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RESEARCH ARTICLE

Vanishing of stable homology with respect to a semidualizing module

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Abstract We investigate stable homology of modules over a commutative noetherian ring R with respect to a semidualzing module C, and give some vanishing results that improve/extend the known results. As a consequence, we show that the balance of the theory forces C to be trivial and R to be Gorenstein.

Keywords Stable homology, semidualizing module, proper resolution
MSC 13D05, 13D07, 16E05

1 Introduction

Stable homology, as a broad generalization of Tate homology to the realm of associative rings, was introduced by Vogel and Goichot [9], and further studied by Celikbas et al. [2,3] and Emmanouil and Manousaki [6]. In [2], it was shown that the vanishing of stable homology over commutative noetherian local rings can detect modules of finite projective (injective) dimension, even of finite Gorenstein dimension, which lead to some characterizations of classical rings such as Gorenstein rings, the original domain of Tate homology. Emmanouil and Manousaki [6] further investigated stable homology of modules, and gave some vanishing results that improve results in [2] by relaxing the conditions on rings and modules.

The study of semidualizing modules was initiated independently by Foxby [8], Golod [10], and Vasconcelos [19]. Over a commutative noetherian ring R, a finitely generated R-module C is semidualizing if

$$\mathrm{Hom}_R(C,C)\cong R,\quad \mathrm{Ext}^i_R(C,C)=0,\ \forall\, i\geqslant 1.$$

Examples include finitely generated projective R-modules of rank 1. Modules of finite homological dimension with respect to a semidualizing module have

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been studied in numerous papers. For example, Takahashi and White [18] and Salimi et al. [15] gave some characterizations for such modules in terms of the vanishing of relative (co)homology. In this paper, we show that the vanishing of stable homology can also detect modules of finite homological dimension with respect to a semidualizing module. Our main results are following two theorems.

Theorem 1.1 *Let R be a commutative noetherian ring, and let C be a semidualizing R-module. For an R-module M, the following conditions are equivalent*:

(i) $\mathscr{F}_C\text{-pd}_R M < \infty$;

(ii) $\widetilde{\mathrm{Tor}}_n^{\mathscr{P}_C\mathscr{I}_C}(M,-) = 0$ *for each* $n \in \mathbb{Z}$;

(iii) $\widetilde{\mathrm{Tor}}_n^{\mathscr{P}_C\mathscr{I}_C}(M,-) = 0$ *for some* $n \geqslant 0$.

Moreover, if M is finitely generated, then (i)–(iii) are equivalent to

(i′) $\mathscr{P}_C\text{-pd}_R M < \infty$.

Theorem 1.2 *Let R be a commutative noetherian ring, and let C be a semidualizing R-module. For an R-module N, the following conditions are equivalent*:

(i) $\mathscr{I}_C\text{-id}_R N < \infty$;

(ii) $\widetilde{\mathrm{Tor}}_n^{\mathscr{P}_C\mathscr{I}_C}(-,N) = 0$ *for each* $n \in \mathbb{Z}$;

(iii) $\widetilde{\mathrm{Tor}}_n^{\mathscr{P}_C\mathscr{I}_C}(-,N) = 0$ *for some* $n < 0$.

The above two results improve the right and left vanishing results in the introduction of [2]. Here, the notations $\mathscr{F}_C\text{-pd}_R M$, $\mathscr{I}_C\text{-id}_R N$, and $\widetilde{\mathrm{Tor}}_n^{\mathscr{P}_C\mathscr{I}_C}(-,-)$ can be found in Sections 2 and 4. As a consequence, we show that the isomorphisms

$$\widetilde{\mathrm{Tor}}_*^{\mathscr{P}_C\mathscr{I}_C}(M,N) \cong \widetilde{\mathrm{Tor}}_*^{\mathscr{P}_C\mathscr{I}_C}(N,M)$$

for all R-modules M and N force C to be trivial and R to be a Gorenstein ring; see Corollary 4.1 below.

We prove these results using the next characterization of stable (unbounded) tensor product inspired by the work of Emmanouil and Manousaki [6].

Theorem 1.3 *Let X be a complex of R°-modules, and let Y be a bounded above complex of R-modules with* $\sup\{i \in \mathbb{Z} \mid Y_i \neq 0\} = k$. *Then there are isomorphisms of complexes of $\mathbb{Z}$-modules*

$$X \overline{\otimes}_R Y \cong \lim_{i\in\mathbb{N}}((X \otimes_R Y)/(X \otimes_R Y_{\leqslant k-i}))$$

and

$$X \widetilde{\otimes}_R Y \cong \lim_{i\in\mathbb{N}}{}^1 (X \otimes_R Y_{\leqslant k-i}).$$

One refers to Section 3 for the definitions of $X\overline{\otimes}_R Y$ and $X\widetilde{\otimes}_R Y$, and $\lim^1$ is the right derived functor of the limit lim; see Section 3.

2 Preliminaries

We begin with some notation and terminology for using throughout this paper.

Throughout this work, all rings are assumed to be associative rings. Let R be a ring; by an R-module we mean a left R-module, and we refer to right R-modules as modules over the opposite ring R°. We denote by $\mathscr{P}$ (resp., $\mathscr{F}$, $\mathscr{I}$) the class of projective R-modules (resp., flat R-modules, injective R-modules).

By an R-complex we mean a complex of R-modules. We frequently (and without warning) identify R-modules with R-complexes concentrated in degree 0. For an R-complex X, we set

$$\sup X = \sup\{i \in \mathbb{Z} \mid X_i \neq 0\}, \quad \inf X = \inf\{i \in \mathbb{Z} \mid X_i \neq 0\}.$$

An R-complex X is *bounded above* if $\sup X < \infty$, and it is *bounded below* if $\inf X > -\infty$. An R-complex X is *bounded* if it is both bounded above and bounded below. The nth *homology* of X is denoted by $\mathrm{H}_n(X)$. For each $k \in \mathbb{Z}$, $\Sigma^k X$ denotes the complex with the degree-n term $(\Sigma^k X)_n = X_{n-k}$ and whose boundary operators are $(-1)^k \partial^X_{n-k}$. We set $\Sigma M = \Sigma^1 M$.

If X and Y are both R-complexes, then by a *morphism* $\alpha\colon X \to Y$ we mean a sequence $\alpha_n\colon X_n \to Y_n$ such that

$$\alpha_{n-1}\partial^X_n = \partial^Y_n \alpha_n, \quad \forall\, n \in \mathbb{Z}.$$

A *quasi-isomorphism*, indicated by the symbol '$\simeq$', is a morphism of complexes that induces an isomorphism in homology.

Let $\mathscr{X}$ be a class of R-modules. Following Enochs and Jenda [7], an $\mathscr{X}$-precover of an R-module M is a homomorphism $X \to M$ with $X \in \mathscr{X}$ such that the homomorphism

$$\mathrm{Hom}_R(X', X) \to \mathrm{Hom}_R(X', M)$$

is surjective for each $X' \in \mathscr{X}$. $\mathscr{X}$ is called a precovering class if each R-module has a $\mathscr{X}$-precover.

For a precovering class $\mathscr{X}$, there is a complex X^+:

$$\cdots \to X_1 \to X_0 \to M \to 0,$$

with each X_i in $\mathscr{X}$, such that $\mathrm{Hom}_R(X', X^+)$ is exact for each $X' \in \mathscr{X}$. The truncated complex X:

$$\cdots \to X_1 \to X_0 \to 0,$$

is called a *proper $\mathscr{X}$-resolution* of M, which is always denoted by $X \to M$. If $\mathscr{X}$ contains all projective R-modules, then the complex X^+ is exact. In this case, we always denote by $X \xrightarrow{\simeq} M$ the proper $\mathscr{X}$-resolution of M.

The $\mathscr{X}$*-projective dimension* of M is the quantity

$$\mathscr{X}\text{-pd}_R M = \inf\{\sup X \mid X \to M \text{ is a proper } \mathscr{X}\text{-resolution of } M\}.$$

We define *preenveloping classes* $\mathscr{Y}$, *proper* $\mathscr{Y}$*-coresolutions*, and $\mathscr{Y}$*-injective dimension* of M (denoted by $\mathscr{Y}\text{-id}_R M$) dually.

When $\mathscr{X}$ is the class of projective (resp., flat) R-modules, $\mathscr{X}\text{-pd}_R M$ is the classical projective (resp., flat) dimension; we refer the reader to [15, Remark 2.6] for the flat case. Also, when $\mathscr{Y}$ is the class of injective R-modules, $\mathscr{Y}\text{-id}_R M$ is the classical injective dimension.

