^{q_k}_{\omega_k}(\mathbb{Q}_p^d)}} \\
&\leqslant Cp^{1/\gamma},
\end{aligned}$$

where C is the constant in (3.7). Letting γ to infinity, by Lebesgue's dominated convergence theorem, we obtain

$$\int_{(\mathbb{Z}_p^\star)^n} \prod_{k=1}^m |s_k(t)|_p^{-(d+\alpha_k)/q_k}\psi(t)\mathrm{d}t \leqslant C. \tag{3.11}$$

From (3.8) and (3.11), we deduce that $\mathscr{A}$ is the best constant in (3.7). $\square$

Theorem 3.2 *Let $1 < q, q_k < \infty$, and let $\lambda, \alpha_k, \lambda_k$ be as in (3.2) and (3.3) such that $-1/q_k \leqslant \lambda_k < 0$ for $k = 1, \ldots, m$. Assume that $(\omega_1, \ldots, \omega_m)$ satisfies $\mathscr{W}_{\boldsymbol{\alpha}}$ condition. Put*

$$\lambda = \frac{d+\alpha_1}{d+\alpha}\lambda_1 + \cdots + \frac{d+\alpha_m}{d+\alpha}\lambda_m.$$

We assume

$$\mathscr{B} = \int_{(\mathbb{Z}_p^\star)^n} \prod_{k=1}^m |s_k(t)|_p^{(d+\alpha_k)\lambda_k} \psi(t)\mathrm{d}t < \infty \tag{3.12}$$

and

$$(\omega(B_0))^{(1+\lambda q)/q} \geqslant \prod_{k=1}^m (\omega_k(B_0))^{(1+\lambda_k q_k)/q_k}. \tag{3.13}$$

Here B_0 is the ball $\{x \in \mathbb{Q}_p^d\colon |x|_p \leqslant 1\}$. Then there exists a constant C such that the inequality

$$\|U_{\psi,\boldsymbol{s}}^{p,m,n}(f_1, \ldots, f_m)\|_{L_\omega^{q,\lambda}(\mathbb{Q}_p^d)} \leqslant C \prod_{k=1}^m \|f_k\|_{L_{\omega_k}^{q_k,\lambda_k}(\mathbb{Q}_p^d)} \tag{3.14}$$

holds for any measurable functions $f_1, \ldots, f_m$. Moreover, the best constant C in (3.14) equals $\mathscr{B}$.

Proof Suppose that $\mathscr{B}$ is finite. Since

$$\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m},$$

by Minkowski's inequality, Hölder's inequality, and p-adic change of variable, we have

$$\begin{aligned}
&\left(\frac{1}{\omega(B_\gamma(a))^{1+\lambda q}} \int_{B_\gamma(a)} |U_{\psi,\boldsymbol{s}}^{p,m,n}(f_1, \ldots, f_m)(x)|^q \omega(x)\mathrm{d}x\right)^{1/q} \\
&\leqslant \int_{(\mathbb{Z}_p^\star)^n} \left(\frac{1}{\omega(B_\gamma(a))^{1+\lambda q}} \int_{B_\gamma(a)} \Big|\prod_{k=1}^m f_k(s_k(t)x)\Big|^q \omega(x)\mathrm{d}x\right)^{1/q} \psi(t)\mathrm{d}t \\
&\leqslant \int_{(\mathbb{Z}_p^\star)^n} \prod_{k=1}^m \left(\frac{1}{\omega_k(B_\gamma(a))^{1+\lambda_k q_k}} \int_{B_\gamma(a)} |f_k(s_k(t)x)|^{q_k} \omega_k(x)\mathrm{d}x\right)^{1/q_k} \psi(t)\mathrm{d}t \\
&= \int_{(\mathbb{Z}_p^\star)^n} \prod_{k=1}^m \left(\int_{s_k(t)B_\gamma(a)} \frac{|f_k(y)|^{q_k} \omega_k(y)\mathrm{d}y}{\omega_k(s_k(t)B_\gamma(a))^{1+\lambda_k q_k}}\right)^{1/q_k} \prod_{j=1}^m |s_j(t)|_p^{(d+\alpha_j)\lambda_j} \psi(t)\mathrm{d}t \\
&\leqslant \mathscr{B} \prod_{k=1}^m \|f_k\|_{L_{\omega_k}^{q_k,\lambda_k}(\mathbb{Q}_p^d)}.
\end{aligned}$$

The last estimate ensures that $U^{p,m,n}_{\psi,\boldsymbol{s}}$ is a bounded operator from $L^{q_1,\lambda_1}_{\omega_1}(\mathbb{Q}^d_p)\times\cdots\times L^{q_m,\lambda_m}_{\omega_m}(\mathbb{Q}^d_p)$ to $L^{q,\lambda}_{\omega}(\mathbb{Q}^d_p)$ and the best constant C in (3.14) is not greater than $\mathscr{B}$.

For the converse, let us take

$$f_{0k}(x)=|x|_p^{(d+\alpha_k)\lambda_k},\quad k=1,\dots,m.$$

Applying Lemma 2.1, we have

$$f_{0k}(x)\in L^{q_k,\lambda_k}_{\omega_k}(\mathbb{Q}^d_p),\quad \|f_{0k}\|_{L^{q_k,\lambda_k}_{\omega_k}(\mathbb{Q}^d_p)}>0,\quad f_0(x)=|x|_p^{(d+\alpha)\lambda}\in L^{q,\lambda}_{\omega}(\mathbb{Q}^d_p).$$

On the other hand, we have

$$\begin{aligned}U^{p,m,n}_{\psi,\boldsymbol{s}}(f_{01},\dots,f_{0m})(x)&=\int_{(\mathbb{Z}^\star_p)^n}\prod_{k=1}^m f_{0k}(s_k(t)x)\psi(t)\mathrm{d}t\\&=\int_{(\mathbb{Z}^\star_p)^n}\prod_{k=1}^m |s_k(t)x|_p^{(d+\alpha_k)\lambda_k}\psi(t)\mathrm{d}t\\&=|x|_p^{(d+\alpha)\lambda}\int_{(\mathbb{Z}^\star_p)^n}\prod_{k=1}^m |s_k(t)|_p^{(d+\alpha_k)\lambda_k}\psi(t)\mathrm{d}t\\&=\mathscr{B}\cdot f_0(x).\end{aligned}$$

Therefore,

$$\|U^{p,m,n}_{\psi,\boldsymbol{s}}(f_{01},\dots,f_{0m})\|_{L^{q,\lambda}_{\omega}(\mathbb{Q}^d_p)}=\mathscr{B}\cdot\|f_0\|_{L^{q,\lambda}_{\omega}(\mathbb{Q}^d_p)}\leqslant C\cdot\|f_0\|_{L^{q,\lambda}_{\omega}(\mathbb{Q}^d_p)}.$$

So $\mathscr{B}$ is not greater than C, the constant in (3.14). Thus, Theorem 3.2 is proved. □