3 Characterization of stable (unbounded) tensor product

We start by recalling the definition of stable (unbounded) tensor product.

Definition 3.1 Let X be an R°-complex, and let Y be an R-complex. The *tensor product* $X \otimes_R Y$ is the $\mathbb{Z}$-complex with degree-n term

$$(X \otimes_R Y)_n = \coprod_{i \in \mathbb{Z}} (X_i \otimes_R Y_{n-i})$$

and differential given by

$$\partial^{X \otimes_R Y}(x \otimes y) = \partial^X(x) \otimes y + (-1)^{|x|} x \otimes \partial^Y(y).$$

Following [2,9], the *unbounded tensor product* $X \overline{\otimes}_R Y$ is the $\mathbb{Z}$-complex with degree-n term

$$(X \otimes_R Y)_n = \prod_{i \in \mathbb{Z}} (X_i \otimes_R Y_{n-i})$$

and differential defined as above. $X \otimes_R Y$ is a subcomplex of $X \overline{\otimes}_R Y$, so we let $X \widetilde{\otimes}_R Y$ denote the quotient complex $(X \overline{\otimes}_R Y)/(X \otimes_R Y)$, which is called the *stable tensor product*.

We notice that if X or Y is bounded, or if both of them are bounded on the same side (above or below), then the unbounded tensor product coincides with the tensor product, and so the stable tensor product $X \widetilde{\otimes}_R Y$ is zero.

Remark 3.1 Let $\{\nu^{uv} \colon X^v \to X^u\}_{u \leqslant v}$ be an $\mathbb{N}$-inverse system of R-complexes. For the morphism

$$1 - \nu \colon \prod_{i \in \mathbb{N}} X^i \to \prod_{i \in \mathbb{N}} X^i$$

given by

$$(1 - \nu)_k (x_i)_{i \in \mathbb{N}} = (x_i - \nu_k^{i,i+1}(x_{i+1}))_{i \in \mathbb{N}}, \quad \forall\, k \in \mathbb{Z},$$

where $(x_i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} X_k^i$, it is well known that

$$\operatorname{Ker}(1 - \nu) = \lim_{i \in \mathbb{N}} X^i, \quad \operatorname{Coker}(1 - \nu) = \lim_{i \in \mathbb{N}}{}^1 X^i.$$

Here, $\lim^1$ is the right derived functor of the limit lim; see, e.g., [5,14,20], for more details. That is, there is an exact sequence of R-complexes:

$$0 \to \lim_{i\in\mathbb{N}} X^i \to \prod_{i\in\mathbb{N}} X^i \to \prod_{i\in\mathbb{N}} X^i \to \lim_{i\in\mathbb{N}}{}^1 X^i \to 0.$$

Let X be an R-complex, and let $X = X^0 \supseteq X^1 \supseteq \cdots$ be a filtration. Then the embeddings $\varepsilon^i\colon X^i \to X^{i-1}$ and the morphisms $\pi^i\colon X/X^i \to X/X^{i-1}$ determine the $\mathbb{N}$-inverse systems

$$\{\varepsilon^{uv}\colon X^v \to X^u\}_{u\leqslant v}, \quad \{\pi^{uv}\colon X/X^v \to X/X^u\}_{u\leqslant v},$$

respectively. For these systems, we have the following result.

Lemma 3.1 *Let X be an R-complex, and let $X = X^0 \supseteq X^1 \supseteq \cdots$ be a filtration. Then $\lim^1_{i\in\mathbb{N}} X/X^i = 0$, and there exists an exact sequence*

$$0 \to \lim_{i\in\mathbb{N}} X^i \to X \to \lim_{i\in\mathbb{N}} X/X^i \to \lim_{i\in\mathbb{N}}{}^1 X^i \to 0.$$

Proof Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \to & \prod_{i\in\mathbb{N}} X^i & \to & \prod_{i\in\mathbb{N}} X & \to & \prod_{i\in\mathbb{N}} X/X^i & \to 0 \\
& \downarrow 1-\varepsilon & & \downarrow 1-\mathrm{id} & & \downarrow 1-\pi & \\
0 \to & \prod_{i\in\mathbb{N}} X^i & \to & \prod_{i\in\mathbb{N}} X & \to & \prod_{i\in\mathbb{N}} X/X^i & \to 0.
\end{array}$$

We notice that the constant $\mathbb{N}$-inverse system $\{X\}$ has $\lim_{i\in\mathbb{N}} X = X$ and $\lim^1_{i\in\mathbb{N}} X = 0$ since $1 - \mathrm{id}$ is surjective. Then by Remark 3.1 and the snake lemma, one gets the desired results. □

Remark 3.2 Let X be an R°-complex, and let Y be an R-complex. For fixed $k \in \mathbb{Z}$, the filtration

$$Y_{\leqslant k} \supseteq Y_{\leqslant k-1} \supseteq Y_{\leqslant k-2} \supseteq \cdots$$

induces a filtration

$$X \otimes_R Y_{\leqslant k} \supseteq X \otimes_R Y_{\leqslant k-1} \supseteq X \otimes_R Y_{\leqslant k-2} \supseteq \cdots.$$

Thus, we have two $\mathbb{N}$-inverse systems

$$\{\varepsilon^{uv}\colon X \otimes_R Y_{\leqslant k-v} \to X \otimes_R Y_{\leqslant k-u}\}_{u\leqslant v},$$
$$\{\pi^{uv}\colon (X \otimes_R Y_{\leqslant k})/(X \otimes_R Y_{\leqslant k-v}) \to (X \otimes_R Y_{\leqslant k})/(X \otimes_R Y_{\leqslant k-u})\}_{u\leqslant v}.$$

Proof of Theorem 1.3 We first prove the case when $k = 0$. In this case, $Y = Y_{\leqslant 0}$. For each $n \in \mathbb{Z}$,

$$(X \otimes_R Y_{\leqslant 0})_n = \coprod_{p\in\mathbb{Z}} (X_{n+p} \otimes_R (Y_{\leqslant 0})_{-p}) = \coprod_{p\geqslant 0} (X_{n+p} \otimes_R (Y_{\leqslant 0})_{-p}),$$

and for each $i \geqslant 1$,

$$(X \otimes_R Y_{\leqslant -i})_n = \coprod_{p\in\mathbb{Z}} (X_{n+p} \otimes_R (Y_{\leqslant -i})_{-p}) = \coprod_{p\geqslant i} (X_{n+p} \otimes_R (Y_{\leqslant 0})_{-p}).$$

Thus, one gets

$$((X \otimes_R Y_{\leqslant 0})/(X \otimes_R Y_{\leqslant -i}))_n \cong \coprod_{p=0}^{i-1} (X_{n+p} \otimes_R (Y_{\leqslant 0})_{-p}) = \prod_{p=0}^{i-1} (X_{n+p} \otimes_R (Y_{\leqslant 0})_{-p}).$$

This implies that

$$\lim_{i\in\mathbb{N}} ((X \otimes_R Y_{\leqslant 0})/(X \otimes_R Y_{\leqslant -i}))_n \cong \prod_{p\in\mathbb{Z}} (X_{n+p} \otimes_R (Y_{\leqslant 0})_{-p}) = (X \overline{\otimes}_R Y_{\leqslant 0})_n.$$

Now, it is straightforward to verify

$$X \overline{\otimes}_R Y_{\leqslant 0} \cong \lim_{i\in\mathbb{N}} ((X \otimes_R Y_{\leqslant 0})/(X \otimes_R Y_{\leqslant -i})).$$

Since $\lim_{i\in\mathbb{N}}(X \otimes_R Y_{\leqslant -i}) = 0$, there is an exact sequence

$$0 \to X \otimes_R Y_{\leqslant 0} \to X \overline{\otimes}_R Y_{\leqslant 0} \to \lim_{i\in\mathbb{N}}{}^1 (X \otimes_R Y_{\leqslant -i}) \to 0$$

by Lemma 3.1 and the isomorphism proved above. Thus, one gets

$$X \widetilde{\otimes}_R Y_{\leqslant 0} \cong \lim_{i\in\mathbb{N}}{}^1 (X \otimes_R Y_{\leqslant -i}).$$

In the general case, when $\sup Y = k \in \mathbb{Z}$, we notice that

$$Y = \Sigma^k (\Sigma^{-k} Y)_{\leqslant 0}, \quad (\Sigma^{-k} Y)_{\leqslant -i} = \Sigma^{-k} Y_{\leqslant k-i}.$$

Then one has

$$\begin{aligned} X \overline{\otimes}_R Y &= \Sigma^k (X \overline{\otimes}_R (\Sigma^{-k} Y)_{\leqslant 0}) \\ &\cong \Sigma^k \lim_{i\in\mathbb{N}} ((X \otimes_R (\Sigma^{-k} Y)_{\leqslant 0})/(X \otimes_R (\Sigma^{-k} Y)_{\leqslant -i})) \\ &\cong \Sigma^k \lim_{i\in\mathbb{N}} ((X \otimes_R \Sigma^{-k} Y_{\leqslant k})/(X \otimes_R \Sigma^{-k} Y_{\leqslant k-i})) \\ &\cong \lim_{i\in\mathbb{N}} ((X \otimes_R Y)/(X \otimes_R Y_{\leqslant k-i})) \end{aligned}$$

and

$$\begin{aligned} X \widetilde{\otimes}_R Y &= \Sigma^k (X \widetilde{\otimes}_R (\Sigma^{-k} Y)_{\leqslant 0}) \\ &\cong \Sigma^k \lim_{i\in\mathbb{N}}{}^1 (X \otimes_R (\Sigma^{-k} Y)_{\leqslant -i}) \\ &\cong \Sigma^k \lim_{i\in\mathbb{N}}{}^1 (X \otimes_R \Sigma^{-k} Y_{\leqslant k-i}) \\ &\cong \lim_{i\in\mathbb{N}}{}^1 (X \otimes_R Y_{\leqslant k-i}), \end{aligned}$$

as desired. □

Corollary 3.1 *Let X be an R°-complex, and let Y be a bounded above R-complex with* $\sup Y = k$. *Then there exists an exact sequence*

$$0 \to \prod_{i\in\mathbb{N}}(X \otimes_R Y_{\leqslant k-i}) \to \prod_{i\in\mathbb{N}}(X \otimes_R Y_{\leqslant k-i}) \to X \,\widetilde{\otimes}_R\, Y \to 0.$$

Proof Since

$$\lim_{i\in\mathbb{N}}(X \otimes_R Y_{\leqslant k-i}) = 0, \quad \lim_{i\in\mathbb{N}}{}^1(X \otimes_R Y_{\leqslant k-i}) \cong X \,\widetilde{\otimes}_R\, Y,$$

by Theorem 1.3, the desired exact sequence now follows from Remark 3.1. We notice that the map from $\prod_{i\in\mathbb{N}}(X \otimes_R Y_{\leqslant k-i})$ to $\prod_{i\in\mathbb{N}}(X \otimes_R Y_{\leqslant k-i})$ in the statement is $1-\varepsilon$, where

$$\varepsilon^{uv}\colon X \otimes_R Y_{\leqslant k-v} \to X \otimes_R Y_{\leqslant k-u}$$

for $u \leqslant v$ is induced by the filtration $Y_{\leqslant k} \supseteq Y_{\leqslant k-1} \supseteq Y_{\leqslant k-2} \supseteq \cdots$; see Remarks 3.1 and 3.2. □

Corollary 3.2 *Let X be an R°-complex, and let Y be a bounded above R-complex with* $\sup Y = k$. *Then, for each $n \in \mathbb{Z}$, there exists an exact sequence*

$$0 \to \lim_{i\in\mathbb{N}}{}^1 \mathrm{H}_{n+1}(X \otimes_R Y_{\leqslant k-i}) \to \mathrm{H}_{n+1}(X \,\widetilde{\otimes}_R\, Y) \to \lim_{i\in\mathbb{N}} \mathrm{H}_n(X \otimes_R Y_{\leqslant k-i}) \to 0.$$

In particular, $\mathrm{H}_{n+1}(X \,\widetilde{\otimes}_R\, Y) = 0$ *if and only if*

$$\lim_{i\in\mathbb{N}}{}^1 \mathrm{H}_{n+1}(X \otimes_R Y_{\leqslant k-i}) = 0 = \lim_{i\in\mathbb{N}} \mathrm{H}_n(X \otimes_R Y_{\leqslant k-i}).$$

Proof By Corollary 3.1, there is an exact sequence

$$0 \to \prod_{i\in\mathbb{N}}(X \otimes_R Y_{\leqslant k-i}) \to \prod_{i\in\mathbb{N}}(X \otimes_R Y_{\leqslant k-i}) \to X \,\widetilde{\otimes}_R\, Y \to 0.$$

Thus, one gets the following exact sequence:

$$\begin{aligned}\cdots \to \prod_{i\in\mathbb{N}} \mathrm{H}_{n+1}(X \otimes_R Y_{\leqslant k-i}) &\to \prod_{i\in\mathbb{N}} \mathrm{H}_{n+1}(X \otimes_R Y_{\leqslant k-i}) \to \mathrm{H}_{n+1}(X \,\widetilde{\otimes}_R\, Y)\\ \to \prod_{i\in\mathbb{N}} \mathrm{H}_n(X \otimes_R Y_{\leqslant k-i}) &\to \prod_{i\in\mathbb{N}} \mathrm{H}_n(X \otimes_R Y_{\leqslant k-i}) \to \cdots,\end{aligned}$$

which yields the desired exact sequence from the definitions of the lim and $\lim^1$ groups. □

Remark 3.3 Recall that an $\mathbb{N}$-inverse system $\{\delta_{uv}\colon M_v \to M_u\}_{u\leqslant v}$ of R-modules satisfies the Mittag-Leffler condition if for each $i \in \mathbb{N}$, there exists an index $j \in \mathbb{N}$ with $j \geqslant i$ such that $\operatorname{Im}\delta_{ij} = \operatorname{Im}\delta_{ik}$ for each $k \in \mathbb{N}$

with $k \geqslant j$. It is clear that if $\delta_{i,i+1}$ is surjective for each $i \gg 0$, then the $\mathbb{N}$-inverse system $\{\delta_{uv}\colon M_v \to M_u\}_{u\leqslant v}$ satisfies the Mittag-Leffler condition. Grothendieck proved in [11] that if the $\mathbb{N}$-inverse system $\{\delta_{uv}\colon M_v \to M_u\}_{u\leqslant v}$ satisfies the Mittag-Leffler condition, then one has $\lim^1_{i\in\mathbb{N}} M_i = 0$. Moreover, following [5, Corollary 6], $\lim^1_{i\in\mathbb{N}} M_i^{(\mathbb{N})} = 0$ if and only if the $\mathbb{N}$-inverse system $\{\delta_{uv}\colon M_v \to M_u\}_{u\leqslant v}$ satisfies the Mittag-Leffler condition.

Corollary 3.3 *Let X be an R°-complex, let Y be a bounded above R-complex with $\sup Y = k$, and let $n \in \mathbb{Z}$. If $\mathrm{H}_n(X^{(\mathbb{N})} \widetilde{\otimes}_R Y) = 0$, then the $\mathbb{N}$-inverse system $\{\delta_{uv}\colon \mathrm{H}_n(X \otimes_R Y_{\leqslant k-v}) \to \mathrm{H}_n(X \otimes_R Y_{\leqslant k-u})\}_{u\leqslant v}$ satisfies the Mittag-Leffler condition.*

Proof If $\mathrm{H}_n(X^{(\mathbb{N})} \widetilde{\otimes}_R Y) = 0$, then by Corollary 3.2,

$$\lim_{i\in\mathbb{N}}{}^1 \mathrm{H}_n(X^{(\mathbb{N})} \otimes_R Y_{\leqslant k-i}) = 0,$$

and so one gets

$$\lim_{i\in\mathbb{N}}{}^1 (\mathrm{H}_n(X \otimes_R Y_{\leqslant k-i}))^{(\mathbb{N})} = 0,$$

which implies that the $\mathbb{N}$-inverse system $\{\delta_{uv}\colon \mathrm{H}_n(X \otimes_R Y_{\leqslant k-v}) \to \mathrm{H}_n(X \otimes_R Y_{\leqslant k-u})\}_{u\leqslant v}$ satisfies the Mittag-Leffler condition; see Remark 3.3. □

Checking the proof of [6, Lemma 4.1], one gets the following result.

Lemma 3.2 *Let $\{\delta_{uv}\colon X_v \to X_u\}_{u\leqslant v}$ be an $\mathbb{N}$-inverse system of R-modules satisfying the Mittag-Leffler condition. If $\lim_{i\in\mathbb{N}} X_i = 0$, then one has*

$$\operatorname*{colim}_{i\in\mathbb{N}} \mathrm{Hom}_{\mathbb{Z}}(X_i, \mathbb{Q}/\mathbb{Z}) = 0.$$

The next proposition will be used to prove our main results advertised in the introduction.