Theorem 3.3 *Let $q,q_k,\lambda,\alpha_k,\lambda_k$ be as in Theorem 3.2 with $q,q_k>1$, and let conditions (3.2) and (3.3) be hold. Assume that $(\omega_1,\dots,\omega_m)$ satisfies $\mathscr{W}_{\boldsymbol{\alpha}}$ condition. Then $U^{p,m,n}_{\psi,\boldsymbol{s}}$ is determined as a bounded operator from $\dot{B}^{q_1,\lambda_1}_{\omega_1}(\mathbb{Q}^d_p)\times\cdots\times\dot{B}^{q_m,\lambda_m}_{\omega_m}(\mathbb{Q}^d_p)$ to $\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}^d_p)$. Moreover,*

$$\|U^{p,m,n}_{\psi,\boldsymbol{s}}\|_{\dot{B}^{q_1,\lambda_1}_{\omega_1}(\mathbb{Q}^d_p)\times\cdots\times\dot{B}^{q_m,\lambda_m}_{\omega_m}(\mathbb{Q}^d_p)\to\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}^d_p)}=\mathscr{B}.\tag{3.15}$$

Proof From the proof of Theorem 3.2, with $a=0$, we obtain that

$$\left(\frac{1}{\omega(B_\gamma)^{1+\lambda q}}\int_{B_\gamma}|U^{p,m,n}_{\psi,\boldsymbol{s}}(f_1,\dots,f_m)(x)|^q\omega(x)\mathrm{d}x\right)^{1/q}\leqslant\mathscr{B}\prod_{k=1}^m\|f_k\|_{\dot{B}^{q_k,\lambda_k}_{\omega_k}(\mathbb{Q}^d_p)}$$

for all $f_k\in\dot{B}^{q_k,\lambda_k}_{\omega_k}(\mathbb{Q}^d_p)$. This implies that $U^{p,m,n}_{\psi,\boldsymbol{s}}$ is bounded on $\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}^d_p)$ if $\mathscr{B}$ is finite.

The converse is similar to the proof of Theorem 3.2 since $f_{0k}\in L^{q_k,\lambda_k}_{\omega_k}(\mathbb{Q}^d_p)$ implies that $f_{0k}\in\dot{B}^{q_k,\lambda_k}_{\omega_k}(\mathbb{Q}^d_p)$ for $k=1,\dots,m$, and $f_0(x)=|x|_p^{(d+\alpha)\lambda}\in\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}^d_p)$, thus we omit the details here. □

4 Sharp estimates for commutator of weighted bilinear Hardy-Cesàro operators

More recently, great attention was paid to the study on commutators of operators. A well-known result of Coifman et al. [10] states that the commutator $T_b f = bTf - T(bf)$ (where T is a Calderón-Zygmund singular integral operator) is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $b \in BMO(\mathbb{R}^n)$. Many results have been generalized to commutators of other operators, not only Calderón-Zygmund singular integral operators. In p-adic settings, commutators of integral operators of Hardy type were recently investigated in various papers (see, e.g., [8,9,13,27–29] and references therein).

In this section, we obtain sharp estimates of the commutator generated by bilinear operator $U^{p,2,n}_{\psi,\boldsymbol{s}}$ with symbols in $CMO^q_\omega(\mathbb{Q}^d_p)$. This commutator can be defined in formally as follows.

Definition 4.1 Let $m, n \in \mathbb{N}$, $\psi\colon (\mathbb{Z}^\star_p)^n \to [0,\infty)$, $s_1, \dots, s_m\colon (\mathbb{Z}^\star_p)^n \to \mathbb{Q}_p$, let $b_1, \dots, b_m$ be locally integrable functions on $\mathbb{Q}^d_p$, and let $f_1, \dots, f_m\colon \mathbb{Q}^d_p \to \mathbb{C}$ be measurable functions. The commutator of weighted multilinear Hardy-Cesàro operator $U^{p,m,n}_{\psi,\boldsymbol{s}}$ is defined as

$$\begin{aligned} &U^{p,m,n,\boldsymbol{b}}_{\psi,\boldsymbol{s}}(f_1, \dots, f_m)(x) \\ &\quad= \int_{(\mathbb{Z}^\star_p)^n} \Big(\prod_{k=1}^m f_k(s_k(t)x) \Big) \Big(\prod_{k=1}^m (b_k(x) - b_k(s_k(t)x)) \Big) \psi(t) \mathrm{d}t. \end{aligned} \tag{4.1}$$

In what follows, we set

$$\mathscr{C}_m = \int_{(\mathbb{Z}^\star_p)^n} \Big(\prod_{k=1}^m |s_k(t)|_p^{(d+\alpha_k)\lambda_k} \Big) \psi(t) \mathrm{d}t, \tag{4.2}$$

$$\mathscr{D}_m = \int_{(\mathbb{Z}^\star_p)^n} \Big(\prod_{k=1}^m |s_k(t)|_p^{(d+\alpha_k)\lambda_k} \Big) \Big(\prod_{k=1}^m |\log_p |s_k(t)|_p| \Big) \psi(t) \mathrm{d}t. \tag{4.3}$$

Notice that in case $m = 1$, we obtained the boundedness of such commutator on p-adic weighted central Morrey space $\dot{B}^{q,\lambda}_\omega(\mathbb{Q}^d_p)$ in [5]. Our purpose is to apply methods which were used in [5,11,14,27] to the case $m = 2$. We have the following result.

Theorem 4.1 *Let*

$$1 < q < q_k < \infty, \quad 1 < p_k < \infty, \quad -\frac{1}{p_k} < \lambda_k < 0, \quad k = 1, 2,$$

such that

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{p_1} + \frac{1}{p_2}$$

and $\lambda = \lambda_1 + \lambda_2$. *Assume*

$$\omega(x) = \omega_1^{\frac{q}{q_1}+\frac{q}{p_1}} \cdot \omega_2^{\frac{q}{q_2}+\frac{q}{p_2}} \quad \alpha = \frac{q\alpha_1}{q_1} + \frac{q\alpha_1}{p_1} + \frac{q\alpha_2}{q_2} + \frac{q\alpha_2}{p_2},$$

and

$$\omega(B_0)^{(1+\lambda q)/q} \geqslant \prod_{k=1}^{2} \omega_k(B_0)^{\frac{1+\lambda_k q_k}{q_k}+\frac{1}{p_k}}.$$

(i) *If both* $\mathscr{C}_2$ *and* $\mathscr{D}_2$ *are finite, then for any* $b = (b_1, b_2) \in CMO^{p_1}_{\omega_1}(\mathbb{Q}_p^d) \times CMO^{p_2}_{\omega_2}(\mathbb{Q}_p^d)$, $U^{p,2,n,\boldsymbol{b}}_{\psi,\boldsymbol{s}}$ *is bounded from* $\dot{B}^{q_1,\lambda_1}_{\omega_1}(\mathbb{Q}_p^d) \times \dot{B}^{q_2,\lambda_2}_{\omega_2}(\mathbb{Q}_p^d)$ *to* $\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}_p^d)$.