Proposition 3.1 *Let X be an R°-complex, let Y be a bounded above R-complex with $\sup Y = k$, and let $n \in \mathbb{Z}$. If*

$$\mathrm{H}_n(X^{(\mathbb{N})} \widetilde{\otimes}_R Y) = 0 = \mathrm{H}_{n+1}(X \widetilde{\otimes}_R Y),$$

then one has

$$\operatorname*{colim}_{i\in\mathbb{N}} \mathrm{H}_{-n}(\mathrm{Hom}_{R^\circ}(X, \mathrm{Hom}_{\mathbb{Z}}(Y, \mathbb{Q}/\mathbb{Z})_{\geqslant i-k})) = 0$$

and

$$\operatorname*{colim}_{i\in\mathbb{N}} \mathrm{H}_{-n}(\mathrm{Hom}_R(Y_{\leqslant k-i}, \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z}))) = 0.$$

Proof The $\mathbb{N}$-inverse system $\{\delta_{uv}\colon \mathrm{H}_n(X \otimes_R Y_{\leqslant k-v}) \to \mathrm{H}_n(X \otimes_R Y_{\leqslant k-u})\}_{u\leqslant v}$ satisfies the Mittag-Leffler condition by Corollary 3.3. The vanishing of $\mathrm{H}_{n+1}(X \widetilde{\otimes}_R Y)$ implies that

$$\lim_{i\in\mathbb{N}} \mathrm{H}_n(X \otimes_R Y_{\leqslant k-i}) = 0;$$

see Corollary 3.2. Thus, by Lemma 3.2, one has

$$\begin{aligned}\operatorname*{colim}_{i\in\mathbb{N}} \mathrm{H}_{-n}(\mathrm{Hom}_{\mathbb{Z}}(X\otimes_R Y_{\leqslant k-i},\mathbb{Q}/\mathbb{Z})) &\cong \operatorname*{colim}_{i\in\mathbb{N}} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_n(X\otimes_R Y_{\leqslant k-i}),\mathbb{Q}/\mathbb{Z})\\ &= 0.\end{aligned}$$

Now, the desired results hold by the adjoint isomorphism. □

We end this section with the following result that will be used in the next section.

Proposition 3.2 *Let X be an R°-complex, let Y be a bounded (R,S°)-complex, and let Z be an S-complex. Then there is an isomorphism of $\mathbb{Z}$-complexes*

$$(X\otimes_R Y)\,\widetilde{\otimes}_S\, Z \to X\,\widetilde{\otimes}_R\,(Y\otimes_S Z), \tag{3.1}$$

which is functorial in X, Y, and Z.

Proof Consider the following commutative diagram of $\mathbb{Z}$-complexes:

$$\begin{array}{ccccccccc} 0 \to & (X\otimes_R Y)\otimes_S Z & \to & (X\otimes_R Y)\overline{\otimes}_S Z & \to & (X\otimes_R Y)\widetilde{\otimes}_S Z & \to 0\\ & \downarrow & & \alpha\downarrow & & & \\ 0\to & X\otimes_R (Y\otimes_S Z) & \to & X\,\overline{\otimes}_R\,(Y\otimes_S Z) & \to & X\,\widetilde{\otimes}_R\,(Y\otimes_S Z) & \to 0. \end{array}$$

We notice that

$$X\,\overline{\otimes}_R\, Y = X\otimes_R Y,\quad Y\overline{\otimes}_S Z = Y\otimes_S Z,$$

since Y is bounded. Then the second vertical map α is an isomorphism by [2, Proposition A4]. The first one is clearly an isomorphism. So one gets an isomorphism (3.1), which is clearly functorial in X, Y, and Z. □

4 Stable homology with respect to semidualizing module

Convention In this section, R is a commutative noetherian ring, and C is a semidualizing R-module.

Definition 4.1 Let $\mathscr{X}$ (resp., $\mathscr{Y}$) be a precovering (resp., preenveloping) class of R-modules. For R-modules M and N, let $X\to M$ be a proper $\mathscr{X}$-resolution of M, and let $N\to Y$ be a proper $\mathscr{Y}$-coresolution of N. For each $n\in\mathbb{Z}$, the nth *stable homology* of M and N with respect to $\mathscr{X}$ and $\mathscr{Y}$ is

$$\widetilde{\mathrm{Tor}}_n^{\mathscr{X}\mathscr{Y}}(M,N) = \mathrm{H}_{n+1}(X\,\widetilde{\otimes}_R\, Y).$$

Following [7, Section 8.2], any two proper $\mathscr{X}$-resolutions of M, and similarly, any two proper $\mathscr{Y}$-coresolutions of N, are homotopy equivalent. Thus, by [2, 1.5(d)], the above definition is independent of the choices of (co)resolutions.

We notice that $\widetilde{\mathrm{Tor}}_n^{\mathscr{P}\mathscr{I}}(M,N)$ is the classical stable homology, $\widetilde{\mathrm{Tor}}_n^{R}(M,N)$, of M and N defined by Goichot [9]; see also [2].

We denote by $\mathscr{P}_C$ (resp., $\mathscr{F}_C$, $\mathscr{I}_C$) the class of R-modules $C \otimes_R P$ (resp., $C \otimes_R F$, $\mathrm{Hom}_R(C,I)$) with P projective (resp., F flat, I injective). Then $\mathscr{P}_C$ and $\mathscr{F}_C$ are precovering and $\mathscr{I}_C$ is preenveloping; see, e.g., Holm and White [12, Proposition 5.3]. In the next lemma, (a) and (b) can be found in [15, Lemma 3.1], (c) can be proved as in [15, Lemma 3.1(c)], and (d) is from [18, Lemma 2.1(b)].

Lemma 4.1 *Let M be an R-module.*

(a) *If $F \xrightarrow{\simeq} \mathrm{Hom}_R(C,M)$ is a proper flat (resp., projective) resolution, then $C \otimes_R F \to M$ is a proper $\mathscr{F}_C$ (resp., $\mathscr{P}_C$)-resolution of M.*

(b) *If $G \to M$ is a proper $\mathscr{F}_C$ (resp., $\mathscr{P}_C$)-resolution of M, then*

$$\mathrm{Hom}_R(C,G) \xrightarrow{\simeq} \mathrm{Hom}_R(C,M)$$

is a proper flat (resp., projective)-resolution of $\mathrm{Hom}_R(C,M)$.

(c) *If $C \otimes_R M \xrightarrow{\simeq} I$ is an injective resolution of $C \otimes_R M$, then $M \to \mathrm{Hom}_R(C,I)$ is a proper $\mathscr{I}_C$-coresolution.*

(d) *If $M \to J$ is a proper $\mathscr{I}_C$-coresolution of M, then $C \otimes_R M \xrightarrow{\simeq} C \otimes_R J$ is an injective resolution of $C \otimes_R M$.*

Proposition 4.1 *Let M and N be R-modules. Then there are isomorphisms*

$$\widetilde{\mathrm{Tor}}_n^{\mathscr{P}_C\mathscr{I}_C}(M,N) \cong \widetilde{\mathrm{Tor}}_n^{R}(\mathrm{Hom}_R(C,M), C \otimes_R N) \cong \widetilde{\mathrm{Tor}}^{\mathscr{F}_C\mathscr{I}_C}(M,N),$$

which are functorial in M and N.

Proof Let $P \xrightarrow{\simeq} \mathrm{Hom}_R(C,M)$ be a projective resolution of $\mathrm{Hom}_R(C,M)$, and let $C \otimes_R N \xrightarrow{\simeq} I$ be an injective resolution of $C \otimes_R N$. Then by Lemma 4.1 (a) and (c), $C \otimes_R P \to M$ is a proper $\mathscr{P}_C$-resolution of M, and $N \to \mathrm{Hom}_R(C,I)$ is a proper $\mathscr{I}_C$-coresolution, and so one gets

$$\begin{aligned}\widetilde{\mathrm{Tor}}_n^{\mathscr{P}_C\mathscr{I}_C}(M,N) &= \mathrm{H}_{n+1}((C \otimes_R P)\,\widetilde{\otimes}_R\,\mathrm{Hom}_R(C,I))\\ &\cong \mathrm{H}_{n+1}(P\,\widetilde{\otimes}_R\,(C \otimes_R \mathrm{Hom}_R(C,I)))\\ &\cong \mathrm{H}_{n+1}(P\,\widetilde{\otimes}_R\,I)\\ &\cong \widetilde{\mathrm{Tor}}_n^{R}(\mathrm{Hom}_R(C,M), C \otimes_R N),\end{aligned}$$

where the first isomorphism follows from Proposition 3.2, and the second one holds since I is a complex of injective R-modules.

The isomorphism

$$\widetilde{\mathrm{Tor}}^{\mathscr{F}_C\mathscr{I}_C}(M,N) \cong \widetilde{\mathrm{Tor}}_n^{R}(\mathrm{Hom}_R(C,M), C \otimes_R N)$$

can be proved similarly by taking a proper flat resolution $F \xrightarrow{\simeq} \mathrm{Hom}_R(C, M)$ and using Lemma 4.1 (a) and [2, Proposition 2.6].

Now, it is straightforward to verify that the desired isomorphisms are functorial in M and N. $\square$

Lemma 4.2 *Let M be an R-module, and let $n \in \mathbb{Z}$.*

(a) *If $\widetilde{\mathrm{Tor}}_{n-1}^{\mathscr{P}_C\mathscr{I}_C}(-, M) = 0$, then $\widetilde{\mathrm{Tor}}_{n}^{\mathscr{P}_C\mathscr{I}_C}(-, M) = 0$.*

(b) *If $\widetilde{\mathrm{Tor}}_{n+1}^{\mathscr{P}_C\mathscr{I}_C}(M, -) = 0$, then $\widetilde{\mathrm{Tor}}_{n}^{\mathscr{P}_C\mathscr{I}_C}(M, -) = 0$.*

Proof (a) For an R-module M', by [12, Proposition 5.3 (b)], there is a complex

$$0 \to K \to P \to M' \to 0$$

with $P \in \mathscr{P}_C$ such that the sequence

$$0 \to \mathrm{Hom}_R(P', K) \to \mathrm{Hom}_R(P', P) \to \mathrm{Hom}_R(P', M') \to 0$$

is exact for each $P' \in \mathscr{P}_C$. In particular, the sequence

$$0 \to \mathrm{Hom}_R(C, K) \to \mathrm{Hom}_R(C, P) \to \mathrm{Hom}_R(C, M') \to 0$$

is exact. Since $\mathrm{Hom}_R(C, P)$ is projective, one gets

$$\widetilde{\mathrm{Tor}}_{n}^{R}(\mathrm{Hom}_R(C, M'), C \otimes_R M) \cong \widetilde{\mathrm{Tor}}_{n-1}^{R}(\mathrm{Hom}_R(C, K), C \otimes_R M),$$

and so by Proposition 4.1,

$$\widetilde{\mathrm{Tor}}_{n}^{\mathscr{P}_C\mathscr{I}_C}(M', M) \cong \widetilde{\mathrm{Tor}}_{n-1}^{\mathscr{P}_C\mathscr{I}_C}(K, M) = 0,$$

which yields $\widetilde{\mathrm{Tor}}_{n}^{\mathscr{P}_C\mathscr{I}_C}(-, M) = 0$.

(b) Let N be an R-module. Then by [12, Proposition 5.3 (c)], there is a complex

$$0 \to N \to I \to K \to 0$$

with $I \in \mathscr{I}_C$ such that the sequence

$$0 \to \mathrm{Hom}_R(K, I') \to \mathrm{Hom}_R(I, I') \to \mathrm{Hom}_R(N, I') \to 0$$

is exact for each $I' \in \mathscr{I}_C$. Since $C^\vee = \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z})$ is in $\mathscr{I}_C$, the sequence

$$0 \to \mathrm{Hom}_R(K, C^\vee) \to \mathrm{Hom}_R(I, C^\vee) \to \mathrm{Hom}_R(N, C^\vee) \to 0$$

is exact, which implies that the sequence

$$0 \to C \otimes_R N \to C \otimes_R I \to C \otimes_R K \to 0$$

is exact. We notice that $C \otimes_R I$ is injective. Then one gets

$$\widetilde{\mathrm{Tor}}_{n}^{R}(\mathrm{Hom}_R(C, M), C \otimes_R N) \cong \widetilde{\mathrm{Tor}}_{n+1}^{R}(\mathrm{Hom}_R(C, M), C \otimes_R K),$$

and so by Proposition 4.1,

$$\widetilde{\mathrm{Tor}}_n^{\mathscr{P}_C\mathscr{I}_C}(M,N)\cong \widetilde{\mathrm{Tor}}_{n+1}^{\mathscr{P}_C\mathscr{I}_C}(M,K)=0,$$

which yields $\widetilde{\mathrm{Tor}}_n^{\mathscr{P}_C\mathscr{I}_C}(M,-)=0$. □

Now, we are in a position to give the proofs of our main results described in the introduction.

Proof of Theorem 1.1 (i) ⇒ (ii) Since $\mathscr{F}_C$-$\mathrm{pd}_R M<\infty$, there is a proper $\mathscr{F}_C$-resolution $F\to M$ with F bounded. Thus, for each R-module N with $N\to I$ a proper $\mathscr{I}_C$-coresolution, one has

$$\widetilde{\mathrm{Tor}}_n^{\mathscr{P}_C\mathscr{I}_C}(M,N)\cong \widetilde{\mathrm{Tor}}^{\mathscr{F}_C\mathscr{I}_C}(M,N)=\mathrm{H}_{n+1}(F\,\widetilde{\otimes}_R\, I)=0$$

by Proposition 4.1.

(ii) ⇒ (iii) It is clear.

(iii) ⇒ (i) We first notice that

$$\widetilde{\mathrm{Tor}}_0^{\mathscr{P}_C\mathscr{I}_C}(M,-)=0=\widetilde{\mathrm{Tor}}_{-1}^{\mathscr{P}_C\mathscr{I}_C}(M,-)$$

by Lemma 4.2.

Let $F\xrightarrow{\simeq}\mathrm{Hom}_R(C,M)$ be a proper flat resolution of $\mathrm{Hom}_R(C,M)$. Then $C\otimes_R F\to M$ is a proper $\mathscr{F}_C$-resolution by Lemma 4.1 (a), and

$$C\otimes_R \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})\cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(C,M),\mathbb{Q}/\mathbb{Z})\xrightarrow{\simeq}\mathrm{Hom}_{\mathbb{Z}}(F,\mathbb{Q}/\mathbb{Z})$$

is an injective resolution of $C\otimes_R \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$, and so

$$\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})\to \mathrm{Hom}_R(C,\mathrm{Hom}_{\mathbb{Z}}(F,\mathbb{Q}/\mathbb{Z}))\cong \mathrm{Hom}_{\mathbb{Z}}(C\otimes_R F,\mathbb{Q}/\mathbb{Z})$$

is a proper $\mathscr{I}_C$-coresolution of $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$ by Lemma 4.1(c).

Let N be an R-module, and let $C\otimes_R N\xrightarrow{\simeq} I$ be an injective resolution of $C\otimes_R N$. Then $N\to \mathrm{Hom}_R(C,I)$ is a proper $\mathscr{I}_C$-coresolution by Lemma 4.1 (c), and

$$C\otimes_R N^{(\mathbb{N})}\cong (C\otimes_R N)^{(\mathbb{N})}\xrightarrow{\simeq} I^{(\mathbb{N})}$$

is an injective resolution of $C\otimes_R N^{(\mathbb{N})}$, and so

$$N^{(\mathbb{N})}\to \mathrm{Hom}_R(C,I^{(\mathbb{N})})\cong (\mathrm{Hom}_R(C,I))^{(\mathbb{N})}$$

is a proper $\mathscr{I}_C$-coresolution by Lemma 4.1 (c).

Since

$$\widetilde{\mathrm{Tor}}_0^{\mathscr{F}_C\mathscr{I}_C}(M,N)=0=\widetilde{\mathrm{Tor}}_{-1}^{\mathscr{F}_C\mathscr{I}_C}(M,N^{(\mathbb{N})})$$

by Proposition 4.1, one gets

$$\mathrm{H}_1((C\otimes_R F)\,\widetilde{\otimes}_R\,\mathrm{Hom}_R(C,I))=0,$$

and

$$\mathrm{H}_0((C \otimes_R F)^{(\mathbb{N})} \widetilde{\otimes}_R \mathrm{Hom}_R(C, I)) \cong \mathrm{H}_0((C \otimes_R F) \widetilde{\otimes}_R (\mathrm{Hom}_R(C, I))^{(\mathbb{N})}) = 0$$

by Proposition 3.2. Now, using Proposition 3.1, one gets

$$\operatorname*{colim}_{i \in \mathbb{N}} \mathrm{H}_0(\mathrm{Hom}_R(\mathrm{Hom}_R(C, I)_{\leqslant -i}, \mathrm{Hom}_{\mathbb{Z}}(C \otimes_R F, \mathbb{Q}/\mathbb{Z}))) = 0,$$

and so

$$\widetilde{\mathrm{Ext}}^0_{\mathscr{I}_C}(N, \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) = 0$$

for each R-module N by Proposition A.9 in Appendix. Thus,

$$\mathscr{I}_C\text{-id}_R \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) < \infty$$

by Proposition A.7 in Appendix, and so $\mathscr{F}_C$-pd$_R M < \infty$; see [16, Lemma 4.2].