(ii) *If for any* $b = (b_1, b_2) \in CMO^{p_1}_{\omega_1}(\mathbb{Q}_p^d) \times CMO^{p_2}_{\omega_2}(\mathbb{Q}_p^d)$, $U^{p,2,n,\boldsymbol{b}}_{\psi,\boldsymbol{s}}$ *is bounded from* $\dot{B}^{q_1,\lambda_1}_{\omega_1}(\mathbb{Q}_p^d) \times \dot{B}^{q_2,\lambda_2}_{\omega_2}(\mathbb{Q}_p^d)$ *to* $\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}_p^d)$, *then* $\mathscr{D}_2$ *is finite.*

It is not hard to see that $\mathscr{D}_2 < \infty$ does not imply $\mathscr{C}_2 < \infty$. For example, if

$$s_k(t) \equiv 1, \quad \psi(t_1, \ldots, t_n) = \frac{1}{|t_1|_p \cdots |t_n|_p},$$

then $\mathscr{D}_2 = 0$ but $\mathscr{C}_2 = \infty$. The example below showed that $\mathscr{C}_2 < \infty$ does not imply $\mathscr{D}_2 < \infty$. Indeed, let

$$s_1(t) = \frac{1}{p}, \quad s_2(t_1, \ldots, t_n) = t_1, \quad \psi(t_1, \ldots, t_n) = \frac{g(t_1)}{\prod_{k=1}^{2} |s_k(t)|_p^{(d+\alpha_k)\lambda_k}},$$

where

$$g(t_1) = \begin{cases} 0, & |t_1|_p = 1, \\ \dfrac{1}{|t_1|_p \log_p^2 |t_1|_p}, & |t_1|_p < 1. \end{cases}$$

Then

$$\mathscr{C}_2 = |\mathbb{Z}_p^\star|^{n-1} \cdot \int_{\mathbb{Z}_p^\star} g(t_1)\mathrm{d}t_1 = |\mathbb{Z}_p^\star|^{n-1} \sum_{j=1}^{\infty} \Big(1 - \frac{1}{p}\Big) \frac{1}{j^2} < \infty.$$

However,

$$\mathscr{D}_2 = |\mathbb{Z}_p^\star|^{n-1} \cdot \int_{\mathbb{Z}_p^\star} |\log_p |t_1|_p| g(t_1)\mathrm{d}t_1 = |\mathbb{Z}_p^\star|^{n-1} \cdot \sum_{j=1}^{\infty} \Big(1 - \frac{1}{p}\Big) \frac{1}{j} = \infty.$$

Corollary 4.1 *Let* $1 < q < q_k < \infty$, $1 < p_k < \infty$, $-1/p_k < \lambda_k < 0$, $k = 1, 2$, *such that*

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{p_1} + \frac{1}{p_2}, \quad \lambda = \lambda_1 + \lambda_2.$$

Furthermore, suppose that $|s_k(t)|_p > 1$ *a.e.* $t \in (\mathbb{Z}_p^\star)^n$ *or* $|s_k(t)|_p < 1$ *a.e.* $t \in (\mathbb{Z}_p^\star)^n$, *for each* $k = 1, 2$. *Then* $U^{p,2,n,\boldsymbol{b}}_{\psi,\boldsymbol{s}}$ *is bounded from* $\dot{B}^{q_1,\lambda_1}_{\omega_1}(\mathbb{Q}_p^d) \times \dot{B}^{q_2,\lambda_2}_{\omega_2}(\mathbb{Q}_p^d)$ *to* $\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}_p^d)$ *if and only if* $\mathscr{D}_2$ *is finite.*

Since $|x|_p > 1$ implies $|x|_p \geqslant p$, we deduce that $\mathscr{D}_2 \geqslant (\log p)\mathscr{C}_2$. Thus, Corollary 4.1 follows immediately from Theorem 4.1. Before proving Theorem 4.1, we need the following lemma.

Lemma 4.1 *Suppose that b is a function in $CMO^{q,\lambda}_\omega(\mathbb{Q}_p^d)$ and γ, γ' are integer numbers. Here, $\lambda \in \mathbb{R}$ so that $\lambda \leqslant 1/d$, $1 < q < \infty$, and $\omega \in \mathscr{W}_\alpha$ with $\alpha > -d$. Then*

$$|b_{B_\gamma,\omega} - b_{B_{\gamma'},\omega}| \leqslant p^{d+\alpha} \cdot |\gamma' - \gamma| \cdot \max\{\omega(B_\gamma)^\lambda, \omega(B_{\gamma'})^\lambda\} \cdot c_\lambda \cdot \|b\|_{CMO^{q,\lambda}_\omega}.$$

Here and after,

$$c_\lambda = \begin{cases} 1, & \lambda = 0, \\ (d+\alpha)\log p \cdot \dfrac{p^{(d+\alpha)\lambda}}{|p^{(d+\alpha)\lambda} - 1|} \cdot |\lambda|, & \lambda \neq 0. \end{cases}$$

Proof It is enough to prove the lemma for $\gamma' > \gamma$. Applying Hölder's inequality, we have

$$\begin{aligned} |b_{B_{\gamma+1},\omega} - b_{B_\gamma,\omega}| &\leqslant \frac{1}{\omega(B_\gamma)} \int_{B_\gamma} |b(x) - b_{B_{\gamma+1},\omega}|\omega(x)\mathrm{d}x \\ &\leqslant \frac{1}{\omega(B_\gamma)} \int_{B_{\gamma+1}} |b(x) - b_{B_{\gamma+1},\omega}|\omega(x)\mathrm{d}x \\ &\leqslant \frac{\omega(B_{\gamma+1})^{(q-1)/q}}{\omega(B_\gamma)} \left(\int_{B_{\gamma+1}} |b(x) - b_{B_{\gamma+1},\omega}|^q \omega(x)\mathrm{d}x \right)^{1/q} \\ &= p^{d+\alpha} \cdot \omega(B_{\gamma+1})^\lambda \cdot \|b\|_{CMO^{q,\lambda}_\omega(\mathbb{Q}_p^d)}. \end{aligned}$$

Therefore,

$$|b_{B_{\gamma+1},\omega} - b_{B_\gamma,\omega}| \leqslant p^{d+\alpha} \cdot \omega(B_{\gamma+1})^\lambda \cdot \|b\|_{CMO^{q,\lambda}_\omega(\mathbb{Q}_p^d)}. \tag{4.4}$$

Now, we have

$$\begin{aligned} |b_{B_{\gamma'},\omega} - b_{B_\gamma,\omega}| &\leqslant \sum_{k=\gamma}^{\gamma'-1} |b_{B_{k+1},\omega} - b_{B_k,\omega}| \\ &\leqslant p^{d+\alpha}\|b\|_{CMO^{q,\lambda}_\omega(\mathbb{Q}_p^d)} \cdot \sum_{k=\gamma}^{\gamma'-1} \omega(B_{k+1})^\lambda \\ &= p^{d+\alpha}\|b\|_{CMO^{q,\lambda}_\omega(\mathbb{Q}_p^d)} \cdot \omega(B_{\gamma'})^\lambda \sum_{j=0}^{\gamma'-\gamma-1} p^{-(d+\alpha)\lambda j}. \end{aligned}$$

Therefore, it suffices to prove the lemma in case $\lambda \neq 0$. For the first case when $\lambda > 0$, by using the elementary inequality $1 - \mathrm{e}^{-x} \leqslant x$ in case $x =$