Finally, if M is finitely generated, then by [15, Theorem 5.5], conditions (i) and (i′) are equivalent. □

Proof of Theorem 1.2 (i) ⇒ (ii) Since $\mathscr{I}_C$-id$_R N < \infty$, there is a proper $\mathscr{I}_C$-coresolution $N \to I$ with I bounded. Thus, for each R-module M with $P \to M$ a proper $\mathscr{P}_C$-resolution, one has

$$\widetilde{\mathrm{Tor}}_n^{\mathscr{P}_C \mathscr{I}_C}(M, N) = \mathrm{H}_{n+1}(P \widetilde{\otimes}_R I) = 0.$$

(ii) ⇒ (iii) It is clear.

(iii) ⇒ (i) We first notice that

$$\widetilde{\mathrm{Tor}}_0^{\mathscr{P}_C \mathscr{I}_C}(-, N) = 0 = \widetilde{\mathrm{Tor}}_{-1}^{\mathscr{P}_C \mathscr{I}_C}(-, N)$$

by Lemma 4.2.

Let M be an R-module, and let $F \xrightarrow{\simeq} \mathrm{Hom}_R(C, M)$ be a proper flat resolution. Then, by Lemma 4.1 (a), $C \otimes_R F \to M$ is a proper $\mathscr{F}_C$-resolution of M. Let $N \to I$ be a proper $\mathscr{I}_C$-coresolution of N. Since

$$\widetilde{\mathrm{Tor}}_0^{\mathscr{F}_C \mathscr{I}_C}(M, N) \cong \widetilde{\mathrm{Tor}}_0^{\mathscr{P}_C \mathscr{I}_C}(M, N) = 0$$

by Proposition 4.1, one gets

$$\mathrm{H}_1((C \otimes_R F) \widetilde{\otimes}_R I) = 0.$$

On the other hand, one has

$$\widetilde{\mathrm{Tor}}_{-1}^{\mathscr{P}_C \mathscr{I}_C}(M^{(\mathbb{N})}, N) = 0,$$

and so by Proposition 4.1,

$$\widetilde{\mathrm{Tor}}_{-1}^{R}((\mathrm{Hom}_R(C, M))^{(\mathbb{N})}, C \otimes_R N) = 0.$$

Note that $F \xrightarrow{\simeq} \mathrm{Hom}_R(C,M)$ is a flat resolution. Then

$$F^{(\mathbb{N})} \xrightarrow{\simeq} (\mathrm{Hom}_R(C,M))^{(\mathbb{N})}$$

is a flat resolution of $(\mathrm{Hom}_R(C,M))^{(\mathbb{N})}$. Since $C \otimes_R N \xrightarrow{\simeq} C \otimes_R I$ is an injective resolution by Lemma 4.1 (d), one gets

$$\mathrm{H}_0(F^{(\mathbb{N})} \widetilde{\otimes}_R (C \otimes_R I)) = 0;$$

see [2, Proposition 2.6]. Thus, we have

$$\mathrm{H}_0((C \otimes_R F)^{(\mathbb{N})} \widetilde{\otimes}_R I) \cong \mathrm{H}_0((C \otimes_R F^{(\mathbb{N})}) \widetilde{\otimes}_R I) \cong \mathrm{H}_0(F^{(\mathbb{N})} \widetilde{\otimes}_R (C \otimes_R I)) = 0,$$

where the second isomorphism follows from Proposition 3.2.

Now, by Proposition 3.1, one gets

$$\operatorname*{colim}_{i \in \mathbb{N}} \mathrm{H}_0(\mathrm{Hom}_R(C \otimes_R F, \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})_{\geqslant i})) = 0.$$

We notice that $C \otimes_R F \to M$ is a proper $\mathscr{F}_C$-resolution of M, and

$$\mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$$

is a proper $\mathscr{F}_C$-resolution of $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$. Then, by Proposition A.8 in Appendix, one gets

$$\widetilde{\mathrm{Ext}}^0_{\mathscr{F}_C}(M, \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})) = 0$$

for each R-module M. Thus,

$$\mathscr{F}_C\text{-pd}_R \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z}) < \infty$$

by Proposition A.6 in Appendix, and so $\mathscr{I}_C$-$\mathrm{id}_R N < \infty$; see [16, Lemma 4.2]. □

As a corollary of the above theorems, we give a balance result for stable homology with respect to a semidualizing module.

Corollary 4.1 *The following conditions are equivalent for a local ring R:*

(i) $\widetilde{\mathrm{Tor}}_n^{\mathscr{P}_C \mathscr{I}_C}(M,N) \cong \widetilde{\mathrm{Tor}}_n^{\mathscr{P}_C \mathscr{I}_C}(N,M)$ *for all R-modules M and N, and for each $n \in \mathbb{Z}$;*

(ii) $\mathscr{I}_C$-$\mathrm{id}_R C < \infty$;

(iii) $C \cong R$ *and R is Gorenstein.*

Proof (i) ⇒ (ii) Since C is C-projective, one gets

$$\widetilde{\mathrm{Tor}}_n^{\mathscr{P}_C \mathscr{I}_C}(M,C) \cong \widetilde{\mathrm{Tor}}_n^{\mathscr{P}_C \mathscr{I}_C}(C,M) = 0$$

for all R-modules M and for each $n \in \mathbb{Z}$, and so $\mathscr{I}_C$-$\mathrm{id}(C) < \infty$ by Theorem 1.2.

(ii) ⇒ (iii) It follows from Sather-Wagstaff and Yassemi [17, Lemma 2.11].

(iii) ⇒ (i) It holds by [2, Corollary 4.7]. □

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Appendix Stable cohomology

The following definitions of bounded and stable Hom-complexes can be found in [1,9].

Definition A.1 For R-complexes X and Y, the *bounded Hom-complex* $\overline{\mathrm{Hom}}_R(X,Y)$ is the subcomplex of $\mathrm{Hom}_R(X,Y)$ with degree-n term

$$\overline{\mathrm{Hom}}_R(X,Y)_n = \coprod_{i\in\mathbb{Z}} \mathrm{Hom}_R(X_i, Y_{n+i}).$$

We denote by $\widetilde{\mathrm{Hom}}_R(X,Y)$ the quotient complex $\mathrm{Hom}_R(X,Y)/\overline{\mathrm{Hom}}_R(X,Y)$, which is called the *stable Hom-complex*.

Proposition A.1 *Let X and Z be an R-complex and an S-complex, respectively, and let Y be a bounded (S,R°)-complex. Then there are isomorphisms of $\mathbb{Z}$-complexes*

$$\overline{\mathrm{Hom}}_S(Y\otimes_R X, Z) \cong \overline{\mathrm{Hom}}_R(X, \mathrm{Hom}_S(Y,Z))$$

and

$$\widetilde{\mathrm{Hom}}_S(Y\otimes_R X, Z) \cong \widetilde{\mathrm{Hom}}_R(X, \mathrm{Hom}_S(Y,Z)),$$

which are functorial in X, Y, and Z.

Proof For every $n\in\mathbb{Z}$, one has

$$\begin{aligned}\overline{\mathrm{Hom}}_S(Y\otimes_R X, Z)_n &= \coprod_{h\in\mathbb{Z}} \mathrm{Hom}_S((Y\otimes_R X)_h, Z_{n+h})\\ &= \coprod_{h\in\mathbb{Z}} \mathrm{Hom}_S\Big(\coprod_{q\in\mathbb{Z}}(Y_q\otimes_R X_{h-q}), Z_{n+h}\Big)\\ &\cong \coprod_{h\in\mathbb{Z}}\coprod_{q\in\mathbb{Z}} \mathrm{Hom}_S(Y_q\otimes_R X_{h-q}, Z_{n+h})\\ &= \coprod_{p\in\mathbb{Z}}\coprod_{q\in\mathbb{Z}} \mathrm{Hom}_S(Y_q\otimes_R X_p, Z_{n+p+q}).\end{aligned}$$

On the other hand, for every $n \in \mathbb{Z}$, one has

$$\begin{aligned}\overline{\mathrm{Hom}}_R(X, \mathrm{Hom}_S(Y,Z))_n &= \prod_{p\in\mathbb{Z}} \mathrm{Hom}_R(X_p, \mathrm{Hom}_S(Y,Z)_{n+p})\\ &= \prod_{p\in\mathbb{Z}} \mathrm{Hom}_R\Big(X_p, \prod_{q\in\mathbb{Z}} \mathrm{Hom}_S(Y_q, Z_{n+p+q})\Big)\\ &\cong \prod_{p\in\mathbb{Z}}\prod_{q\in\mathbb{Z}} \mathrm{Hom}_R(X_p, \mathrm{Hom}_S(Y_q, Z_{n+p+q})),\end{aligned}$$

where the isomorphism holds since Y is bounded.

We notice that there is a natural isomorphism of $\mathbb{Z}$-modules:

$$\rho_{Y_qX_pZ_{n+p+q}}\colon \mathrm{Hom}_S(Y_q \otimes_R X_p, Z_{n+p+q}) \to \mathrm{Hom}_R(X_p, \mathrm{Hom}_S(Y_q, Z_{n+p+q})).$$

Then one gets an isomorphism of $\mathbb{Z}$-complexes:

$$\rho_{YXZ}\colon \overline{\mathrm{Hom}}_S(Y\otimes_R X, Z) \to \overline{\mathrm{Hom}}_R(X, \mathrm{Hom}_S(Y,Z)).$$

It is straightforward to verify that ρ_{YXZ} is functorial in X, Y, and Z.

For the second isomorphism in the statement, consider the following commutative diagram of $\mathbb{Z}$-complexes:

$$\begin{array}{ccccccc} 0 \to & \overline{\mathrm{Hom}}_S(Y\otimes_R X, Z) & \to & \mathrm{Hom}_S(Y\otimes_R X, Z) & \to & \widetilde{\mathrm{Hom}}_S(Y\otimes_R X, Z) & \to 0\\ & \rho\downarrow & & \varrho\downarrow & & & \\ 0 \to & \overline{\mathrm{Hom}}_R(X, \mathrm{Hom}_S(Y,Z)) & \to & \mathrm{Hom}_R(X, \mathrm{Hom}_S(Y,Z)) & \to & \widetilde{\mathrm{Hom}}_R(X, \mathrm{Hom}_S(Y,Z)) & \to 0.\end{array}$$

Since ρ and ϱ are isomorphisms, one gets an isomorphism

$$\widetilde{\mathrm{Hom}}_S(Y\otimes_R X, Z) \to \widetilde{\mathrm{Hom}}_R(X, \mathrm{Hom}_S(Y,Z)),$$

which is clearly functorial in X, Y, and Z. □

Let $\mathscr{X}$ be a precovering class of R-modules, and let $X_M \to M$ and $X_N \to N$ be proper $\mathscr{X}$-resolutions of R-modules M and N, respectively. For each $n \in \mathbb{Z}$, the nth *stable cohomology* of M and N with respect to $\mathscr{X}$ is

$$\widetilde{\mathrm{Ext}}^n_{\mathscr{X}}(M,N) = \mathrm{H}_{-n}(\widetilde{\mathrm{Hom}}_R(X_M, X_N)).$$

Dually, let $\mathscr{Y}$ be a preenveloping class of R-modules, and let $M \to Y_M$ and $N \to Y_N$ be proper $\mathscr{Y}$-coresolutions of M and N, respectively. For each $n \in \mathbb{Z}$, the nth *stable cohomology* of M and N with respect to $\mathscr{Y}$ is

$$\widetilde{\mathrm{Ext}}^n_{\mathscr{Y}}(M,N) = \mathrm{H}_{-n}(\widetilde{\mathrm{Hom}}_R(Y_M, Y_N)).$$

Any two proper $\mathscr{X}$-resolutions of M, and similarly, any two proper $\mathscr{Y}$-coresolutions of N, are homotopy equivalent; see [7, Section 8.2]. Thus, the above definitions are independent of the choices of (co)resolutions. We notice that $\widetilde{\mathrm{Ext}}^n_{\mathscr{P}}(M,N)$ is the classical stable cohomology, $\widetilde{\mathrm{Ext}}^n_R(M,N)$, of M and N; see [1] and [9]. Also $\widetilde{\mathrm{Ext}}^n_{\mathscr{I}}(M,N)$ is the cohomology given by Nucinkis [13].

A.1 Stable cohomology with respect to proper flat (injective) resolutions

The proof of the next result can be modelled along the argument in the proof of [1, Proposition 2.2], when the argument is applied to the functor $\mathrm{Ext}^i_{\mathscr{F}}(M,-)$, that is computed by $\mathrm{H}_{-i}(\mathrm{Hom}_R(F,-))$, where $F \xrightarrow{\simeq} M$ is a proper flat resolution.

Proposition A.2 *For an R-module M, the following conditions are equivalent*:

(i) $\mathrm{fd}_R M < \infty$;

(ii) $\widetilde{\mathrm{Ext}}^n_{\mathscr{F}}(M,-) = 0 = \widetilde{\mathrm{Ext}}^n_{\mathscr{F}}(-,M)$ *for each* $n \in \mathbb{Z}$;

(iii) $\widetilde{\mathrm{Ext}}^0_{\mathscr{F}}(M,M) = 0$.

Dually, we have the following result that was proved by Nucinkis [13, Theorem 3.7].

Proposition A.3 *For an R-module N, the following conditions are equivalent*:

(i) $\mathrm{id}_R N < \infty$;

(ii) $\widetilde{\mathrm{Ext}}^n_{\mathscr{I}}(N,-) = 0 = \widetilde{\mathrm{Ext}}^n_{\mathscr{I}}(-,N)$ *for each* $n \in \mathbb{Z}$;

(iii) $\widetilde{\mathrm{Ext}}^0_{\mathscr{I}}(N,N) = 0$.

Proposition A.4 *Let M and N be R-modules with proper flat resolutions $F \xrightarrow{\simeq} M$ and $F' \xrightarrow{\simeq} N$, respectively. For every $n \in \mathbb{Z}$, there is an isomorphism*

$$\widetilde{\mathrm{Ext}}^n_{\mathscr{F}}(M,N) \cong \operatorname*{colim}_{i\in\mathbb{N}} \mathrm{H}_{-n}(\mathrm{Hom}_R(F, F'_{\geqslant i})).$$

Proof Set

$$\Omega_s M = \mathrm{Coker}(F_{s+1} \to F_s), \quad \Omega_s N = \mathrm{Coker}(F'_{s+1} \to F'_s).$$

Using a similar proof as in [13, Theorem 3.6], one gets a natural isomorphism

$$\operatorname*{colim}_{i\in\mathbb{N}} \mathrm{Ext}^i_{\mathscr{F}}(M, \Omega_{i-n}N) \cong \operatorname*{colim}_{i\in\mathbb{N}} \mathrm{Hom}_R(\Omega_i M, \Omega_{i-n}N)/\mathrm{FHom}_R(\Omega_i M, \Omega_{i-n}N).$$

Here, $\mathrm{FHom}_R(\Omega_i M, \Omega_{i-n}N)$ denotes the set of all homomorphisms of R-modules $f \in \mathrm{Hom}_R(\Omega_i M, \Omega_{i-n}N)$ factoring through a flat R-module. As proved in [13, Theorem 4.4] (see also [3, B.2]), one gets an isomorphism

$$\widetilde{\mathrm{Ext}}^n_{\mathscr{F}}(M,N) \cong \operatorname*{colim}_{i\in\mathbb{N}} \mathrm{Hom}_R(\Omega_i M, \Omega_{i-n}N)/\mathrm{FHom}_R(\Omega_i M, \Omega_{i-n}N).$$

On the other hand, we notice that

$$\Sigma^{-i} F'_{\geqslant i} \xrightarrow{\simeq} \Omega_i N$$

is a proper flat resolution. Thus, one has

$$\begin{aligned}\operatorname*{colim}_{i\in\mathbb{N}} \mathrm{Ext}^{i}_{\mathscr{F}}(M,\Omega_{i-n}N) &\cong \operatorname*{colim}_{i\in\mathbb{N}} \mathrm{Ext}^{i+n}_{\mathscr{F}}(M,\Omega_{i}N)\\ &\cong \operatorname*{colim}_{i\in\mathbb{N}} \mathrm{H}_{-i-n}(\mathrm{Hom}_R(F,\Sigma^{-i}F'_{\geqslant i}))\\ &\cong \operatorname*{colim}_{i\in\mathbb{N}} \mathrm{H}_{-n}(\mathrm{Hom}_R(F,F'_{\geqslant i})),\end{aligned}$$

where the second isomorphism follows from [4, Proposition 2.6]. Now, one gets the isomorphism in the statement. □

Dually, one gets the following result, which was proved in [6, Proposition 1.1 (iii)].