$(d+\alpha)\lambda(\gamma'-\gamma)\log p$, we obtain

$$
\begin{aligned}
&|b_{B_{\gamma'},\omega}-b_{B_\gamma,\omega}|\\
&\leqslant p^{d+\alpha}\cdot\frac{p^{(d+\alpha)\lambda}}{p^{(d+\alpha)\lambda}-1}\cdot(d+\alpha)\lambda(\gamma'-\gamma)\log p\|b\|_{CMO_\omega^{q,\lambda}(\mathbb{Q}_p^d)}\cdot\omega(B_{\gamma'})^\lambda\\
&=p^{d+\alpha}\cdot|\gamma'-\gamma|\cdot\max\{\omega(B_\gamma)^\lambda,\omega(B_{\gamma'})^\lambda\}\cdot c_\lambda\cdot\|b\|_{CMO_\omega^{q,\lambda}(\mathbb{Q}_p^d)}.
\end{aligned}
$$

For the rest case when $\lambda<0$, the proof is similar, so we omit it here. □

Proof of Theorem 4.1 Suppose first that $\mathscr{C}_2$ and $\mathscr{D}_2$ are finite. Let $\boldsymbol{b}=(b_1,b_2)\in CMO_{\omega_1}^{p_1}(\mathbb{Q}_p^d)\times CMO_{\omega_2}^{p_2}(\mathbb{Q}_p^d)$. Then Minkowski's inequality implies that

$$
\begin{aligned}
I&=\left(\frac{1}{\omega(B_\gamma)^{1+\lambda q}}\int_{B_\gamma}|U_{\psi,\boldsymbol{s}}^{p,2,n,\boldsymbol{b}}(f_1,f_2)(x)|^q\omega(x)\mathrm{d}x\right)^{1/q}\\
&\leqslant\left(\frac{1}{\omega(B_\gamma)^{1+\lambda q}}\int_{B_\gamma}\left(\int_{(\mathbb{Z}_p^\star)^n}\prod_{j=1}^2|f_j(s_j(t)x)|\right.\right.\\
&\qquad\left.\left.\cdot\prod_{k=1}^2|b_k(x)-b_k(s_k(t)x)|\psi(t)\mathrm{d}t\right)^q\omega(x)\mathrm{d}x\right)^{1/q}\\
&\leqslant\int_{(\mathbb{Z}_p^\star)^n}\left(\frac{1}{\omega(B_\gamma)^{1+\lambda q}}\int_{B_\gamma}\left(\prod_{j=1}^2|f_j(s_j(t)x)|\right.\right.\\
&\qquad\left.\left.\cdot\prod_{k=1}^2|b_k(x)-b_k(s_k(t)x)|\right)^q\omega(x)\mathrm{d}x\right)^{1/q}\psi(t)\mathrm{d}t.
\end{aligned}
$$

For any $x_i,y_i,z_i,t_i\in\mathbb{C}$ with $i=1,2$, we have the following elementary inequality:

$$
\begin{aligned}
\prod_{k=1}^2|x_i-y_i|\leqslant&\prod_{k=1}^2|x_i-z_i|+\prod_{k=1}^2|y_i-t_i|+\prod_{k=1}^2|z_i-t_i|\\
&+|(x_1-z_1)(z_2-t_2)|+|(x_2-z_2)(z_1-t_1)|\\
&+|(x_1-z_1)(y_2-t_2)|+|(x_2-z_2)(y_1-z_1)|\\
&+|(z_1-t_1)(y_2-t_2)|+|(z_2-t_2)(y_1-t_1)|.
\end{aligned}
$$

It is convenient to denote

$$
b_{i,B_\gamma,\omega}:=\int_{B_\gamma}\frac{1}{\omega(B_\gamma)}b_i(x)\omega(x)\mathrm{d}x,\quad i=1,2.
$$

Now, applying the inequality above with

$$
x_i=b_i(x),\quad y_i=b_i(s_i(t)x),\quad z_i=b_{i,B_\gamma},\quad t_i=b_{i,s_i(t)B_\gamma},
$$

and using Minkowski's inequality, we get

$$I \leqslant I_1 + \cdots + I_6,$$

where, if we set

$$\overline{f}(x) = \prod_{k=1}^{2} |f_k(s_k(t)x)|,$$

then

$$I_1 = \int_{(\mathbb{Z}_p^\star)^n} \left(\int_{B_\gamma} \left(\overline{f}(x) \prod_{k=1}^{2} |b_k(x) - b_{k,B_\gamma,\omega_k}| \right)^q \frac{\omega(x)\mathrm{d}x}{\omega(B_\gamma)^{1+\lambda q}} \right)^{1/q} \psi(t)\mathrm{d}t,$$

$$I_2 = \int_{(\mathbb{Z}_p^\star)^n} \left(\int_{B_\gamma} \left(\overline{f}(x) \prod_{k=1}^{2} |b_k(s_k(t)x) - b_{k,s_k(t)B_\gamma,\omega_k}| \right)^q \frac{\omega(x)\mathrm{d}x}{\omega(B_\gamma)^{1+\lambda q}} \right)^{1/q} \psi(t)\mathrm{d}t,$$

$$I_3 = \int_{(\mathbb{Z}_p^\star)^n} \left(\int_{B_\gamma} \left(\overline{f}(x) \prod_{k=1}^{2} |b_{k,B_\gamma,\omega_k} - b_{k,s_k(t)B_\gamma,\omega_k}| \right)^q \frac{\omega(x)\mathrm{d}x}{\omega(B_\gamma)^{1+\lambda q}} \right)^{1/q} \psi(t)\mathrm{d}t,$$

$$I_4 = \int_{(\mathbb{Z}_p^\star)^n} \left(\int_{B_\gamma} \left(\overline{f}(x) \sum_{1\leqslant i\neq j\leqslant 2} |(b_i(x) - b_{i,B_\gamma})(b_{j,B_\gamma} - b_{j,s_j(t)B_\gamma,\omega_j})| \right)^q \cdot \frac{\omega(x)\mathrm{d}x}{\omega(B_\gamma)^{1+\lambda q}} \right)^{1/q} \psi(t)\mathrm{d}t,$$

$$I_5 = \int_{(\mathbb{Z}_p^\star)^n} \left(\int_{B_\gamma} \left(\overline{f}(x) \sum_{1\leqslant i\neq j\leqslant 2} |(b_i(x) - b_{i,B_\gamma})(b_j(s_j(t)x) - b_{j,s_j(t)B_\gamma,\omega_j})| \right)^q \cdot \frac{\omega(x)\mathrm{d}x}{\omega(B_\gamma)^{1+\lambda q}} \right)^{1/q} \psi(t)\mathrm{d}t,$$

$$I_6 = \int_{(\mathbb{Z}_p^\star)^n} \left(\int_{B_\gamma} \left(\overline{f}(x) \sum_{1\leqslant i\neq j\leqslant 2} |(b_{i,B_\gamma} - b_{i,s_i(t)B_\gamma})(b_j(s_j(t)x) - b_{j,s_j(t)B_\gamma,\omega_j})| \right)^q \cdot \frac{\omega(x)\mathrm{d}x}{\omega(B_\gamma)^{1+\lambda q}} \right)^{1/q} \psi(t)\mathrm{d}t.$$