Proposition A.5 *Let M and N be R-modules with injective resolutions $M \xrightarrow{\simeq} I$ and $N \xrightarrow{\simeq} I'$, respectively. For every $n \in \mathbb{Z}$, there is an isomorphism*

$$\widetilde{\mathrm{Ext}}^{n}_{\mathscr{I}}(M,N) \cong \operatorname*{colim}_{i\in\mathbb{N}} \mathrm{H}_{-n}(\mathrm{Hom}_R(I_{\leqslant -i},I')).$$

A.2 Stable cohomology with respect to a semidualizing module

In this subsection, we assume that R is a commutative noetherian ring, and let C be a semidualizing R-module.

Lemma A.1 *Let M and N be R-modules. Then there is an isomorphism*

$$\widetilde{\mathrm{Ext}}^{n}_{\mathscr{F}_C}(M,N) \cong \widetilde{\mathrm{Ext}}^{n}_{\mathscr{F}}(\mathrm{Hom}_R(C,M),\mathrm{Hom}_R(C,N)),$$

which is functorial in M and N.

Proof Let

$$F \xrightarrow{\simeq} \mathrm{Hom}_R(C,M), \quad F' \xrightarrow{\simeq} \mathrm{Hom}_R(C,N),$$

be proper flat resolutions of $\mathrm{Hom}_R(C,M)$ and $\mathrm{Hom}_R(C,N)$, respectively. Then by Lemma 4.1 (a), $C\otimes_R F \to M$ and $C\otimes_R F' \to N$ are proper $\mathscr{F}_C$-resolutions of M and N, respectively. Thus, one has

$$\begin{aligned}\widetilde{\mathrm{Ext}}^{n}_{\mathscr{F}_C}(M,N) &= \mathrm{H}_{-n}(\widetilde{\mathrm{Hom}}_R(C\otimes_R F, C\otimes_R F'))\\ &\cong \mathrm{H}_{-n}(\widetilde{\mathrm{Hom}}_R(F,\mathrm{Hom}_R(C,C\otimes_R F')))\\ &\cong \mathrm{H}_{-n}(\widetilde{\mathrm{Hom}}_R(F,F'))\\ &\cong \widetilde{\mathrm{Ext}}^{n}_{\mathscr{F}}(\mathrm{Hom}_R(C,M),\mathrm{Hom}_R(C,N)),\end{aligned}$$

where the first isomorphism follows from Proposition A.1, and the second one holds since F' is a complex of flat R-modules. It is straightforward to verify that the desired isomorphism is functorial in M and N. □

The next result can be proved dually using Lemma 4.1 (c) and Proposition A.1.

Lemma A.2 *Let M and N be R-modules. Then there is an isomorphism*

$$\widetilde{\operatorname{Ext}}^n_{\mathscr{I}_C}(M,N) \cong \widetilde{\operatorname{Ext}}^n_{\mathscr{I}}(C\otimes_R M, C\otimes_R N),$$

which is functorial in M and N.

Proposition A.6 *For an R-module M, the following conditions are equivalent:*

(i) $\mathscr{F}_C$-$\operatorname{pd}_R M < \infty$;

(ii) $\widetilde{\operatorname{Ext}}^n_{\mathscr{F}_C}(M,-) = 0 = \widetilde{\operatorname{Ext}}^n_{\mathscr{F}_C}(-,M)$ *for each* $n\in\mathbb{Z}$;

(iii) $\widetilde{\operatorname{Ext}}^0_{\mathscr{F}_C}(M,M) = 0$.

Proof (i) $\Rightarrow$ (ii) Since $\mathscr{F}_C$-$\operatorname{pd}_R M < \infty$, there is a proper $\mathscr{F}_C$-resolution $F\to M$ with F bounded, and so

$$\widetilde{\operatorname{Hom}}_R(F,-) = 0 = \widetilde{\operatorname{Hom}}_R(-,F).$$

Thus, one gets

$$\widetilde{\operatorname{Ext}}^n_{\mathscr{F}_C}(M,-) = 0 = \widetilde{\operatorname{Ext}}^n_{\mathscr{F}_C}(-,M), \quad \forall\, n\in\mathbb{Z}.$$

(ii) $\Rightarrow$ (iii) It is clear.

(iii) $\Rightarrow$ (i) By Lemma A.1, one gets

$$\widetilde{\operatorname{Ext}}^0_{\mathscr{F}}(\operatorname{Hom}_R(C,M),\operatorname{Hom}_R(C,M)) \cong \widetilde{\operatorname{Ext}}^0_{\mathscr{F}_C}(M,M) = 0,$$

and so $\operatorname{fd}_R\operatorname{Hom}_R(C,M) < \infty$ by Proposition A.2. Thus, one gets $\mathscr{F}_C$-$\operatorname{pd}_R M < \infty$; see [15, Proposition 5.2(b)]. □

The next result can be proved dually using Proposition A.3, Lemma A.2, and [18, Theorem 2.11(b)].

Proposition A.7 *For an R-module N, the following conditions are equivalent:*

(i) $\mathscr{I}_C$-$\operatorname{id}_R N < \infty$;

(ii) $\widetilde{\operatorname{Ext}}^n_{\mathscr{I}_C}(N,-) = 0 = \widetilde{\operatorname{Ext}}^n_{\mathscr{I}_C}(-,N)$ *for each* $n\in\mathbb{Z}$;

(iii) $\widetilde{\operatorname{Ext}}^0_{\mathscr{I}_C}(N,N) = 0$.

Proposition A.8 *Let M and N be R-modules with proper $\mathscr{F}_C$-resolutions $F\to M$ and $F'\to N$, respectively. For every $n\in\mathbb{Z}$, there is an isomorphism*

$$\widetilde{\operatorname{Ext}}^n_{\mathscr{F}_C}(M,N) \cong \operatorname*{colim}_{i\in\mathbb{N}} \operatorname{H}_{-n}(\operatorname{Hom}_R(F,F'_{\geqslant i})).$$

Proof By Lemma 4.1 (b), $\operatorname{Hom}_R(C,F) \xrightarrow{\simeq} \operatorname{Hom}_R(C,M)$ and $\operatorname{Hom}_R(C,F') \xrightarrow{\simeq} \operatorname{Hom}_R(C,N)$ are proper flat resolutions of $\operatorname{Hom}_R(C,M)$ and $\operatorname{Hom}_R(C,N)$, respectively. Thus, we have

$$\begin{aligned}\widetilde{\operatorname{Ext}}^n_{\mathscr{F}_C}(M,N) &\cong \widetilde{\operatorname{Ext}}^n_{\mathscr{F}}(\operatorname{Hom}_R(C,M),\operatorname{Hom}_R(C,N))\\ &\cong \operatorname*{colim}_{i\in\mathbb{N}} \mathrm{H}_{-n}(\operatorname{Hom}_R(\operatorname{Hom}_R(C,F),\operatorname{Hom}_R(C,F')_{\geqslant i}))\\ &= \operatorname*{colim}_{i\in\mathbb{N}} \mathrm{H}_{-n}(\operatorname{Hom}_R(\operatorname{Hom}_R(C,F),\operatorname{Hom}_R(C,F'_{\geqslant i})))\\ &\cong \operatorname*{colim}_{i\in\mathbb{N}} \mathrm{H}_{-n}(\operatorname{Hom}_R(C\otimes_R \operatorname{Hom}_R(C,F),F'_{\geqslant i}))\\ &\cong \operatorname*{colim}_{i\in\mathbb{N}} \mathrm{H}_{-n}(\operatorname{Hom}_R(F,F'_{\geqslant i})),\end{aligned}$$

where the first isomorphism follows from Lemma A.1, the second one follows from Proposition A.4, and the last one holds since F is a complex of C-flat R-modules. □

Dually, we have the following result.

Proposition A.9 *Let M and N be R-modules with proper $\mathscr{I}_C$-coresolutions $M\to I$ and $N\to I'$, respectively. For every $n\in\mathbb{Z}$, there is an isomorphism*

$$\widetilde{\operatorname{Ext}}^n_{\mathscr{I}_C}(M,N)\cong \operatorname*{colim}_{i\in\mathbb{N}} \mathrm{H}_{-n}(\operatorname{Hom}_R(I_{\leqslant -i},I')).$$

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