Choose now $q < s_1, s_2 < \infty$ such that

$$\frac{1}{s_1} = \frac{1}{p_1} + \frac{1}{q_1}, \quad \frac{1}{s_2} = \frac{1}{p_2} + \frac{1}{q_2}.$$

Notice that

$$\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{q}.$$

Then, by Hölder's inequality, we have

$$\begin{aligned}
I_1 &\leqslant \int_{(\mathbb{Z}_p^\star)^n} \prod_{j=1}^{2} \left(\frac{1}{\omega_j(B_\gamma)^{1+\lambda_j q_j}} \int_{B_\gamma} |f_j(s_j(t)x)|^{q_j} \omega_j(x) \mathrm{d}x \right)^{1/q_j} \\
&\quad \cdot \prod_{k=1}^{2} \left(\frac{1}{\omega_k(B_\gamma)} \int_{B_\gamma} |b_k(x) - b_{k,B_\gamma,\omega_k}|^{p_k} \omega_k(x) \mathrm{d}x \right)^{1/p_k} \psi(t) \mathrm{d}t \\
&\leqslant \int_{(\mathbb{Z}_p^\star)^n} \prod_{j=1}^{2} |s_j(t)|_p^{(d+\alpha_j)\lambda_j} (\omega_j(B_\gamma))^{\lambda_j} \prod_{k=1}^{2} \left(\int_{s_k(t)B_\gamma} \frac{|f_k(y)|^{q_k} \omega_k(y) \mathrm{d}y}{\omega_k(s_k(t)B_\gamma)^{1+\lambda_k q_k}} \right)^{1/q_k} \\
&\quad \cdot \prod_{l=1}^{2} \left(\frac{1}{\omega_l(B_\gamma)} \int_{B_\gamma} |b_l(x) - b_{l,B_\gamma,\omega_l}|^{p_l} \omega_l(x) \mathrm{d}x \right)^{1/p_l} \psi(t) \mathrm{d}t \\
&\leqslant \int_{(\mathbb{Z}_p^\star)^n} \prod_{j=1}^{2} |s_j(t)|_p^{(d+\alpha_j)\lambda_j} (\omega_j(B_\gamma))^{\lambda_j} \prod_{k=1}^{2} \|f_k\|_{\dot{B}^{q_k,\lambda_k}_{\omega_k}(\mathbb{Q}_p^d)} \\
&\quad \cdot \prod_{l=1}^{2} \|b_l\|_{CMO^{p_l}_{\omega_l}(\mathbb{Q}_p^d)} \psi(t) \mathrm{d}t \\
&= \prod_{i=1}^{2} (\omega_i(B_\gamma))^{\lambda_i} \prod_{j=1}^{2} \|b_j\|_{CMO^{p_j}_{\omega_j}(\mathbb{Q}_p^d)} \prod_{k=1}^{2} \|f_k\|_{\dot{B}^{q_k,\lambda_k}_{\omega_k}(\mathbb{Q}_p^d)} \\
&\quad \cdot \int_{(\mathbb{Z}_p^\star)^n} \prod_{l=1}^{2} |s_l(t)|_p^{(d+\alpha_l)\lambda_l} \psi(t) \mathrm{d}t.
\end{aligned}$$

Similar to the estimate of I_1, we have

$$\begin{aligned}
I_2 &\leqslant \int_{(\mathbb{Z}_p^\star)^n} \prod_{j=1}^{2} \left(\frac{1}{\omega_j(B_\gamma)^{1+\lambda_j q_j}} \int_{B_\gamma} |f_j(s_j(t)x)|^{q_j} \omega_j(x) \mathrm{d}x \right)^{1/q_j} \\
&\quad \cdot \prod_{k=1}^{2} \left(\frac{1}{\omega_k(B_\gamma)} \int_{B_\gamma} |b_k(s_k(t)x) - b_{k,s_k(t)B_\gamma,\omega_k}|^{p_k} \omega_k(x) \mathrm{d}x \right)^{1/p_k} \psi(t) \mathrm{d}t \\
&\leqslant \int_{(\mathbb{Z}_p^\star)^n} \prod_{j=1}^{2} |s_j(t)|_p^{(d+\alpha_j)\lambda_j} (\omega_j(B_\gamma))^{\lambda_j} \prod_{k=1}^{2} \left(\int_{s_k(t)B_\gamma} \frac{|f_k(y)|^{q_k} \omega_k(y) \mathrm{d}y}{\omega_k(s_k(t)B_\gamma)^{1+\lambda_k q_k}} \right)^{1/q_k} \\
&\quad \cdot \prod_{l=1}^{2} \left(\frac{1}{\omega_l(s_l(t)B_\gamma)} \int_{s_l(t)B_\gamma} |b_l(y) - b_{l,s_l(t)B_\gamma,\omega_l}|^{p_l} \omega_l(x) \mathrm{d}x \right)^{1/p_l} \psi(t) \mathrm{d}t \\
&= \prod_{i=1}^{2} (\omega_i(B_\gamma))^{\lambda_i} \prod_{j=1}^{2} \|b_j\|_{CMO^{p_j}_{\omega_j}(\mathbb{Q}_p^d)} \prod_{k=1}^{2} \|f_k\|_{\dot{B}^{q_k,\lambda_k}_{\omega_k}(\mathbb{Q}_p^d)} \\
&\quad \cdot \int_{(\mathbb{Z}_p^\star)^n} \prod_{l=1}^{2} |s_l(t)|_p^{(d+\alpha_l)\lambda_l} \psi(t) \mathrm{d}t.
\end{aligned}$$

Now, we give the estimate for I_3. Applying Hölder's inequality, we get

$$\begin{aligned}
I_3 &= \int_{(\mathbb{Z}_p^\star)^n} \left(\frac{1}{\omega(B_\gamma)^{1+\lambda q}} \int_{B_\gamma} \left(\prod_{k=1}^{2} |f_k(s_k(t)x)|\omega(x)\right)^q \mathrm{d}x\right)^{1/q} \\
&\quad \cdot \prod_{k=1}^{2} |b_{k,B_\gamma,\omega_k} - b_{k,s_k(t)B_\gamma,\omega_k}|\psi(t)\mathrm{d}t \\
&\leqslant \int_{(\mathbb{Z}_p^\star)^n} \prod_{j=1}^{2} \left(\frac{1}{\omega_j(B_\gamma)^{1+\lambda_j s_j}} \int_{B_\gamma} |f_j(s_j(t)x)|^{s_j}\omega_j(x)\mathrm{d}x\right)^{1/s_j} \\
&\quad \cdot \prod_{k=1}^{2} |b_{k,B_\gamma,\omega_k} - b_{k,s_k(t)B_\gamma,\omega_k}|\psi(t)\mathrm{d}t \\
&\leqslant \int_{(\mathbb{Z}_p^\star)^n} \prod_{j=1}^{2} |s_j(t)|_p^{(d+\alpha_j)\lambda_j} (\omega_j(B_\gamma))^{\lambda_j} \prod_{k=1}^{2} \left(\int_{s_k(t)B_\gamma} \frac{|f_k(y)|^{q_k}\omega_k(y)\mathrm{d}y}{\omega_k(s_k(t)B_\gamma)^{1+\lambda_k q_k}}\right)^{1/q_k} \\
&\quad \cdot \prod_{l=1}^{2} |b_{l,B_\gamma,\omega_l} - b_{l,s_l(t)B_\gamma,\omega_l}|\psi(t)\mathrm{d}t \\
&\leqslant \prod_{i=1}^{2}(\omega_i(B_\gamma))^{\lambda_i} \prod_{j=1}^{2} \|f_j\|_{\dot{B}^{q_j,\lambda_j}_{\omega_j}(\mathbb{Q}_p^d)} \\
&\quad \cdot \int_{(\mathbb{Z}_p^\star)^n} \prod_{k=1}^{2} |s_k(t)|_p^{(d+\alpha_k)\lambda_k} \prod_{l=1}^{2} |b_{l,B_\gamma,\omega_l} - b_{l,s_l(t)B_\gamma,\omega_l}|\psi(t)\mathrm{d}t.
\end{aligned}$$

From the hypothesis of the theorem, it follows that, for almost everywhere $t \in \mathbb{Z}_p^\star$, there exists an integer γ' such that $|s(t)|_p = p^{\gamma'}$. Using Lemma 4.1 with $\lambda = 0$, we get

$$\begin{aligned}
|b_{k,B_\gamma,\omega_k} - b_{k,s_k(t)B_\gamma,\omega_k}| &= |b_{k,B_\gamma,\omega_k} - b_{k,B_{\gamma+\gamma'},\omega_k}| \\
&\leqslant p^{d+\alpha_k}|\gamma'|\, \|b_k\|_{CMO^{p_k}_{\omega_k}(\mathbb{Q}_p^d)} \\
&= p^{d+\alpha_k}|\log_p |s_k(t)|_p|\, \|b_k\|_{CMO^{p_k}_{\omega_k}(\mathbb{Q}_p^d)}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
I_3 &\leqslant \prod_{i=1}^{2} p^{d+\alpha_i} \prod_{j=1}^{2}(\omega_j(B_\gamma))^{\lambda_j} \prod_{k=1}^{2} \|b_k\|_{CMO^{p_k}_{\omega_k}(\mathbb{Q}_p^d)} \prod_{l=1}^{2} \|f_l\|_{\dot{B}^{q_l,\lambda_l}_{\omega_l}(\mathbb{Q}_p^d)} \\
&\quad \cdot \int_{(\mathbb{Z}_p^\star)^n} \prod_{r=1}^{2} |s_r(t)|_p^{(d+\alpha_r)\lambda_r} |\log_p |s_r(t)|_p|\psi(t)\mathrm{d}t \\
&\leqslant p^{2d+\alpha} \prod_{i=1}^{2}(\omega_i(B_\gamma))^{\lambda_i} \prod_{j=1}^{2} \|b_j\|_{CMO^{p_j}_{\omega_j}(\mathbb{Q}_p^d)} \prod_{k=1}^{2} \|f_k\|_{\dot{B}^{q_k,\lambda_k}_{\omega_k}(\mathbb{Q}_p^d)}
\end{aligned}$$

$$\cdot \int_{(\mathbb{Z}_p^\star)^n} \prod_{l=1}^{2} |s_l(t)|_p^{(d+\alpha_l)\lambda_l} |\log_p |s_l(t)|_p| \psi(t)\mathrm{d}t.$$

Now, we give the estimate for I_4. Similarly, we choose $s \in (1,\infty)$ such that $\frac{1}{s} = \frac{1}{p_1} + \frac{1}{p_2}$. Let

$$\overline{b}_{i,j}(x) = |(b_i(x) - b_{i,B_\gamma})(b_{j,B_\gamma} - b_{j,s_j(t)B_\gamma}, \omega_j)|.$$

Then, by Minkowski's inequality and Hölder's inequality, we have

$$\begin{aligned}
I_4 &\leqslant C \int_{(\mathbb{Z}_p^\star)^n} \sum_{1\leqslant i\neq j\leqslant 2} \left(\int_{B_\gamma} \left(\prod_{k=1}^{2} |f_k(s_k(t)x)|\overline{b}_{i,j}(x) \right)^q \frac{\omega(x)\mathrm{d}x}{\omega(B_\gamma)^{1+\lambda q}} \right)^{1/q} \psi(t)\mathrm{d}t \\
&\leqslant C \int_{(\mathbb{Z}_p^\star)^n} \prod_{k=1}^{2} \left(\frac{1}{\omega_k(B_\gamma)^{1+\lambda_k q_k}} \int_{B_\gamma} |f_k(s_k(t)x)|^{q_k}\omega_k(x)\mathrm{d}x \right)^{1/q_k} \\
&\quad \cdot \sum_{1\leqslant i\neq j\leqslant 2} \left(\int_{B_\gamma} \frac{1}{\omega_i(B_\gamma)} \right)^{1/s} |b_{j,B_\gamma} - b_{j,s_j(t)B_\gamma,\omega_j}|\psi(t)\mathrm{d}t \\
&\leqslant C \prod_{r=1}^{2} (\omega_r(B_\gamma))^{\lambda_r} \int_{(\mathbb{Z}_p^\star)^n} \prod_{l=1}^{2} |s_l(t)|_p^{(d+\alpha_l)\lambda_l} \\
&\quad \cdot \prod_{k=1}^{2} \left(\frac{1}{\omega_k(s_k(t)B_\gamma)^{1+\lambda_k q_k}} \int_{s_k(t)B_\gamma} |f_k(y)|^{q_k}\omega_k(y)\mathrm{d}y \right)^{1/q_k} \\
&\quad \cdot \sum_{1\leqslant i\neq j\leqslant 2} \left(\int_{B_\gamma} \frac{|b_i(x) - b_{i,B_\gamma}|^s \omega_i(x)\mathrm{d}x}{\omega_i(B_\gamma)} \right)^{1/s} |b_{j,B_\gamma} - b_{j,s_j(t)B_\gamma,\omega_j}|\psi(t)\mathrm{d}t \\
&\leqslant C \prod_{k=1}^{2} (\omega_k(B_\gamma))^{\lambda_k} \|f_k\|_{\dot{B}^{q_k,\lambda_k}_{\omega_k}(\mathbb{Q}_p^d)} \int_{(\mathbb{Z}_p^\star)^n} \prod_{l=1}^{2} |s_l(t)|_p^{(d+\alpha_l)\lambda_l} \psi(t) \\
&\quad \cdot \sum_{1\leqslant i\neq j\leqslant 2} \left(\frac{1}{\omega_i(B_\gamma)} \int_{B_\gamma} |b_i(x) - b_{i,B_\gamma}|^s \omega_i(x)\mathrm{d}x \right)^{1/s} |b_{j,B_\gamma} - b_{j,s_j(t)B_\gamma,\omega_j}|\mathrm{d}t.
\end{aligned}$$

From the estimates of I_1 and I_3, we deduce that

$$\begin{aligned}
I_4 &\leqslant C \prod_{r=1}^{2} (\omega_r(B_\gamma))^{\lambda_r} \|f_r\|_{\dot{B}^{q_r,\lambda_r}_{\omega_r}(\mathbb{Q}_p^d)} \prod_{l=1}^{2} \|b_l\|_{CMO^{p_l}_{\omega_l}(\mathbb{Q}_p^d)} \\
&\quad \cdot \int_{(\mathbb{Z}_p^\star)^n} \prod_{k=1}^{2} |s_k(t)|_p^{(d+\alpha_k)\lambda_k} \left(\sum_{1\leqslant i\neq j\leqslant 2} p^{d+\alpha_j} |\log_p |s_j(t)|_p| \right) \psi(t)\mathrm{d}t \\
&\leqslant C \prod_{r=1}^{2} (\omega_r(B_\gamma))^{\lambda_r} \|f_r\|_{\dot{B}^{q_r,\lambda_r}_{\omega_r}(\mathbb{Q}_p^d)} \prod_{l=1}^{2} \|b_l\|_{CMO^{p_l}_{\omega_l}(\mathbb{Q}_p^d)} \\
&\quad \cdot \int_{(\mathbb{Z}_p^\star)^n} \prod_{k=1}^{2} |s_k(t)|_p^{(d+\alpha_k)\lambda_k} \left(\sum_{j=1}^{2} p^{d+\alpha_j} |\log_p |s_j(t)|_p| \right) \psi(t)\mathrm{d}t.
\end{aligned}$$

It can be deduced from the estimates of $I_1, \ldots, I_4$ that

$$I_5 \leqslant C \int_{(\mathbb{Z}_p^\star)^n} \sum_{1\leqslant i\neq j\leqslant 2} \Bigg(\frac{1}{\omega(B_\gamma)^{1+\lambda q}} \int_{B_\gamma} \Bigg(\prod_{k=1}^{2} |f_k(s_k(t)x)|$$

$$\cdot |(b_i(x) - b_{i,B_\gamma})(b_j(s_j(t)x) - b_{j,s_j(t)B_\gamma,\omega_j})| \Bigg)^q \omega(x)\mathrm{d}x \Bigg)^{1/q} \psi(t)\mathrm{d}t$$

$$\leqslant C \int_{(\mathbb{Z}_p^\star)^n} \sum_{1\leqslant i\neq j\leqslant 2} \prod_{k=1}^{2} \left(\frac{1}{\omega_k(B_\gamma)^{1+\lambda_k q_k}} \int_{B_\gamma} |f_k(s_k(t)x)|^{q_k} \omega_k(x)\mathrm{d}x \right)^{1/q_k}$$

$$\cdot \left(\frac{1}{\omega_i(B_\gamma)} \int_{B_\gamma} |b_i(x) - b_{i,B_\gamma}|^{p_i} \omega_i(x)\mathrm{d}x \right)^{1/p_i}$$

$$\cdot \left(\frac{1}{\omega_j(B_\gamma)} \int_{B_\gamma} |b_j(s_j(t)x) - b_{j,s_j(t)B_\gamma,\omega_j}|^{p_j} \omega_j(x)\mathrm{d}x \right)^{1/p_j} \psi(t)\mathrm{d}t$$

$$\leqslant C \prod_{k=1}^{2} (\omega_k(B_\gamma))^{\lambda_k} \|f_k\|_{\dot{B}^{q_k,\lambda_k}_{\omega_k}(\mathbb{Q}_p^d)}$$

$$\cdot \int_{(\mathbb{Z}_p^\star)^n} \prod_{l=1}^{2} |s_l(t)|_p^{(d+\alpha_l)\lambda_l} \sum_{1\leqslant i\neq j\leqslant 2} \|b_i\|_{CMO^{p_i}_{\omega_i}(\mathbb{Q}_p^d)} \|b_j\|_{CMO^{p_j}_{\omega_j}(\mathbb{Q}_p^d)} \psi(t)\mathrm{d}t$$

$$\leqslant C \prod_{k=1}^{2} (\omega_k(B_\gamma))^{\lambda_k} \|f_k\|_{\dot{B}^{q_k,\lambda_k}_{\omega_k}(\mathbb{Q}_p^d)} \prod_{l=1}^{2} \|b_l\|_{CMO^{p_l}_{\omega_l}(\mathbb{Q}_p^d)}$$

$$\cdot \int_{(\mathbb{Z}_p^\star)^n} \prod_{j=1}^{2} |s_j(t)|_p^{(d+\alpha_j)\lambda_j} \psi(t)\mathrm{d}t$$

and

$$I_6 = \int_{(\mathbb{Z}_p^\star)^n} \Bigg(\frac{1}{\omega(B_\gamma)^{1+\lambda q}} \int_{B_\gamma} \Bigg(\overline{f}(x) \sum_{1\leqslant i\neq j\leqslant 2} |(b_{i,B_\gamma} - b_{i,s_i(t)B_\gamma})$$

$$\cdot (b_j(s_j(t)x) - b_{j,s_j(t)B_\gamma,\omega_j})| \Bigg)^q \omega(x)\mathrm{d}x \Bigg)^{1/q} \psi(t)\mathrm{d}t$$

$$\leqslant C \int_{(\mathbb{Z}_p^\star)^n} \sum_{1\leqslant i\neq j\leqslant 2} \Bigg(\frac{1}{\omega(B_\gamma)^{1+\lambda q}} \int_{B_\gamma} \Bigg(\prod_{k=1}^{2} |f_k(s_k(t)x)|$$

$$\cdot |(b_{i,B_\gamma} - b_{i,s_i(t)B_\gamma})(b_j(s_j(t)x) - b_{j,s_j(t)B_\gamma,\omega_j})| \Bigg)^q \omega(x)\mathrm{d}x \Bigg)^{1/q} \psi(t)\mathrm{d}t$$

$$\leqslant C \int_{(\mathbb{Z}_p^\star)^n} \sum_{1\leqslant i\neq j\leqslant 2} \prod_{k=1}^{2} \left(\frac{1}{\omega_k(B_\gamma)^{1+\lambda_k q_k}} \int_{B_\gamma} |f_k(s_k(t)x)|^{q_k} \omega_k(x)\mathrm{d}x \right)^{1/q_k}$$

$$
\begin{aligned}
&\quad\cdot\left(\frac{1}{\omega_j(B_\gamma)}\int_{B_\gamma}|b_j(s_j(t)x)-b_{j,s_j(t)B_\gamma,\omega_j}|^s\omega_j(x)\mathrm{d}x\right)^{1/s}\\
&\quad\cdot|b_{i,B_\gamma}-b_{i,s_i(t)B_\gamma}|\psi(t)\mathrm{d}t\\
&=C\int_{(\mathbb{Z}_p^\star)^n}\prod_{k=1}^{2}\left(\frac{1}{\omega_k(B_\gamma)^{1+\lambda_k q_k}}\int_{B_\gamma}|f_k(s_k(t)x)|^{q_k}\omega_k(x)\mathrm{d}x\right)^{1/q_k}\\
&\quad\cdot\Bigg(\sum_{1\leqslant i\neq j\leqslant 2}\left(\frac{1}{\omega_j(B_\gamma)}\int_{B_\gamma}|b_j(s_j(t)x)-b_{j,s_j(t)B_\gamma,\omega_j}|^s\omega_j(x)\mathrm{d}x\right)^{1/s}\\
&\quad\cdot|b_{i,B_\gamma}-b_{i,s_i(t)B_\gamma}|\Bigg)\psi(t)\mathrm{d}t\\
&\leqslant C\prod_{r=1}^{2}(\omega_r(B_\gamma))^{\lambda_r}\|f_r\|_{\dot{B}^{q_r,\lambda_r}_{\omega_r}(\mathbb{Q}_p^d)}\prod_{l=1}^{2}\|b_l\|_{CMO^{p_l}_{\omega_l}(\mathbb{Q}_p^d)}\\
&\quad\cdot\int_{(\mathbb{Z}_p^\star)^n}\prod_{k=1}^{2}|s_k(t)|_p^{(d+\alpha_k)\lambda_k}\sum_{1\leqslant i\neq j\leqslant 2}p^{d+\alpha_i}|\log_p|s_i(t)|_p|\psi(t)\mathrm{d}t\\
&=C\prod_{r=1}^{2}(\omega_r(B_\gamma))^{\lambda_r}\|f_r\|_{\dot{B}^{q_r,\lambda_r}_{\omega_r}(\mathbb{Q}_p^d)}\prod_{l=1}^{2}\|b_l\|_{CMO^{p_l}_{\omega_l}(\mathbb{Q}_p^d)}\\
&\quad\cdot\int_{(\mathbb{Z}_p^\star)^n}\prod_{k=1}^{2}|s_k(t)|_p^{(d+\alpha_k)\lambda_k}\sum_{j=1}^{2}p^{d+\alpha_j}|\log_p|s_j(t)|_p|\psi(t)\mathrm{d}t.
\end{aligned}
$$

Combining the estimates of $I_1,\dots,I_6$ gives

$$
\begin{aligned}
I&=\left(\frac{1}{\omega(B_\gamma)}\int_{B_\gamma}|U^{p,2,n,\boldsymbol{b}}_{\psi,\boldsymbol{s}}(f_1,f_2)(x)|^q\omega(x)\mathrm{d}x\right)^{1/q}\\
&\leqslant\prod_{i=1}^{2}(\omega_i(B_\gamma))^{\lambda_i}\|f_i\|_{\dot{B}^{q_i,\lambda_i}_{\omega_i}(\mathbb{Q}_p^d)}\prod_{j=1}^{2}\|b_j\|_{CMO^{p_j}_{\omega_j}(\mathbb{Q}_p^d)}\int_{(\mathbb{Z}_p^\star)^n}\prod_{k=1}^{2}|s_k(t)|_p^{(d+\alpha_k)\lambda_k}\\
&\quad\cdot\left(3+C+p^{2d+\alpha}\prod_{l=1}^{2}|\log_p|s_l(t)|_p|+2C\sum_{r=1}^{2}p^{d+\alpha_r}|\log_p|s_r(t)|_p|\right)\psi(t)\mathrm{d}t.
\end{aligned}
$$

This proves the first part of Theorem 4.1.

Now, we assume that $U^{p,2,n,\boldsymbol{b}}_{\psi,\boldsymbol{s}}$ is bounded from $\dot{B}^{q_1,\lambda_1}_{\omega_1}(\mathbb{Q}_p^d)\times\dot{B}^{q_2,\lambda_2}_{\omega_2}(\mathbb{Q}_p^d)$ to $\dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}_p^d)$. Take

$$b_1(x)=b_2(x)=\log_p|x|_p.$$

Then $b_1,b_2\in BMO_\omega(\mathbb{Q}_p^d)$ (see [13, Lemma 6.1]). Since

$$BMO_\omega(\mathbb{Q}_p^d)\subset CMO^q_\omega(\mathbb{Q}_p^d),\quad\forall\, q\in(1,\infty),$$

we obtain $b_k\in CMO^{p_k}_{\omega_k}(\mathbb{Q}_p^d)$. Let

$$f_k(x)=|x|_p^{(d+\alpha_k)\lambda_k}.$$

Then Lemma 2.2 implies that $f_k(x)$ belongs to $\dot{B}^{q_k,\lambda_k}_{\omega_k}(\mathbb{Q}_p^d)$, and

$$f_0(x) = |x|_p^{(d+\alpha)\lambda} \in \dot{B}^{q,\lambda}_{\omega}(\mathbb{Q}_p^d).$$

We have

$$\begin{aligned}
&U^{p,2,n,\boldsymbol{b}}_{\psi,\boldsymbol{s}}(f_1,f_2)(x)\\
&\quad= \int_{(\mathbb{Z}_p^\star)^n} \prod_{j=1}^{2} |s_j(t)x|_p^{(d+\alpha_j)\lambda_j} \prod_{k=1}^{2} (\log_p |x|_p - \log_p |s_k(t)x|_p)\psi(t)\mathrm{d}t\\
&\quad= \prod_{j=1}^{2} |x|_p^{(d+\alpha_j)\lambda_j} \int_{(\mathbb{Z}_p^\star)^n} \prod_{k=1}^{2} |s_k(t)|_p^{(d+\alpha_k)\lambda_k} \prod_{l=1}^{2} \Big(\log_p \frac{1}{|s_l(t)|_p}\Big)\psi(t)\mathrm{d}t\\
&\quad= f_0(x) \int_{(\mathbb{Z}_p^\star)^n} \prod_{k=1}^{2} |s_k(t)|_p^{(d+\alpha_k)\lambda_k} \prod_{l=1}^{2} \Big(\log_p \frac{1}{|s_l(t)|_p}\Big)\psi(t)\mathrm{d}t.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|U^{p,2,n,\boldsymbol{b}}_{\psi,\boldsymbol{s}}(f_1,f_2)\|_{\dot{B}^{q_k,\lambda_k}_{\omega_k}(\mathbb{Q}_p^d)}\\
&\quad= \bigg(\frac{1}{\omega(B_\gamma)^{1+\lambda q}} \int_{B_\gamma} |f_0(x)|^q \omega(x)\mathrm{d}x\bigg)^{1/q}\\
&\qquad\cdot \int_{(\mathbb{Z}_p^\star)^n} \prod_{k=1}^{2} |s_k(t)|_p^{(d+\alpha_k)\lambda_k} \prod_{l=1}^{2} \bigg|\log_p \frac{1}{|s_l(t)|_p}\bigg|\psi(t)\mathrm{d}t\\
&\quad= \|f_0\|_{\dot{B}^{q_k,\lambda_k}_{\omega_k}(\mathbb{Q}_p^d)} \int_{(\mathbb{Z}_p^\star)^n} \prod_{k=1}^{2} |s_k(t)|_p^{(d+\alpha_k)\lambda_k} \prod_{l=1}^{2} |\log_p |s_l(t)|_p|\psi(t)\mathrm{d}t\\
&\quad= \mathscr{D}_2 \cdot \|f_0\|_{\dot{B}^{q_k,\lambda_k}_{\omega_k}(\mathbb{Q}_p^d)}.
\end{aligned}$$

Therefore, $\mathscr{D}_2$ is finite. □

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